Sharp quantitative Talenti's inequality in particular cases

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Abstract

In this paper, we focus on the famous Talenti's symmetrization inequality, more precisely its L^p corollary asserting that the L^p-norm of the solution to $-\Delta v = f^\sharp$ is higher than the L^p-norm of the solution to $-\Delta u = f$ (we are considering Dirichlet boundary conditions, and $f^\sharp$ denotes the Schwarz symmetrization of $f : \Omega \to \mathbb{R}_+$). We focus on the particular case where functions f are defined on the unit ball, and are characteristic functions of a subset of this unit ball. We show in this case that stability occurs for the L^p-Talenti inequality with the sharp exponent 2.

1 Introduction

In this paper, we will investigate some stability versions of Talenti's inequality in particular cases. Given $\Omega \subset \mathbb{R}^n$ a bounded open set and $f \in L^2(\Omega)$, we will denote by u_f the unique weak solution to

$$\begin{cases} -\Delta u_f = f & \text{in } \Omega, \\ u_f = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The aim of Talenti's inequality is to compare u_f and $u_{f^\sharp}$ where $f^\sharp : \Omega^\sharp \to \mathbb{R}$ is the Schwarz symmetrization of f, defined on $\Omega^\sharp$ the centered ball of same volume as Ω (see Definition 2.3): more precisely Talenti's inequality (Theorem 2.4) states that if $f \geq 0$, then

$$\forall x \in \Omega^\sharp, \;\; u_f^\sharp(x) \leq v(x),$$

where $v = u_{f^\sharp}$ solves

$$\begin{cases} -\Delta v = f^\sharp & \text{in } \Omega^\sharp \\ v = 0 & \text{on } \partial\Omega^\sharp. \end{cases} \tag{1.2}$$

and $u_f^\sharp$ is the Schwarz symmetrization of u_f. This ponctual comparison implies in particular the following: for every $p \in [1, +\infty]$,

$$\|u_f\|_{L^p(\Omega)} \leq \|v\|_{L^p(\Omega^\sharp)}. \tag{1.3}$$

Moreover, as shown in [2] (see Theorem 2.5), equality is realized in (1.3) for some $p \in [1, +\infty]$ only if Ω is a ball and $f = f^\sharp$ up to translations, see also [16].

In this paper we are interested in quantitative versions of (1.3). Notice that there are two parameters that are symmetrized in these inequalities: Ω and f. Therefore the stability inequalities we may be seeking for should take into account both the distance from f to $f^\sharp$ and from Ω to $\Omega^\sharp$: to evaluate the asymmetry of Ω for example, we denote

$$\alpha(\Omega) = \min_{x_0 \in \mathbb{R}^n} \left\{ \frac{|\Omega \Delta B_r(x_0)|}{|B_r|}, \; |B_r| = |\Omega| \right\}$$

the Fraenkel asymmetry of Ω ($B_r(x)$ denotes the ball of radius $r > 0$ and centered at $x \in \mathbb{R}^n$; we also denote $B_r = B_r(0)$ the centered ball of radius r).

As far as we know, there are only two partial results in this direction:

1. in [9], the author focuses on the case $f \equiv 1$ and shows that for every $p \in [1, +\infty]$, there exists $c = c(n, p)$ such that for every Ω bounded open set in $\mathbb{R}^n$,

$$\|v\|^p_{L^p(\Omega^\sharp)} - \|u_1\|^p_{L^p(\Omega)} \geq c\alpha(\Omega)^{2+p} \qquad \text{if } p \in [1, +\infty),$$
$$\|v\|_{L^\infty(\Omega^\sharp)} - \|u_1\|_{L^\infty(\Omega)} \geq c\alpha(\Omega)^3.$$

2. in [3] the authors focus on the case $p = \infty$: it is shown that for any Ω bounded open set of $\mathbb{R}^n$ and any nonnegative $f \in L^2(\Omega)$, there exists $c = c(n, |\Omega|, f^\sharp)$ and $\theta = \theta(n)$ such that

$$\|v - u^\sharp\|_{L^\infty(\Omega^\sharp)} \geq c\left(\alpha(\Omega)^3 + \inf_{x_0 \in \mathbb{R}^n} \left\|f - f^\sharp(\cdot + x_0)\right\|^\theta_{L^1(\mathbb{R}^n)}\right).$$

   Note that the proof of Talenti's inequality shows that $v - u^\sharp$ is radially decreasing, so that $\|v - u^\sharp\|_{L^\infty(\Omega^\sharp)} = \|v\|_{L^\infty(\Omega^\sharp)} - \|u\|_{L^\infty(\Omega)}$.

   In the same paper, the authors also obtain a partial result in the case $p = 2$, namely:

$$\|v\|^2_{L^2(\Omega^\sharp)} - \|u\|^2_{L^2(\Omega)} \geq c\alpha(\Omega)^4.$$

In both of these results, the obtained exponents are not expected to be sharp (also in the second result, θ does not seem to be explicit).

The goal of this paper is to provide a first quantitative result for (1.3) with a sharp exponent. In order to start this investigation with only one parameter (similarly to [9] where the author assumed $f \equiv 1$), we will assume that $\Omega = B_1$ is a centered ball of radius 1, so that $\Omega = \Omega^\sharp$ and the only parameter is f. Moreover, we will assume $f = \chi_E$ to be the characteristic function of a set $E \subset B_1$: in this case we denote u_E for u_{χ_E}, and $f^\sharp = \chi_{B_*}$ where B_* is the centered ball of same volume as E. This will allow us to use a geometric approach and the framework of shape derivatives. In this setting, we obtain the following stability result:

Theorem 1.1. *Let $n \geq 1$, $p \in [1, +\infty]$, and $m \in (0, |B_1|)$. Then there exists $c = c(p, m, n) > 0$ such that for every measurable set $E \subset B_1$ with $|E| = m$ we have*

$$\|u_{B_*}\|_{L^p(B_1)} - \|u_E\|_{L^p(B_1)} \geq c|E\Delta B_*|^2, \tag{1.4}$$

where B_* is the centered ball of volume m.

Moreover, we show in Propositions 2.8, 2.14 and 2.17 that the exponent 2 obtained in Theorem 1.1 is sharp. In a forthcoming work [1] we will generalize the strategy introduced in the present paper to investigate the more general case where f is not assumed to be a characteristic function.

The proof of Theorem 1.1 will be divided in three cases depending on the value of p. In the special case $p = 1$, the proof is quite straightforward and relies only on Talenti's inequality: we provide the proof in Section 2.2, which is based on the study of the following shape optimization problem:

$$\max\left\{ \|u_E\|_1 \;\middle|\; \begin{array}{c} |E| = m, \\ |E\Delta B_*| = \delta \end{array} \right\}$$

for $m \in (0, |B_1|)$ and $\delta > 0$, which can be seen as looking for the worst set of given asymmetry, with regard of the Talenti deficit $\|u_{B_*}\|_1 - \|u_E\|_1$. We identify explicitely the solution of this problem as a target-shaped set (it is the union of a ball and an annulus, see Definition 2.6).

In the case $p > 1$, we were not able to solve explicitly $\max\{\|u_E\|_p, |E| = m, |E\Delta B_*| = \delta\}$, we therefore follow a strategy developed in [18, 19, 5, 6] to prove (1.4). In fact the case $1 < p < \infty$ falls into the following framework: given a function $j : \mathbb{R}_+ \to \mathbb{R}$, we study the stability for the following shape optimization problem:

$$\max\{\,\mathcal{J}(E) \mid |E| = m\,\}, \qquad \mathcal{J}(E) := \int_{B_1} j(u_E(x))dx \tag{1.5}$$

which is solved by the centered ball B_* if j is non-decreasing. Classically, under convexity assumption on j, such problem can be relaxed in the class

$$\mathcal{M}_m := \left\{ V \in L^\infty(B_1) \;\middle|\; \begin{array}{l} 0 \le V \le 1, \\ \int_{B_1} V = m \end{array} \right\}$$

where $m \in (0, |B_1|)$, and we define $\mathcal{J}(V) := \int_{B_1} j(u_V)$ for $V \in L^\infty(B_1)$ non-negative. More precisely we prove:

Theorem 1.2. *Let $n \ge 1$, $m \in (0, |B_1|)$, and $j \in C^1(\mathbb{R}_+) \cap C^2((0, +\infty))$ be such that $j'(0) \ge 0$ and $j''(s) > 0$ for every $s > 0$, and such that*

$$\limsup_{s \to 0^+} s^\alpha j''(s) < +\infty \tag{1.6}$$

for some $\alpha < 1$. Then there exists a positive constant $c = c(j, m, n)$ such that for every $V \in \mathcal{M}_m$ we have

$$\mathcal{J}(B_*) - \mathcal{J}(V) \ge c\|V - \chi_{B_*}\|_1^2.$$

where B_* is the centered ball of volume m.

Remark 1.3. Note that assumption (1.6) implies that j' is locally $\beta := (1 - \alpha)$-Hölder continuous. This follows easily from the equality $j'(y) - j'(x) = \int_x^y j''(t)dt$.

Proof of Theorem 1.1 from Theorem 1.2 when $p \in (1, \infty)$: Let $p \in (1, \infty)$: we consider $j : s \in [0, \infty) \mapsto s^p$, which is C^2 in $(0, \infty)$ and C^1 up to 0, with $j'(0) = 0$ and $j'' > 0$ on $(0, \infty)$, and satisfies (1.6).

Therefore, Theorem 1.2 applies, and there exists $c = c(p, m, n)$ such that

$$\|u_{B_*}\|_p^p \left(1 - \left(\frac{\|u_E\|_p}{\|u_{B_*}\|_p}\right)^p\right) \ge c|E\Delta B_*|^2.$$

Then the result follows by using the inequality

$$1 - x^p \le p(1 - x) \qquad \forall x \ge 0.$$

□

Finally, we give a seperate proof of Theorem 1.1 when $p = \infty$, see below for more details.

Outline of the paper

In the following section, we recall basic definitions on Schwarz symmetrization, and then produce a proof of Theorem 1.1 in the particular case $p = 1$. We then provide useful results to prepare the proof of Theorem 1.2: in particular we prove that under suitable assumptions on j,

$$\mathcal{J}(B_*) = \max\left\{ \mathcal{J}(V) := \int_{B_1} j(u_V(x))dx \;\middle|\; V \in \mathcal{M}_m \right\},$$

that B_* is the unique maximizer, and that for any small $\delta > 0$ there exists E_δ such that $|E_\delta| = m$, $|E_\delta \Delta B_*| = \delta$ and

$$\mathcal{J}(E_\delta) = \max \left\{ \mathcal{J}(V) \;\middle|\; \begin{array}{c} V \in \mathcal{M}_m, \\ \|V - \chi_{B_*}\|_1 = \delta \end{array} \right\}.$$

We also show that the exponent 2 in the stability result of Theorem 1.1 and Theorem 1.2 is sharp.

We then focus on the proof of Theorem 1.2, which is split in the next two sections: in Section 3, we show that the proof of Theorem 1.2 reduces to a Fuglede type of result (Theorem 3.1) asserting that stability occurs for smooth deformations of B_*. This strategy falls into the strategy of "selection principle" first introduced in the setting of shape optimization by Cicalese and Leonardi [8] to prove the quantitative isoperimetric inequality: the main tool here is the quantitative baththub principle proved in [19]. Section 4, the most technical part of the paper, is devoted to the proof of Theorem 3.1. It requires a computation of first and second order shape derivative of the energy (Section 4.1), a proof of the coercivity of the second order optimality condition (Section 4.2), and a continuity property of the second order derivative to control the Taylor expansion (Section 4.3).

In Section 5 we provide the proof of Theorem 1.1 for $p = +\infty$. The proof uses a representation of the solution with the Green function of the ball, and relies on adapting the shape derivative approach of Section 3 and Section 4 to the L^∞ norm. This requires to compute a second order shape derivative of such norm, which up to our knowledge, is new (see [14] for a computation of a first order shape derivative in a similar context).

Finally, we detailed in the appendix some results about the way convergence of functions may imply the convergence of level sets, that did not seem completely new but for which the literature felt a bit elliptic, so we provided precise proofs for these statements used in Section 3.

Acknowledgement: the authors would like to warmly thank Idriss Mazari, for several fruitful discussions about this paper, which in particular lead to a simplification of the proof of Theorem 1.2 in subsection 3.2. This work was also partially supported by the project ANR STOIQUES financed by the French Agence Nationale de la Recherche (ANR). This work was partially supported by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of Istituto Nazionale di Alta Matematica (INdAM). The author Paolo Acampora was partially supported by the 2025 project GNAMPA 2024: "Analisi di Problemi Inversi nelle Equazioni alle Derivate Parziali", CUP_E5324001950001 .

2 Tools and first results

2.1 Symmetrization and Talenti's inequality

We refer the reader to [22] or [16] for an overview about definitions and properties of rearrangements.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $u : \Omega \to \mathbb{R}$ be a measurable function. We define the *distribution function* $\mu_u : [0, +\infty) \to [0, +\infty)$ of u as the function

$$\mu_u(t) = |\{ x \in \Omega \,,\, |u(x)| > t \}|.$$

Definition 2.2. Let $u : \Omega \to \mathbb{R}$ be a measurable function. We define the *decreasing rearrangement* u^* of u as

$$u^*(s) = \inf \{ t > 0 \,,\, \mu_u(t) \leq s \} .$$

Definition 2.3. Let $u : \Omega \to \mathbb{R}$ be a measurable function. We define the *Schwarz rearrangement* $u^\sharp$ of u as

$$u^\sharp(x) = u^*(\omega_n |x|^n) \qquad\qquad x \in \Omega^\sharp,$$

where $\Omega^\sharp$ denotes the centered ball having the same volume as Ω.

The next result is the famous symmetrization result obtained by G. Talenti, see [21, Theorem I].

Theorem 2.4 (Talenti's comparison)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $f \in L^2(\Omega)$ and let $v_f \in H_0^1(\Omega)$ and $v_{f^\sharp} \in H_0^1(\Omega^\sharp)$ be the unique solutions to*

$$\begin{cases} -\Delta u_f &= f & \text{in } \Omega, \\ u_f &= 0 & \text{on } \partial\Omega, \end{cases} \qquad\qquad \begin{cases} -\Delta u_{f^\sharp} &= f^\sharp & \text{in } \Omega^\sharp, \\ u_{f^\sharp} &= 0 & \text{on } \partial\Omega^\sharp. \end{cases}$$

Then

$$u_f^\sharp \leq u_{f^\sharp}.$$

In particular, for every set E, letting $u_E = u_{\chi_E}$ and $u_{E^\sharp} = u_{\chi_{E^\sharp}}$, we have

$$u_E^\sharp \leq u_{E^\sharp}.$$

The following result solves the case of equality in Talenti's inequality and was first proved in [2, Theorem 1].

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and open set, and let $f \in L^2(\Omega)$ a non-negative function such that $f \not\equiv 0$. If $u_f^\sharp = u_{f^\sharp}$ almost everywhere, then there exists $x_0 \in \mathbb{R}^n$ such that up to negligible sets $\Omega = x_0 + \Omega^\sharp$, $f(\cdot) = f^\sharp(\cdot + x_0)$, and $u_f(\cdot) = u_{f^\sharp}(\cdot + x_0)$ almost everywhere.*

2.2 Proof of Theorem 1.1 for $p = 1$

In the case $p = 1$, the proof of Theorem 1.1 is quick and relies only of Talenti's symmetrization result. It also shows the importance of the following annulus.

Definition 2.6. Let $m \in (0, |B_1|)$. For every $\delta \in (0, \min\{2m, 2(|B_1| - m)\})$ we define the *optimal δ-asymmetric radial open set* A_δ as

$$A_\delta = B_{r_1(\delta)} \cup \{\, x \in \mathbb{R}^n \mid r_* < |x| < r_2(\delta) \,\},$$

where r_* is the radius of B_* the ball of volume m, and $r_1(\delta)$ and $r_2(\delta)$ are chosen in such a way that

$$|A_\delta \Delta B_*| = \delta.$$

Namely, we have

$$r_1(\delta) = \frac{1}{|B_1|^{1/n}} \left(m - \frac{\delta}{2} \right)^{\frac{1}{n}} \qquad \text{and} \qquad r_2(\delta) = \frac{1}{|B_1|^{1/n}} \left(m + \frac{\delta}{2} \right)^{\frac{1}{n}}.$$

The next result shows that among sets E of given volume and such that $|E \Delta B_*| = \delta$, A_δ is the worst set with regard to the deficit $\|u_{B_*}\|_1 - \|u_E\|_1$.

Proposition 2.7. *Let $j : s \in \mathbb{R}_+ \mapsto s$, i.e. for every measurable set $E \subset B_1$,*

$$\mathcal{J}(E) = \int_{B_1} u_E \, dx.$$

Let $m \in (0, |B_1|)$ and $\delta \in (0, \inf\{2m, 2(|B_1| - m)\})$. Let $E \subset B_1$ be a measurable set such that $|E| = m$ and $|E\Delta B_*| = \delta$ where B_* is the centered ball of volume m. Then

$$\mathcal{J}(E) \leq \mathcal{J}(A_\delta), \quad \text{i.e.} \quad \int_{B_1} u_E\, dx \leq \int_{B_1} u_{A_\delta}\, dx.$$

where A_δ is defined in Definition 2.6.

Proof. By linearity we may write

$$u_E = u_{E\cap B_*} + u_{E\cup B_*} - u_{B_*}.$$

Using Talenti's comparison result (see Theorem 2.4), we know that

$$u^\sharp_{E\cap B_*} \leq u_{B_{r_1(\delta)}}, \qquad u^\sharp_{E\cup B_*} \leq u_{B_{r_2(\delta)}}.$$

Integrating over B_1 the two inequalities, and using the equi-measurability of the Schwarz rearrangement, we get:

$$\begin{aligned}\int_{B_1} u_E\, dx &= \int_{B_1} u^\sharp_{E\cap B_*}\, dx + \int_{B_1} u^\sharp_{E\cup B_*}\, dx - \int_{B_1} u_{B_*}\, dx \\ &\leq \int_{B_1} u_{B_{r_1(\delta)}}\, dx + \int_{B_1} u_{B_{r_2(\delta)}}\, dx - \int_{B_1} u_{B_*}\, dx = \int_{B_1} u_{A_\delta}\, dx.\end{aligned}$$

□

Proposition 2.8. *With the same notations as in Proposition 2.7, there exists a positive constant $c = c(m,n)$ such that*

$$\mathcal{J}(B_*) - \mathcal{J}(E) \geq c|E\Delta B_*|^2,$$

for every measurable set $E \subseteq B_1$ such that $|E| = m$. Moreover, the exponent 2 is optimal, in the sense that the inequality cannot be valid for any lower exponent.

