- $f(x)$ DL en ~~b~~ à l'ordre n.

$$f(x) = a_0 + a_1(x-b) + a_2 \cdot (x-b)^2 + a_3 \cdot (x-b)^3 + \cdots + a_n (x-b)^n + o_b((x-b)^n)$$

si $b=0$

$$f(x) = a_0 + a_1 \cdot x + a_2 x^2 + a_3 x^3 + \cdots + a_n \cdot x^n + o_0(x^n)$$

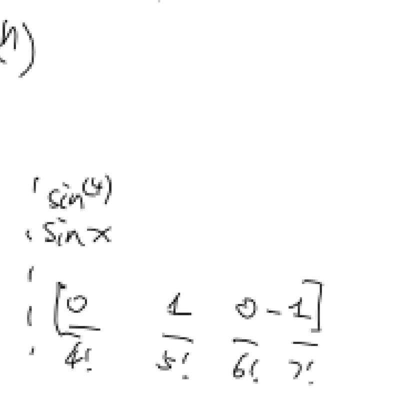
- Taylor-Young $\Rightarrow$ $a_n = \frac{f^{(n)}(b)}{n!}$ où $n! = n \times (n-1) \times \cdots \times 2 \times 1$

$f^{(n)}$ est la n-ième dérivée

$$f(x) = f(b) + f'(b)(x-b) + \cdots$$

DL à 0 jusqu'à l'ordre n.
$\downarrow$
$DL_n(0)$

si $b=0$ $a_n = \frac{f^{(n)}(0)}{n!}$



- $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^6)$.

$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^5)$.

$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots + \frac{1}{n!}x^n + o_0(x^n)$

(TY) $a_n = \frac{f^{(n)}(0)}{n!}$

$\sin x$ $\quad \sin^{(n)}(x)\big|_{x=0}$

	$\sin x$	$\sin'$ $\cos x$	$\sin''$ $-\sin x$	$\sin^{(3)}$ $-\cos x$	$\sin^{(4)}$ $\sin x$			
$x=0$ $f^{(n)}(0)$	$\frac{0}{0!}$	$\frac{1}{1!}$	$\frac{0}{2!}$	$\frac{-1}{3!}$	$\frac{0}{4!}$	$\frac{1}{5!}$	$\frac{0}{6!}$	$\frac{-1}{7!}$
	/	$1 \cdot x$	/	$-\frac{1}{6}x^3$	/	$\frac{1}{120}x^5$		

f

(TY) $\Rightarrow$ DL

$\tan(x)$	$\tan'$	$\tan''$	$\tan'''$		
	$\frac{1}{\cos^2(x)}$	$(-2)\frac{-\sin}{\cos^3} = \frac{2\sin}{\cos^3}$ ($\cos x$)	$\frac{2\sin}{\cos^3} = 2\left(\frac{\cos \cdot \cos^3 - \sin 3\cos^2(-\sin)}{\cos^6}\right) = 2\left(\frac{\cos^4 + 3\sin^2\cos^2}{\cos^6}\right) = \boxed{2\left(\frac{\cos^2 + 3\sin^2}{\cos^4}\right)}$		
0	1	0	2	$\tan^{(4)}$	$\tan^{(5)}$

$x=0$

$$DL_3(0)(\tan x) = \cancel{0} + 1x + \cancel{\frac{0}{2!}x^2} + 2 \cdot \frac{1}{3!}x^3 + o_0(x^3)$$
$$= x + \frac{1}{3}x^3 + o_0(x^3) \qquad \frac{2}{3!} = \frac{2}{6} = \frac{1}{3}$$

$\tan x = \frac{\sin x}{\cos x} =$

- $\sin x$ $\cos x$ on fait par calculer directement les dérivées.
- $\tan x$
  - on fait pareil.

$\tan x$ $\quad \tan' x = \frac{1}{\cos^2 x}$ $\quad \tan'' x = \frac{2\sin}{\cos^3}$ $\quad \tan''' x = \frac{2(\cos^2 + 3\sin^2)}{\cos^4}$

on met $x=0$.

