# Regularity of optimal spectral domains

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**Abstract**

In this paper, we review known results and open problems on the question of *regularity of the optimal shapes* for minimization problems of the form

$$\min\{\lambda_k(\Omega),\ \ \Omega \subset D, |\Omega| = a\},$$

where $D$ is an open set in $\mathbb{R}^d$, $a \in (0, |D|), k \in \mathbb{N}^*$ and $\lambda_k(\Omega)$ denotes the $k$-th eigenvalue of the Laplacian with homogeneous Dirichlet boundary conditions. We also discuss some related problems involving $\lambda_k$, but leading to singular optimal shapes.

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## Contents



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# 1 Introduction

The main goal of this paper is to review known results and open problems about the regularity of optimal shapes for the minimization problems

$$\min\{\lambda_k(\Omega),\ \ \Omega\subset D, |\Omega|=a\}\,, \tag{1}$$

where $D$ is a given open subset of $\mathbb{R}^d$, $a\in(0,|D|), k\in\mathbb{N}^*$ and $\lambda_k(\Omega)$ is the $k$-th eigenvalue of the Laplace operator on $\Omega$ with homogeneous Dirichlet boundary conditions. We will also consider the regularity question for penalized versions of (1), and discuss as well the possible appearance of singularities for optimal shapes, either for (1) or for related problems involving convexity constraints.

We refer to [27, Chapter 2] for all necessary definitions and for the question of existence of optimal shapes. It is recalled in particular that, if $D$ is bounded or if $D=\mathbb{R}^d$, then Problem (1) has a solution (say $\Omega^*$) in the family of *quasi-open* subsets of $\mathbb{R}^d$ (*as explained in [27, Chapter 2], the eigenvalues $\lambda_k(\Omega)$ may be well-defined for all quasi-open sets $\Omega$ with finite measure as well as the space $H^1_0(\Omega)$*).

**Here we analyze the question of the regularity of this optimal shape $\Omega^*$.**

As it will appear in this paper, this turns out to be a difficult and still widely open question. Even deciding whether $\Omega^*$ is open, is itself a difficult question and is not completely understood yet.

**Is $\Omega^*$ always open? Is at least one of the optimal $\Omega^*$ open? What is the regularity of the optimal $k$-th eigenfunctions $u_{\Omega^*}$?**

As recalled in [27, Chapter 2], if $D=\mathbb{R}^d$ or more generally if $D$ is 'large enough':

- $\Omega^*$ is a ball if $k=1$,
- $\Omega^*$ is the union of two disjoint identical balls if $k=2$,

with uniqueness in both cases up to translations (and sets of zero-capacity). *Here, $D$ 'large enough' means, when $k=1$, that it can contain a ball of volume $a$, and when $k=2$ that it can contain two disjoints identical balls whose total volume is $a$.* Thus full regularity holds for the optimal shape in these two cases. But, the question remains for 'large' $D$ with $k\geq 3$ and for any $k$ with 'small' $D$. Then, the regularity analysis of the optimal shapes in (1) is very similar in several ways to the analysis of the optimal shapes for the Dirichlet energy, namely

$$\min\{G_f(\Omega),\ \Omega\subset D,\ |\Omega|=a\}\,, \tag{2}$$

where $f\in L^\infty(D)$ is given and

$$G_f(\Omega)=\int_\Omega\left[\frac{1}{2}|\nabla u_\Omega|^2-f\,u_\Omega\right],\ \ u_\Omega\in H^1_0(\Omega),\ -\Delta u_\Omega=f\ in\ \Omega. \tag{3}$$

(*The solution $u_\Omega$ of this Dirichlet problem is classically defined when $\Omega$ is an open set with finite measure. As explained in [27, Chapter 2], this definition may be extended to the case when $\Omega$ is only a quasi-open set with finite measure.*)

Actually, for these two problems (1) and (2), the analysis of the regularity follows the same main steps and offers the following main features. They will provide the content of Sections 2 and 3.

1. **The situation is easier when the state function is nonnegative!** For the Dirichlet energy case (2), for instance in dimension two, full regularity of the boundary holds for positive data $f$, inside $D$ (see [8] and Paragraph 2.3.1 below). On the other hand, even in dimension two, it is easily seen that *singularities do necessarily occur* at each point of the boundary of the optimal set

$\Omega^*$ where the state function $u_{\Omega^*}$ (as defined in (3)) vanishes and changes sign in a neighborhood. The change of sign of $u_{\Omega^*}$ does imply that its gradient has to be discontinuous and, therefore, that the boundary cannot be regular near these points. For instance, cusps will then generally occur in dimension two (see e.g. [37]).

For the eigenvalue problem (1), state functions are the $k$-th eigenfunctions on $\Omega^*$ of the Laplace-Dirichlet operator. Thus the situation (and the analysis) will be quite different if $k = 1$ where the first eigenfunction is nonnegative and if $k \geq 2$ where the eigenfunction changes sign. This partly explains why we devote the specific Section 2 to Problem (1) with $k = 1$. Another reason is that the problem is then equivalent to a minimization problem where the variable are functions rather than domains. We are then led to a free boundary formulation (see Paragraph 2.1) where one has to understand the regularity of the boundary of $[u_{\Omega^*} > 0]$. One can essentially obtain as good regularity results as for the Dirichlet energy case with nonnegative data $f$, see [10]. Here we strongly rely on the seminal paper [3] by Alt-Caffarelli about regularity of free boundaries.

On the other hand, the case $k \geq 2$ is far from being so well understood and we will try to describe what is the current state of the art (see Section 3).

2. **A first step: regularity of the state function**. For the Dirichlet energy case, the analysis starts by studying the regularity of $u_{\Omega^*}$ as defined in (3). It is proved (see [9]) that $u_{\Omega^*}$ *is locally Lipschitz continuous* on $D$, this for any optimal shape $\Omega^*$ and no matter the sign of $u_{\Omega^*}$. This Lipschitz continuity is the optimal regularity we can expect for $u_{\Omega^*}$, as it vanishes on $D \setminus \Omega^*$, and is expected to have a non vanishing gradient on $\partial\Omega^*$ from inside $\Omega^*$. As expected, the proof in the case $u_{\Omega^*}$ changes sign is quite more involved and requires for instance the Alt-Caffarelli-Friedman Monotonicity Lemma (proved in [4], [20], see Lemma 3.4 below).