Proof. Let $E \subset B_1$ of volume m. Then $\delta = |E\Delta B_*| \in [0, \min\{2m, 2(|B_1| - m)\})$. If $\delta = 0$ then $E = B_*$ a.e. and $\mathcal{J}(B_*) = \mathcal{J}(E)$. If however $\delta > 0$, then we have by Proposition 2.7:

$$\mathcal{J}(B_*) - \mathcal{J}(E) \geq \mathcal{J}(B_*) - \mathcal{J}(A_\delta).$$

We introduce $w \in H^1_0(B_1)$ the unique solution to $-\Delta w = 1$ in B_1. Classical computations lead to $w(x) = w(|x|) = \frac{1-|x|^2}{2n}$. Moreover

$$\begin{aligned}\mathcal{J}(B_*) - \mathcal{J}(A_\delta) &= \int_{B_1} (-\Delta w)[u_{B_*} - u_{A_\delta}] = \int_{B_1} w[\chi_{B_*} - \chi_{A_\delta}] \\ &= |\mathbb{S}^{n-1}| \left(\int_{r_1(\delta)}^{r_*} w(r) r^{n-1} dr - \int_{r_*}^{r_2(\delta)} w(r) r^{n-1} dr \right) \\ &= \frac{|\mathbb{S}^{n-1}|}{2n(n+2)} \left(r_1(\delta)^{n+2} + r_2(\delta)^{n+2} - 2r_*^{n+2} \right)\end{aligned}$$

When $n = 1$ we compute explicitely $r_1(\delta)^3 + r_2(\delta)^3 - 2r_*^3 = \frac{3}{2^4} m\delta^2$ so that $c = m/2^4$ works. In the case $n \geq 2$, we use that the function $\varphi : x \in (0,1) \mapsto x^{(n+2)/n}$ is strongly convex ($\varphi'' \geq \frac{2(n+2)}{n^2}$), so that

$$\begin{aligned} r_1(\delta)^{n+2} + r_2(\delta)^{n+2} - 2r_*^{n+2} &= \varphi\left(\frac{m - \frac{\delta}{2}}{|B_1|}\right) + \varphi\left(\frac{m + \frac{\delta}{2}}{|B_1|}\right) - 2\varphi\left(\frac{m}{|B_1|}\right) \\ &\geq \frac{2(n+2)}{n^2} \left(\frac{\delta}{2|B_1|}\right)^2 \end{aligned}$$

hence the result with $c = \frac{1}{4n^2\omega_n}$. Finally, to show that the exponent is optimal, we use a Taylor expansion in the previous computation to obtain:

$$\mathcal{J}(B_*) - \mathcal{J}(A_\delta) = \frac{m^{2/n-1}}{4n^2\omega_n^{2/n}}\delta^2 + o(\delta^2).$$

Therefore, if $\alpha \in (0,2)$ then

$$\frac{\mathcal{J}(B_*) - \mathcal{J}(A_\delta)}{|B_*\Delta A_\delta|^\alpha} \xrightarrow[\delta\to 0]{} 0.$$

□

Remark 2.9. We notice that the case $p = 1$ implies Theorem 1.2 when j is in $C^1(\mathbb{R}_+)$, convex and $j'(0) > 0$. Indeed, we have by convexity of j

$$\mathcal{J}(B_*) - \mathcal{J}(E) \geq \int_{B_1} j'(u_E)(u_{B_*} - u_E)\,dx \geq j'(0)(\|u_{B_*}\|_1 - \|u_E\|_1),$$

so the stability inequality for $\mathcal{J}$ follows from Proposition 2.8. Of course, this does not apply to the case $j : s \mapsto s^p$ $(p > 1)$ whose derivative vanishes at 0.

2.3 Optimization among densities

In this section, we prove the following:

Proposition 2.10. *Let $j \in C^0(\mathbb{R}_+)$ be a convex non-decreasing function and $m \in (0, |B_1|)$. Then the functional $\mathcal{J} : V \in \mathcal{M}_m \mapsto \int_{B_1} j(u_V)$ is well-defined, convex, and*

$$\mathcal{J}(B_*) = \max_{V\in\mathcal{M}_m} \mathcal{J}(V). \tag{2.1}$$

Moreover, the following properties hold:

1. *for any $\delta \in (0, \min\{2m, 2(|B_1| - m)\})$, we define the set of* fixed asymmetry *weights*

   $$\mathcal{M}_m^\delta := \{\, V \in \mathcal{M}_m\ ,\ \|V - \chi_{B_*}\|_1 = \delta \,\}\,.$$

   Then the optimization problem

   $$\sup_{V\in\mathcal{M}_m^\delta} \mathcal{J}(V) \tag{2.2}$$

   admits a bang-bang solution, which means there exists a set E_δ such that

   $$\mathcal{J}(E_\delta) = \max_{V\in\mathcal{M}_m^\delta} \mathcal{J}(V).$$

2. *if j is strictly convex and strictly increasing in $(0,\infty)$, then χ_{B_*} is the unique maximizer to (2.1).*

Before proving this result, we state first a classical elliptic regularity theorem that will be used often throughout the paper. We refer for instance to [13, Theorem 9.13, Theorem 8.16].

Theorem 2.11. *Let $q > \max\{1, n/2\}$. Then there exists $C = C(n,q)$ such that for every $f \in L^q(B_1)$,*

$$\|u_f\|_{2,q} \leq C\|f\|_q,$$

where $u_f \in H_0^1(B_1)$ is the unique solution to (1.1).

Also, we need a few technical results: classically, weak-$*$ L^∞ convergence does not imply convergence of the L^1-norm, but if one has some control on the sign of the involved functions, then one can retrieve such convergence:

Lemma 2.12. *Let $h \in L^\infty(B_1)$ and (h_k) be a sequence of functions in $L^\infty(B_1)$ such that*

$$h_k \xrightarrow{*-L^\infty(B_1)} h.$$

We assume that there exists $E \subset B_1$ such that

$$\forall k \in \mathbb{N}, \qquad h_k \geq 0 \text{ in } E, \qquad \text{and} \qquad h_k \leq 0 \text{ in } B_1 \setminus E.$$

Then

$$\lim_k \int_{B_1} |h_k|\, dx = \int_{B_1} |h|. \tag{2.3}$$

Proof. It is sufficient to notice that under these assumptions,

$$h_k^+ = h_k \chi_E, \qquad h_k^- = h_k \chi_{E^c},$$

where h_k^+ and h_k^- are the positive part and the negative part of h_k respectively. In particular, by weak convergence of h_k, we get

$$h_k^+ \xrightarrow{*-L^\infty(B_1)} h^+ = h\chi_E,$$

$$h_k^- \xrightarrow{*-L^\infty(B_1)} h^- = h\chi_{E^c},$$

which implies (2.3) □

Lemma 2.13. *Let $m \in (0, |B_1|)$ and $\delta \in (0, \min\{2m, 2(|B_1| - m)\})$. Then the sets $\mathcal{M}_m, \mathcal{M}_m^\delta$ are convex and compact with respect to the weak-$*$ L^∞ topology, and their extremals are characteristic functions.*

Proof. • The convexity for $\mathcal{M}_m$ is immediate by definition, while for $\mathcal{M}_m^\delta$, we need to show that if $W_0, W_1 \in \mathcal{M}_m^\delta$ and

$$W_\alpha = \alpha W_1 + (1-\alpha) W_0,$$

for $\alpha \in (0,1)$, then $\|W_\alpha - V_0\|_1 = \delta$. This is true since $0 \leq W_\alpha \leq 1$, and by definition of V_0 we have

$$W_\alpha \leq V_0 \text{ in } B_*, \qquad W_\alpha \geq V_0 \text{ in } B_1 \setminus B_*,$$

so that explicitly computing the L^1 norms,

$$\|W_\alpha - V_0\|_1 = \alpha\|W_1 - V_0\|_1 + (1-\alpha)\|W_0 - V_0\|_1 = \delta.$$

• $\mathcal{M}_m$ is compact because if W_k weakly-$*$ converges in L^∞ to some W, then

$$m = \lim_k \int_{B_1} W_k\, dx = \int_{B_1} W\, dx.$$

For what regards $\mathcal{M}_m^\delta$, if we take a sequence $W_k \in \mathcal{M}_m^\delta$ converging to W weakly-$*$ in L^∞, then the functions $\chi_{B_*} - W_k$ satisfy the assumptions of Lemma 2.12, and then

$$\delta = \lim_k \|W_k - \chi_{B_*}\|_1 = \|W - \chi_{B_*}\|.$$

• The fact that extremal points of $\mathcal{M}_m$ are characteristic functions is classical (see for instance [15, Prop. 7.2.17]). Let us detail the same result for $\mathcal{M}_m^\delta$: let $W \in \mathcal{M}_m^\delta$, and assume that

$$|\{\, 0 < W < 1 \,\}| > 0;$$

we must show that W is not extremal. Since $\|W\|_1 = m = \|\chi_{B_*}\|_1$, we have

$$0 < \delta = \|W - \chi_{B_*}\|_1 = 2\int_{B_*} (1 - W)\, dx \le 2|\{\, 0 < W < 1 \,\} \cap B_*|.$$

Analogously we prove $|\{\, 0 < W < 1 \,\} \setminus B_*| > 0$. Therefore, for small $\varepsilon > 0$ we can split the set $\{\, \varepsilon < W < 1 - \varepsilon \,\}$ in four pairwise disjoint subsets S_1, S_2, S_3, S_4 of B_1 such that

$$|S_1| = |S_2| > 0, \qquad\qquad |S_3| = |S_4| > 0,$$

and

$$S_1 \cup S_2 = \{\, \varepsilon < W < 1 - \varepsilon \,\} \cap B_* \qquad\qquad S_3 \cup S_4 = \{\, \varepsilon < W < 1 - \varepsilon \,\} \setminus B_*.$$

Under these assumptions we can write

$$W = \frac{1}{2}(W - \varepsilon\chi_{S_1\cup S_3} + \varepsilon\chi_{S_2\cup S_4}) + \frac{1}{2}(W + \varepsilon\chi_{S_1\cup S_3} - \varepsilon\chi_{S_2\cup S_4}),$$

with $W \mp \varepsilon\chi_{S_1\cup S_3} \pm \varepsilon\chi_{S_2\cup S_4} \in \mathcal{M}_m^\delta$. This proves that a non-bang-bang function is not an extreme point for $\mathcal{M}_m^\delta$.

□

Proof of Proposition 2.10. For $V \in \mathcal{M}_m$, $u_V \ge 0$ is bounded, so $j(u_V) \in L^1(B_1)$ and $\mathcal{J}$ is well-defined.

• **Convexity of $\mathcal{J}$:** given $W_0, W_1 \in \mathcal{M}_m$ and $\alpha \in [0, 1]$, if we take $W_\alpha = \alpha W_1 + (1 - \alpha)W_0$, then by linearity with respect to V of the equation (1.1), we get

$$u_{W_\alpha} = \alpha u_{W_1} + (1 - \alpha)u_{W_0},$$

which, joint with the convexity of j gives

$$\mathcal{J}(W_\alpha) \le \alpha\mathcal{J}(W_1) + (1 - \alpha)\mathcal{J}(W_0).$$

• **$\mathcal{J}(\cdot)$ is continuous with respect to the weak-$*$ L^∞ convergence:** let W_k be a sequence converging in the weak-$*$ L^∞ sense to some function W_∞. Let $u_k := u_{W_k} \in H_0^1(B_1)$ solution to (1.1). We show that u_k converges to u_{W_∞}. Let $q > n$; since (W_k) are equibounded in L^∞, Theorem 2.11 applies, and we have that (u_k) are equi-bounded in $W^{2,q}(B_1)$, namely there exists some constant $C = C(n, q) > 0$ such that

$$\|u_k\|_{2,q} \le C.$$

Therefore, by Kondrachov Theorem, there exists a subsequence (not relabelled) such that

$$u_k \xrightarrow{W^{1,q}(B_1)} u$$

for some $u \in W^{1,q}(B_1)$. In particular, by Sobolev's imbeddings, the convergence happens strongly in L^∞, and $u \in H_0^1(B_1)$. Moreover, using the weak-$*$ convergence of W_k, we get, passing to the limit in the weak formulation of (1.1),

$$\int_{B_1} \nabla u \cdot \nabla\varphi\, dx = \int_{B_1} W_\infty\, \varphi\, dx \qquad\qquad \forall\varphi \in H_0^1(B_1),$$

which implies $u = u_{W_\infty}$. Finally, since j is continuous and u_k converges strongly in L^∞ to u, then

$$\lim_k \mathcal{J}(W_k) = \lim_k \int_{B_1} j(u_k)\,dx = \mathcal{J}(W_\infty).$$

Since the argument is valid for every choice of the subsequence, this proves the continuity.

- **Solution to (2.1) and existence to (2.2):** by Lemma 2.13 about compactness of $\mathcal{M}_m$ and $\mathcal{M}_m^\delta$ and the continuity proved in the previous step, we get the existence of solutions to (2.1) and (2.2). As extremal points of $\mathcal{M}_m$ and $\mathcal{M}_m^\delta$ are bang-bang functions, by convexity of $\mathcal{J}$ we get that there exist bang-bang solutions to (2.1) and (2.2). Moreover, the monotonicity of j implies that $j(u^\sharp) = j(u)^\sharp$, so that, using Talenti's inequality, we deduce that B_* solves (2.1).
- **Uniqueness:** since j is strictly convex, also $\mathcal{J}$ is strictly convex and any maximizer of $\mathcal{J}$ is necessarily bang-bang. In particular, if E is an optimal set such that χ_E maximizes $\mathcal{J}$ we have that

$$\mathcal{J}(B_*) = \mathcal{J}(E) = \int_{B_1} j(u_E)\,dx = \int_{B_1} j(u_E^\sharp)\,dx, \tag{2.4}$$

where we used that the monotonicity of j implies $j(u_E^\sharp) = j(u_E)^\sharp$. By Talenti's inequality we know that $j(u_E^\sharp) \le j(u_{B_*})$. The strict monotonicity of j and (2.4) ensure that $u_E^\sharp = u_{B_*}$ almost everywhere. By the rigidity of Talenti's inequality (see Theorem 2.5) we get $E = B_*$. □

2.4 Sharpness of the exponent

We show that the exponent 2 is sharp in Theorem 1.2, in the sense that for $\alpha \in (0, 2)$, one can find a sequence of sets $E_k \in \mathcal{M}_m \setminus \{B_*\}$ such that

$$\frac{\mathcal{J}(B_*) - \mathcal{J}(E_k)}{|B_* \Delta E_k|^\alpha} \xrightarrow[k\to\infty]{} 0.$$

We proceed as in Section 2.2 by using the annulus A_δ (who is a candidate as being the worst asymmetric set with regard to Talenti's deficit, though we are not in position to prove it except in the case of the L^1-norm). More precisely, we show the following result:

Proposition 2.14. *Let $m \in (0, |B_1|)$, and let $j \in C^1(\mathbb{R}_+)$ be a non-constant, non-decreasing, convex function. Then there exists $\bar\delta > 0$ and a positive constant $C = C(j, m, n)$ such that for every $\delta \in (0, \bar\delta)$,*

$$\frac{1}{C}\delta^2 \le \mathcal{J}(B_*) - \mathcal{J}(A_\delta) \le C\delta^2. \tag{2.5}$$

For the proof of Proposition 2.14 and in several other places in this paper, we need to define the notion of adjoint state (see also [15, Section 5.8]):

Definition 2.15 (Adjoint State)**.** Let $j \in C^1(\mathbb{R}_+)$. For every $V \in L^2(B_1)$ nonnegative, we define the *adjoint state* w_V of u_V as the unique function solving in the weak sense the *adjoint problem*

$$\begin{cases} -\Delta w_V = j'(u_V) & \text{in } B_1, \\ w = 0 & \text{on } \partial B_1, \end{cases}$$

namely,

$$\int_{B_1} \nabla w_V \cdot \nabla \varphi\,dx = \int_{B_1} j'(u_V)\varphi \qquad \forall \varphi \in H_0^1(B_1). \tag{2.6}$$

When $V = \chi_E$ we will write $w_E := w_{\chi_E}$.

Proof of Proposition 2.14. For every $\delta \in [0, \min\{2m,\, 2|B_1| - 2m\}\,)$, let us define $u_\delta = u_{A_\delta}$, $u_0 = u_{B_*}$, and $w_\delta = w_{A_\delta}$, $w_0 = w_{B_*}$. Moreover, since these functions are all radial, we will identify, with a slight abuse of notation

$$u_\delta(x) = u_\delta(|x|), \qquad\qquad w_\delta(x) = w_\delta(|x|).$$

By convexity of j, we have

$$\int_{B_1} j'(u_\delta)(u_0 - u_\delta)\,dx \le \mathcal{J}(B_*) - \mathcal{J}(A_\delta) \le \int_{B_1} j'(u_0)(u_0 - u_\delta)\,dx.$$

After two integration by parts, we may rewrite

$$\int_{B_1} w_\delta(\chi_{B_*} - \chi_{A_\delta})\,dx \le \mathcal{J}(B_*) - \mathcal{J}(A_\delta) \le \int_{B_1} w_0(\chi_{B_*} - \chi_{A_\delta})\,dx. \tag{2.7}$$

Noticing that $\chi_{B_*} - \chi_{A_\delta}$ is non-zero only between the radii $r_1(\delta)$ and $r_2(\delta)$, we focus our attention on the values of w_δ and w_0 near $|x| = r_*$, where we recall that $B_* = B_{r_*}$.

- **Estimates for w_0:** by Taylor expansion,

$$w_0(r) = w_0(r_*) + \partial_r w_0(r_*)(r - r_*) + R_1(r), \tag{2.8}$$

with

$$R_1(r) = \int_{r_*}^{r} (\partial_r w_0(s) - \partial_r w_0(r_*))\,dr.$$

Since w_0 solves the equation $-\Delta w_0 = j'(u_0)$, then by classical elliptic regularity (Theorem 2.11) we have that $w_0 \in C^{1,\beta}$ for some $\beta \in (0,1)$, so that

$$|\partial_r w_0(r_*) - \partial_r w_0(s)| \le C|s - r_*|^\beta,$$

and

$$|R_1(r)| \le C|r - r_*|^{1+\beta}. \tag{2.9}$$

Therefore, noticing that (see Definition 2.6 for the definition of $r_1 = r_1(\delta), r_2 = r_2(\delta)$)

$$\int_{B_1} (\chi_{B_*} - \chi_{A_\delta})\,dx = 0 \qquad\qquad \chi_{B_*}(r) - \chi_{A_\delta}(r) = \operatorname{sgn}(r_* - r)\chi_{(r_1,r_2)}(r),$$

we obtain by (2.8) and (2.9)

$$\int_{B_1} w_0(\chi_{B_*} - \chi_{A_\delta})\,dx = \partial_r w_0(r_*)|\mathbb{S}^{n-1}| \int_{r_1}^{r_2} r^{n-1}|r - r_*|\,dr + R_2(\delta)$$

with

$$|R_2(\delta)| \le C|r_2 - r_1|^{2+\beta}.$$

In particular, since $r_2 - r_1 = O(\delta)$, we have

$$\int_{B_1} w_0(\chi_{B_*} - \chi_{A_\delta})\,dx \le C\delta^2 + o(\delta^2), \tag{2.10}$$

so that, with (2.7), the upper bound in (2.5) is proven for a suitable choice of $\bar{\delta}$.

• **Estimates for w_δ:** we now need to adapt the previous estimate with w_δ instead of w_0: to that end, we first notice that w_δ smoothly converges to w_0 for δ that goes to 0. Indeed, since $-\Delta u_\delta = \chi_{A_\delta}$, we have from classical elliptic regularity (see Theorem 2.11) that

$$u_\delta \xrightarrow{L^\infty(B_1)} u_0.$$

In particular, the continuity of j' ensures that $j'(u_\delta)$ converges in L^∞ to $j'(u_0)$ and, for any $\beta \in (0,1)$,

$$w_\delta \xrightarrow{C^{1,\beta}(B_1)} w_0.$$

This convergence, joint with (2.8) and (2.9) ensures that for some uniform positive constant C we have

$$w_\delta(r) = w_\delta(r_*) + \partial_r w_\delta(r_*)(r - r_*) + R_3(r), \tag{2.11}$$

with

$$|R_3(r)| \le C|r - r_*|^{1+\beta}. \tag{2.12}$$

If $\bar\delta$ is small enough, since $j' \ge 0$ and $j' \not\equiv 0$ we know by Hopf's lemma that w_0 is strictly radially decreasing and we have

$$-\partial_r w_\delta(r_*) \ge -\frac{1}{2}\partial_r w_0(r_*) > 0. \tag{2.13}$$

We now compute similarly to (2.10), using (2.11), (2.12), and (2.13) we obtain

$$\int_{B_1} w_\delta(\chi_{B_*} - \chi_{A_\delta})\, dx \ge c\delta^2 + o(\delta^2),$$

for $c > 0$ small enough, thus concluding the proof.