$0 \qquad 1 \qquad 0 \qquad 2 \qquad \cdots$

$$DL_3(0)(\tan x) = 0 + 1 \cdot \frac{x}{1!} + 0 \cdot \frac{x^2}{2!} + 2 \cdot \frac{x^3}{3!} + o_0(x^3)$$
$$= x + \frac{1}{3}x^3 + o_0(x^3) \qquad \frac{2}{3!} = \frac{2}{6} = \frac{1}{3}$$

- (Methode alternative)

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^5)}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^5)}$$

$\frac{A}{1-B} = A(1 + B + B^2 + B^3 \cdots)$

$B = \frac{1}{2}x^2 - \frac{1}{24}x^4$

$$= \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^5)\right) \cdot \left(1 + B + B^2 + \cancel{B^3} + o(x^5)\right)$$
$$= (\qquad\qquad) \left(1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \left(\frac{1}{2}x^2 - \frac{1}{24}x^4\right)^2 + o(x^5)\right)$$
$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^5)$$

$B = \frac{1}{2}x^2 - \frac{1}{24}x^4$
$\overset{0}{\sim} \frac{1}{2}x^2$
$B \sim \frac{1}{2}x^2$
$B^3 \sim \left(\frac{1}{2}x^2\right)^3$
$\parallel$
$\frac{1}{8}x^6$
$= o_0(x^5)$

- e^x $\quad e^x \quad (e^x)' = e^x \quad \cdots\cdots$

$1 \qquad 1 \qquad 1 \qquad 1 \qquad 1$

$f(x)^{g(x)}$
$\parallel$
$e^{\ln f(x) \cdot g(x)}$

$$DL_n(0)(e^x) = 1 + 1 \cdot x + 1 \cdot \frac{x^2}{2!} + 1 \cdot \frac{x^3}{3!} + 1 \cdot \frac{x^4}{4!} + \cdots + 1 \cdot \frac{x^n}{n!} + o_0(x^n)$$

- $\frac{1}{1+x}$ $\quad \frac{1}{1-x}$ $\qquad \boxed{\frac{1}{1-x}} = 1 + x + x^2 + x^3 + \cdots + x^n + o_0(x^n)$.

- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)$.

$\ln(1+x)$ $\quad (\ln(1+x))' = \frac{1}{1+x}$ $\quad f^{(2)} = -\frac{1}{(1+x)^2}$ $\quad f^{(3)} = -(-2) \cdot \frac{1}{(1+x)^3}$

$f^{(4)} = -(-2) \cdot (-3) \cdot \frac{1}{(1+x)^4}$

$f^{(5)} = (-1) \cdot (-2) \cdot (-3) \cdot (-4) \cdot \frac{1}{(1+x)^5}$

on met $x=0$

0	1	2	3	4	5	
$\ln 1 = 0$,	1,	(-1),	$(-1)\cdot(-2)$,	$(-1)\cdot(-2)\cdot(-3)$,	$(-1)(-2)(-3)(-4)$,	$\cdots$

$$f^{(n)}(0) = (-1)(-2)\cdots(-(n-1))$$
$$= (-1)^{n-1}\, 1 \cdot 2 \cdot 3 \cdots (n-1).$$

$$a_n \cdot x^n = \frac{f^{(n)}(0)}{n!} \cdot x^n$$
$$= (-1)^{n-1} \frac{\cancel{1 \cdot 2 \cdot 3 \cdots (n-1)}}{\cancel{1 \cdot 2 \cdot 3 \cdots (n-1)}\, n} \cdot x^n = (-1)^{n-1} \cdot \frac{x^n}{n}$$

