Ex 20 ② $\lim\limits_{x\to 1} \dfrac{\sqrt{3x-2} - x^{1/4}}{1-x^{2/3}}$

je note $x = 1+y$ comme $x \to 1$, $y \to 0$

$\sqrt{3x-2} = \sqrt{3(1+y)-2} = \sqrt{3+3y-2} = (1+3y)^{1/2}$

$x^{1/4} = (1+y)^{1/4}$ $\qquad (1+y)^{\alpha} =$

$x^{2/3} = (1+y)^{2/3}$

$(1+3y)^{1/2} = 1 + 1/2 \cdot (3y) + \dfrac{1/2 \times (1/2 - 1)}{2\times 1}(3y)^2 + o_0(y^2)$ $\quad + o_0(y)$

$(1+y)^{1/4} = 1 + 1/4 \cdot y + \dfrac{1/4\times(1/4-1)}{2\times 1} y^2 + o_0(y^2)$

$(1+y)^{2/3} = 1 + 2/3 \cdot y + \dfrac{2/3 \times (2/3 - 1)}{2\times 1} y^2 + o_0(y^2)$

$\dfrac{(1+3y)^{1/2} - (1+y)^{1/4}}{1-(1+y)^{2/3}} = \dfrac{(\cancel{1} + \frac{3}{2}y) - (\cancel{1} + \frac{y}{4}) + o_0(y)}{\cancel{1} - (\cancel{1} + \frac{2}{3}y) + o_0(y)}$

$= \dfrac{(\frac{3}{2} - \frac{1}{4})y + o(y)}{-\frac{2}{3}y + o(y)}$ $\qquad \dfrac{\frac{3}{2}-\frac{1}{4}}{-\frac{2}{3}} = -\dfrac{15}{8}$

$\lim\limits_{y\to 0} \dfrac{(\frac{3}{2}-\frac{1}{4})y + o_0(y)}{-\frac{2}{3}y + o_0(y)} = \lim\limits_{y\to 0} \dfrac{(\frac{3}{2}-\frac{1}{4}) + \frac{o_0(y)}{y}}{-\frac{2}{3} + \frac{o_0(y)}{y}}$

comme $\lim\limits_{y\to 0} \dfrac{o_0(y)}{y} = 0$

on obtient $\dfrac{\frac{3}{2}-\frac{1}{4}}{-\frac{2}{3}} = -\dfrac{15}{8}$

le DL de $(1+y)^{\alpha}$ pour $\alpha \in \mathbb{R}$ en $y=0$

$(1+y)^{\alpha} = 1 + \alpha \cdot y + \dfrac{\alpha(\alpha-1)}{2!} y^2 + \dfrac{\alpha(\alpha-1)(\alpha-2)}{\underbrace{3\times 2\times 1}_{3!}} y^3 + \dfrac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{\underbrace{4\times 3\times 2\times 1}_{4!}} y^4 + \dots + o(y^n)$

- $\alpha \in \mathbb{N}$ $\quad (1+y)^2 = 1 + 2y + y^2$

  $(1+y)^3 = 1 + 3y + 3y^2 + y^3$

  $(1+y)^4 = 1 + 4y + 6y^2 + 4y^3 + y^4 = \sum\limits_{k=0}^{4} \binom{4}{k} y^k$ $\qquad \binom{4}{k} = \dfrac{4(4-1)\cdots(4-(k-1))}{k\times(k-1)\cdots\times 1}$

- $\alpha \in \mathbb{R}$

  $f(y) = (1+y)^{\alpha}$ $\qquad f'(y) = \alpha \cdot (1+y)^{\alpha-1}$

  $f^{(2)}(y) = \alpha(\alpha-1)(1+y)^{\alpha-2}$

  $f^{(3)}(y) = \alpha(\alpha-1)(\alpha-2)(1+y)^{\alpha-3} \dots$

on peut mettre $y=0$

$\Rightarrow f(0) = 1 \quad f'(0) = \alpha \quad f''(0) = \alpha(\alpha-1) \quad f^{(3)}(0) = \alpha(\alpha-1)(\alpha-2)$