□

Remark 2.16. We notice that the convexity assumption in Proposition 2.14 is only needed to make the proof simpler. As in [18, Proof of Formula (32)] we could have used a parametric derivative approach to obtain the same result without assuming j to be convex.

In the following we show the exponent 2 is sharp also for $p = \infty$.

Proposition 2.17. *Let $m \in (0, |B_1|)$. Then there exists $\bar\delta > 0$ and a positive constant $C = C(m,n)$ such that for every $\delta \in (0, \bar\delta)$,*

$$\frac{1}{C}\delta^2 \le \|u_{B_*}\|_\infty - \|u_{A_\delta}\|_\infty \le C\delta^2. \tag{2.14}$$

Proof. Since both u_{B_*} and u_{A_δ} are radially decreasing, if G denotes the Green's function on the ball (see subsection 5.1 for the definition), we have

$$\|u_{B_*}\|_\infty - \|u_{A_\delta}\|_\infty = \int_{B_1} G(0,y)(\chi_{B_*} - \chi_{A_\delta})\, dy.$$

Using a Taylor expansion of $G(0,\cdot)$, the fact that $|\nabla_y G(0,\cdot)| > 0$ near ∂B_*, and using the volume constraint, we obtain the result as in the proof of Proposition 2.14. □

3 Proof of Theorem 1.2

In this section, we provide the first part of the proof of Theorem 1.2 (which implies Theorem 1.1 in the case $p \in (1,\infty)$ as shown in Section 1). In fact, in the spirit of the Selection principle used in [8] to deduce the sharp quantitative isoperimetric inequality from a stability result by Fuglede ([11]), we show that Theorem 1.2 is a consequence of the following Theorem 3.1, asserting stability among smooth deformations of B_*. In what follows, for every set $E \subset B_1$ and every vector field $\Phi : B_1 \to \mathbb{R}^n$, we use the notation

$$E^\Phi = (\mathrm{Id} + \Phi)(E).$$

Theorem 3.1. *Let $m \in (0, |B_1|)$ and $j \in C^1(\mathbb{R}_+) \cap C^2((0,\infty))$ be such that $j'(0) \geq 0$, $j''(s) > 0$ for every $s > 0$ and*

$$\limsup_{s \to 0^+} s^\alpha j''(s) < +\infty \tag{3.1}$$

for some $\alpha < 1$. Then there exist positive constants $c = c(j,m,n)$, $\eta = \eta(j,m,n)$ such that for every $\Phi \in W^{2,q}(B_1, \mathbb{R}^n)$ orthogonal to ∂B_* with

$$\|\Phi\|_{W^{2,q}} \leq \eta, \qquad |B_*^\Phi| = |B_*| = m,$$

we have

$$\mathcal{J}(B_*) - \mathcal{J}(B_*^\Phi) \geq c|B_* \Delta B_*^\Phi|^2.$$

We show how one can deduce Theorem 1.2 from Theorem 3.1 in 2 subsequent steps. The proof of Theorem 3.1 is postponed to Section 4. Let us stress that the orthogonality assumption in Theorem 3.1 can be removed, since every deformation Φ can be replaced with a deformation Ψ normal to ∂B_* in a way that $B_*^\Phi = B_*^\Psi$ and $\|\Psi\|_{W^{2,q}}$ is equivalent to $\|\Phi\|_{W^{2,q}}$ (see for instance [15, Lemma 5.9.5]).

3.1 Step 1: local stability implies global stability

In this first step, we show that it is enough to prove Theorem 1.2 in the regime $\|V - \chi_{B^*}\|_1 \to 0$: for $m \in (0, |B_1|)$ given and δ small enough, we recall

$$\mathcal{M}_m^\delta := \left\{ V \in L^\infty(B_1) \;\middle|\; \begin{array}{c} 0 \leq V \leq 1, \\ \int_{B_1} V = m, \\ \|V - \chi_{B_*}\|_1 = \delta \end{array} \right\}.$$

Proposition 3.2. *Let $j \in C^0(\mathbb{R}_+)$ be strictly convex and strictly increasing. If*

$$\liminf_{\delta \to 0} \inf_{V \in \mathcal{M}_m^\delta} \frac{\mathcal{J}(B_*) - \mathcal{J}(V)}{\delta^2} > 0, \tag{3.2}$$

then Theorem 1.2 holds true.

Proof. We denote

$$c = \liminf_{\delta \to 0} \inf_{V \in \mathcal{M}_m^\delta} \frac{\mathcal{J}(B_*) - \mathcal{J}(V)}{\delta^2}$$

that is assumed to be positive, and we let V_k be a minimizing sequence for the problem

$$\inf_{\substack{V \in \mathcal{M}_m \\ V \neq \chi_{B_*}}} G(V) := \frac{\mathcal{J}(B_*) - \mathcal{J}(V)}{\|V - \chi_{B_*}\|_1^2}.$$

By compactness of $\mathcal{M}_m$ (see Lemma 2.13) we may assume that V_k weakly-$*$ L^∞ converges to some function $V_\infty \in \mathcal{M}_m$.

First case: if $V_\infty = \chi_{B_*}$, then by weak-* convergence and the sign constraints on $V_k - \chi_{B_*}$ we can apply Lemma 2.12 and we get:

$$\lim_k \|V_k - \chi_{B_*}\|_1 = 0,$$

and so by definition of c,

$$\lim_k G(V_k) \geq c > 0.$$

Second case: if $V_\infty \neq \chi_{B_*}$ then again by Lemma 2.12 $\lim_k \|V_k - \chi_{B_*}\|_1 = \|V_\infty - \chi_{B_*}\|_1$ and by continuity of $\mathcal{J}$ (see Lemma 2.13),

$$\lim_k G(V_k) = G(V_\infty).$$

But by the uniqueness stated in Proposition 2.10, $G(V_\infty) > 0$, hence the result. □

3.2 Step 2: Theorem 3.1 implies Theorem 1.2

To conclude the proof of Theorem 1.2, we will use the following result from [19]:

Theorem 3.3 (Quantitative bathtub principle)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set, $m \in (0, |\Omega|)$, and $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$. We assume*

$$|\{ u > t_* \}| = |\{ u \geq t_* \}| = m \qquad \text{and} \qquad \min_{\partial\{u>t_*\}} |\nabla u| > 0$$

for some unique $t_* \in \mathbb{R}$. Then there exists a positive constant $c = c(\|u\|_{C^{1,\alpha}})$ such that for every $V \in L^1(\Omega)$ non-negative with $\int_\Omega V\,dx = m$,

$$\int_\Omega uV \leq \int_\Omega u\chi_{\{u>t_*\}} - c\left(\min_{\partial\{u>t_*\}} |\nabla u|\right) \frac{\mathcal{H}^{n-1}(\partial B_*)}{\mathcal{H}^{n-1}(\partial\{u > t_*\})} \|V - \chi_{\{u>t_*\}}\|_1^2,$$

where B_* is the ball of volume m.

Proof of Theorem 1.2 from Theorem 3.1. By Proposition 2.10 we have that for every $\delta > 0$ small there exists a set E_δ solution to the problem

$$\mathcal{J}(E_\delta) = \max_{V \in \mathcal{M}_m^\delta} \mathcal{J}(V).$$

Thanks to Proposition 3.2 it is sufficient to prove that

$$\liminf_{\delta \to 0} \frac{\mathcal{J}(B_*) - \mathcal{J}(E_\delta)}{\delta^2} > 0.$$

Let us assume by contradiction, up to extracting a subsequence, that

$$\lim_{\delta \to 0} \frac{\mathcal{J}(B_*) - \mathcal{J}(E_\delta)}{\delta^2} = 0.$$

For every δ we define $u_\delta := u_{E_\delta}$, and $w_\delta := w_{E_\delta}$ the adjoint states (see Definition 2.15).

• We first claim that there exists t_δ such that

$$\widetilde{E}_\delta := \{w_\delta > t_\delta\}$$

is of volume m. To that end we prove the function $t \in (0,\infty) \mapsto |\{w_\delta > t\}|$ is continuous, which is the same as proving that $|\{w_\delta = t\}| = 0$ for every $t > 0$. Let $t > 0$: we argue similarly to [7, page 7]. Because w_δ is in $H^2(B_1)$, we have $\Delta w_\delta = 0$ a.e. in $\{w_\delta = t\}$, and therefore $j'(u_\delta) = 0$ a.e. in $\{w_\delta = t\}$. By assumption $j'(s) > 0$ for every $s > 0$ (j' is strictly increasing as $j'' > 0$ in $(0,\infty)$), and since $\{w_\delta = t\} \subset B_1$ and $u_\delta > 0$ in B_1, we obtain $|\{w_\delta = t\}| = 0$. This shows the existence of t_δ, and also that $|\{w_\delta > t_\delta\}| = |\{w_\delta \geq t_\delta\}|$. Finally, we show that t_δ is unique: let $\widehat{t}$ be such that $|\{w > \widehat{t}\}|$ is also equal to m, and assume for example $t_\delta \leq \widehat{t}$ (the other case being similar). Then the set $\{t_\delta < w < \widehat{t}\}$ is open with zero measure, so it is empty. But as $m \in (0, |B_1|)$, $\min w_\delta < t_\delta \leq \widehat{t} < \max w_\delta$, so by connectedness of B_1 and continuity of w_δ, the set $\{t_\delta < w < \widehat{t}\}$ can be empty only if $t_\delta = \widehat{t}$.
• Second, we show that $\widetilde{E}_\delta$ is a $W^{2,q}$ deformation of B_*. First, by classical elliptic estimates (Theorem 2.11) u_δ converges to u_0 in $W^{2,q}(B_1)$ for every $q < \infty$. As j is locally $\beta := (1-\alpha)$-Hölder continuous (see Remark 1.3) this implies that w_δ converges to w_0 in $W^{2,q}$ for every $q < \infty$. On the other hand, $j'(u_0)$ is C^1 far from ∂B_1, so that $w_0 \in W^{3,q}(B_{1-\varepsilon};\mathbb{R}^n)$ for every small ε. Moreover, since $0 < m < |B_1|$, there exists a positive constant c such that $|\nabla w_0| \geq c$ on ∂B_*.
Therefore, we are in position to apply Lemma A.1 to w_δ and Proposition A.2 to $w_\delta - t_\delta$, so that the following holds: for every fixed $\varepsilon > 0$ and for $\delta < \delta_0(\varepsilon)$ we can find deformations $\Phi_\delta \in W^{2,q}(B_1;\mathbb{R}^n)$ such that $\|\Phi_\delta\|_{W^{2,q}} \leq \varepsilon$, Φ_δ are orthogonal to ∂B_*, and

$$\widetilde{E}_\delta = B_*^{\Phi_\delta}.$$

• Third, by the convergence of w_δ and the regularity of Φ_δ, we find constants $c, C > 0$ such that for δ small enough,

$$|\nabla w_\delta| \geq c \qquad \text{on } \partial B_* \qquad \text{and} \qquad \mathcal{H}^{n-1}(\{w_\delta = t_\delta\}) = \int_{\partial B_*} \mathrm{Jac}^{\partial B_*}(\mathrm{Id} + \Phi_\delta)\, d\mathcal{H}^{n-1} \leq C. \tag{3.3}$$

We now compute, using the convexity of j,

$$\mathcal{J}(E_\delta) = \int_{B_1} j(u_\delta)\, dx \leq \mathcal{J}(\widetilde{E}_\delta) - \int_{B_1} j'(u_\delta)(\widetilde{u}_\delta - u_\delta),$$

where $\widetilde{u}_\delta = u_{\widetilde{E}_\delta}$. We integrate by parts two times using $j'(u_\delta) = -\Delta w_\delta$, so that

$$\mathcal{J}(E_\delta) \leq \mathcal{J}(\widetilde{E}_\delta) - \int_{B_1} w_\delta(\chi_{\widetilde{E}_\delta} - \chi_{E_\delta}).$$

We now notice that thanks to the uniform estimates (3.3) and the quantitative bathtub principle (Theorem 3.3) we get the existence of a uniform constant c such that

$$\mathcal{J}(\widetilde{E}_\delta) - \mathcal{J}(E_\delta) \geq c|\widetilde{E}_\delta \Delta E_\delta|^2. \tag{3.4}$$

On the other hand, since $\|\Phi\|_{W^{2,q}} < \varepsilon$, we can apply Theorem 3.1 when ε is small enough, which gives

$$\mathcal{J}(B_*) - \mathcal{J}(\widetilde{E}_\delta) \geq c|\widetilde{E}_\delta \Delta B_*|^2, \tag{3.5}$$

also for a positive constant c. Finally, noticing that

$$\delta = |E_\delta \Delta B_*| \leq |E_\delta \Delta \widetilde{E}_\delta| + |\widetilde{E}_\delta \Delta B_*|,$$

we join (3.4) and (3.5), and we get the existence of a constant c such that

$$c\delta^2 \leq 2c\Big(|E_\delta \Delta \widetilde{E}_\delta|^2 + |\widetilde{E}_\delta \Delta B_*|^2\Big) \leq \mathcal{J}(B_*) - \mathcal{J}(E_\delta),$$

which is a contradiction and concludes the proof. □

4 Proof of Theorem 3.1

So far, we have proved Theorem 1.1 in the special case $p = 1$, and we have reduced the proof of Theorem 1.2 (which contains Theorem 1.1 for $p \in (1, \infty)$) to the proof of Theorem 3.1 asserting stability of B^* among smooth deformations. This section is dedicated to the proof of this last result, which is based on a shape derivative approach (see for example [10] for references and details). The section is divided in 4 paragraphs:

1. first, the computation of first and second order shape derivatives, that leads to the expression of the Lagrangian for problem (2.1),
2. then the proof of coercivity for the second order derivative of this Lagrangian,
3. as a classical third step, we then need to prove an improved continuity property of the second order derivative,
4. and finally we combine all of these ingredients to conclude the proof.

4.1 Computation of shape derivatives

In this section we compute the shape derivatives of the shape functional

$$E \longmapsto \mathcal{J}(E) = \int_{B_1} j(u_E).$$

We start by recalling some basic facts about shape derivatives.

Definition 4.1 (Shape derivative)**.** Let $s \in (0,1)$, let $\mathcal{O}$ be a family of $C^{1,s}$ sets compactly supported in B_1, and let

$$\mathcal{F} : \mathcal{O} \to X$$

with X a Banach space. Let $q > n$ large enough to have $W^{2,q}(B_1; \mathbb{R}^n) \hookrightarrow C^{1,s}(B_1; \mathbb{R}^n)$. For every set $E \in \mathcal{O}$, we say that $\mathcal{F}$ is shape differentiable at first or second order if the functional

$$\Phi \in W^{2,q}(B_1; \mathbb{R}^n) \longmapsto \mathcal{F}((\mathrm{Id} + \Phi)(E)) \in X. \tag{4.1}$$

admits Fréchet derivatives (of first or second order) at 0, and in this case, we define the *shape derivatives* (denoted $\mathcal{F}'(E)[\Phi]$ and $\mathcal{F}''(E)[\Phi, \Psi]$) as the Fréchet derivatives at 0 of (4.1).

We will follow the classical approach (see for instance [15, Theorem 5.3.2]) to prove the existence and compute the shape derivative of u_E, w_E and finally $\mathcal{J}(E)$, in the next three results respectively. In the whole section, E will denote a set of class $C^{1,s}$ such that $\overline{E} \subset B_1$, and p large enough so that $W^{1,q}(B_1) \subset C^{0,s}(\overline{B_1})$. Then for every $\Phi \in W^{2,q}(B_1; \mathbb{R}^n)$

$$E^\Phi := (\mathrm{Id} + \Phi)(E)$$

is also of class $C^{1,s}$, and $W^{1,q}(B_1, \mathbb{R}^n)$ is an algebra.

We then define the functions

$$u_\Phi := u_{E^\Phi} \qquad\qquad w_\Phi := w_{E^\Phi}$$
$$\widehat{u}_\Phi := u_{E^\Phi} \circ (\mathrm{Id} + \Phi) \qquad\qquad \widehat{w}_\Phi := w_{E^\Phi} \circ (\mathrm{Id} + \Phi),$$

where w_E is the adjoint state defined in Definition 2.15. Let us recall the classical Hadamard formula that can be found for instance in [15, Corollary 5.2.8].

Lemma 4.2 (Hadamard formula). *Let $T > 0$ and $F \in C^1([0,T); W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n))$ and we denote $F_t(x) = F(t,x)$. We assume F_t invertible for every $t \in [0,T)$. Then for every function $f \in C^1([0,T); L^1(\mathbb{R}^n)) \cap C^0([0,T); W^{1,1}(\mathbb{R}^n))$*

$$\frac{d}{dt}\left(\int_{F_t(E)} f(t,x)dx\right) = \int_{F_t(E)} \left(\partial_t f(t,x) + \operatorname{div}(f(t,x)\, V_t(x))\right) dx, \tag{4.2}$$

where $V_t(x) = \partial_t F_t(F_t^{-1}(x))$.

Remark 4.3. In particular, when $F_t = \mathrm{Id} + t\Phi$ with $\Phi \in W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n)$, we define

$$\widetilde{\Phi}_t(x) := V_t(x) = \Phi \circ F_t^{-1}(x).$$

Proposition 4.4 (Shape differentiability of u_Φ). *The application*

$$\Phi \in W^{2,q}(B_1;\mathbb{R}^n) \longmapsto u_\Phi \in W_0^{1,q}(B_1)$$

is of class C^1 in a neighborhood of 0. In particular, if $\Phi \in W^{2,q}(B_1;\mathbb{R}^n)$ is small enough and we denote by $u_t := u_{t\Phi}$, then for every $t \in [0,1]$ we have that $u_t' \in W_0^{1,q}(B_1)$ is the unique solution in $H_0^1(B_1)$ to

$$-\Delta u_t' = (\widetilde{\Phi}_t \cdot \nu_t)\, d\mathcal{H}^{n-1} \llcorner_{\partial E^{t\Phi}}, \tag{4.3}$$

(in the distributional sense), where $\widetilde{\Phi}_t = \Phi \circ (\mathrm{Id} + t\Phi)^{-1}$ and ν_t is the exterior unit normal to $E^{t\Phi}$.

Proof. • If $\|\Phi\|_{2,q} < 1$, we can invert the matrix $I_n + D\Phi$, so that, defining

$$J_\Phi := \det(I_n + D\Phi) \qquad\qquad A_\Phi := J_\Phi\,(I_n + D\Phi)^{-1}(I_n + D\Phi)^{-T},$$

we have in the distributional sense

$$-\operatorname{div}(A_\Phi \nabla \widehat{u}_\Phi) = \chi_E\, J_\Phi.$$

• We introduce $X = H_0^1(B_1) \cap W^{2,q}(B_1)$ and

$$\begin{aligned} \mathcal{F}:\ \ W^{2,q}(B_1;\mathbb{R}^n) \times X &\longrightarrow L^q(B_1) \\ (\Phi, \widehat{u}) &\longmapsto -\operatorname{div}(A_\Phi \nabla \widehat{u}) - \chi_E\, J_\Phi. \end{aligned}$$

Then $\mathcal{F}$ is of class C^∞ in a neighbourhood of 0. Indeed, we first notice that the application

$$(A, \widehat{u}) \in \mathbb{R}^{n\times n} \times X \longmapsto -\operatorname{div}(A\nabla \widehat{u}) \in L^p(B_1)$$

is of class C^∞ because it is linear and continuous in both variables. Analogously, using that $W^{1,q}$ is an algebra, multilinearity and continuity in the origin imply that the applications

$$\Phi \in W^{2,q}(B_1) \longmapsto (I_n + D\Phi)^{-1} = \sum_{k=1}^{+\infty} (D\Phi)^k \in W^{1,q}(B_1)$$

$$\Phi \in W^{2,q}(B_1) \longmapsto J_\Phi = \det(I_n + D\Phi) \in W^{1,q}(B_1)$$

are of class C^∞ in a neighborhood of 0. Therefore, the application

$$\Phi \in W^{2,q}(B_1) \longmapsto A_\Phi \in W^{1,q}(B_1)$$

is C^∞ in a neighborhood of 0, as well as $\mathcal{F}$.