   For the optimal eigenvalue problem (1) with $k = 1$, it can be proved as well that the corresponding eigenfunction on $\Omega^*$ is locally Lipschitz continuous on $D$ (see Theorem 2.15). For $k \geq 2$ and $D = \mathbb{R}^d$, it has been proved in [14] that one of the $k$-th eigenfunction is Lipschitz continuous (see Theorem 3.3) (note that the optimal eigenvalue is generally expected to be multiple). However, the case $D$ bounded and $k \geq 2$ is still to be understood. The main difference is that, when $D = \mathbb{R}^d$, Problem (1) is equivalent to the penalized version

$$\min\left\{\lambda_k(\Omega) + \mu|\Omega|, \;\; \Omega \subset \mathbb{R}^d\right\}, \tag{4}$$

   for some convenient $\mu \in (0, \infty)$ (see Proposition 3.1). And, as explained below, more regularity information may be derived on optimal state functions for penalized versions.

3. **Penalized versions.** In order to obtain information on the regularity of $\Omega^*$ or $u_{\Omega^*}$, a natural tool is to make admissible perturbations of $\Omega^*$ and use its minimization property. Obviously, there is more freedom to choose perturbations on the penalized version (4) where the volume constraint $|\Omega| = a$ is relaxed, rather than on the constrained initial version (1). Actually, the analysis of (1) when $k = 1$ starts by showing that (1) is *equivalent to the penalized version*

$$\min\left\{\lambda_1(\Omega) + \mu[|\Omega| - a]^+, \;\; \Omega \subset D\right\}, \tag{5}$$

   for $\mu$ large enough (see Proposition 2.6). Then, analysis of the regularity may be more easily made on the optimal shapes of (5). In Paragraph 2.3.2, we make an heuristic analysis of this *"exact penalty"* property for general optimization problems where not only the penalized version converges to the constrained problem as the penalization coefficient $\mu \to \infty$, but more precisely that the two problems are *equivalent* for $\mu$ large enough. Optimal such factors $\mu$ play the role of Lagrange multipliers. Actually, this approach is used again, in a local way in Paragraph 2.3.3, to prove that the 'pseudo'-Lagrange multiplier does not vanish (see Proposition 2.23). It is also used in [27, Chapter 7] to study the regularity of optimal shapes for similar functionals.

4. **How to obtain the regularity of the boundary of $\Omega^*$?** Knowing that the state function is Lipschitz continuous is a first main step in the study of the regularity of the boundary of the optimal set, but obviously not sufficient.

   For example for $k = 1$, this boundary can be seen as the boundary of the set $[u_{\Omega^*} > 0]$. If we were in a regular situation (say $u$ is $C^1$ on $\overline{\Omega^*}$), then knowing that the gradient of $u_{\Omega^*}$ does not vanish at the boundary would imply regularity of this boundary by the implicit function theorem.

Indeed, the next main step is (heuristically) to prove that the gradient of the state function *does not degenerate at the boundary.* This is what is done and then used in Paragraph 2.3.3 for the optimal sets of (1) when $k = 1$. Full regularity of the boundary is proved in dimension two and regularity of the reduced boundary is proved in any dimension (see Theorem 2.19). Here we strongly rely on the seminal paper [3] by Alt-Caffarelli as explained in details in Section 2.3.1. Note that it is also used in [27, Chapter 7] as mentioned at the end of Point 3 above. Nothing like this is known when $k \geq 2$. It is already a substantial piece of information to sometimes know that $\Omega^*$ is an open set ! (see Section 3).

In Section 4, we partially analyze the regularity of $\Omega^*$ solution of (1) up to the boundary of the box $D$, when $k = 1$. We notice in particular that it is natural to expect the contact to be tangential (although this is not proved anywhere as far as we know), but we cannot expect in general that the contact be very smooth; we prove indeed, for example when $D$ is a strip (too narrow to contain a disc of volume $a$), that the optimal shape is $C^{1,1/2}$ and not $C^{1,1/2+\varepsilon}$ with $\varepsilon > 0$. In order to show that this behavior is not exceptional and is not only due to the presence of a box constraint, we show that a similar property is valid for solutions to the problem

$$\min \{\lambda_2(\Omega), \Omega \text{ open and convex}, |\Omega| = a\}.$$

This last problem enters the general framework of *convexity constraint*, which is quite challenging from the point of view of calculus of variations. We conclude this paper with Section 5 where we discuss some problems in this framework. They are of the form

$$\min \{J(\Omega), \Omega \text{ open and convex}\},$$

where $J$ involves $\lambda_1$, and possibly other geometrical quantities (such as the volume $|\Omega|$ or the perimeter $P(\Omega)$), and which lead to *singular optimal shapes, such as polygons (in dimension 2).* Thanks to the convexity constraint, it is allowed to consider the question of maximizing the perimeter and/or the first Dirichlet eigenvalue, and in this direction we discuss a few recent results about some reverse Faber-Krahn inequality.

**Remark 1.1.** The question of regularity could also be considered for the following optimization problems:

$$\min \{\lambda_k(\Omega),\ \ \Omega \subset D, P(\Omega) = p\}, \qquad \min \{P(\Omega) + \lambda_k(\Omega),\ \ \Omega \subset D, |\Omega| = a\}$$

where $P$ denotes the perimeter (in the sense of geometric measure theory), and $D$ is either a bounded smooth box, or $\mathbb{R}^d$. In these cases, as it has been shown in [24, 23], the regularity of optimal shapes is driven by the presence of the perimeter term. More precisely it can be shown that they exist (which is not trivial if $D = \mathbb{R}^d$) and that they are quasi-minimizers of the perimeter, and therefore smooth up outside a singular set of dimension less than $d - 8$.

# 2 Minimization for $\lambda_1$

In this section, we focus on the regularity of the optimal shapes of the following problem:

$$\min \{\lambda_1(\Omega),\ \ \Omega \subset D,\ \Omega \text{ quasi} - \text{open},\ |\Omega| = a\}, \tag{6}$$

where $D$ is an open set in $\mathbb{R}^d$, $a \in (0, |D|)$ and $k \in \mathbb{N}^*$.

Thanks to the Faber-Krahn inequality, it is well-known that, if $D$ contains a ball of volume $a$, then this ball is a solution of the problem, and is moreover unique, up to translations (and to sets of zero-capacity). Therefore, the results of this section are relevant only if such a ball does not exist.

## 2.1 Free boundary formulation

We first give an equivalent version of problem (6) as a free boundary problem, namely an optimization problem in $H_0^1(D)$ where domains are level sets of functions.