④ $\lim\limits_{x\to 1} \dfrac{x^x - x}{(x-1)^2} \leftarrow 1 - 2x + x^2$ $\qquad (x^x)' = ((e^{\ln x})^x)' \quad (x = e^{\ln x})$ $\quad = (e^{x \cdot \ln x})'$

$x^x - x = x\cdot(x^{x-1} - 1) = x\cdot(e^{\ln x \cdot (x-1)} - 1)$

$x = 1+y$ avec $y \to 0$

$\dfrac{x^x - x}{(x-1)^2} = \dfrac{(1+y)(e^{\ln(1+y)\cdot y} - 1)}{y^2} \overset{0}{\sim} \dfrac{e^{\ln(1+y)\cdot y} - 1}{y^2}$ $\qquad = a_0 + a_1 y + a_2 y^2$

$e^{\ln(1+y)\cdot y} - 1$ $\qquad \ln(1+y) = y - \dfrac{y^2}{2} + \dfrac{y^3}{3} + o(y^3)$ $\qquad e^u - 1 = (1 + u + \frac{u^2}{2} + \dots) - 1 = u + \frac{u^2}{2} + \dots$

$\ln(1+y) = y + o(y)$ $\qquad e^u - 1 \sim u$

$e^{\ln(1+y)\cdot y} - 1 = e^{(y+o(y))\cdot y} - 1 = e^{y^2 + o(y^2)} - 1$ $\qquad \ln(1+y) \sim y$

$(e^u = 1 + u + o(u))$ $\quad = 1 + (y^2 + o(y^2)) + o(y^2) - 1$ $\qquad e^{\ln(1+y)\cdot y} - 1 \sim \ln(1+y)\cdot y \sim y^2$

$= y^2 + o(y^2)$ $\qquad y^2 + o(y^2)$

$\lim\limits_{x\to 1} \dfrac{x^x - x}{(x-1)^2} \overset{(x=1+y)}{=} \lim\limits_{y\to 0} \dfrac{\cancel{(1+y)}((1+y)^y - 1)}{y^2}$

$= \lim\limits_{y\to 0} \dfrac{e^{\ln(1+y)\cdot y} - 1}{y^2}$ $\qquad e^u - 1 \overset{0}{\sim} u$ $\qquad \ln(1+y) \overset{0}{\sim} y$

$e^{\ln(1+y)\cdot y} - 1 \overset{0}{\sim} \ln(1+y)\cdot y \overset{0}{\sim} y^2$

$= 1$

⑤ $\boxed{\lim\limits_{x\to 0} \dfrac{\ln\cos x + \operatorname{sh}^2 x/2}{\sin^4 x}}$ $\qquad a + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$ $\qquad x^4$

$\lim \dfrac{0}{\cancel{\sin x}}$ $\quad \dfrac{\sin x}{x}$ $\quad \sin x \overset{0}{\sim} x$

$\operatorname{sh} x = \dfrac{e^x - e^{-x}}{2} = \dfrac{1}{2}\Big((\cancel{1} + x + \cancel{\tfrac{x^2}{2}} + \tfrac{x^3}{6} + \cancel{\tfrac{x^4}{24}} + \dots)$
$\qquad - (\cancel{1} + (-x) + \cancel{\tfrac{(-x)^2}{2}} + \tfrac{(-x)^3}{6} + \cancel{\tfrac{(-x)^4}{24}} + \dots)\Big)$

$= \dfrac{1}{2}\left(2x + 2\cdot\dfrac{x^3}{6} + 2\dfrac{x^5}{5!} + 2\dfrac{x^7}{7!} + \dots\right)$

$= x + \dfrac{x^3}{6} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + o_0(x^7)$ $\qquad (0,1,0,1,0,1 \dots)$

$\sin x = x - \dfrac{x^3}{6} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + o_0(x^7)$ $\qquad (0,1,0,-1 \,|\, 0,1,0,-1 \,|\, \dots)$