• We are now in position to use the implicit function theorem to deduce from the previous point that the application

$$\Phi \in W^{2,q}(B_1;\mathbb{R}^n) \longmapsto \widehat{u}_\Phi \in X,$$

that satisfies $\mathcal{F}(\Phi, \widehat{u}_\Phi) = 0$ is of class C^∞ in a neighborhood of 0. Indeed, $\widehat{u}_0 = u_E$ and the map

$$\xi \in X \longmapsto D_{\widehat{u}}\mathcal{F}(0, u_E)[\xi] = -\Delta\xi - \chi_E,$$

is a diffeomorphism from X onto $L^q(B_1)$ thanks to the classical elliptic regularity (see for instance [13, Theorem 9.14]).

• Then, by noticing that $u_\Phi = \widehat{u}_\Phi \circ (\mathrm{Id} + \Phi)^{-1}$, we know that the application

$$\Phi \in W^{2,q}(B_1;\mathbb{R}^n) \longmapsto u_\Phi \in W^{1,q}_0(B_1)$$

is C^1 in a neighborhood of 0 (for instance, we can apply [15, Lemma 5.3.3] to $\widehat{u}_\Phi \circ (\mathrm{Id} + \Phi)^{-1}$ and $\nabla\widehat{u}_\Phi \circ (\mathrm{Id} + \Phi)^{-1}$). Let us also recall that u_t solves the weak equation:

$$\int_{B_1} \nabla u_t \cdot \nabla\varphi \, dx = \int_{E^{t\Phi}} \varphi \, dx \qquad \forall \varphi \in H^1_0(B_1). \tag{4.4}$$

Using Lemma 4.2 to differentiate (4.4), we get that u_t' solves

$$\int_{B_1} \nabla u_t' \cdot \nabla\varphi \, dx = \int_{\partial E^{t\Phi}} \varphi \, (\widetilde{\Phi}_t \cdot \nu_t) \, d\mathcal{H}^{n-1} \qquad \forall \varphi \in H^1_0(B_1),$$

which is (4.3).

□

Proposition 4.5 (Shape differentiability of w_Φ)**.** *Let $j \in C^1(\mathbb{R}_+) \cap C^2((0,\infty))$ such that*

$$\limsup_{s\to 0^+} s^\alpha j''(s) < +\infty$$

for some $\alpha \in (0,1)$, and let $\varepsilon \in (0,1)$. Then the application

$$\Phi \in W^{2,q}(B_1;\mathbb{R}^n) \longmapsto w_\Phi \in W^{1,q}_0(B_1) \cap W^{2,q}(B_{1-\varepsilon})$$

is of class C^1 in a neighborhood of 0. In particular, if $\Phi \in W^{2,q}(B_1;\mathbb{R}^n)$ is small enough and we denote by $w_t := w_{t\Phi}$, then for $t \in [0,1]$ we have that $w_t' \in W^{1,q}_0(B_1)$ is the unique solution in $H^1_0(B_1)$ to

$$-\Delta w_t' = j''(u_t)u_t', \tag{4.5}$$

where u_t' is defined in Proposition 4.4.

Proof. We proceed as in the proof of Proposition 4.4, with the extra difficulty that j is not C^2 up to 0.

• First we claim that the map

$$\mathcal{G} : \Phi \in W^{2,q}(B_1;\mathbb{R}^n) \longmapsto j'(\widehat{u}_\Phi) \in L^q(B_1) \cap W^{1,q}(B_{1-\varepsilon/2})$$

is of class C^1 near 0. Since j' is C^1 far from 0, we immediately have by Proposition 4.4 that $\Phi \mapsto j'(\widehat{u}_\Phi) \in W^{1,q}(B_{1-\varepsilon/2})$ is of class C^1. It remains to show that $\Phi \mapsto j'(\widehat{u}_\Phi) \in L^q(B_1)$ is C^1. As j' is not assumed to be C^1 up to 0, as it is done for instance in [12, Lemma 2.5], we approximate the functional $\mathcal{G}$: for small $\varepsilon > 0$ we define

$$\mathcal{G}_\varepsilon : \Phi \in W^{2,q}(B_1;\mathbb{R}^n) \longmapsto j'(\varepsilon + \widehat{u}_\Phi) \in L^q(B_1).$$

From the proof of Proposition 4.4, recalling

$$X = H_0^1(B_1) \cap W^{2,q}(B_1) \subset C^1(\overline{B_1}),$$

there exists $\mathcal{U}$ neighborhood of 0 in $W^{2,q}(B_1;\mathbb{R}^n)$ such that the map $\Phi \in \mathcal{U} \longmapsto \widehat{u}_\Phi \in X$ is C^1. In particular $\|\widehat{u}_\Phi\|_\infty$ is equibounded for $\Phi \in \mathcal{U}$, and since $j' \in C^1([\varepsilon, +\infty))$, we get that $\mathcal{G}_\varepsilon$ is C^1 in $\mathcal{U}$ and that $\mathcal{G}_\varepsilon(0)$ converges to $\mathcal{G}(0)$ in $L^q(B_1)$. Once we prove that

$$\begin{array}{rccl} \mathcal{G}'_\varepsilon : & \mathcal{U} & \to & \mathcal{L}(W^{2,q}(B_1;\mathbb{R}^n), L^q(B_1)) \\ & \Phi & \mapsto & (\eta \mapsto j''(\varepsilon + \widehat{u}_\Phi)\, \widehat{u}'_\Phi[\eta]) \end{array}$$

converges uniformly in Φ, from [4, Theorem 3.6.1] we can conclude that $\mathcal{G}$ is C^1 near 0. If we let $h(t) = tj''(t)$, then by assumption $h \in C^0(\mathbb{R}_+)$ and by uniform continuity, $h(\varepsilon + \widehat{u}_\Phi)$ converges to $h(\widehat{u}_\Phi)$ uniformly in $\Phi \in \mathcal{U}$. Therefore, it remains to check that

$$\lim_{\varepsilon \to 0^+} \sup_{\substack{\Phi \in \mathcal{U} \\ \|\eta\|_{2,q} \le 1}} \left\| \widehat{u}'_\Phi[\eta] \left(\frac{1}{\varepsilon + \widehat{u}_\Phi} - \frac{1}{\widehat{u}_\Phi} \right) \right\|_q = 0. \tag{4.6}$$

Up to choosing a smaller $\mathcal{U}$, there exist uniform positive constants c, C such that $\|\nabla \widehat{u}'_\Phi[\eta]\|_\infty \le C$, and $|\nabla \widehat{u}_\Phi(y)| \ge c$ for every $y \in B_1 \setminus B_{1-c}$ (here we used that $|\nabla u_0| > 0$ near ∂B_1). This leads to the following uniform estimate: for every $x \in B_1$, and for every $\varepsilon \in (0,1)$,

$$\left| \frac{\varepsilon\, \widehat{u}'_\Phi[\eta](x)}{(\varepsilon + \widehat{u}_\Phi(x))\, \widehat{u}_\Phi(x)} \right| \le \frac{C\varepsilon(1-|x|)}{\varepsilon + c\, \max\{1-|x|,\, c\}} \le \frac{C}{1 + c\, \max\{1-|x|,\, c\}},$$

which, by dominated convergence, ensures (4.6).

- As in the proof of Proposition 4.4, we obtain

$$\begin{array}{rccl} \mathcal{F} : & W^{2,q}(B_1;\mathbb{R}^n) \times X & \longrightarrow & L^q(B_1) \\ & (\Phi, \widehat{u}) & \longmapsto & -\operatorname{div}(A_\Phi \nabla \widehat{u}) - j'(\widehat{u}_\Phi)\, J_\Phi. \end{array}$$

is C^1. As

$$D_{\widehat{w}} \mathcal{F}(0, w_E)[\xi] = -\Delta \xi - j'(u_E)$$

is a diffeomorphism of X onto $L^q(B_1)$, the implicit function theorem applies, and $\Phi \in W^{2,q}(B_1;\mathbb{R}^n) \mapsto \widehat{w}_\Phi \in X$ is of class C^1 in a neighborhood of 0, and

$$\Phi \in W^{2,q}(B_1;\mathbb{R}^n) \longmapsto w_\Phi \in W_0^{1,q}(B_1)$$

is C^1 in a neighborhood of 0. Moreover, the function w_t solves the weak equation (2.6), namely

$$\int_{B_1} \nabla w_t \cdot \nabla \varphi \, dx = \int_{B_1} j'(u_t) \varphi \, dx \qquad \forall \varphi \in H_0^1(B_1). \tag{4.7}$$

Using Proposition 4.4, we can differentiate (4.7) to obtain that w'_t solves

$$\int_{B_1} \nabla w'_t \cdot \nabla \varphi \, dx = \int_{B_1} j''(u_t) u'_t \, \varphi \, dx \qquad \forall \varphi \in H_0^1(B_1),$$

which proves (4.5).

□

Proposition 4.6 (Shape derivative of $\mathcal{J}$)**.** *Let $j \in C^1(\mathbb{R}_+) \cap C^2((0,+\infty))$ be such that*

$$\limsup_{s \to 0^+} s^\alpha j''(s) < +\infty$$

for some $\alpha \in (0,1)$. Then the application

$$\Phi \in W^{2,q}(B_1;\mathbb{R}^n) \longmapsto \mathcal{J}(E^\Phi) \in \mathbb{R}$$

is of class C^2 in a neighborhood of 0. Moreover, if $\Phi \in W^{2,q}(B_1;\mathbb{R}^n)$ is small enough, then $J(t) = \mathcal{J}(E^{t\Phi})$ satisfies for $t \in [0,1]$,

$$J'(t) = \int_{\partial E^{t\Phi}} w_t\,(\widetilde{\Phi}_t \cdot \nu_t)\, d\mathcal{H}^{n-1}, \tag{4.8}$$

where $w_t = w_{E^{t\Phi}}$ is the adjoint state defined in Definition 2.15, $\nu_t = \nu_{E^{t\Phi}}$ is the outer unit normal to $E^{t\Phi}$, and $\widetilde{\Phi}_t = \Phi \circ (\mathrm{Id} + t\Phi)^{-1}$. Moreover, letting $g_t = \widetilde{\Phi}_t \cdot \nu_t$,

$$J''(t) = \int_{\partial E^{t\Phi}} \Big(w_t' g_t + g_t(\nabla w_t \cdot \widetilde{\Phi}_t) \Big)\, d\mathcal{H}^{n-1} + a_t(\Phi,\Phi), \tag{4.9}$$

where

$$a_t(\Phi,\Phi) = \int_{\partial E^{t\Phi}} w_t \Big(g_t \operatorname{div}(\widetilde{\Phi}_t) - ((D\widetilde{\Phi}_t)\widetilde{\Phi}_t) \cdot \nu_t \Big) d\mathcal{H}^{n-1}.$$

Proof. For every $\Phi \in W^{2,q}(B_1;\mathbb{R}^n)$

$$\mathcal{J}(E^\Phi) = \int_{B_1} j(u_\Phi)\, dx,$$

so Proposition 4.4, joint with $j \in C^1(\mathbb{R}_+)$, gives that $\Phi \mapsto \mathcal{J}(E^\Phi)$ is C^1. Let us denote by $u_t := u_{t\Phi}$. We begin by noticing that

$$J'(t) = \int_{B_1} j'(u_t) u_t'\, dx = \int_{B_1} \nabla w_t \cdot \nabla u_t'\, dx = \int_{\partial E^{t\Phi}} w_t\, g_t\, d\mathcal{H}^{n-1}$$

where we used (2.6) and (4.3). Therefore, we have the following shape derivative

$$\mathcal{J}'(E_\Phi)[\Psi] = \int_{\partial E^\Phi} w_{E^\Phi} (\Psi \cdot \nu_{E^\Phi})\, d\mathcal{H}^{n-1} = \int_{E^\Phi} \operatorname{div}(w_{E^\Phi} \Psi)\, dx.$$

By Proposition 4.5 we know that the map $\Phi \mapsto w_{E^\Phi} \in W^{2,q}(B_{1-\varepsilon})$ is of class C^1 in a neighbourhood of 0. In particular, if Φ and ε are small, we have $E^\Phi \subset B_{1-\varepsilon}$. Finally, since for every fixed $f \in W^{1,1}$ the map $\Phi \mapsto \int_{E^\Phi} f\, dx$ is of class C^1 in a neighbourhood of 0 (see for instance [15, Theorem 5.2.2]), then $\mathcal{J}$ is of class C^2. We now compute again $J'(t)$ as

$$J'(t) = \int_{E^{t\Phi}} \operatorname{div}(w_t \widetilde{\Phi}_t)\, dx,$$

so that by the Hadamard's formula Lemma 4.2 we get

$$J''(t) = \int_{\partial E^{t\Phi}} \operatorname{div}(w_t \widetilde{\Phi}_t)\widetilde{\Phi}_t \cdot \nu_t\, d\mathcal{H}^{n-1} + \int_{\partial E^{t\Phi}} (w_t' \widetilde{\Phi}_t + w_t\, \partial_t \widetilde{\Phi}_t) \cdot \nu_t\, d\mathcal{H}^{n-1}.$$

Formula (4.9) finally follows by differentiating in t the equation $\widetilde{\Phi}_t \circ (\mathrm{Id} + t\Phi) = \Phi$, leading to

$$\partial_t \widetilde{\Phi}_t = -(D\widetilde{\Phi}_t)\widetilde{\Phi}_t.$$

□

Remark 4.7. When we evaluate (4.8) in $t=0$, we get

$$J'(0) = \int_{\partial E} w_E(\Phi\cdot\nu_E)\,d\mathcal{H}^{n-1}.$$

When Φ is orthogonal to ∂E (i.e. $\Phi = g_0\nu_0$ on ∂E), then (4.9) in $t=0$ reads

$$J''(0) = \int_{\partial E}\left(w_0'\,g_0 + w_0 H_E g_0^2 + \frac{\partial w_0}{\partial \nu_0}g_0^2\right)d\mathcal{H}^{n-1},$$

with H_E the mean curvature of ∂E. Indeed, notice that in this case

$$g_0\operatorname{div}(\Phi) - ((D\Phi)\Phi)\cdot\nu_0 = g_0^2\operatorname{div}^{\partial E}(\nu_0) = g_0^2 H_E.$$

The previous computations allows us to write the Lagrangian for the maximizing problem

$$\max_{|E|=m}\mathcal{J}(E). \tag{4.10}$$

In the following, for x close to ∂B_*, we let $\pi_{\partial B_*}(x)$ be the unique projection of x onto ∂B_*, and we let $\nu_0(x) = \nu_0(\pi_{\partial B_*}(x))$ be the extended unit normal to ∂B_*.

Corollary 4.8. *For $\tau\in\mathbb{R}$ and $E\subset B_1$, we define*

$$\mathcal{L}_\tau(E) := \mathcal{J}(E) + \tau|E|.$$

For $m\in(0,|B_1|)$ and $E=B_*$ the centered ball of volume m, we set

$$\tau = -w_0\big|_{\partial B_*}. \tag{4.11}$$

Then $\Phi\in W^{2,q}(B_1;\mathbb{R}^n)\mapsto\mathcal{L}_\tau(B_*^\Phi)$ is of class C^2 near 0, and

(i) $\mathcal{L}_\tau'(B_*)\equiv 0$

(ii) for every $\Phi\in W^{2,q}(B_1;\mathbb{R}^n)$ small enough such that Φ is normal on ∂B_* and $\widetilde{\Phi}_t=\Phi$ for $t\in[0,1]$, if we denote $L(t)=\mathcal{L}_\tau(B_*^{t\Phi})$ then for every t,

$$L''(t) = \int_{\partial B_*^{t\Phi}}\left(w_t'g + g^2\frac{\partial w_t}{\partial\nu_0}\right)(\nu_t\cdot\nu_0)\,d\mathcal{H}^{n-1} + \int_{\partial B_*^{t\Phi}}(w_t+\tau)(\nu_t\cdot\nu_0)g^2\operatorname{div}(\nu_0)\,d\mathcal{H}^{n-1}, \tag{4.12}$$

where $g=\Phi\cdot\nu_0$.

Proof. Let $V(t)=|B_*^{t\Phi}|$. From Lemma 4.2 we have

$$V'(0) = \int_{\partial B_*}(\Phi\cdot\nu)\,d\mathcal{H}^{n-1},$$

so *(i)* follows from (4.8).

Using two times Lemma 4.2 gives (see also [10, Proof of Proposition 4.1]),

$$V''(t) = \int_{\partial B_*^{t\Phi}}\operatorname{div}(\Phi)(\Phi\cdot\nu_t)d\mathcal{H}^{n-1}$$

so with Proposition 4.6 and the fact that $0=\partial_t\widetilde{\Phi}_t$, we get, letting $g_t=\Phi\cdot\nu_t$,

$$L''(t) = \int_{\partial B_*^{t\Phi}} w_t'g_t + (\nabla w_t\cdot\Phi)g_t + \int_{\partial B_*^{t\Phi}}(w_t+\tau)g_t\operatorname{div}(\Phi).$$

Now, as Φ is normal on ∂B_* and $\Phi\circ(\operatorname{Id}+t\Phi)=\Phi$, we have for every t that $\Phi_{|\partial B_*^{t\Phi}} = g\nu_0$, and we conclude using $0=\partial_t\widetilde{\Phi}_t = -(D\Phi)\Phi = -g\nabla g\cdot\nu_0$ so that $g\operatorname{div}(\Phi)=g^2\operatorname{div}(\nu_0)$. □

Remark 4.9. The general scheme of proof of Theorem 1.2 was strongly inspired by [18]; nevertheless, it seems to us that [18, Section 2.5.7 and formula before (73)] are incomplete (note that these computations are also used in [19, 5]). Compared to our formula (4.12), the first discrepancy is due to the assumption $\widetilde{\Phi}_t = \Phi$. This seems to be missing in [18], but without such assumption we should have

$$L''(t) = \mathcal{L}_\tau''(B_*^{t\Phi})[\widetilde{\Phi}_t, \widetilde{\Phi}_t] + \mathcal{L}_\tau'(B_*^{t\Phi})[\partial_t \widetilde{\Phi}_t]$$

while in [18] it is used $L''(t) = \mathcal{L}_\tau''(B_*^{t\Phi})[\Phi, \Phi]$, which seems incorrect to us without assuming $\widetilde{\Phi}_t$ constant in t. The second discrepancy is about the outer unit normal appearing in the equality: ν in [18, Formula before (73)] represents the outer unit normal to ∂B_*, but it should be replaced with $\widehat{\nu}_t = \nu_t \circ (\mathrm{Id} + t\Phi)$.