**Notation.** For $w \in H_0^1(D)$, we will denote $\Omega_w = \{x \in D; w(x) \neq 0\}$.

Recall that for a bounded quasi-open subset $\Omega$ of $D$ (see [27, Chapter 2])

$$\lambda_1(\Omega) = \min\left\{\int_\Omega |\nabla v|^2; v \in H_0^1(\Omega),\ \int_\Omega v^2 = 1\right\}. \tag{7}$$

**Definition 2.1.** In this section, we denote by $u_\Omega$ any nonnegative minimizer in (7), i.e. such that

$$u_\Omega \in H_0^1(\Omega),\ \int_\Omega |\nabla u_\Omega|^2 = \lambda_1(\Omega),\quad \int_\Omega u_\Omega^2 = 1.$$

**Remark 2.2.** Choosing in (7) $v = v(t) := (u_\Omega + t\varphi)/\|u_\Omega + t\varphi\|_{L^2(\Omega)}$ with $\varphi \in H_0^1(\Omega)$, and using that the derivative at $t = 0$ of $t \to \int_\Omega |\nabla v(t)|^2$ vanishes leads to

$$\forall\, \varphi \in H_0^1(\Omega),\ \int_\Omega \nabla u_\Omega \nabla \varphi = \lambda_1(\Omega) \int_\Omega u_\Omega \varphi. \tag{8}$$

If $\Omega$ is an open set, (8) means exactly that $-\Delta u_\Omega = \lambda_1 u_\Omega$ in the sense of distributions in $\Omega$.

Note that if $u_\Omega$ is a minimizer in (7), so is $|u_\Omega|$. Therefore, with no loss of generality, we can assume that $u_\Omega \geq 0$ and we will always do it in this section on the minimization of $\lambda_1(\Omega)$. If $\Omega$ is a connected open set, then $u_\Omega > 0$ on $\Omega$. This is a consequence of the maximum principle applied to $-\Delta u_\Omega = \lambda_1(\Omega) u_\Omega \geq 0$ on $\Omega$. This extends (quasi-everywhere) to the case when $\Omega$ is a *quasi-connected* quasi-open set, but the proof requires a little more computation.

Since $\Omega \mapsto \lambda_1(\Omega)$ is nonincreasing with respect to inclusion, any solution of (6) is also solution of

$$\min\left\{\lambda_1(\Omega),\ \ \Omega \subset D,\ \Omega \text{ quasi} - \text{open},\ |\Omega| \leq a\right\}. \tag{9}$$

The converse is true in most situations, in particular if $D$ is connected, see Remark 2.5, Corollary 2.17 and the discussion in Section 2.4.1. Note that it may happen that if $D$ is not connected, then a solution to (9) does not satisfy $|\Omega| = a$.

Nevertheless, we will first consider Problem (9) and this will provide a complete understanding of (6) as well. We start by proving that (9) is equivalent to a free boundary problem.

**Proposition 2.3.** *1. Let $\Omega^*$ be a quasi-open solution of the minimization problem (9) and let $u = u_{\Omega^*}$. Then*

$$\int_D |\nabla u|^2 = \min\left\{\int_D |\nabla v|^2; v \in H_0^1(D); \int_D v^2 = 1, |\Omega_v| \leq a\right\}. \tag{10}$$

*2. Let $u$ be solution of the minimization problem (10). Then $\Omega_u$ is solution of (9).*

*Proof.* For the first point, we choose $v \in H_0^1(D)$ with $|\Omega_v| \leq a$ and we apply (9) to $\Omega = \Omega_v$. This gives $\int_D |\nabla u|^2 = \lambda_1(\Omega^*) \leq \lambda_1(\Omega_v)$ and we use the property (7) for $\lambda_1(\Omega_v)$ so that

$$\int_D |\nabla u|^2 \leq \min\left\{\int_D |\nabla v|^2; v \in H_0^1(D), \int_D v^2 = 1, |\Omega_v| \leq a\right\}.$$

But equality holds since $u \in H_0^1(\Omega^*) \subset H_0^1(D)$, and $|\Omega_u| = |\Omega^*| \leq a$.

For the second point, let $u$ be a solution of (10). Then, $|\Omega_u| \leq a, \int_D u^2 = 1$. Let $\Omega \subset D$ quasi-open with $|\Omega| \leq a$ and let $u_\Omega$ as in Definition 2.1. Then

$$\lambda_1(\Omega_u) \leq \int_D |\nabla u|^2 \leq \int_D |\nabla u_\Omega|^2 = \lambda_1(\Omega).$$

□

**Remark 2.4.** We will now work with the functional problem (10) rather than (9). Note that if $D$ is bounded (or with finite measure), then existence of the minimum $u$ follows easily from the compactness of $H_0^1(D)$ into $L^2(D)$ applied to a minimizing sequence (that we may assume to be weakly convergent in $H_0^1(D)$ and strongly in $L^2(D)$).

**Remark 2.5.** As we explain below, the two different following situations may occur. If $D$ is connected and $\Omega^*$ solves (9), then $a^* := |[u_{\Omega^*} > 0]| = a$ and $\Omega^* = [u_{\Omega^*} > 0]$. If however $D$ is not connected, it may happen that $a^* < a$; then $u_{\Omega^*} > 0$ on some of the connected components of $D$ and identically zero on the others.

Indeed, if $a^* < a$, then for all ball $B \subset D$ with measure less than $a - a^*$ and all $\varphi \in H_0^1(B)$, we may choose $v = v(t) = (u + t\varphi)/\|u + t\varphi\|_{L^2(D)}$ with $u := u_{\Omega^*} \geq 0$ in (10). Writing that the derivative at $t = 0$ of $t \mapsto \int_D |\nabla v(t)|^2$ vanishes gives

$$\int_D \nabla u \nabla \varphi = \lambda_a \int_D u\,\varphi \quad \text{with} \quad \lambda_a := \int_D |\nabla u|^2,$$

and this implies: $-\Delta u = \lambda_a u$ <u>in $D$</u>. By strict maximum principle in each connected component of $D$, either $u > 0$ or $u \equiv 0$. If $D$ is connected, we get a contradiction since $a < |D|$. Therefore necessarily $a^* = a$ if $D$ is connected.

We refer to Corollary 2.17 and Proposition 2.28 for a complete description of the regularity when $D$ is not connected.

## 2.2 Existence and Lipschitz regularity of the state function

### 2.2.1 Equivalence with a penalized version

As announced in the introduction, we will first prove that (10) is equivalent to a penalized version.