$\operatorname{ch} x = \dfrac{e^x + e^{-x}}{2} = 1 + \dfrac{x^2}{2} + \dfrac{x^4}{24} + \dfrac{x^6}{6!} + o_0(x^6)$ $\qquad (1,0,1,0,1,0,\dots)$

$\cos x = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \dfrac{x^6}{6!} + o_0(x^6)$ $\qquad (1,0,-1,0 \,|\, 1,0,-1,0 \,|\, \dots)$

$\ln\cos x + \operatorname{sh}^2 x/2 = a_0 + \dots + a_4 x^4 + o_0(x^4)$

$\ln\cos x = \ln\left(1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} + o(x^4)\right)$ $\qquad \ln(1+u) = u - \dfrac{u^2}{2} + \left(\dfrac{u^3}{3} + o(u^3)\right)$ $\quad o(u^2)$

$u = -\dfrac{x^2}{2} + \dfrac{x^4}{24} + o(x^4) \overset{0}{\sim} -\dfrac{x^2}{2}$ $\qquad o(u) = o(x^2)$ $\quad o(u^2) = o(x^4)$

$= u - \dfrac{u^2}{2} + o(u^2)$

$= \left(-\dfrac{x^2}{2} + \dfrac{x^4}{24} + o(x^4)\right) - \dfrac{1}{2}\left(-\dfrac{x^2}{2} + \dfrac{x^4}{24} + o(x^4)\right)^2 + o(x^4)$

$= -\dfrac{x^2}{2} + \dfrac{x^4}{24} - \dfrac{1}{2}\left(\dfrac{x^4}{4} + o(x^4)\right) + o(x^4)$

$= -\dfrac{x^2}{2} - \dfrac{1}{12}x^4 + o(x^4)$

$\operatorname{sh}^2 x/2 = \dfrac{1}{2}\left(x + \dfrac{x^3}{6} + o(x^4)\right)^2 = \dfrac{1}{2}\left(x^2 + 2x\cdot\dfrac{x^3}{6} + \cancel{\left(\tfrac{x^3}{6}\right)^2} + o(x^4)\right)$

$= \dfrac{1}{2}x^2 + \dfrac{1}{6}x^4 + o(x^4)$

$\ln\cos x + \operatorname{sh}^2 x/2 = \left(-\dfrac{1}{2} + \dfrac{1}{2}\right)x^2 + \left(-\dfrac{1}{12} + \dfrac{1}{6}\right)x^4 + o(x^4) = \dfrac{1}{12}x^4 + o(x^4)$

Donc $\lim\limits_{x\to 0} \dfrac{\ln\cos x + \operatorname{sh}^2 x/2}{\sin^4 x} = \lim\limits_{x\to 0} \dfrac{\frac{1}{12}x^4 + o(x^4)}{x^4 + o(x^4)} = \dfrac{1}{12}$

$\sin^4 x \overset{0}{\sim} x^4$ $\qquad \sin x \overset{0}{\sim} x$ $\quad \sin^4 x \overset{0}{\sim} x^4$

$\dfrac{\bigcirc}{\sin x} \sim \dfrac{\bigcirc}{e^x - 1} \sim \dfrac{\bigcirc}{\ln(1+x)} \sim \dfrac{\bigcirc}{x}$

⑥ $\lim\limits_{x\to 0} \dfrac{2\tan x - \operatorname{sh} 2x}{(1-\cos 3x)\arctan x}$ $\quad$ on peut remplacer par x $\qquad x = \arctan y$ $\qquad \sin x \overset{0}{\sim} x \overset{0}{\sim} e^x - 1 \overset{0}{\sim} \ln(1+x)$

$= \lim\limits_{x\to 0} \dfrac{2\tan x - \operatorname{sh} 2x}{x\cdot(1-\cos 3x)}$ $\qquad \tan x \overset{0}{\sim} x$ $\qquad \tan(\arctan y) = y \overset{0}{\sim} \arctan y$