Remark 4.10. We also point out a computation mistake in [10, Proof of Proposition 4.1] where H should be replaced with $\widehat{\operatorname{div} \nu_0} = (\operatorname{div} \nu_0) \circ (\mathrm{Id} + t\Phi)$. Indeed, $\operatorname{div} \nu_0(x)$ is not the mean curvature of ∂E in $\pi_{\partial E}(x)$, it represents, up to the sign, the mean curvature in x of the boundary of the outer parallel set

$$\partial \{ \, y \in \mathbb{R}^n \mid d(y, \partial E) \le d(x, \partial E) \, \} .$$

4.2 Step 2: coercivity of the second order shape derivative in B_*

The goal of this section is to prove a sufficient optimality condition for problem (4.10):

Proposition 4.11. *Let $m \in (0, |B_1|)$, B_* the centered ball of volume m, and $j \in C^1(\mathbb{R}_+) \cap C^2((0, +\infty))$ such that $j'(0) \ge 0$, $j'' > 0$ on $(0, \infty)$ and*

$$j''(u_0) \in L^1(B_1).$$

Then there exist positive constants $c = c(j, m), \eta = \eta(j, m)$ such that for every $\Phi \in W^{2,q}(B_1; \mathbb{R}^n)$ such that $|B_*^\Phi| = |B_*|$ and $\|\Phi\|_\infty < \eta$ we have

$$\mathcal{L}_\tau''(B_*)[\Phi, \Phi] \le -c \|\Phi \cdot \nu_0\|^2_{L^2(\partial B_*)},$$

where $\mathcal{L}_\tau$ is the Lagrangian defined in Corollary 4.8.

The rest of this section is devoted to the proof of Proposition 4.11, see the two substeps below. To simplify the computation of $\mathcal{L}_\tau''(B_*)$, we first remark that this quadratic form depends only on the trace of Φ on ∂B_* (this is a well known fact, see for instance [17, proof of Theorem 2.3]). We can therefore replace Φ by $\Psi(x) := \Phi(\pi_{\partial B_*}(x))$ in a neighborhood of ∂B_* and extend it smoothly to B_1. It is also well-known (see [15, Theorem 5.9.2]) that as $\mathcal{L}_\tau'(B_*) = 0$, $\mathcal{L}_\tau''(B_*)$ only depends on $g = (\Phi \cdot \nu_0)_{|\partial B_*}$, so we can also assume Φ to be normal on ∂B_*. With these adaptations, we have $\widetilde{\Phi}_t = \Phi$ for every t, and we can apply Corollary 4.8, and $\mathcal{L}_\tau''(B_*)$ will reduce to a quadratic form on $L^2(\partial B_*)$ (see also [5] and [6, Lemma 35]).

Substep 1: Rewriting $\mathcal{L}_\tau''(B_*)$

We will compute $\mathcal{L}_\tau''(B_*)$ in terms of $g = (\Phi \cdot \nu_0)_{\partial B_*}$, defining a suitable eigenvalue problem that has been introduced in [5] (see in particular [5, Theorem III] and [6, Proposition 34]): from Proposition 4.4 and Proposition 4.5 we can consider $u_0' := u_{B_*}'[\Phi]$ and $w_0' := w_{B_*}'[\Phi]$ that are the unique solution in $H_0^1(B_1)$ to the equations

$$-\Delta u_0' = g \, d\mathcal{H}^{n-1} \llcorner_{\partial B_*} \qquad\qquad -\Delta w_0' = j''(u_0) u_0'.$$

Using Corollary 4.8 we get

$$\mathcal{L}_\tau''(B_*)[\Phi,\Phi] = \int_{\partial B_*}\left(gw_0' - \left|\frac{\partial w_0}{\partial \nu_0}\right| g^2\right) d\mathcal{H}^{n-1}.$$

For every $g \in L^2(\partial B_*)$ we consider $(U_g, W_g) \in H_0^1(B_1;\mathbb{R}^2)$ the unique solution to the coupled boundary value problems

$$-\Delta U_g = g d\mathcal{H}^{n-1}\llcorner_{\partial B_*} \qquad\qquad -\Delta W_g = j''(u_0)U_g.$$

Denoting $\mathrm{tr}_{\partial B_*} : W^{1,2}(B_1) \to L^2(\partial B_*)$ the trace operator on ∂B_*, we also define the operator

$$T : g \in L^2(\partial B_*) \longmapsto \mathrm{tr}_{\partial B_*}(W_g) \in L^2(\partial B_*),$$

which is symmetric, as $\int_{\partial B_*} hW_g d\mathcal{H}^{n-1} = \int_{B_1} j''(u_0)U_gU_h$. Then for $h, g \in L^2(\partial B_*)$ we define

$$l_2(h,g) := \int_{\partial B_*} h(W_g - |\partial_\nu w_0| g)\, d\mathcal{H}^{n-1} = \int_{\partial B_*} hTg\, d\mathcal{H}^{n-1} - |\partial_\nu w_0|_{\partial B_*}\int_{\partial B_*} hg\, d\mathcal{H}^{n-1}$$

Note that, when $g = g_\Phi = (\Phi\cdot\nu_0)_{|\partial B_*}$, we have $U_g = u_0'$ and $W_g = w_0'$, and therefore

$$\mathcal{L}_\tau''(B_*)[\Phi,\Phi] = l_2(g,g).$$

Substep 2: diagonalization of T

In the following, when $n \geq 2$ (the case $n = 1$ will be dealt with separately, see below in the proof of Proposition 4.11) and $k \in \mathbb{N}$ we denote $(Y_{k,m})_{1\leq m\leq M(k)}$ the real spherical harmonics of degree k (that is to say, a basis of the space of homogeneous, real and harmonic polynomials of degree k), and it is well known that these are eigenfunctions of the opposite of the spherical Laplacian $-\Delta_{\mathbb{S}^{n-1}}$, with eigenvalue $\Lambda_k = k(k+n-2)$ ($M(k)$ being its multiplicity).

In this step we show that T is diagonalizable and that the spherical harmonics are its eigenfunctions:

Proposition 4.12. *Let $n \geq 2$, and let $j \in C^1(\mathbb{R}_+)\cap C^2((0,\infty))$ be such that $j'(0) \geq 0$, $j'' > 0$ and*

$$j''(u_0) \in L^1(B_1).$$

Then for every $k \in \mathbb{N}^*$, the spherical harmonics of degree k, $(Y_{k,m})_{1\leq m\leq M(k)}$ are eigenfunctions of T with the same corresponding eigenvalue $1/\lambda_k$. Moreover, the sequence $(\lambda_k)_{k\in\mathbb{N}^*}$ is non-decreasing in k and we have

$$\lambda_1 > \left(|\partial_{\nu_0} w_0|_{|\partial B_*}\right)^{-1}, \tag{4.13}$$

where $w_0 = w_{B_*}$ is the adjoint state of $u_0 = u_{B_*}$.

Remark 4.13. The previous result is also valid for $k = 0$, but the computations are slightly different, and we don't need it as we naturally work in the space $\left\{ g \in L^2(\partial B_*) \,\middle|\, \int_{\partial B_*} g d\mathcal{H}^{n-1} = 0\right\}$.

Proof. • Let $k \geq 1$ be fixed. Let us define $|x| = r$ and $|x|^{-1}x = \theta$; in the following computations we will write $u_0(r)$ for $u_0(x)$ with $|x| = r$. To show that $Y_{k,m}(\theta)$ are eigenfunctions for T, we look for a nontrivial function

$$W(x) = \varphi(r)Y_{k,m}(\theta),$$

solving

$$\begin{cases} (-\Delta)\left(\frac{1}{j''(u_0)}(-\Delta)\right)W = \lambda W d\mathcal{H}^{n-1}\llcorner_{\partial B_*} & \text{in} \quad B_1, \\ W = \left(\frac{1}{j''(u_0)}\right)\Delta W = 0 & \text{on} \quad \partial B_1, \end{cases} \tag{4.14}$$

	ψ			$\frac{1}{j''(u_0)}\mathcal{D}_r^k\psi$			$\partial_r(r^{-k}\psi)$
	$r=0$	$r=r_*$	$r=1$	$r=0$	$r=r_*$	$r=1$	$r=r_*$
ψ_1	0	r_*^k	1	0	0	0	0
ψ_2	$+\infty$			0	0		$(2-n-2k)r_*^{1-n-2k}$
$\widetilde{\psi}_1$	0	0	$\widetilde{\psi}_1(1)$	0	r_*^k	1	0
$\widetilde{\psi}_2$		0	$\widetilde{\psi}_2(1)$	$+\infty$	r_*^{2-n-k}	1	0

Table 1: Evaluation table for ψ_β and $\widetilde{\psi}_\beta$. Empty cells are not needed for the computation.

for some $\lambda \in \mathbb{R}$.
Let us denote by Δ_r and $\mathcal{D}_r^k$ the differential operators on functions of one variable, defined as

$$\Delta_r\varphi = r^{1-n}\partial_r(r^{n-1}\partial_r\varphi), \qquad \mathcal{D}_r^k\varphi = \Delta_r\varphi - \frac{\Lambda_k}{r^2}\varphi = r^{1-n-k}\partial_r\left(r^{n-1+2k}\partial_r\left(r^{-k}\varphi\right)\right), \quad (4.15)$$

so that

$$\Delta(\varphi(r)Y_{k,m}(\theta)) = \left(\Delta_r + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}}\right)(\varphi(r)Y_{k,m}(\theta)) = Y_{k,m}(\theta)\mathcal{D}_r^k\varphi(r).$$

As a consequence, we look for φ solution of the ODE:

$$\begin{cases} \mathcal{D}_r^k\left(\frac{1}{j''(u_0)}\mathcal{D}_r^k\varphi\right)(r) = \lambda\varphi\delta_{r_*} \text{ in } (0,1), \\ \varphi(1) = \left(\frac{1}{j''(u_0)}\right)\mathcal{D}_r^k\varphi(1) = 0 \end{cases}$$

and such that the behavior of φ at 0 makes W smooth enough near the origin.
Using the second formulation of $\mathcal{D}_r^k$ in (4.15), we introduce four independent functions

$$\psi_1(r) = r^k, \qquad \psi_2(r) = r^{2-n-k},$$

$$\text{and} \qquad \widetilde{\psi}_\beta(r) = r^k\int_{r_*}^r s^{-(n-1+2k)}\int_{r_*}^s j''(u_0(t))\,\psi_\beta(t)t^{n+k-1}\,dt\,ds, \qquad \beta = 1,2,$$

which respectively solve the equations

$$\mathcal{D}_r^k\psi_\beta = 0, \qquad \mathcal{D}_r^k\widetilde{\psi}_\beta = j''(u_0)\psi_\beta, \qquad \beta = 1,2. \quad (4.16)$$

We now define φ in the following way

$$\varphi(r) = \begin{cases} c_1^-\psi_1 + c_2^-\psi_2 + c_3^-\widetilde{\psi}_1 + c_4^-\widetilde{\psi}_2 & r \le r_*, \\ c_1^+\psi_1 + c_2^+\psi_2 + c_3^+\widetilde{\psi}_1 + c_4^+\widetilde{\psi}_2 & r > r_*. \end{cases}$$

The behavior of φ near $0, r_*$ and 1 will add 7 independent conditions (see Table 1 for the values of ψ_β and its derivatives):

- near 0, we require $\varphi(0) = 0$: indeed, if this is not the case, as $Y_{k,m}$ is not constant, then $\varphi(r)Y_{k,m}(\theta)$ cannot be continuous at 0. Similarly, we require $\mathcal{D}_r^k\varphi(0) = 0$ so that ΔW is continuous at 0.
  As $\mathcal{D}_r^k\psi_1(0) = \mathcal{D}_r^k\psi_2(0) = \mathcal{D}_r^k\widetilde{\psi}_1(0) = 0$ while $\mathcal{D}_r^k\widetilde{\psi}_2(0) = +\infty$, the first condition is $c_4^- = 0$.
  And as $\psi_1(0) = \widetilde{\psi}_1(0) = 0$ while $\psi_2(0) = +\infty$, the second condition reads $c_2^- = 0$.

– near r_*, we require three continuity conditions, namely for φ, $\partial_r(r^{-k}\varphi)$ and $\frac{\mathcal{D}_r^k\varphi}{j''(u_0)}$: as we have chosen $\widetilde{\psi}_\beta$ so that they vanish at r_*, these respectively lead to the three equations:

$$\begin{cases} c_1^- = c_1^+ \\ 0 = c_2^- = c_2^+ \\ c_3^- r_*^k = c_3^+ r_*^k + c_4^+ r_*^{2-n-k} \end{cases}$$

– finally, the two boundary conditions $\varphi(1) = \frac{1}{j''(u_0)}\mathcal{D}_r^k\varphi(1) = 0$ write:

$$\begin{cases} c_1^+ + c_3^+\widetilde{\psi}_1(1) + c_4^+\widetilde{\psi}_2(1) = 0 \\ c_3^+ + c_4^+ = 0 \end{cases}$$

where the summability assumption on $j''(u_0)$ ensures that $\widetilde{\psi}_1(1)$ and $\widetilde{\psi}_2(1)$ are well defined and finite.

We therefore have 7 linear conditions for 8 variables, so there is at least a one dimensional vectorial space of solutions, and we choose φ a nontrivial one. Notice also that the 7 conditions are independent: if $c_1^- = 0$, then all the constants $c_i^\pm$ need to be zero, as by direct computation ($t^k < t^{2-n-k}$ for $t \in (0,1)$)

$$\widetilde{\psi}_1(1) - \widetilde{\psi}_2(1) < 0.$$

To conclude that $W = \varphi(r)Y_{k,m}(\theta)$ is a solution to (4.14), it remains to observe that $\varphi(r_*) \neq 0$, so that

$$\lambda := \frac{\left[\partial_r\left(\frac{\mathcal{D}_r^k\varphi}{j''(u_0)}\right)\right](r_*)}{\varphi(r_*)}$$

is well defined, where $[v](r_*)$ denotes the jump of the function v at r_*; this is the case as $\varphi(r_*) = c_1^- r_*^k$, and we have just observed that if $c_1^- = 0$, then $\varphi \equiv 0$.

• We now prove the monotonicity of $(\lambda_k)_k$:

For $k \geq 1$, let φ_k one of the solutions computed in the previous item. We choose the scaling so that

$$\lambda_k\varphi_k(r_*) = 1. \tag{4.17}$$

We make the proof in two steps: first we show that $\varphi_k \geq 0$, and then we study the difference $\psi_k := \varphi_{k+1} - \varphi_k$:

Step 1: let

$$\Phi_k = \frac{1}{j''(u_0)}\mathcal{D}_r^k\varphi_k,$$

where φ_k is one solution computed in the previous item, and so that

$$\begin{cases} \Delta_r\Phi_k = \dfrac{\Lambda_k}{r^2}\Phi_k + \delta_{r_*} \quad \text{in } (0,1), \\ \Phi_k(0) = \Phi_k(1) = 0, \end{cases} \tag{4.18}$$

where δ_{r_*} denotes the Dirac mass concentrated in r_*. We claim that $\Phi_k \leq 0$. Indeed, let us assume that $\bar{r}$ is a maximum point for Φ_k. If $\bar{r} = 1$ or $\bar{r} = 0$, the claim is proved. On the other hand, the Dirac mass in (4.18) implies that

$$\Phi_k'(r_*^+) - \Phi_k'(r_*^-) = \lambda_k\varphi_k(r_*) > 0,$$

which ensures that r_* cannot be a maximum point. Therefore, we may evaluate the ODE (4.18) in the maximum point $\bar{r}$ obtaining, after the expansion of Δ_r,

$$0 \geq \Phi_k''(\bar{r}) = \Delta_r \Phi_k(\bar{r}) = \frac{\Lambda_k}{\bar{r}^2}\Phi_k(\bar{r}),$$

which proves the claim.
Since $\Phi_k \leq 0$, then

$$\begin{cases} \Delta_r \varphi_k \leq \dfrac{\Lambda_k}{r^2}\varphi_k & r \in (0,1), \\ \varphi_k(0) = \varphi_k(1) = 0. \end{cases}$$

With the same argument as above for Φ_k, simplified by the fact that φ_k is of class C^2 in the whole interval $(0,1)$, we get (by looking at its minimum) that $\varphi_k \geq 0$.
Step 2: let $\psi_k := \varphi_{k+1} - \varphi_k$, and let $\Psi_k := \Phi_{k+1} - \Phi_k$, and note that both are $C^2(0,1)$. We claim that $\Psi_k \geq 0$. First we notice that, since $\Phi_k \leq 0$ and $\Lambda_k < \Lambda_{k+1}$,

$$\left(\Delta_r - \frac{\Lambda_{k+1}}{r^2}\right)\Phi_{k+1} = \delta_{r_*} = \left(\Delta_r - \frac{\Lambda_k}{r^2}\right)\Phi_k \leq \left(\Delta_r - \frac{\Lambda_{k+1}}{r^2}\right)\Phi_k,$$

so that

$$\left(\Delta_r - \frac{\Lambda_{k+1}}{r^2}\right)(\Psi_k) \leq 0. \tag{4.19}$$

As done for φ_k in Step 1, (4.19) ensures that $\Psi_k \geq 0$.
Finally, $\Psi_k \geq 0$ reads

$$0 \leq \left(\Delta_r - \frac{\Lambda_{k+1}}{r^2}\right)\varphi_{k+1} - \left(\Delta_r - \frac{\Lambda_k}{r^2}\right)\varphi_k \leq \left(\Delta_r - \frac{\Lambda_{k+1}}{r^2}\right)(\varphi_{k+1} - \varphi_k),$$

where we have used in the last inequality $\Lambda_k\varphi_k \leq \Lambda_{k+1}\varphi_k$. As before, this implies $\varphi_{k+1} \leq \varphi_k$, which because of (4.17) leads to $\lambda_k \leq \lambda_{k+1}$.
• We can now compute the eigenvalue λ_1: we drop the exponent k in the notations as its value will remain 1 in this paragraph. Using equations (4.16) and $c_2^- = c_4^- = 0$, we obtain

$$\lambda_1 = \frac{\left[\partial_r\left(\frac{\mathcal{D}_r\varphi_1}{j''(u_0(r))}\right)\right](r_*)}{\varphi_1(r_*)} = \frac{c_3^+ - c_3^- + \frac{1-n}{r_*^n}c_4^+}{c_1^- r_*}.$$

Let us recall the previous system when $k = 1$:

$$\begin{cases} c_2^- = c_2^+ = c_4^- = 0 \\ c_1^- = c_1^+ \\ c_3^- r_* = c_3^+ r_* + c_4^+ r_*^{1-n} \\ c_1^+ + c_3^+\widetilde{\psi}_1(1) + c_4^+\widetilde{\psi}_2(1) = 0 \\ c_3^+ + c_4^+ = 0 \end{cases}$$

From the third equation, we compute $c_3^+ - c_3^-$ in terms of c_4^+ and therefore

$$\lambda_1 = -\frac{n}{r_*^{n+1}}\frac{c_4^+}{c_1^-}.$$

The second equation joint with the fourth one and the fifth one expresses c_1^- in terms of c_4^+, which eventually leads to

$$\lambda_1 = \frac{n}{r_*^{n+1}\Big[\widetilde{\psi}_2(1) - \widetilde{\psi}_1(1)\Big]}. \tag{4.20}$$

Now observe that by Fubini's theorem

$$\begin{aligned}
\widetilde{\psi}_2(1) - \widetilde{\psi}_1(1) &= \int_{r_*}^1 s^{-n-1} \int_{r_*}^s j''(u_0(t))\,(t^{1-n} - t)t^n \,dt\,ds \\
&= \int_{r_*}^1 j''(u_0(t))\,t(1-t^n) \int_t^1 s^{-n-1}\,ds\,dt \qquad (4.21)\\
&= \frac{1}{n}\int_{r_*}^1 j''(u_0(t))\,t^{1-n}(1-t^n)^2\,dt
\end{aligned}$$