**Proposition 2.6.** *Assume $|D| < +\infty$. Let $u$ be a solution of (10) and $\lambda_a := \int_D |\nabla u|^2$. Then, there exists $\mu > 0$ such that*

$$\int_D |\nabla u|^2 \leq \int_D |\nabla v|^2 + \lambda_a \left[1 - \int_D v^2\right]^+ + \mu\,[|\Omega_v| - a]^+, \quad \forall v \in H_0^1(D). \tag{11}$$

**Remark 2.7.** Given a quasi-open set $\Omega \subset D$, and choosing $v = u_\Omega$ in (11), we obtain the penalized 'domain' version of (9), where $\Omega^*$ is solution of (9)

$$\lambda_1(\Omega^*) \leq \lambda_1(\Omega) + \mu[|\Omega] - a]^+, \;\; \forall\, \Omega \subset D,\; \Omega \text{ quasi} - \text{open}. \tag{12}$$

*Proof of Proposition 2.6.* Note first that, by definition of $u$ and of $\lambda_a$, for all $v \in H_0^1(D)$ with $|\Omega_v| \leq a$, we have $\int_D |\nabla v|^2 - \lambda_a \int_D v^2 \geq 0$, or

$$\int_D |\nabla u|^2 \leq \int_D |\nabla v|^2 + \lambda_a \left[1 - \int_D v^2\right]. \tag{13}$$

Let us now denote by $J_\mu(v)$ the right-hand side of (11) and let $u_\mu$ be a minimizer of $J_\mu(v)$ for $v \in H_0^1(D)$ (its existence follows by compactness of $H_0^1(D)$ into $L^2(D)$, see also Remark 2.4). Up to replacing $u_\mu$ by $|u_\mu|$, we may assume $u_\mu \geq 0$. Using that $J_\mu(u_\mu) \leq J_\mu(u_\mu/\|u_\mu\|_2)$, we also deduce that $\|u_\mu\|_2^2 = \int_D u_\mu^2 \leq 1$.

For the conclusion of the proposition, it is sufficient to prove $|\Omega_{u_\mu}| \leq a$ since then

$$J_\mu(u_\mu) \leq J_\mu(u) = \int_D |\nabla u|^2 \leq J_\mu(u_\mu),$$

the last inequality coming from (13).

Assume by contradiction that $|\Omega_{u_\mu}| > a$ and introduce $u^t := (u_\mu - t)^+$. Then $J_\mu(u_\mu) \leq J_\mu(u^t)$. This implies, using that $|\Omega_{u^t}| > a$ for $t$ small,

$$\int_{[0<u_\mu<t]} |\nabla u_\mu|^2 + \mu\,|[0 < u_\mu < t]| \leq \lambda_a \int_{[0<u_\mu<t]} u_\mu^2 + 2t\lambda_a \int_D u_\mu.$$

Using the coarea formula (see e.g. [28], [30]), this may be rewritten for $t \leq t_0 \leq \sqrt{\mu/\lambda_a}$ as

$$\int_0^t ds \int_{[u_\mu=s]} \left[|\nabla u_\mu| + \frac{\mu - \lambda_a s^2}{|\nabla u_\mu|}\right] d\mathcal{H}^{d-1} \leq 2t\lambda_a \int_D u_\mu \leq 2t\lambda_a |\Omega_{u_\mu}|^{1/2}.$$

But the function $x \in (0,\infty) \mapsto x + (\mu - \lambda_a s^2)x^{-1} \in [0,\infty)$ is bounded from below by $2\sqrt{\mu - \lambda_a s^2}$ and also by $2\sqrt{\mu - \lambda_a t_0^2}$ as soon as $s^2 \le t^2 \le t_0^2 \le \mu/\lambda_a$. It follows that

$$\forall\, t \in [0,t_0),\ \ 2\sqrt{\mu - \lambda_a t_0^2}\int_0^t \int_{[u_\mu = s]} d\mathcal{H}^{d-1} \le 2t\lambda_a |\Omega_{u_\mu}|^{1/2}. \tag{14}$$

We now use the isoperimetric inequality: $\int_{[u_\mu=s]} d\mathcal{H}^{d-1} \ge C(d)\,|[u_\mu > s]|^{\frac{d-1}{d}}$. We divide the inequality by $t$ and we let $t \to 0$, then $t_0 \to 0$, to deduce

$$2\sqrt{\mu}\,C(d)|\Omega_{u_\mu}|^{\frac{d-1}{d}} \le 2\lambda_a|\Omega_\mu|^{1/2}, \ \text{ and finally } \ 2\sqrt{\mu}\,C(d)\,a^{\frac{d-2}{2d}} \le 2\lambda_a.$$

Thus, if $d \ge 2$, $|\Omega_{u_\mu}| > a$ is impossible if $\mu > \mu^* := \lambda_a^2 C(d)^{-2} a^{(2-d)/d}$. Therefore the conclusion of Proposition 2.6 holds for any $\mu > \mu^*$.

If $d = 1$, we have $\sqrt{\mu}C(1) \le \lambda_a|\Omega_\mu|^{1/2}$. On the other hand, by definition of $u_\mu$ we also have $|\Omega_{u_\mu}| \le a + \lambda_1(\Omega_1)/\mu$ for some fixed $\Omega_1 \subset D$ with $|\Omega_1| = a$. We deduce an upper bound for $\mu$ as well. □

**Remark 2.8.** With respect to the heuristic remarks made in Paragraph 2.3.2, it is interesting to notice that our problem here is not in a 'differentiable setting'. However, we do perform some kind of differentiation in the direction of the perturbations $t \mapsto (u_\mu - t)^+$. This provides the upper bound $\mu^*$ on $\mu$ which plays the role of a Lagrange multiplier. This remark is a little more detailed in Paragraph 2.3.2. Note that $\mu^*$ does not depend on $|D|$. The assumption $|D| < \infty$ was used only to prove existence of the minimizer $u_\mu$.

**Remark 2.9** (Sub- and super-solutions)**.** Note that to prove Proposition 2.6, we only use perturbations of the optimal domain $\Omega_u$ *from inside*. This means that the same result is valid for -sometimes called- *shape subsolutions where (10) is assumed only for functions $v$ for which $\Omega_v \subset \Omega_u$.*

Next, we will prove Lipschitz continuity of the functions $u$ solutions of the penalized problem (11). Actually, even more interestingly, Lipschitz continuity will hold for *super-solutions of (11) that is when the inequality (11) is valid only for perturbations* from outside, *i.e. such that $\Omega_u \subset \Omega_v$.*

### 2.2.2 A general sufficient condition for Lipschitz regularity

We now state a general result to prove Lipschitz regularity of functions, independently of shape optimization. It applies to signed functions as well and will be used again in the minimization of the $k$-th eigenvalue.