$\left(\tan x = \dfrac{\sin x}{\cos x},\ \sin x \overset{0}{\sim} x,\ \cos x \overset{0}{\sim} 1\right)$

$\lim\limits_{x\to 0} \dfrac{\tan x}{x} = \lim\limits_{x\to 0} \dfrac{\sin x}{x}\cdot\dfrac{1}{\cos x} = 1$

$\tan x = x + \frac{1}{3}x^3 + \dots = x + o_0(x)$

$\cos 3x = 1 - \dfrac{1}{2}(3x)^2 + o(x^2)$

$\cos u = 1 - \dfrac{u^2}{2} + o(u^2)$

$1 - \cos 3x = \dfrac{1}{2}(3x)^2 + o(x^2) = \dfrac{9}{2}x^2 + o(x^2)$ $\qquad \lim\limits_{x\to 0} \dfrac{2\tan x - \operatorname{sh} 2x}{\frac{9}{2}x^3 + o(x^3)}$

$2\tan x = 2\left(x + \dfrac{x^3}{3} + o(x^3)\right) = 2x + \dfrac{2}{3}x^3 + o(x^3)$ $\qquad 2\tan x - \operatorname{sh} 2x = -\dfrac{2}{3}x^3 + o(x^3)$

$\operatorname{sh} 2x = 2x + \dfrac{(2x)^3}{6} + o(x^3) = 2x + \dfrac{4}{3}x^3 + o(x^3)$ $\qquad \lim\limits_{x\to 0} \dfrac{-\frac{2}{3}x^3 + o(x^3)}{\frac{9}{2}x^3 + o(x^3)} = \dfrac{-\frac{2}{3}}{\frac{9}{2}} = -\dfrac{4}{27}$

$\operatorname{sh} u = u + \dfrac{u^3}{6} + o(u^3)$

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le DL(0) de $\arctan x$ $\left(\overset{0}{\sim} x \quad x + o(x)\right)$. $\qquad \dfrac{1}{1-u} = 1 + u + u^2 + o(u^2)$

- $\arctan x$ $\qquad (\arctan x)' = \dfrac{1}{1+x^2} \overset{u=(-x^2)}{=} 1 + (-x^2) + x^4 + o(x^4)$

  $= 1 - x^2 + x^4 + o(x^4)$ $\qquad$ on intégrale comme si c'était un polynôme

  $\arctan x = x - \dfrac{1}{3}x^3 + \dfrac{1}{5}x^5 + o(x^5)$

- $\tan x = x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + o(x^5)$ $\qquad y = \arctan x \overset{0}{\sim} x$

$y = \tan(\arctan y) = \arctan y + \dfrac{\arctan y^3}{3} + \dfrac{2}{15}\arctan y^5 + o(y^5)$

$\arctan y = y + a_2 y^2 + a_3 y^3 + a_4 y^4$

$(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$, $\qquad (1-x)^{-1/2}$

Trouver a, b t.q.

Ex 19 $\quad f(x) = \cos x - \dfrac{1+ax^2}{1+bx^2}$ $\qquad f(x) = 0 + 0x + 0x^2 + \dots + 0x^5 + o(x^5)$

$\cos x = 1 - \dfrac{1}{2}x^2 + \dfrac{1}{24}x^4 + o(x^5)$ $\qquad \dfrac{1+ax^2}{1+bx^2} = (1+ax^2)\cdot\dfrac{1}{1+bx^2}$

$\dfrac{1+ax^2}{1+bx^2} = (1+ax^2)(1 - bx^2 + (bx^2)^2 + o(x^5))$ $\qquad \dfrac{1}{1+u} = 1 - u + u^2 + o(u^2)$

$= (1 + ax^2) - (bx^2 + abx^4) + b^2x^4 + o(x^5)$ $\qquad \dfrac{1}{2+u} = \dfrac{1}{2}\cdot\dfrac{1}{1+\frac{u}{2}}$

$= 1 + (a-b)x^2 + (b^2 - ab)x^4 + o(x^5)$.