This leads to

$$\frac{1}{\lambda_1} = \frac{1}{n^2 r_*^{n-1}} \int_{r_*}^1 j''(u_0(t))\,r_*^{2n}\,(1-t^n)^2\,t^{1-n}\,dt \tag{4.22}$$

On the other hand, we compute $|\partial_\nu w_0|_{|\partial B_*}$: we first notice

$$\begin{aligned}
\left|\frac{\partial w_0}{\partial \nu}\right|_{|\partial B_*} &= \frac{1}{P(B_*)}\int_{\partial B_*}\left(-\frac{\partial w_0}{\partial \nu}\right)d\mathcal{H}^{n-1} = \frac{1}{P(B_*)}\int_{B_*}(-\Delta w_0)\,dx \\
&= \frac{1}{P(B_*)}\int_{B_1}(-\Delta u_0) j'(u_0)\,dx \geq \frac{1}{P(B_*)}\int_{B_1} j''(u_0)|\nabla u_0|^2\,dx,
\end{aligned}$$

as $j'(u_0) \in W^{1,1}(B_1)$ and $j' \geq 0$. Then, identifying $u_0(x) = u_0(|x|)$, we observe that $\nabla u_0(x) = \partial_r u_0(|x|)$ and that u_0 solves

$$\begin{cases}
-\partial_r(r^{n-1}\partial_r u_0(r)) = r^{n-1}\chi_{(0,r_*)}(r) & r \in (0,1), \\
\partial_r u_0(0) = 0, \\
u_0(1) = 0.
\end{cases}$$

Hence,

$$\partial_r u_0(r) = -r^{1-n}\int_0^{r\wedge r_*} s^{n-1}\,ds = -\frac{(r\wedge r_*)^n}{n r^{n-1}},$$

which implies

$$|\partial_\nu w_0|_{|\partial B_*} \geq \frac{1}{n^2 r_*^{n-1}}\int_0^1 j''(u_0(t))(t\wedge r_*)^{2n}\,t^{1-n}\,dt. \tag{4.23}$$

By comparing the integrands in (4.22) and (4.23), we easily conclude $\lambda_1 > (|\partial_\nu w_0|_{|\partial B_*})^{-1}$. □

Conclusion of the proof of Proposition 4.11

Proof of Proposition 4.11. • In the case $n = 1$, $L^2(\partial B_*)$ is a two dimensional space, B_*^Φ is an interval, and because we assume $|B_*^\Phi| = |B_*|$, we can restrict to Φ being a translation, in which case $g(x) = \mathrm{sgn}(x)\alpha$ for some $\alpha \in (r_* - 1, 1 - r_*)$. Let us denote by $W = W_{\mathrm{sgn}}$. We notice that the functions W and $\partial_\nu w_0$ are odd, so that the shape derivative computed in Corollary 4.8 reads

$$\mathcal{L}_\tau''(B_*)[\Phi,\Phi] = l_2(g,g) = 2\alpha^2(W(r_*) - |\partial_\nu w_0|(r_*)),$$

Finally, reproducing the proof of (4.13) in Proposition 4.12 we get that $W(r_*) < |\partial_\nu w_0|(r_*)$, and this gives

$$\mathcal{L}''_\tau(B_*)[\Phi,\Phi] \leq -c\|\Phi\cdot\nu_0\|^2_{L^2(\partial B_*)}.$$

• Assume now $n \geq 2$. Let Φ be as in the proposition. We denote $g = \Phi\cdot\nu_{0|\partial B_*}$, and recall that

$$\mathcal{L}''_\tau(B_*)[\Phi,\Phi] = l_2(g,g) = \int_{\partial B_*} gTg - |\partial_\nu w_0|_{|B_*}\int_{\partial B_*} g^2.$$

From the volume constraint, if $\|g\|_\infty \leq \eta$ then

$$0 = \frac{1}{n}\int_{\partial B_*}((1+g)^n - 1)\,d\mathcal{H}^{n-1} = \alpha_0 + \frac{1}{n}\sum_{k=2}^{n}\binom{n}{k}\int_{\partial B_*} g^k\,d\mathcal{H}^{n-1} \geq |\alpha_0| - C\eta\|g\|_2,$$

where $\alpha_0 = \int_{\partial B_*} g$. We define $\widetilde{g} = g - \alpha_0$ and we first study $l_2(\widetilde{g},\widetilde{g})$: as $(Y_{k,m})_{k\in\mathbb{N}^*,1\leq m\leq M(k)}$ is an orthogonal basis of $\{\, h\in L^2(\partial B_*) \mid \int_{\partial B_*} hd\mathcal{H}^{n-1} = 0\,\}$, we can decompose $\widetilde{g} = \sum_{k\geq1,m}\alpha_{k,m}Y_{k,m}$ and we get from Proposition 4.12:

$$\begin{aligned} l_2(\widetilde{g},\widetilde{g}) &= \sum_{k\geq1,m}\frac{1}{\lambda_k}\alpha^2_{k,m}\|Y_{k,m}\|^2_{L^2(\partial B_*)} - |\partial_\nu w_0|_{|\partial B_*}\sum_{k\geq1,m}\alpha^2_{k,m}\|Y_{k,m}\|^2_{L^2(\partial B_*)} \\ &\leq \left(\frac{1}{\lambda_1} - |\partial_\nu w_0|_{|\partial B_*}\right)\|\widetilde{g}\|^2_{L^2(\partial B_*)}, \end{aligned}$$

so from (4.13) there exists $c > 0$ such that

$$l_2(\widetilde{g},\widetilde{g}) \leq -c\|\widetilde{g}\|^2_{L^2(\partial B_*)}.$$

Now, as l_2 is continuous on $L^2(\partial B_*)$, we get

$$\begin{aligned} \mathcal{L}''_\tau(B_*)[\Phi,\Phi] = l_2(g,g) &= l_2(\widetilde{g},\widetilde{g}) + 2l_2(g,\alpha_0) - l_2(\alpha_0,\alpha_0) \\ &\leq -c\|g-\alpha_0\|^2_{L^2(\partial B_*)} + 2|\alpha_0|\|g\|_{L^2(\partial B_*)} + |\alpha_0|^2 \end{aligned}$$

and we conclude using $\|g-\alpha_0\|_{L^2(\partial B_*)} \geq \|g\|_{L^2(\partial B_*)} - |\alpha_0|$ and $|\alpha_0| \leq C\eta\|g\|_{L^2(\partial B_*)}$ with $\eta > 0$ small enough.

□

4.3 Step 3: improved continuity of the second order shape derivative

We introduce here some useful notations.

Definition 4.14. Let X, Y be two normed vector spaces, and $J : X \to Y$. We write

$$J(x) = \omega^X_Y(x)$$

to indicate that

$$\lim_{\|x\|_X\to0}\|J(x)\|_Y = 0.$$

In particular, when $Y = \mathbb{R}$ we only write $\omega^X_Y = \omega^X$. Moreover, when $Y = W^{k,p}(B_1;\mathbb{R}^n)$ and $X = W^{j,q}(B_1;\mathbb{R}^n)$ then we write

$$\omega^X_Y = \omega^{j,q}_{k,p}.$$

When $Y = L^p(B_1)$ we write $\omega^X_Y = \omega^X_p$.

Also, when $J : [0,1]\times X \to Y$ depends on the extra parameter $t \in [0,1]$, $J(x) = \omega^X_Y(x)$ is meant in a uniform sense in t, namely $\lim_{\|x\|_X\to0}\sup_t\|J(t,x)\|_Y = 0$.

The main aim of this section is to prove the following.

Proposition 4.15. *Let $m \in (0, |B_1|)$, $q > n$ and $j \in C^1(\mathbb{R}_+) \cap C^2((0,\infty))$ such that*

$$\limsup_{s\to 0^+} s^\alpha j''(s) < +\infty.$$

Then
$$L''(t) = L''(0) + \omega^{2,q}(\Phi)\,\|\Phi\|^2_{L^2(\partial B_*)},$$
that is to say, for every $\varepsilon > 0$ there exists $\eta > 0$ such that for every $t \in [0,1]$ and for every $\Phi \in W^{2,q}(B_1;\mathbb{R}^n)$ orthogonal to ∂B_* such that $\|\Phi\|_{2,q} \le \eta$, then

$$|L''(t) - L''(0)| \le \varepsilon \|\Phi\|^2_{L^2(\partial B_*)},$$

where L is defined in Corollary 4.8.

As explained in the introduction of the proof of Proposition 4.11, it is not restrictive to assume Φ to be constant in the normal direction to ∂B_* (in a neighborhood of ∂B_*), which allows to apply Corollary 4.8, leading to the following formula:

$$L''(t) = \int_{\partial B_*} J_t^\tau \left(\widehat{w_t'} g + g^2 \widehat{\nabla w_t} \cdot \widehat{\nu}_t\right) d\mathcal{H}^{n-1} + b_t(\Phi,\Phi), \tag{4.24}$$

where for every function h we have denoted by $\widehat{h} = h \circ (\mathrm{Id} + t\Phi)$, $J_t^\tau = \mathrm{Jac}^{\partial B_*}(\mathrm{Id} + t\Phi)$, $g = \Phi \cdot \nu_0$ and

$$b_t(\Phi,\Phi) = \int_{\partial B_*} J_t^\tau (\widehat{w}_t - w_0)(\widehat{\nu}_t \cdot \nu_0) g^2 \,\widehat{\mathrm{div}(\nu_0)}\, d\mathcal{H}^{n-1}. \tag{4.25}$$

Notice also that we used $\widehat{g} = g$ and $\widehat{\nu_0} = \nu_0$. To prove Proposition 4.15 we need some geometric estimates on $\widehat{\nu}_t$ and $I_n + tD\Phi$, and estimates on $\widehat{w}_t$ and $\widehat{w_t'}$ that will be resumed in the following lemmas.

Lemma 4.16 ([10, Lemma 4.3, Lemma 4.7],[20, Lemma 3.7])**.** *Let $q > n$ and $\Phi \in W^{2,q}(B_1;\mathbb{R}^n)$. Then*

$$(I_n + tD\Phi)^{-1} = I_n + \omega^{2,q}_\infty(\Phi) \qquad \det(I_n + tD\Phi) = 1 + \omega^{2,q}_\infty(\Phi)$$
$$\widehat{\nu}_t = \nu_0 + \omega^{2,q}_\infty(\Phi), \qquad J_t^\tau = 1 + \omega^{2,q}_\infty(\Phi),$$

Lemma 4.17. *Let $m \in (0, |B_1|)$, $\Phi \in W^{2,q}(B_1;\mathbb{R}^n)$ with $q > n$. We denote $u_t := u_{B_*^{t\Phi}}$, and $\widehat{u_t'} = u_t' \circ (\mathrm{Id} + t\Phi)$. Then there exist constants $C = C(m,n,q)$ and $\delta = \delta(m,n,q)$ such that if $\|\Phi\|_{2,q} \le \delta$ then*

$$\|\widehat{u_t'}\|_{1,2} \le C \|\Phi\|_{L^2(\partial B_*)}. \tag{4.26}$$

Moreover, we have

$$\widehat{u_t'} = u_0' + \omega^{2,q}_{1,2}(\Phi)\|\Phi\|_{L^2(\partial B_*)}. \tag{4.27}$$

Proof. Let us recall that, by Proposition 4.4, u_t' solves the equation

$$\int_{B_1} \nabla u_t' \cdot \nabla \varphi \, dx = \int_{\partial B_*^{t\Phi}} \varphi\, \widetilde{\Phi}_t \cdot \nu_t \, d\mathcal{H}^{n-1} \qquad \forall \varphi \in H^1_0(B_1).$$

In particular, with the change of variables $x = (\mathrm{Id} + t\Phi)(y)$, we get

$$\int_{B_1} A_t \nabla \widehat{u_t'} \cdot \nabla \varphi \, dy = \int_{\partial B_*} J_t^\tau \varphi\, \widehat{g}_t \, d\mathcal{H}^{n-1} \qquad \forall \varphi \in H^1_0(B_1), \tag{4.28}$$

where

$$J_t^\tau := \mathrm{Jac}^{\partial B_*}(\mathrm{Id}+t\Phi), \qquad A_t := \det(I_n + tD\Phi)(I_n+tD\Phi)^{-1}(I_n+tD\Phi)^{-T}.$$

Choosing $\varphi = \widehat{u_t'}$, and noticing that by Lemma 4.16 we have that A_t is uniformly elliptic for small $\|\Phi\|_{2,q}$, then there exists a positive constant C such that

$$\|\nabla \widehat{u_t'}\|_2^2 \le C \int_{\partial B_*} |\widehat{g}_t \widehat{u_t'}|\, d\mathcal{H}^{n-1}.$$

Using Poincaré inequality, Young inequality, and the embedding $W^{1,2}(B_1) \hookrightarrow L^2(\partial B_*)$, we get for every $\eta > 0$

$$\|\widehat{u_t'}\|_{1,2}^2 \le C\Big(\eta \|\widehat{g}_t\|_2^2 + \frac{1}{\eta}\|\widehat{u_t'}\|_{1,2}^2\Big).$$

Formula (4.26) follows by choosing a suitable η, and by recalling $|\widehat{g}_t| = |\Phi\cdot\widehat{\nu}_t| \le |\Phi|$.

Analogously, subtracting the weak equations (4.34) solved by $\widehat{u_t'}$ and u_0', we have

$$\int_{B_1} (A_t \nabla \widehat{u_t'} - \nabla u_0')\cdot\nabla\varphi\, dy = \int_{\partial B_*} (J_t^\tau \widehat{g}_t - g_0)\varphi\, d\mathcal{H}^{n-1} \qquad \forall \varphi \in H_0^1(B_1). \tag{4.29}$$

In the rest of the proof, we evaluate (4.29) with the test function $\varphi = \widehat{u_t'} - u_0'$. We estimate the left-hand side of (4.29) by adding and subtracting $A_t\nabla u_0'\cdot\nabla\varphi$, and using the uniform ellipticity of A_t joint with Lemma 4.16, so that for some constant $c > 0$

$$\int_{B_1} (A_t \nabla \widehat{u_t'} - \nabla u_0')\cdot\nabla\varphi\, dy \ge c\|\nabla(\widehat{u_t'} - u_0')\|_2^2 - \int_{B_1} \omega_\infty^{2,q}(\Phi)\Big|\nabla u_0' \cdot \nabla(\widehat{u_t'} - u_0')\Big|\, dy.$$

Thanks to (4.26), we get

$$\int_{B_1} (A_t \nabla \widehat{u_t'} - \nabla u_0')\cdot\nabla\varphi\, dy \ge c\|\nabla(\widehat{u_t'} - u_0')\|_2^2 - \omega^{2,q}(\Phi)\|\Phi\|_{2,\partial B_*}^2. \tag{4.30}$$

On the other hand, we may estimate the right-hand side of (4.29) as before and get

$$\begin{aligned}\int_{\partial B_*} (J_t^\tau \widehat{g}_t - g_0)\varphi\, d\mathcal{H}^{n-1} &= \int_{\partial B_*} (1+\omega_\infty^{2,q}(\Phi))\Phi\cdot\omega_\infty^{2,q}(\Phi)(\widehat{u_t'} - u_0')\, d\mathcal{H}^{n-1} \\ &\le \omega^{2,q}(\Phi)\Big(\eta\|\Phi\|_{L^2(\partial B_*)}^2 + \frac{1}{\eta}\|\widehat{u_t'} - u_0'\|_{1,2}^2\Big)\end{aligned} \tag{4.31}$$

for every $\eta > 0$. Joining (4.29), (4.30), (4.31), and the Poincaré inequality, with the right choice of η we get

$$\|\widehat{u_t'} - u_0'\|_{1,2}^2 \le \omega^{2,q}(\Phi)\|\Phi\|_{L^2(\partial B_*)}^2.$$

□

Lemma 4.18. *Let $m \in (0, |B_1|)$ and $j \in C^1(\mathbb{R}_+)\cap C^2((0,\infty))$ such that*

$$\limsup_{s\to 0^+} s^\alpha j''(s) < +\infty.$$

Let $\Phi \in W^{2,q}(B_1;\mathbb{R}^n)$ with $q > n$, $w_t := w_{B_*^{t\Phi}}$ the adjoint state defined in Definition 2.15, and $\widehat{w_t'} = w_t'\circ(\mathrm{Id}+t\Phi)$. Then there exist constants $C = C(j,m,n,q)$ and $\delta = \delta(j,m,n,q)$ such that if $\|\Phi\|_{2,q} \le \delta$ then

$$\|\widehat{w_t'}\|_{1,2} \le C\|\Phi\|_{L^2(\partial B_*)}. \tag{4.32}$$

Moreover, we have

$$\widehat{w_t'} = w_0' + \omega_{1,2}^{2,q}(\Phi)\|\Phi\|_{L^2(\partial B_*)}. \tag{4.33}$$

Proof. As in the proof of Lemma 4.17, we have

$$\int_{B_1} A_t \nabla \widehat{w_t'} \cdot \nabla\varphi \, dy = \int_{B_1} \det(I_n + tD\Phi)\, j'(\widehat{u}_t)\widehat{u_t'}\varphi \, dy \qquad \forall \varphi \in H_0^1(B_1). \tag{4.34}$$

By standard elliptic estimates joint with the continuity of j', the equi-boundedness of u_t, and the geometric estimates Lemma 4.16, we get

$$\|\widehat{w_t'}\|_{1,2} \le C\|\widehat{u_t'}\|_2.$$

Therefore, (4.26) in Lemma 4.17 implies (4.32).