**Proposition 2.10.** *Let $U \in H_0^1(D)$, bounded and continuous on $D$ and let $\omega := \{x \in D; U(x) \ne 0\}$. Assume $\Delta U$ is a measure such that $\Delta U = g$ on $\omega$ with $g \in L^\infty(\omega)$ and*

$$|\Delta|U||\,(B(x_0,r)) \le Cr^{d-1} \tag{15}$$

*for all $x_0 \in D$ with $B(x_0,2r) \subset D, r \le 1$ and $U(x_0) = 0$. Then $U$ is locally Lipschitz continuous on $D$. If moreover $D = \mathbb{R}^d$, then $U$ is globally Lipschitz continuous.*

**Remark 2.11.** Note that if $U$ is locally Lipschitz continuous on $D$ with $\Delta U \ge 0$, then for a test function $\varphi$ with

$$\begin{gathered}\varphi \in C_0^\infty(B(x_0,2r)),\ B(x_0,2r) \subset D,\ 0 \le \varphi \le 1,\\ \varphi \equiv 1 \text{ on } B(x_0,r),\ \|\nabla\varphi\|_{L^\infty(B)} \le C/r,\end{gathered} \tag{16}$$

we have

$$\begin{gathered}\Delta U(B(x_0,r)) \le \int_D \varphi d(\Delta U) = -\int_D \nabla\varphi\nabla U \le\\ \|\nabla U\|_{L^\infty}|\Omega_\varphi|\|\nabla\varphi\|_{L^\infty} \le C\|\nabla U\|_{L^\infty}\, r^{d-1}.\end{gathered}$$

This indicates that the estimate (15) is essentially a necessary condition for the Lipschitz continuity of $U$. This theorem states that the converse holds in some cases which are relevant for our analysis as it will appear in the next paragraph.

**Remark 2.12.** In the proof of Proposition 2.10, as in [9], we will use the following identity which is useful to estimate the variation of functions:

$$\fint_{\partial B(x_0,r)} U(x)d\sigma(x) - U(x_0) = C(d)\int_0^r s^{1-d}\left[\int_{B(x_0,s)} d(\Delta U)\right] ds. \tag{17}$$

This is easily proved for regular functions $U$ by integration in $s$ of

$$\frac{d}{ds}\fint_{\partial B(0,1)} U(x_0+s\xi)d\sigma(\xi) = \fint_{\partial B(0,1)} \nabla U(x_0+s\xi)\cdot\xi = C(d)s^{1-d}\int_{B(x_0,s)} \Delta U,$$

which implies that for *a.e.* $0 < r_1 < r_2$,

$$\fint_{\partial B(x_0,r_2)} U(x)d\sigma(x) - \fint_{\partial B(x_0,r_1)} U(x)d\sigma(x) = C(d)\int_{r_1}^{r_2}\left[s^{1-d}\int_{B(x_0,s)} \Delta U\right] ds.$$

It extends to functions $U \in H^1(D)$ where $\Delta U$ is a measure with $\int_0^r s^{1-d}\int_{B(x_0,s)} d(|\Delta U|)ds < \infty$. We may then consider that $U$ is precisely defined at $x_0$ as:

$$U(x_0) = \lim_{r\to 0^+}\fint_{\partial B(x_0,r)} U(x)d\sigma(x), \tag{18}$$

and (17) holds with this precise definition of $U(x_0)$.

*Proof of Proposition 2.10.* We want to prove that $\nabla U \in L^\infty_{loc}(D)$. We can first claim that $\nabla U = 0$ a.e. on $D\setminus\omega$. On the open set $\omega$, we have $\Delta U = g \in L^\infty(\omega)$ so that at least $U \in C^1(\omega)$.

Let us denote $D_\delta = \{x \in D; d(x,\partial D) > \delta\}$ (we start with the case $D \neq \mathbb{R}^d$). We will bound $\nabla U(x_0)$ for $x_0 \in \omega\cap D_\delta$. The meaning of the constant $C$ will vary but always depend only on $\delta$, $\|U\|_{L^\infty(D)}$, $\|g\|_{L^\infty(D)}$, $d$ and on the constant $C$ in the assumption (15).

Let $y_0 \in \partial\omega$ be such that $|x_0 - y_0| = d(x_0,\partial\omega) := r_0$. Then $r_0 > 0$ and $B(x_0,r_0) \subset \omega$. We have $U(y_0) = 0$ since $y_0 \in \partial\omega$ and $U$ is continuous. Let us introduce $s_0 := \min\{r_0, 1\}, B_0 := B(x_0,s_0)$ and $V \in H^1_0(B_0)$ such that $\Delta V = g$ on $B_0$. Since $g \in L^\infty$, by scaling we obtain

$$\|V\|_{L^\infty(B_0)} \le Cs_0^2,\ \|\nabla V\|_{L^\infty(B_0)} \le Cs_0,\ C = C(\|g\|_{L^\infty}).$$

Since $U - V$ is harmonic on $B_0$, we also have $|\nabla(U-V)(x_0)| \le \frac{d}{s_0}\|U-V\|_{L^\infty(B_0)}$ so that

$$|\nabla U(x_0)| \le |\nabla V(x_0)| + ds_0^{-1}\|U-V\|_{L^\infty(B_0)} \le C\left[s_0 + s_0^{-1}\|U\|_{L^\infty(B_0)}\right]. \tag{19}$$

If $s_0 \ge \delta/16$, we deduce from (19): $|\nabla U(x_0)| \le C(\delta, \|U\|_{L^\infty}, \|g\|_{L^\infty})$. We now assume $\delta \le 16$.
If $s_0 < \delta/16$ i.e. $r_0 = s_0 < \delta/16$, since $x_0 \in D_\delta$, $d(y_0,\partial D) \ge d(x_0,\partial D) - d(x_0,y_0) \ge \delta - r_0 \ge 15r_0$ which implies $B(x_0,r_0) \subset B(y_0,2r_0) \subset B(y_0,8r_0) \subset D$. Thanks to assumption (15), $U(y_0) = 0$ and to formula (17) applied with $U$ replaced by $|U|$, we deduce $\fint_{\partial B(y_0,4r_0)} |U(z)|d\sigma(z) \le C\,r_0$. Finally, using the representation $(U-V)(x) = \fint_{B(y_0,4r_0)} U(z)P_x(z)d\sigma(z)$ for all $x \in B(y_0,2r_0)$ where $P_x(\cdot)$ is the Poisson kernel at $x$, we have

$$\|U-V\|_{L^\infty(B_0)} \le \|U-V\|_{L^\infty(B(y_0,2r_0))} \le C\fint_{\partial B(y_0,4r_0)} |U(z)|\,d\sigma(z) \le C\,r_0.$$

This together with (19) (where $s_0 = r_0$) and $\|V\|_{L^\infty(B_0)} \le Cr_0^2$, this implies $|\nabla u(x_0)| \le C$.