- "$\ln(1+x)$ $\Longleftrightarrow$ $e^y - 1$" $\quad$ ln et exp $\quad \ln(1+x) \Leftrightarrow e^y - 1$ $\qquad e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24}$

si je note

$y = \ln(1+x) \Rightarrow e^y = 1 + x$, $\quad x = \boxed{e^y - 1} = y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} + \cdots$

je cherche le DL pour $\ln(1+x)$

je cherche les a_0, a_1, a_2

$$y = \ln(1+x) = a_0 + a_1 \cdot x + a_2 x^2 + a_3 x^3 + o(x^3)$$
$$= a_0 + a_1\left(y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} + \cdots\right)$$
$$+ a_2\left(y + \frac{y^2}{2} + \cdots\right)^2 \left[= a_2\left(y^2 + \boxed{2y \cdot \frac{y^2}{2}} + \cdots\right)\right]$$
$$+ a_3 \cdot \left(y + \frac{y^2}{2} + \frac{y^3}{6} + \cdots\right)^3 + \cdots$$

$$y = a_0 + a_1 y + y^2\left(\frac{a_1}{2} + a_2\right) + y^3\left(\frac{a_1}{6} + a_2 + a_3\right) + \cdots$$

$a_0 = 0 \qquad a_1 = 1 \qquad \frac{a_1}{2} + a_2 = 0 \qquad \frac{a_1}{6} + a_2 + a_3 = 0 \qquad \cdots$

$a_2 = -\frac{1}{2} \qquad \frac{1}{6} + \left(-\frac{1}{2}\right) + a_3 = 0 \quad \boxed{a_3 = \frac{1}{3}}$

$$a_0 + a_1\left(y + \frac{y^2}{2} + \frac{y^3}{6}\right)$$
$$+ a_2\left(y + \frac{y^2}{2} + \frac{y^3}{6}\right)^2 + a_3\left(y + \frac{y^2}{2} + \frac{y^3}{6}\right)^3$$

$$y = a_0 + a_1 y + \frac{a_1}{2}y^2 + \frac{a_1}{6}y^3$$
$$a_2 y^2 + a_2 \cdot 2y \cdot \frac{y^2}{2} + \cancel{a_2\left(\left(\frac{y^2}{2}\right)^2 + 2y \cdot \frac{y^3}{6}\right)} + \cdots$$
$$a_3 y^3 + \cancel{a_3\, 3y^2 \cdot \frac{y^2}{2}} + \cdots$$

$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$
$0 \qquad 1 \qquad 0 \qquad 0 \qquad \cdots$

$\rightsquigarrow$ on peut maintenant résoudre tous les $a_0, a_1, a_2, \cdots$

on a donc calculé le $DL_n(0)(\ln(1+x))$

Ex 18 ① sin/cos/tan

② $\frac{x}{\sin x}$ $\quad DL_5(0)$

$$\frac{x}{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^6)} = \frac{1}{1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + o(x^5)}$$

$B = \frac{1}{6}x^2 - \frac{1}{120}x^4 + o(x^5)$ $\qquad B \sim \frac{1}{6}x^2 \quad B^3 \sim \frac{1}{6^3}x^6 = o(x^5)$

$\left(\frac{1}{1-B}\right)$
$\parallel$
$1 + B + B^2 + B^3$

$$= 1 + B + B^2 + \cancel{B^3} + o(x^5)$$
$$= 1 + \left(\frac{1}{6}x^2 - \frac{1}{120}x^4\right) + \left(\boxed{\frac{1}{6}x^2} - \frac{1}{120}x^4\right)^2 + o(x^5)$$

$\frac{7}{360}$ $\qquad \frac{1}{36}x^4 + o(x^5)$ $\qquad (A-B)^2 = A^2 - 2AB + B^2$

$$= 1 + \frac{1}{6}x^2 + \left(\frac{1}{36} - \frac{1}{120}\right)x^4 + o(x^5)$$

③ $\sin(x + x^3)$ $\quad DL_5(0)$ $\qquad (y = x + x^3 \overset{0}{\sim} x)$

on est au voisinage de 0
$x^{n+k} = o(x^n)$ pour $k > 0$

$\sin y = y - \frac{y^3}{6} + \frac{y^5}{120} + o(y^5)$

$$\sin(x + x^3) = (x + x^3) - \frac{\boxed{(x+x^3)^3}}{6} + \frac{(x+x^3)^5}{120} + o(x^5)$$