Analogously, we now estimate the norm of $\widehat{w_t'} - w_0'$ rewriting the equation as

$$\underbrace{\int_{B_1} (A_t\nabla\widehat{w_t'} - \nabla w_0') \cdot \nabla\varphi \, dy}_{I_1} = \underbrace{\int_{B_1} (J_t\, j'(\widehat{u}_t)\widehat{u_t'} - j'(u_0)u_0')\varphi \, dy}_{I_2} \qquad \forall \varphi \in H_0^1(B_1). \tag{4.35}$$

As done in the proof of Lemma 4.17, we evaluate (4.35) with the test function $\varphi = \widehat{w_t'} - w_0'$ and get

$$I_1 \ge c\|\nabla(\widehat{w_t'} - w_0')\|_2^2 - \omega^{2,q}(\Phi)\|\Phi\|^2_{L^2(\partial B_*)}. \tag{4.36}$$

We also notice that by the shape differentiability of $\widehat{u}_t$ (see the proof of Proposition 4.4) we have

$$\widehat{u}_t = u_0 + \omega_\infty^{2,q}(\Phi).$$

Also, the Hölder continuity of j' (see Remark 1.3) ensures that

$$j'(\widehat{u}_t) = j'(u_0) + \omega_\infty^{2,q}(\Phi).$$

Therefore, as in Lemma 4.17,

$$\begin{aligned} I_2 &\le C\int_{B_1} \left|\left(j'(u_0) + \omega_\infty^{2,q}(\Phi)\right)\widehat{u_t'} - j'(u_0)u_0'\right|\left|\widehat{w_t'} - w_0'\right| dy \\ &\le C\int_{B_1} |j'(u_0)|\left|\widehat{u_t'} - u_0'\right|\left|\widehat{w_t'} - w_0'\right| dy + \omega^{2,q}(\Phi)\int_{B_1}\left|\widehat{u_t'}\right|\left|\widehat{w_t'} - w_0'\right| dy \\ &\le C\omega^{2,q}(\Phi)\left(\eta\|\Phi\|^2_{2,\partial B_*} + \frac{1}{\eta}\|\widehat{w_t'} - w_0'\|^2_{1,2}\right) \end{aligned} \tag{4.37}$$

for every $\eta > 0$ and a suitable constant $C > 0$. Joining (4.35), (4.36), (4.37), and the Poincaré inequality, with the right choice of η we get

$$\|\widehat{w_t'} - w_0'\|^2_{1,2} \le \omega^{2,q}(\Phi)\|\Phi\|^2_{2,\partial B_*},$$

which implies (4.33). □

Proof of Proposition 4.15. **Step 1:** we first prove

$$b_t(\Phi,\Phi) = \omega^{2,q}(\Phi)\|\Phi\|^2_{L^2(\partial B_*)}. \tag{4.38}$$

Since $\nu_0(x) = |x|^{-1}x$, then we can rewrite (4.25) as

$$b_t(\Phi,\Phi) = \int_{\partial B_*} J_t^\tau(\widehat{w}_t - w_0)(\widehat{\nu}_t\cdot\nu_0)g^2\,\frac{n-1}{r_* + tg}\, d\mathcal{H}^{n-1}.$$

By Lemma 4.16 there exists a constant $C > 0$ such that if $\|\Phi\|_{2,q}$ is small enough, then

$$|J_t^\tau| + |\widehat{\nu}_t \cdot \nu_0| + \frac{1}{r_* + tg} \leq C.$$

Moreover, by shape differentiability of $\widehat{w}_t$ proved in Proposition 4.6, we have

$$\widehat{w}_t = w_0 + \omega_{1,\infty}^{2,q}(\Phi). \tag{4.39}$$

These estimates yield

$$b_t(\Phi, \Phi) = \omega^{2,q}(\Phi)\|\Phi\|_{L^2(\partial B_*)}^2. \tag{4.40}$$

Step 2: we now prove that

$$L''(t) - b_t(\Phi, \Phi) = L''(0) + \omega^{2,q}(\Phi)\|\Phi\|_{L^2(\partial B_*)}^2.$$

By (4.24)

$$L''(t) - b_t(\Phi, \Phi) = \int_{\partial B_*} J_t^\tau \left(\widehat{w_t'} g + g^2 \widehat{\nabla w_t} \cdot \nu_0\right)(\widehat{\nu}_t \cdot \nu_0)\, d\mathcal{H}^{n-1} = L''(0) + I_1 + I_2,$$

where

$$I_1 = \int_{\partial B_*} (J_t^\tau \widehat{w_t'}(\widehat{\nu}_t \cdot \nu_0) - w_0')g\, d\mathcal{H}^{n-1},$$

$$I_2 = \int_{\partial B_*} g^2 \left(J_t^\tau (\widehat{\nabla w_t} \cdot \nu_0)(\widehat{\nu}_t \cdot \nu_0) - (\nabla w_0 \cdot \nu_0)\right) d\mathcal{H}^{n-1}.$$

To estimate I_1, we use Lemma 4.16, and Lemma 4.18 to get, dropping the dependence on Φ inside the infinitesimal notation (i.e. $\omega_X^Y = \omega_X^Y(\Phi)$),

$$\begin{aligned} I_1 &= \int_{\partial B_*} g\left((1 + \omega_\infty^{2,q})\left(w_0' + \omega_{2,\partial B_*}^{2,q}\|\Phi\|_{L^2(\partial B_*)}\right) - w_0'\right) d\mathcal{H}^{n-1} \\ &= \omega^{2,q}\|\Phi\|_{L^2(\partial B_*)}^2, \end{aligned}$$

where we used Hölder inequality, $\|g\|_2 = \|\Phi\|_{L^2(\partial B_*)}$, and (4.32).

Finally, to estimate I_2 we use again (4.39) and we notice that

$$\widehat{\nabla w_t} = (I_n + tD\Phi)^{-T}\nabla\widehat{w}_t. \tag{4.41}$$

Hence, using (4.41), (4.39), and Lemma 4.16, we get

$$I_2 = \int_{\partial B_*} g^2 \left((I_n + \omega_\infty^{2,p})(\nabla w_0 \cdot \nu_0 + \omega_\infty^{2,q})(1 + \omega_\infty^{2,q}) - (\nabla w_0 \cdot \nu_0)\right) d\mathcal{H}^{n-1} = \omega^{2,q}\|\Phi\|_{L^2(\partial B_*)}^2.$$

□

4.4 Conclusion to the proof of Theorem 3.1

We are now in position to prove Theorem 3.1: let m, q, j as in the statement of the result. Let also $\Phi \in W^{2,q}(B_1; \mathbb{R}^n)$ orthogonal to ∂B_* such that $|B_*^\Phi| = |B_*|$. We apply Corollary 4.8 providing the Lagrangian $\mathcal{L}_\tau$ for some $\tau \in \mathbb{R}$ and

$$\forall t \in [0, 1],\ L(t) = \mathcal{L}_\tau(B_*^{t\Phi}).$$

By Proposition 4.11 there exists $c_1 > 0$ such that

$$\mathcal{L}_\tau''(B_*)[\Phi, \Phi] = L''(0) \leq -c_1\|\Phi \cdot \nu_0\|_{L^2(\partial B_*)}^2 = -c_1\|\Phi\|_{L^2(\partial B_*)}^2. \tag{4.42}$$

By Proposition 4.15, we get the existence of $\eta > 0$ such that if $\|\Phi\|_{2,q} \le \eta$,

$$\forall t \in [0,1], \qquad L''(t) \le L''(0) + \frac{c_1}{2}\|\Phi\|^2_{L^2(\partial B_*)}. \tag{4.43}$$

Joining (4.42), (4.43), and the optimality condition $L'(0) = 0$, we obtain for some $t_0 \in (0,1)$

$$\mathcal{L}_\tau(B_*) - \mathcal{L}_\tau(B_*^\Phi) = L(0) - L(1) = -\frac{L''(t_0)}{2} \ge \frac{c_1}{4}\|\Phi\|^2_{L^2(B_*)}.$$

The result now follows by noticing that, since Φ is orthogonal to ∂B_*, and $\|\Phi\|_\infty$ is arbitrarily small, with $g = (\Phi \cdot \nu_0)_{|\partial B_*}$,

$$|B_* \Delta B_*^\Phi| = \frac{1}{n}\int_{\partial B_*} |(1+g)^n - 1|\, d\mathcal{H}^{n-1} \le C(n)\|g\|_{L^1(\partial B_*)} \le C(n,m)\|\Phi\|_{L^2(\partial B_*)}.$$

5 The case $\mathcal{J}(E) = \|u_E\|_\infty$

To deal with the case $p = \infty$ in Theorem 1.1, we will follow a similar strategy to that of Theorem 1.2, but there are significant modification to the proofs. In this section we prove

Theorem 5.1. *Let $n \ge 1$ and $m \in (0, |B_1|)$. Then there exists a positive constant $c = c(m,n)$ such that for every $V \in \mathcal{M}_m$ we have*

$$\|u_{B_*}\|_\infty - \|u_V\|_\infty \ge c\|V - \chi_{B_*}\|_1^2.$$

where B_* is the centered ball of volume m.

In this section we denote $\mathcal{J}(V) = \|u_V\|_\infty$.

5.1 Proof of Theorem 5.1 from a Fuglede-type result

As done for the case $1 < p < +\infty$, we start by showing that Theorem 5.1 will follow from the following Fuglede-type result:.

Theorem 5.2. *Let $m \in (0, |B_1|)$, $q > n$. Then there exist positive constants $c = c(m,n)$, $\eta = \eta(m,n)$ such that for every $\Phi \in W^{2,q}(B_1, \mathbb{R}^n)$ orthogonal to ∂B_* with*

$$\|\Phi\|_{2,q} \le \eta, \qquad |B_*^\Phi| = |B_*| = m,,$$

we have

$$\mathcal{J}(B_*) - \mathcal{J}(B_*^\Phi) \ge c|B_* \Delta B_*^\Phi|^2.$$

We postpone the proof of Theorem 5.2 to the next sections, and we show that Theorem 5.1 holds. For $r \in (0,1)$, let

$$\zeta(r) = \begin{cases} -\dfrac{1}{2\pi}\ln(r) & n = 2, \\ \dfrac{1}{n(n-2)\omega_n} r^{2-n} & n \ne 2, \end{cases}$$

be the fundamental solution in $\mathbb{R}^n$ to the Laplace equation. We identify $\zeta(x)$ with $\zeta(|x|)$ and we recall the Green's function for the ball B_1 given by

$$G(x,y) = \zeta(y-x) - \zeta(|x|(y - \tilde{x})), \qquad \tilde{x} = \frac{x}{|x|^2},$$

and

$$G(0,y) = \zeta(y) - \zeta(1).$$

Often we will use the notation $G_x(y) = G(x, y)$. Let $E \subset B_1$ and let x_E be a maximum point of u_E. Then

$$\mathcal{J}(E) = \|u_E\|_\infty = u_E(x_E) = \int_E G(x_E, y)\, dy.$$

The function $G(x_E, \cdot)$ will play the role of the adjoint state w_E in the case $1 < p < +\infty$, as we will see later.

Proof of Theorem 5.1. • Similarly to Proposition 2.10, we can prove that $V \in L^\infty(B_1) \mapsto \mathcal{J}(V)$ is convex (not strictly) and weakly-$*$ continuous in $L^\infty(B_1)$. Therefore, there exists a bang-bang maximizer of $\mathcal{J}$ both in $\mathcal{M}_m$ and in $\mathcal{M}_m^\delta$ for every $\delta > 0$. Moreover, the maximizer in $\mathcal{M}_m$ is the ball and it is unique. Indeed, B^* is a maximizer by Talenti's inequality and the fact that there exists a bang-bang maximizer. About uniqueness, let $f \in \mathcal{M}_m$ be a maximizer for $\mathcal{J}$ and notice that, using the Talenti's inequality

$$\int_{B_1} G(0, y) f^\sharp(y)\, dy = \mathcal{J}(f^\sharp) \geq \mathcal{J}(f) = \mathcal{J}(B_*) = \int_{B_1} G(0, y)\chi_{B_*}(y)\, dy;$$

by the rigidity of the bathtub principle, since $\mathcal{J}(B_*) \geq \mathcal{J}(f^\sharp)$, the equality implies that $f^\sharp = \chi_{B_*}$ and then $f = \chi_E$ for some measurable set E. Therefore, $E = B_*$ because B_* is the unique maximizer of $\mathcal{J}$ among characteristic functions because of the rigidity of the Talenti's inequality for the L^∞ norm (see [2, Corollary 1]). Therefore, Proposition 3.2 also holds for $\mathcal{J}(V) = \|u_V\|_\infty$, and proving the theorem is equivalent to prove that

$$\liminf_{\delta\to 0} \inf_{V \in \mathcal{M}_m^\delta} \frac{\mathcal{J}(B_*) - \mathcal{J}(V)}{\delta^2} > 0.$$

• Let us assume by contradiction that

$$\lim_{\delta\to 0} \frac{\mathcal{J}(B_*) - \mathcal{J}(E_\delta)}{\delta^2} = 0,$$

where E_δ is the maximizer of $\mathcal{J}$ in $\mathcal{M}_m^\delta$. Since for every measurable set $E \subseteq B_1$,

$$\mathcal{J}(E) = \int_E G(x_E, y)\, dx, \qquad\qquad x_E \in \arg\max u_E,$$

we can reproduce the proof of Theorem 1.2 from subsection 3.2, provided that $x_\delta = x_{E_\delta}$ is uniquely determined and x_δ converges to $0 = x_{B_*}$. Indeed, in that case, since $G \in C^\infty(B_1 \times B_1 \setminus \mathrm{diag}(B_1 \times B_1))$, we have that for every fixed small $\varepsilon > 0$, the functions G_δ converge to G_0 in $C^2(B_1 \setminus B_\varepsilon)$. Therefore, we have non-degeneracy of ∇G_δ and the result will follow by retracing the proof of Theorem 1.2: with similar computations as in subsection 3.2, we obtain

- for every δ small there exists a level set $\widetilde{E}_\delta$ of G_δ of measure m;
- by the quantitative bathtub principle applied to G_δ,

$$\mathcal{J}(\widetilde{E}_\delta) - \mathcal{J}(E_\delta) \geq u_{\widetilde{E}_\delta}(x_\delta) - u_{E_\delta}(x_\delta) = \int_{B_1} G(x_\delta, \cdot)(\chi_{\widetilde{E}_\delta} - \chi_{E_\delta}) \geq c|E_\delta \Delta \widetilde{E}_\delta|^2$$

- by Proposition A.2 $\widetilde{E}_\delta$ are smooth deformations of B_*, leading to a contradiction with Theorem 5.2.

• We let $u_\delta = u_{E_\delta}$. We now show that x_δ is uniquely defined and it converges to $x_{B_*} = 0$. Since $|\nabla u_0| = 0$ only in the origin, we have that any critical point of u_δ has to be uniformly close to the origin. Thanks to the Schauder's estimates we have that for every $K \subset\subset B_1 \setminus \partial B_*$

$$u_\delta \xrightarrow{C^2(K)} u_0,$$

and since u_0 has a unique maximum, and by explicit computations $-D^2 u_0(0)$ is positively definite, we apply the implicit function theorem to the equation $\nabla u_\delta(x_\delta) = 0$, so that for small δ, the functions u_δ also have a unique maximum x_δ converging to 0.

□

5.2 Computation of shape derivative

We will now study optimality conditions when the objective functional is $E \mapsto \|u_E\|_\infty$. A similar study was also performed in the paper [14], where the authors deal with the shape derivative of the L^∞ norm of the torsion function of a set Ω; our analysis goes further as we compute second order derivatives (which requires a computation of the derivative of the maximum point).

We fix $q > n$. As we saw in the proof of Theorem 5.1, for $\Phi \in W^{2,q}(B_1;\mathbb{R}^n)$ small, u_Φ has a unique maximum point x_Φ, and we now show that we can differentiate x_Φ with respect to Φ.

Proposition 5.3 (Shape differentiability of x_Φ)**.** *The application*

$$\Phi \in W^{2,q}(B_1;\mathbb{R}^n) \longmapsto x_\Phi = \arg\max u_{B_*^\Phi} \in \mathbb{R}^n$$

is of class C^1 in a neighborhood of 0. In particular, if $\Phi \in W^{2,q}(B_1;\mathbb{R}^n)$ is small enough and we denote by $x_t := x_{t\Phi}$, then we have that for every $t \in [0,1]$,

$$x_t' = -D^2 u_t(x_t) \nabla u_t'(x_t).$$

Proof. Since u_Φ depends only on the values of Φ on ∂B_*, without loss of generality we may assume that $\mathrm{supp}(\Phi) \subset B_1 \setminus B_{r_*/2}$. Let us recall that x_Φ is uniquely defined as the solution to the equation

$$\nabla u_\Phi(x_\Phi) = 0. \tag{5.1}$$

As shown in the proof of Theorem 5.1, we know that x_Φ is converging to 0 as Φ goes to 0, and we will assume to have $x_\Phi \in B_{r_*/2}$. Since for small Φ we have $\chi_{B_*^\Phi} = 1$ in $B_{r_*/2}$, then using the Schauder's estimate $\|u_\Phi - u_0\|_{C^2(B_{r_*/2})} \le C_\varepsilon \|u_\Phi - u_0\|_\infty$ we can easily adapt the proof of Proposition 4.4 to show that

$$\Phi \in W^{2,q}(B_1;\mathbb{R}^n) \cap H_0^1(B_1 \setminus B_{r_*/2}) \longmapsto u_\Phi \in W_0^{1,q}(B_1) \cap C^2(B_{r_*/2})$$

is of class C^1. The conclusion follows by the implicit function theorem applied to equation (5.1).

□

Proposition 5.4 (Representation formula for u_t')**.** *Let $\Phi \in W^{2,q}(B_1;\mathbb{R}^n)$, let $u_t := u_{t\Phi}$. Then for every $x \in B_1 \setminus \partial E_t$ we have*

$$u_t'(x) = \int_{\partial E_t} G_x \left(\widetilde{\Phi}_t \cdot \nu_t\right) d\mathcal{H}^{n-1}. \tag{5.2}$$

where $\widetilde{\Phi}_t = \Phi \circ (\mathrm{Id} + t\Phi)^{-1}$.

Proof. We recall that by Proposition 4.4,

$$-\Delta u'_t = (\widetilde{\Phi}_t \cdot \nu)\, d\mathcal{H}^{n-1} \llcorner_{\partial B_*^{t\Phi}}.$$

The result then follows by writing u'_t with the Green's representation formula. □

In the following we denote by $\nabla_x G_{x_t}(y) = \nabla_x G(x_t, y)$ and $\nabla_y G_{x_t}(y) = \nabla_y G(x_t, y)$. Moreover, we let $G'_t(y) = \nabla_x G_{x_t}(y) \cdot x'_t$.

Proposition 5.5. *The application*

$$\Phi \in W^{2,q}(B_1; \mathbb{R}^n) \longmapsto \mathcal{J}(B_*^\Phi) \in \mathbb{R}$$

is of class C^2 in a neighborhood of 0. If $\Phi \in W^{2,q}(B_1; \mathbb{R}^n)$ is small enough and $J(t) = \mathcal{J}(B_*^{t\Phi})$, then for $t \in [0, 1]$,

$$J'(t) = \int_{\partial B_*^{t\Phi}} G_{x_t}\, (\widetilde{\Phi}_t \cdot \nu_t)\, d\mathcal{H}^{n-1}, \tag{5.3}$$

where $\nu_t = \nu_{E^{t\Phi}}$ is the outer unit normal to $B_*^{t\Phi}$, and $\widetilde{\Phi}_t = \Phi \circ (\mathrm{Id} + t\Phi)^{-1}$. Moreover, letting $g_t = \widetilde{\Phi}_t \cdot \nu_t$,

$$J''(t) = \int_{\partial B_*^{t\Phi}} \Big(g_t G'_t + g_t \nabla_y G_{x_t} \cdot \widetilde{\Phi}_t \Big)\, d\mathcal{H}^{n-1}(y) + a_t(\Phi, \Phi), \tag{5.4}$$

where

$$a_t(\Phi, \Phi) = \int_{\partial B_*^{t\Phi}} G_{x_t} \Big(g_t \operatorname{div}(\widetilde{\Phi}_t) - ((D\widetilde{\Phi}_t)\widetilde{\Phi}_t) \cdot \nu_t \Big) d\mathcal{H}^{n-1}.$$

Proof. In the following we denote by $E_t = B_*^{t\Phi}$. We first notice that $J(t) = u_t(x_t)$, so that by Proposition 5.4

$$J'(t) = u'_t(x_t) + \nabla u_t(x_t) \cdot x'_t.$$

Since x_t is the maximum point for u_t, then $\nabla u_t(x_t) = 0$. Using the representation formula for u'_t, we get

$$J'(t) = \int_{\partial E_t} G_{x_t} (\widetilde{\Phi}_t \cdot \nu_t)\, dx = -\int_{B_1 \setminus E_t} \operatorname{div}(G_{x_t} \widetilde{\Phi}_t)\, dy.$$

Since G_{x_t} is $C^\infty(B_1 \setminus B_{r_*/2})$, we can apply the Hadamard's formula to conclude the proof. □

As done in Section 4.1, the previous results lead to the following one about the Lagrangian.