Now if $D = \mathbb{R}^d$, either $\omega = \mathbb{R}^d$ and (19) gives the estimate ($r_0 = +\infty, s_0 = 1$), or $\omega \neq \mathbb{R}^d$: then we argue just as above, replacing $\delta/16$ by 1 in the discussion. □

In Proposition 2.10, the fonction $U$ is assumed to be continuous on $D$. For our optimal eigenfunctions, this will be a consequence of the following lemma.

**Lemma 2.13.** *Let $U \in H^1_0(D)$ such that $\Delta U$ is a measure satisfying*

$$|\Delta U|\,(B(x_0,r)) \le Cr^{d-1}, \tag{20}$$

*for all $x_0 \in D$ with $B(x_0,2r) \subset D, r \le 1$. Then $U$ is continuous on $D$.*

*Proof.* Assumption (20) implies that $\int_0^r s^{1-d}|\Delta U|(B(x_0,s)) < \infty$ so that (18) and (17) hold. Let $x_0, y_0 \in D$ and $r > 0$ small enough so that $B(x_0, 2r) \subset D, B(y_0, 2r) \subset D$. We deduce, using (20) again and the representation (18):

$$|U(x_0) - U(y_0)| \leq \left| \fint_{\partial B(x_0,r)} U - \fint_{\partial B(y_0,r)} U \right| + C\, r \leq \\ \fint_{\partial B(0,r)} |U(x_0+\xi) - U(y_0+\xi)| d\sigma(\xi) + C\, r.$$

But by continuity of the trace operator from $H^1(B(0,r))$ into $L^1(B(0,r))$, this implies

$$|U(x_0) - U(y_0)| \leq C(r) \|U(x_0 + .) - U(y_0 + .)\|_{H^1(B(0,r))} + C\, r.$$

Thus

$$\limsup_{y_0 \to x_0} |U(x_0) - U(y_0)| \leq C\, r.$$

This being valid for all small enough $r$, continuity of $U$ at $x_0$ follows and therefore continuity on $D$ as well. □

**Remark 2.14.** Looking at the proof, we easily see that the assumptions could be weakened in Lemma 2.13: $U \in W_0^{1,1}(D)$ would be sufficient and $r^{d-1}$ could be replaced in (20) by $r^{d-2}\varepsilon(r)$ with $\varepsilon(r)/r$ integrable on $(0,1)$.

### 2.2.3 Lipschitz continuity of the optimal eigenfunction

**Theorem 2.15.** *Let $u$ be a solution of (10). Then $u$ is locally Lipschitz continuous on $D$.*

*Proof.* Up to replacing $u$ by $|u|$, we may assume that $u \geq 0$. We will first show that $U = u$ satisfies the assumptions of Lemma 2.13. It will follow that $u$ is continuous on $D$. Therefore, we will have $-\Delta u = \lambda_a u$ on the open set $\omega = [u > 0]$ (see Remark 2.2). Then we will prove (see also Remark 2.16 below) that

$$-\Delta u \leq \lambda_a u \text{ in } D. \tag{21}$$

It will imply that $\Delta u$ is a measure and also, by an easy bootstrap that $u \in L^\infty(D)$. Thus the assumptions of Proposition 2.10 will be satisfied and local Lipschitz continuity on $D$ will follow.

By Proposition 2.6, $u$ is also solution of Problem (11). We apply this inequality with $v = u + t\varphi$ where $t > 0, \varphi \in H_0^1(D)$. Then

$$0 \leq \int_D 2\nabla u \nabla \varphi + t|\nabla \varphi|^2 + \lambda_a \left[-2u\varphi - t\varphi^2\right]^+ + \frac{\mu}{t} \left[|\Omega_{u+t\varphi}| - a\right]^+. \tag{22}$$

Choosing first $\varphi = -p_n(u)\psi$ where $\psi \in C_0^\infty(D)$, $\psi \geq 0$, and $p_n(r) = \min\{r^+/n, 1\}$, we obtain with $q_n(r) = \int_0^r p_n(s)ds$ and after letting $t \to 0$ (note that $|\Omega_{u+t\varphi}| = |\Omega_u| \leq a$)

$$0 \leq \int_D -2p_n'(u)|\nabla u|^2\psi - 2\nabla q_n(u)\nabla\psi + 2\lambda_a u p_n(u)\psi.$$

Note that $up_n(u) \to u^+ = u, q_n(u) \to u^+ = u$ in a nondecreasing way as $n$ increases to $+\infty$. Using $p_n'(u)|\nabla u|^2 \geq 0$, we obtain at the limit that $\Delta u + \lambda_a u \geq 0$ in the sense of distributions in $D$, whence (21).

Choosing $\varphi \in C_0^\infty(D)^+$ in (22) leads to $-2\int_D \nabla u \nabla \varphi \leq \int_D t|\nabla\varphi|^2 + \frac{\mu}{t}|\Omega_\varphi|$ or

$$2\langle \Delta u + \lambda_a u, \varphi\rangle \leq \int_D 2\lambda_a u\varphi + t|\nabla\varphi|^2 \quad + \frac{\mu}{t}|\Omega_\varphi|. \tag{23}$$

Minimizing over $t \in (0,\infty)$ gives

$$\langle \Delta u + \lambda_a, \varphi\rangle \leq \int_D \lambda_a u\varphi + \|\nabla\varphi\|_{L^2} \left[\mu|\Omega_\varphi|\right]^{1/2}. \tag{24}$$

Let now $x_0 \in D$ such that $B(x_0, 2r) \subset D$ and let $\varphi \in C_0^\infty(B(x_0,2r))^+$ as in (16). Using also $u \in L^\infty$, we deduce that

$$|\Delta u|\left(B(x_0,r)\right) \leq (\Delta u + \lambda_a u)\left((B(x_0,r)\right) + \lambda_a \int_{B(x_0,r)} u \leq Cr^{d-1},$$

whence the estimate (15).

□

**Remark 2.16.** Here, we strongly use the positivity of $u$. Actually, $u$ is an eigenfunction for the eigenvalue $\lambda_a$ on $\Omega_u$. Since it is open, we have $\Delta u + \lambda_a u = 0$ on $\Omega_u$ (see Remark 2.2). Knowing moreover that $u \geq 0$, only with that much information, one can prove that $\Delta u + \lambda_a u \geq 0$ on $D$. For this, note that to prove it above, we used the test functions $\varphi = -p_n(u)\psi$ which satisfies $\Omega_\varphi \subset \Omega_u$ and therefore belong to $H_0^1(\Omega_u)$. Thus applying (8) in Remark 2.2 with this $\varphi$ is sufficient (and we finish as above).