$\lim \frac{x^3}{x} = \lim x^2 = 0$
$\Rightarrow x^3 = o(x)$

$x = o_0(1)$
$x^2 = o_0(x) = o_0(1)$
$x^3 = o_0(x^2) = o_0(x)$

$=$

$(A+B)^3$
$= A^3 + 3A^2B + 3AB^2 + B^3$

$(A+B)^5$
$= A^5 + 5A^4B + 10A^3B^2$
$+ 10A^2B^3 + 5AB^4 + B^5$

$(x + x^3)^3$
$= x^3 + 3 \cdot x^2 \cdot x^3 + \cancel{3x(x^3)^2 + (x^3)^3}$
$= x^3 + 3x^5 + o(x^5)$

$(x + x^3)^5$
$= x^5 + \cancel{5x^4 \cdot x^3 + 10x^3 \cdot x^6 \cdots}$
$= x^5 + o(x^5)$

$$= x + x^3 - \frac{1}{6}(x^3 + 3x^5) + \frac{1}{120}x^5 + o(x^5)$$
$$= x + \frac{5}{6}x^3 - \boxed{\frac{59}{120}}x^5 + o(x^5)$$

④ $\ln\left(\frac{\sin x}{x}\right) = \ln\left(\frac{x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)}{x}\right)$

$1 + u$

$\ln(1+u) = u - \frac{u^2}{2} + \cancel{\frac{u^3}{3} - \frac{u^4}{4} + o(u^4)}$
$+ o(u^2)$

$$= \ln\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + o(x^4)\right)$$

$u = -\frac{x^2}{6} + \frac{x^4}{120} + \boxed{o(x^4)}$ $\qquad u \sim -\frac{x^2}{6}$, $u^2 \sim \frac{1}{36}x^4$, $u^3 = o(x^4)$

$$= u - \frac{u^2}{2} + o(u^2)$$

$o(x^4)\left(a_0 + a_1 x + a_2 x^2\right) = a_0 \cdot o(x^4) \,[= o(x^4)]$
$+ a_1 x\, o(x^4) \,[= o(x^5) = o(x^4)]$
$+ a_2 x^2 \cdot o(x^4) \,[= o(x^6) = o(x^4)]$

$$= -\frac{x^2}{6} + \frac{x^4}{120} - \frac{\left(-\frac{x^2}{6} + \frac{x^4}{120}\right)^2}{2} + o(x^4)$$
$$= -\frac{x^2}{6} + \frac{x^4}{120} - \frac{\frac{1}{36}x^4}{2} + o(x^4)$$
$$= -\frac{1}{6}x^2 - \frac{1}{180}x^4 + o(x^4)$$

Ex 20 ① $\lim_{x \to 0} \frac{\boxed{\cos(\sin x) - \cos x}}{x^4}$

$0 + 0x + 0x^2 + 0 \cdot x^3 + l \cdot x^4 + o(x^4)$

$4x^3 \quad 12x^2 \quad 24x \quad \frac{0}{24}$ $\qquad$ (règle de l'Hôpital)

$\cos y = 1 - \frac{1}{2}y^2 + \frac{1}{24}y^4 + o(y^4)$

$\sin x = x - \frac{1}{6}x^3 + o(x^4)$ $\qquad y = x - \frac{1}{6}x^3 + o(x^4) \sim x$

$$\cos(\sin x) = 1 - \frac{1}{2}\left(x - \frac{1}{6}x^3\right)^2 + \frac{1}{24}\left(x - \frac{1}{6}x^3\right)^4 + o(x^4)$$
$$= 1 - \frac{1}{2}x^2 + \boxed{\frac{1}{6}x^4} + \frac{1}{24}x^4 + o(x^4)$$

$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)$

$\lim_{x \to 0} \frac{\frac{1}{6}x^4 + o(x^4)}{x^4} = \frac{1}{6}$

$\cos(\sin x) - \cos x = \frac{1}{6}x^4 + o(x^4)$

$\lim_{x \to 0} \frac{\frac{1}{6}x^4 + o(x^4)}{x^4} = \frac{1}{6} + \lim_{x \to 0} \frac{o(x^4)}{x^4} = \frac{1}{6}$