$\cos x - \dfrac{1+ax^2}{1+bx^2} = 0 + \left(-\dfrac{1}{2} - (a-b)\right)x^2 + \left(\dfrac{1}{24} - (b^2 - ab)\right)x^4 + o(x^5)$

$\begin{cases} a - b = -\frac{1}{2} \\ b^2 - ab = \frac{1}{24} \end{cases} \Rightarrow \begin{cases} b - a = \frac{1}{2} \\ b(b-a) = \frac{1}{24} \end{cases} \Rightarrow \begin{cases} a = b - \frac{1}{2} = -\frac{5}{12} \\ b = \frac{1}{12} \end{cases}$

Ex 26 $\quad f(x) = e^{\cos x} \overset{(TY)}{=} f(0) + f'(0)\cdot x + f''(0)\cdot\dfrac{x^2}{2} + f^{(3)}(0)\dfrac{x^3}{6} + f^{(4)}(0)\dfrac{x^4}{24} + o(x^4)$
$\qquad\qquad e \qquad 0 \qquad -\frac{e}{2} \qquad 0 \qquad \frac{e}{6}$

$\| \quad a_0 + a_1 x + a_2 x^2 + \dots + o(x^4)$ $\quad \rightsquigarrow$ on obtient ainsi les $f^{(k)}(0)$

$\begin{cases} f(0) = e \\ f'(0) = 0 \\ f''(0) = -e \\ f^{(3)}(0) = 0 \\ f^{(4)}(0) = 4e \end{cases}$

$\boxed{e^{\cos x}} = e^{1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)}$

$f'(x) = e^{\cos x}\cdot(-\sin x)$ $\qquad = e^1 \cdot e^{-\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)}$ $\qquad e^u = 1 + u + \dots$ est valable si $u \to 0$.

$f''(x) = \dots$ $\qquad \left(u = -\dfrac{x^2}{2} + \dfrac{x^4}{24} + o(x^4)\right)$ $\qquad u \overset{0}{\sim} -\dfrac{x^2}{2}$ $\quad o(u^2) = o(x^4)$

$= e\cdot\left(1 + u + \dfrac{u^2}{2} + o(u^2)\right)$

$= e\left(1 + \left(-\dfrac{x^2}{2} + \dfrac{x^4}{24}\right) + \dfrac{x^4}{8} + o(x^4)\right)$

$= e\left(1 - \dfrac{x^2}{2} + \dfrac{1}{6}x^4 + o(x^4)\right) = e - \dfrac{e}{2}x^2 + \dfrac{e}{6}x^4 + o(x^4)$

Ex 21 $\quad f(x) = (1+x)^{3/2}\cdot\ln(1+x)$

$(1+x)^{\alpha} \qquad \ln(1+x)$

$= \left(1 + 3/2\cdot x + \dfrac{3/2(3/2 - 1)}{2\times 1}x^2 + o(x^2)\right)\left(x - \dfrac{x^2}{2} + o(x^2)\right)$

$= \left(1 + \dfrac{3}{2}x + \dfrac{3}{8}x^2 + o(x^2)\right)\left(x - \dfrac{x^2}{2} + o(x^2)\right)$

$= \left(x + \dfrac{3}{2}x^2 + \cancel{\tfrac{3}{8}x^3}\right) + \left(-\dfrac{x^2}{2} - \cancel{\tfrac{3}{4}x^3}\right) + o(x^2)$

$= x + x^2 + o(x^2)$

on a calculé le DL en 0 $\qquad f(0) = 0 \quad f'(0) = 1$ $\qquad f(x) = a + bx + o(x)$ $\quad f(x) = a + bx + cx^2 + o(x^2)$

- $f(x) = a + bx + o(x) \Rightarrow$ l'équation de la droite tangente
- $f(x) = a + bx + cx^2 + o(x^2)$ $\quad$ le signe de c détermine la position de f par rapport à la droite tangente

si $c > 0$ $\quad cx^2 > 0$

si $c < 0$ $\quad cx^2 < 0$

si $c = 0$