Next this positivity of the measure $\Delta u + \lambda_a u$ allows to directly estimate the mass of $|\Delta u|$ on balls only with the information (24). This will not be the case when dealing with $k$-th eigenfunctions when $k \geq 2$ (see the remarks and comments on the use of the Monotonicity Lemma 3.4).

Let us now state a corollary of Proposition 2.15 for the initial actual shape optimization problem (6).

**Corollary 2.17.** *Assume $D$ is open and with finite measure. Then there exists an open set $\Omega^*$ which is solution of (6). Moreover, for any (quasi-open) solution $\Omega^*$ of (6), $u_{\Omega^*}$ is locally Lipschitz continuous on $D$. If $D$ is connected, then all solutions $\Omega^*$ of (6) are open.*

**Remark 2.18.** If $D$ is not connected, then it may happen that $\Omega^*$ is not open: we refer for instance to Example 2.27. However $u_{\Omega^*}$ is always locally Lipschitz continuous. Let us mention that the existence of an optimal open set for (6) had first been proved in [34]. A different penalization was used and the corresponding state function was proved to converge to a Lipschitz optimal eigenfunction.

*Proof of Corollary 2.17.* If $D$ is of finite measure, as already seen, Problem (10) has a solution $u$. By Theorem 2.15, it is locally Lipschitz continuous on $D$. In particular $\Omega_u$ is open. If $|\Omega_u| = a$, then $\Omega^* := \Omega_u$ is an open solution of (6). If $|\Omega_u| < a$, then any open set $\Omega^*$ satisfying $\Omega_u \subset \Omega^* \subset D, |\Omega^*| = a$ is also a solution since then, by monotonicity $\lambda_1(\Omega^*) \leq \lambda_1(\Omega_u)$ (and there are such $\Omega^*$ like for instance $\Omega^* := \Omega_u \cup B(x_0, r) \cap D$ where $x_0 \in D$ and $r$ is chosen so that $|\Omega^*| = a$ ).

Now let $\Omega^*$ be a solution of (6). Then, by Proposition 2.3, $u_{\Omega^*}$ is solution of the minimization problem (10). By Theorem 2.15, it is locally Lipschitz continuous in $D$. As proved in Remark 2.5, if $D$ is connected, then $\Omega^* = [u_{\Omega^*} > 0]$. Therefore $\Omega^*$ is open. □

## 2.3 Regularity of the boundary

In this section, we go deeper in the analysis of the free boundary problem (10), and we explain the strategy to prove the regularity of the boundary of any solution to (6). Here is the main result of this section:

**Theorem 2.19** (Briançon-Lamboley [10])**.** *Assume $D$ is open, bounded and connected. Then any solution of (6) satisfies:*

1. *$\Omega^*$ has locally finite perimeter in $D$ and*

$$\mathcal{H}^{d-1}((\partial\Omega^* \setminus \partial^*\Omega^*) \cap D) = 0, \tag{25}$$

   *where $\mathcal{H}^{d-1}$ is the Hausdorff measure of dimension $d-1$, and $\partial^*\Omega^*$ is the reduced boundary (in the sense of sets with finite perimeter, see [28] or [30]).*

2. *There exists $\Lambda > 0$ such that*

$$\Delta u_{\Omega^*} + \lambda_1(\Omega^*) u_{\Omega^*} = \sqrt{\Lambda}\mathcal{H}^{d-1}\lfloor\partial\Omega^*,$$

   *in the sense of distribution in $D$, where $\mathcal{H}^{d-1}\lfloor\partial\Omega^*$ is the restriction of the $(d-1)$-Hausdorff measure to $\partial\Omega^*$.*

3. *$\partial^*\Omega^*$ is an analytic hypersurface in $D$.*

4. *If $d = 2$, then the whole boundary $\partial\Omega^* \cap D$ is analytic.*

### 2.3.1 Regularity for free boundary problems

In this section, we give a small overview of the literature about regularity of free boundaries, related to the problem (6) or (9); we insist that it is by far not exhaustive as the literature on the subject is huge.

It has been seen in Section 2.1 that problem (9) is equivalent to a free boundary problem: this will allow us to use some of the deep and well-known results on the subject. The seminal paper [3] is dealing with the following model problem:

$$\min\left\{\int_D |\nabla u|^2 + \int_D g(x)^2 \mathbb{1}_{\Omega_u},\ \ u \in H^1(D),\ \ u = u_0 \text{ on } \partial D\right\} \tag{26}$$

where $u_0 \in H^1(D)$ is a given positive boundary data, $D$ is an open bounded set, and we recall that $\Omega_u = \{u \neq 0\}$. In [3], different results were given about the regularity of the free boundary $\partial\Omega_u$ where $u$ solves (26). Let us first notice that it is expected to obtain some regularity from the optimality condition, which is

$$\Delta u = 0 \text{ in } \Omega_u, \quad |\nabla u| = g \text{ on } \partial\Omega_u \cap D, \tag{27}$$

where this has to be understood in a weak sense (see below). In the paper [42], it is for example shown that if $\partial\Omega_u$ is assumed to be $C^{1,\alpha}$ (locally inside $D$) for some $\alpha > 0$ and $g$ is smooth (say analytic), then $\partial\Omega_u$ is actually locally analytic. As usual though, the most difficult part is to obtain regularity for $\partial\Omega_u$ from scratch, and in particular, to give sense to (27). In [32], these results have been adapted to a different situation, namely

$$\min\left\{\int_D |\nabla u|^2 - 2fu + g^2 \mathbb{1}_{\Omega_u},\ \ u \in H^1(D), u = u_0 \text{ on } \partial D\right\} \tag{28}$$

where $f$ is a given nonnegative bounded function (in [32] they actually have $D = \mathbb{R}^d$ and no boundary condition: nevertheless, their results can easily be adapted to this situation, and for more clarity we prefer to deal with this problem here). In that case, the Euler-Lagrange equation is given by

$$\left\{\begin{array}{rcll} -\Delta u & = & f & \text{in} \quad \Omega_u, \\ |\nabla u| & = & g & \text{on} \quad \partial\Omega_u \cap D. \end{array}\right. \tag{29}$$

These equations, especially the second one that defines the free boundary condition, makes sense a priori only if it is known that $u \in H^1(D)$ and $\partial\Omega_u$ are smooth. Therefore, there are different way of formulating this boundary condition in a weak sense:

- Shape derivative formulation (see [3, Theorem 2.5]): for any $\xi \in C_0^\infty(D)$,

$$\lim_{\varepsilon\to 0}\int_{\partial\{u>\varepsilon\}} (|\nabla u|^2 - g^2)(\xi\cdot\nu)d\mathcal{H}^{d-1} = 0, \tag{30}$$

  where $\nu$ is the outward normal vector.