Corollary 5.6. *For $\tau \in \mathbb{R}$ and $E \subset B_1$, we define*

$$\mathcal{L}_\tau(E) := \mathcal{J}(E) + \tau |E|.$$

For $m \in (0, |B_1|)$ and $E = B_*$ the centered ball of volume m, we set

$$\tau = -G_0\big|_{\partial B_*}. \tag{5.5}$$

Then $\Phi \in W^{2,q}(B_1; \mathbb{R}^n) \mapsto \mathcal{L}_\tau(B_*^\Phi)$ is of class C^2 near 0, and

(i) $\mathcal{L}'_\tau(B_*) \equiv 0$

(ii) for every $\Phi \in W^{2,q}(B_1; \mathbb{R}^n)$ small enough such that Φ is normal on ∂B_* and $\widetilde{\Phi}_t = \Phi$ for $t \in [0, 1]$, if we denote $L(t) = \mathcal{L}_\tau(B_*^{t\Phi})$ then for every t,

$$\begin{aligned} L''(t) = &\int_{\partial B_*^{t\Phi}} \Big(G'_t g + g^2 \frac{\partial G_{x_t}}{\partial \nu_0} \Big) (\nu_t \cdot \nu_0)\, d\mathcal{H}^{n-1}(y) \\ &+ \int_{\partial B_*^{t\Phi}} (w_t - G_0(r_*))(\nu_t \cdot \nu_0) g^2 \operatorname{div}(\nu_0)\, d\mathcal{H}^{n-1}(y), \end{aligned} \tag{5.6}$$

where $G_0(r_*) = G_0(y)$ for any $y \in \partial B_*$ and $g = \Phi \cdot \nu_0$ (recall that we consider an extension of ν_0 through the projection onto ∂B_*).

5.3 Coercivity in 0

As done for Theorem 3.1, we now prove that the Lagrangian is coercive.

Proposition 5.7. *Let $m \in (0, |B_1|)$, B_* the centered ball of volume m. Then there exist positive constants $c = c(n,m), \eta = \eta(n,m)$ such that for every $\Phi \in W^{2,q}(B_1;\mathbb{R}^n)$ such that $|B_*^\Phi| = |B_*|$ and $\|\Phi\|_\infty < \eta$ we have*

$$\mathcal{L}_\tau''(B_*)[\Phi,\Phi] \leq -c\|\Phi\cdot\nu_0\|^2_{L^2(\partial B_*)},$$

where $\mathcal{L}_\tau$ is the Lagrangian defined in Corollary 5.6.

Proof. Let Φ be as in Corollary 5.6 (as explained in subsection 4.2, this is not restrictive), so that

$$\mathcal{L}_\tau''(B_*)[\Phi,\Phi] = L''(0) = \int_{\partial B_*}\left(g\left(\nabla_x G_0\cdot x_0'\right) + g^2\left(\nabla_y G_0\cdot\frac{y}{|y|}\right)\right)d\mathcal{H}^{n-1}(y). \tag{5.7}$$

By Proposition 5.3

$$x_0' = \frac{1}{n}\int_{\partial B_*}\nabla_x G(0,z)\,g(z)\,d\mathcal{H}^{n-1},$$

where we used again that $D^2u_0(0) = -1/n\,I_n$ and Proposition 5.4. By the definition of G and its symmetries, we may write

$$\nabla_x G(0,y) = \nabla_y G(y,0) = -(\zeta'(r_*) - \zeta'(1)r_*)\frac{y}{|y|}, \qquad \nabla_y G(0,y) = \zeta'(r_*)\frac{y}{|y|}.$$

If $n=1$, then the constraint $|B_*^\Phi| = |B_*|$ gives that g has to be odd. Therefore, direct computations give $\nabla_x G_0(y) = \frac{1-r_*}{2}\operatorname{sgn}(y)$, and $x_0' = (1-r_*)g(r_*)$, so that (5.7) reads

$$L''(0) = -\big(1-(1-r_*)^2\big)g(r_*)^2,$$

which concludes the proof. Let us now assume $n \geq 2$, and let us decompose g in spherical harmonics $g = \sum_{k,m}\alpha_{k,m}Y_{k,m}$, recalling that

$$Y_{1,j}(z) = \sqrt{\frac{n}{P(B_1)}}\frac{z_j}{|z|}, \qquad \alpha_{1,j} = \frac{1}{\|Y_{1,j}\|^2_{L^2(\partial B_*)}}\int_{\partial B_*} g(z)Y_{1,j}\,d\mathcal{H}^{n-1}.$$

We notice that we have

$$\nabla_x G_0\cdot x_0' = \frac{P(B_1)}{n^2}(\zeta'(r_*) - \zeta'(1)r_*)^2\sum_{j=1}^n \alpha_{1,j}\|Y_{1,j}\|^2_{L^2(\partial B_*)}Y_{1,j}, \qquad \nabla_y G_0\cdot\frac{y}{|y|} = \zeta'(r_*)$$

Hence, substituting in (5.7), we obtain

$$L''(0) = \sum_{j=1}^n \alpha_{1,j}^2\|Y_{1,j}\|^4_{L^2(\partial B_*)}\frac{P(B_1)}{n^2}\big(\zeta'(r_*) - r_*\zeta'(1)\big)^2 + \sum_{k=1}^{+\infty}\sum_{j=1}^{M(k)}\alpha_{k,j}^2\|Y_{k,j}\|^2_{L^2(\partial B_*)}\zeta'(r_*).$$

Using $\zeta'(r_*) = -(P(B_*))^{-1}$ and $P(B_1)\|Y_{1,j}\|^2_{L^2(\partial B_*)} = P(B_*)$, we rewrite

$$\begin{aligned} L''(0) &= \sum_{j=1}^n \alpha_{1,j}^2\|Y_{1,j}\|^2_{L^2(\partial B_*)}\zeta'(r_*)\left(\frac{(1-r_*^n)^2}{n^2} - 1\right) + \sum_{k=2}^{+\infty}\sum_{j=1}^{M(k)}\alpha_{k,j}^2\|Y_{k,j}\|^2_{L^2(\partial B_*)}\zeta'(r_*) \\ &\leq -\frac{1}{P(B_*)}\left(1 - \frac{(1-r_*^n)^2}{n^2}\right)\|g\|_2^2, \end{aligned}$$

where we used that $\|g\|_2^2 = \sum_{k,j}\alpha_{k,j}^2\|Y_{1,j}\|^2_{L^2(\partial B_*)}$. ◻

5.4 Improved continuity

Also in this case, we need to show the following improved continuity result.

Proposition 5.8. *Let $m \in (0, |B_1|)$ and $q > n$. Then*

$$L''(t) = L''(0) + \omega^{2,q}(\Phi) \, \|\Phi\|^2_{L^2(\partial B_*)},$$

where $L(t) := \mathcal{L}_\tau(B_*^{t\Phi})$, i.e. for every $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\forall \Phi \in W^{2,q}(B_1; \mathbb{R}^n) \text{ with } \|\Phi\|_{2,q} \le \eta, \qquad \forall t \in [0,1], \qquad |L''(t) - L''(0)| \le \varepsilon \|\Phi\|^2_{L^2(\partial B_*)}.$$

Proof. Since the expression of $L''(t)$ computed in Corollary 5.6 is quite similar to the one computed in Corollary 4.8, we can perform a proof analogous to the one of Proposition 4.15. Thanks to Proposition 5.4 and the smoothness of $G(\cdot, y)$ far from y, we know that $\Phi \mapsto G_{x_\Phi} \in W^{2,q}(B_1 \setminus B_{r_*/2})$ is of class C^1 in a neighborhood of 0, and in particular

$$\widehat{G_{x_t}} = G_0 + \omega^{2,q}_{W^{2,q}(B_1 \setminus B_{r_*/2})}(\Phi),$$

where we recall that for every function h we denoted $\widehat{h} = h \circ (\mathrm{Id} + t\Phi)$. It remains to prove that, letting $G'_t = \nabla_x G_{x_t} \cdot x'_t$,

$$\widehat{G'_t} = G'_0 + \omega^{2,q}_{L^\infty(\partial B_*)} \|\Phi\|_{L^2(\partial B_*)}, \qquad \|G'_0\|_{L^\infty(\partial B_*)} \le C \|\Phi\|_{L^2(\partial B_*)}. \tag{5.8}$$

Indeed, we have that

$$G'_t = \nabla_x G_{x_t} \cdot D^2 u_t(x_t) \nabla u'_t(x_t). \tag{5.9}$$

Since $G \in C^\infty(B_1 \times B_1 \setminus \mathrm{diag}(B_1 \times B_1))$ and since we may assume $\operatorname{supp} \Phi \subset\subset B_1 \setminus B_{r_*/2}$ (as also done in Proposition 5.4) we get

$$\nabla_x G(x_t, (\mathrm{Id} + t\Phi)^{-1}(y)) = \nabla_x G(0, y) + \omega^{2,q}_{L^\infty(\partial B_*)}(\Phi), \tag{5.10}$$

where we also used the continuity of $\Phi \mapsto x_\Phi$. By the Schauder's estimates we get

$$D^2 u_t(x_t) = D^2 u_0(0) + \omega^{2,q}(\Phi), \tag{5.11}$$

and by the representation formula

$$\nabla u'_t(x) = \int_{\partial B_*} \nabla_x G(x, y) g(\nu_0 \cdot \nu_t) \, d\mathcal{H}^{n-1}$$

we finally obtain

$$|\nabla u'_t(x_t)| \le C \|g\|_{L^2(\partial B_*)}. \tag{5.12}$$

Joining (5.9), (5.10), (5.11), and (5.12) we obtain (5.8). The proof now follows by the same computations in Proposition 4.15. □

Proof of Theorem 5.2. The proof is the same as Section 4.4. □

A Appendix

A.1 About the convergence of level sets

In this section, we add some details to [18, After formula (68)] or [19, Formula (Def)] where the authors deduce a strong convergence of level sets from the convergence of functions, as used in Section 3.2. We felt the proofs to be a bit elliptic, and we decided to expand them in this section.

Lemma A.1. *Let $\Omega \subset \mathbb{R}^n$ be a connected and bounded open set, (u_k) a sequence of functions in $L^\infty(\Omega)$ and $u \in C^0(\Omega) \cap L^\infty(\Omega)$ such that $|\{ u = \operatorname{essinf}_\Omega u \}| = 0$. For $t_k, t_* \in \mathbb{R}$ we assume*

$$\forall k \in \mathbb{N}, \quad |\{ u_k > t_k \}| = |\{ u > t_* \}| \in (0, |\Omega|), \qquad \text{and} \qquad u_k \xrightarrow{L^\infty(\Omega)} u.$$

Then $t_* = \lim_k t_k$.

Proof. We define $m = |\{u > t_*\}| = |\{u_k > t_k\}|$. Since functions u_k converge uniformly, they are equi-bounded. In particular, this ensures that the sequence t_k is bounded in $\mathbb{R}$, otherwise we would either get for large k that $t_k > \sup_j \|u_j\|_\infty$, and $|\{ u_k > t_k \}| = 0$, or $t_k \leq \inf_j \operatorname{essinf} u_j$ and $|\{ u_k > t_k \}| = |\Omega|$. Therefore, up to passing to a subsequence, we can define $\bar{t} = \lim_k t_k$, and let $\varepsilon > 0$. By L^∞ convergence we get that for large k

$$\{ u > \bar{t} + 2\varepsilon \} \subseteq \{ u_k > \bar{t} + \varepsilon \} \subseteq \{ u_k > t_k \},$$

and therefore defining $\mu(t) = |\{u > t\}|$ (this differs from Definition 2.1 as we did not assume u to be non-negative), we get

$$\mu(\bar{t} + 2\varepsilon) \leq m.$$

Analogously

$$\{ u > \bar{t} - 2\varepsilon \} \supseteq \{ u_k > \bar{t} - \varepsilon \} \supseteq \{ u_k > t_k \},$$

and so

$$\mu(\bar{t} + 2\varepsilon) \leq m \leq \mu(\bar{t} - 2\varepsilon). \tag{A.1}$$

We now observe that μ is strictly decreasing in $(\operatorname{essinf}_\Omega u, \|u\|_\infty)$. Indeed, if μ is constant in (t_1, t_2) with $\operatorname{essinf} u < t_1 < t_2 < \|u\|_\infty$, then $|\{ t_1 < u \leq t_2 \}| = 0$. On the other hand, since Ω is connected and u is continuous, $u^{-1}((t_1, t_2))$ is an open non-empty set, which has positive measure, thus a contradiction. Therefore, as by assumption $t_* \in (\operatorname{essinf} u, \|u\|_\infty)$, (A.1) implies

$$\bar{t} - 2\varepsilon \leq t_* \leq \bar{t} + 2\varepsilon,$$

and since ε is arbitrary, $\bar{t} = t_*$. The argument does not depend on the choice of the subsequence, and the conclusion follows. □

Proposition A.2. *Let $q > n$, let $\Omega \subset \mathbb{R}^n$ be an open bounded connected set, and let $u \in W^{3,q}(\Omega)$ be a function such that for some positive constant k*

$$|\nabla u|(x) \geq k, \qquad \forall x \in u^{-1}(0).$$

Assume in addition that $d(u^{-1}(0), \partial\Omega) > 0$. Let $u_j \in C^{2,\alpha}(\Omega)$ be a sequence of functions such that

$$u_j \xrightarrow{W^{2,q}(\Omega)} u.$$

Then for every j large enough $u_j^{-1}(0)$ is a $C^{1,s}$ hypersurface, with $s = 1 - n/q$, and there exist deformations $\Phi_j \in W^{2,q}(\Omega; \mathbb{R}^n)$ such that *(i)* Φ_j *are orthogonal to* $u^{-1}(0)$*;*

(ii) $u_j^{-1}(0) = (\mathrm{Id} + \Phi_j)(u^{-1}(0))$;

(iii) $\lim_j \|\Phi_j\|_{2,q} = 0$

Proof. • **Step 1:** we first notice that for every $\varepsilon > 0$ there exists t_0 such that if $|t| \leq t_0$ then

$$u^{-1}(t) \subseteq (u^{-1}(0))^{\varepsilon},$$

where we use the notation $(K)^t = \{ p \in \mathbb{R}^n \mid d(p, K) \leq t \}$, for the *outer parallel* set. Indeed, let us assume by contradiction that there exist $\varepsilon_0 > 0$ and points $y_k \in u^{-1}(t_k)$ with $\lim_k t_k = 0$ such that

$$\forall k \in \mathbb{N}, \qquad d(y_k, u^{-1}(0)) > \varepsilon_0.$$

As Ω is bounded, up to a subsequence, we may assume that y_k converge to some point $\bar{y} \in \overline{\Omega}$. The continuity of u gives $u(\bar{y}) = 0$, so that $\bar{y} \in u^{-1}(0)$. On the other hand,

$$d(\bar{y}, u^{-1}(0)) \geq \varepsilon_0,$$

which is a contradiction.

• **Step 2:** We now want to extend the non-degeneracy property of the gradient of u to the functions u_j on their level sets $u_j^{-1}(0)$. We notice that by uniform convergence and by the previous step, for every $\varepsilon > 0$ there exist $t_0 > 0$ and j_0 such that

$$\forall j \geq j_0, \qquad u_j^{-1}(0) \subset u^{-1}((-t_0, t_0)) \subset (u^{-1}(0))^{\varepsilon}. \tag{A.2}$$

On the other hand, ∇u is uniformly continuous in Ω so there exists $\alpha_0 > 0$ such that

$$\forall (x, z) \in \Omega \text{ s.t. } |z - x| \leq \alpha_0, \qquad |\nabla u(z) - \nabla u(x)| \leq \frac{k}{4}.$$

In particular,

$$\forall x \in (u^{-1}(0))^{\alpha_0}, \qquad |\nabla u(x)| \geq \frac{k}{2}.$$

Also, ∇u_j uniformly converge to ∇u, so for j large enough

$$|\nabla u_j(x) - \nabla u(x)| \leq \frac{k}{8} \qquad \forall x \in \Omega.$$

From the previous estimates, we get $|\nabla u_j|(x) > 0$ for every $x \in u_j^{-1}(0)$ and j large enough, which implies that the sets $u_j^{-1}(0)$ are $C^{1,s}$ hypersurfaces.

• **Step 3:** Let us denote $\Omega_1 = (u^{-1}(0))^{\alpha_0}$ and

$$\forall x \in \Omega_1, \qquad \nu(x) = -\frac{\nabla u}{|\nabla u|}(x).$$

We show that for every $x \in \Omega_1$, the line

$$t \in [-\alpha_0, \alpha_0] \mapsto x + t\nu(x)$$

intersects $u^{-1}(0)$. Let $x \in \Omega_1$. Up to choosing a smaller α_0, using that $u^{-1}(0)$ is far from $\partial\Omega$, we may assume that $(\Omega_1)^{\alpha_0} \subset \Omega$. By Lagrange theorem applied to the function $\alpha \mapsto u(x + \alpha\nu(x)) - u(x) - \alpha\nabla u(x) \cdot \nu(x)$, there exists z on the segment between x and $x + \alpha_0\nu(x)$ such that

$$u(x + \alpha_0\nu(x)) = u(x) - \alpha_0|\nabla u|(x) + \alpha_0(\nabla u(z) - \nabla u(x)) \cdot \nu(x) \leq u(x) - \frac{\alpha_0 k}{4}.$$

Similarly we get

$$u(x - \alpha_0\nu(x)) \geq u(x) + \frac{\alpha_0 k}{4}.$$

• **Step 4:** Now we construct the diffeomorphisms Φ_j. For every $x \in \Omega_1$, let us define

$$F_j(x,t) = u_j(x + t\nu(x)) - u(x).$$

We show that F_j is strictly monotone in t and that it always admits a zero. First we notice that

$$\partial_t F_j(x,t) = \nabla u_j(x + t\nu(x)) \cdot \nu(x).$$

and from Step 2 we deduce

$$\forall t \in [-\alpha_0, \alpha_0], \quad \forall j \geq j_0, \qquad \partial_t F_j(x,t) \leq -k/4.$$

By uniform convergence and the previous step, for j large enough we have $F_j(x,\alpha_0) < 0 < F_j(x,-\alpha_0)$. Therefore, there exists a unique $t_j(x) \in [-\alpha_0, \alpha_0]$ such that $F_j(x, t_j(x)) = 0$, or, equivalently

$$u_j(x + t_j(x)\nu(x)) = u(x). \tag{A.3}$$

• **Step 5:** We claim that $\Phi_j(x) := t_j(x)\nu(x)$ works (up to multiplying it by a cutoff function). Property *(i)* follows by noticing that by construction ν is orthogonal to $u^{-1}(0)$.
Let us now prove *(ii)*. First (A.3) for $x \in u^{-1}(0)$ implies

$$(\mathrm{Id} + \Phi_j)(u^{-1}(0)) \subset u_j^{-1}(0).$$

For the converse inclusion, let $y \in u_j^{-1}(0)$. The point y can be written as

$$y = x + d(y, u^{-1}(0))\bar{\nu}(x)$$

with $x \in u^{-1}(0)$ and either $\bar{\nu} = \nu(x)$ or $\bar{\nu} = -\nu(x)$. On the other hand, by (A.2), we have for large j

$$u_j^{-1}(0) \subset (u^{-1}(0))^{\alpha_0},$$

so that $d(y, u^{-1}(0)) < \alpha_0$. As a consequence, $d(y, u^{-1}(0)) = |t_j(x)|$ and we have proved that

$$(\mathrm{Id} + \Phi_j)(u^{-1}(0)) = u_j^{-1}(0).$$

Finally, *(iii)* follows by noticing that (A.2) implies that t_j converges uniformly to 0 and by computing

$$\begin{aligned} \nabla t_j(x) &= -\frac{\partial_x F_j(x, t_j(x))}{\partial_t F_j(x, t_j(x))} \\ &= |\nabla u|(x) \frac{\nabla u_j(x + \Phi_j(x)) - t_j(x) D\nu(x) \nabla u_j(x + \Phi_j(x)) - \nabla u(x)}{\nabla u_j(x + \Phi_j(x)) \cdot \nabla u(x)}. \end{aligned}$$

Since $W^{2,q}$ is an algebra, computing the derivatives of t_j we iteratively get the convergence of higher order derivatives.

□

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