- Weak solution ([3, 32]): $u \in H^1(D)$ is called a weak solution of (29) if

  1. $u$ is continuous and *nonnegative*,
  2. $u$ satisfies, in the sense of distribution in $D$:

$$\Delta u + f = g\mathcal{H}^{d-1}\lfloor\partial^*\Omega_u \text{ in } D. \tag{31}$$

  3. there exists $0 < c \leq C$ such that for all balls $B_r(x) \subset D$ with $x \in \partial\Omega_u$,

$$c \leq \frac{1}{r}\fint_{\partial B_r(x)} u d\mathcal{H}^{d-1} \leq C,$$

  As stated in [3, Theorem 5.5], if $g$ is continuous, then local minimum for (26) are weak solutions in this sense, and in particular $\mathcal{H}^{d-1}(\partial\Omega_u \setminus \partial^*\Omega_u) = 0$, so in (31) the term $\mathcal{H}^{d-1}\lfloor\partial^*\Omega_u$ can be replaced by $\mathcal{H}^{d-1}\lfloor\partial\Omega_u$ although this is not true in general (note that in [32] they use a weaker notion of weak solutions, though this one is more suitable for regularity results).

- Viscosity formulation: though we will not use this framework here, let us emphasize the complete regularity theory developed for *viscosity solutions to* (29) (see [17, 19, 18, 16]). Lots of proofs have been drastically simplified compared to the original paper [3], see also [25].

For our purpose, we focus on the use of the notion of weak solutions, and here are the two main results about the regularity for these solutions:

**Theorem 2.20** (Theorem 8.2 in [3], Theorem 2.17(a) in [32]). *Suppose $f \in L^\infty(D)$, $f \geq 0$, $g$ is Hölder continuous in $D$ and there exists $c > 0$ such that $g \geq c$ in $D$. Then if $u$ is a weak solution (see the definition above), $\partial^*\Omega_u$ is locally $C^{1,\alpha}$ in $D$, for some $\alpha > 0$, and moreover*

$$\mathcal{H}^{d-1}((\partial\Omega_u \setminus \partial^*\Omega_u) \cap D) = 0. \tag{32}$$

In the two-dimensional case, we can improve this statement.

**Theorem 2.21** (Theorem 8.3 in [3]). *Under the same assumption as Theorem 2.20, if moreover $d = 2$ and*

$$\fint_{B_r \cap \Omega_u} (g^2 - |\nabla u|^2)_+ \underset{r \to 0}{\longrightarrow} 0$$

*then*

$$\partial\Omega_u = \partial^*\Omega_u \quad (\text{in } D)$$

*and therefore $\partial\Omega_u$ is locally $C^{1,\alpha}$ in $D$.*

These two results are the main achievement of [3], see Sections 5 to 8 in this paper. This relies in particular on the proof of the fact that "flatness implies regularity", in other words if the boundary is assumed to be a bit flat (therefore avoiding the singularity described in Paragraph 2.4.2), then it is actually smooth.

Different generalizations or simplifications of the proofs of these results did appear in the literature after the paper [3]. As we noticed just before, a complete regularity theory for viscosity solutions of (29) has been developped, and we can find in [25] a simplified proof of a version Theorem 2.20 for these viscosity solutions. See also Section 2.4.2 for further results and in particular an improvement of (32).

As we already noticed at the beginning of this paragraph, once a $C^{1,\alpha}$ regularity is obtained for $\partial^*\Omega_u$ (or $\partial\Omega_u$), then depending on the assumption on the regularity of the data $f$ and $g$, we can easily improve this regularity: namely as proven in [42], if $f$ is $C^{m,\beta}$ and $g$ is $C^{m+1,\beta}$, then $\partial^*\Omega_u$ (or $\partial\Omega_u$ is $d = 2$) is $C^{m+2,\beta}$; moreover, if $f$ and $g$ are analytic, so is $\partial^*\Omega_u$.

### 2.3.2 Some heuristic remarks on "exact penalty" property

As seen in Paragraph 2.2.1 and as it will be used again in the next Paragraph 2.3.3 and in Section 3, regularity results are proved by strongly using the so-called "exact penalty" property. It says that our constrained problems (like (9), (48)) are *equivalent* to penalized versions (like (12), (39), (40), (49) or (51))) at least for well chosen or large enough penalty factors: this property goes quite beyond the weaker property that the penalized version 'converges' to the constrained one as the penalty factor tends to $\infty$), see Propositions 2.6, 2.23, 3.1 and Remark 3.2. This general question is analyzed in the literature of optimization theory: we refer for instance to [39, Section VII] for the basic theory or to [6] for some interesting results in this direction. Though our optimization problems do not seem to fit in their demanding framework (in particular for differentiability or convexity assumptions), they nevertheless present the same features.

Let us describe some main questions and answers, together with 'heuristical proofs' inspired from this literature, for the following abstract optimization problem

$$\min \{f(x), \;\; g(x) = a\} \tag{33}$$

where $f, g : X \to [0, \infty)$, $a > 0$ and $X$ is a real Banach space. Let us say we want to prove that any solution $x_0$ of (33) is also solution of the penalized following version:

$$\min \{f(x) + \mu\,[g(x) - a]\,, \;\; g(x) \geq a\}\,, \tag{34}$$

(the same problem with the constraint $g(x) \leq a$ instead, is similar). We concentrate on the two following questions:

- If $x_0$ is a solution of (33), is it still a (local) solution of (34) for some (finite) value of $\mu$? Or in the terminology introduced in Remark 2.9, is $x_0$ a super-solution of the penalized version (34) for some $\mu > 0$?

- If so, what is the link between the best (=smallest) possible $\mu$ and the Lagrange multiplier $\Lambda$ associated with $x_0$, that is the real number $\Lambda$ such that

$$f'(x_0) + \Lambda\, g'(x_0) = 0\ ? \tag{35}$$

  Again the proof of the regularity result of Theorem 2.19 and in particular its point 2) strongly relies on this comparison and on the main fact that $\Lambda > 0$.

Let us start with the first question: in many situation, the answer to this question is yes. As a heuristic proof of this fact, let us assume that $f, g$ are smooth enough (say $C^1$), and that there exists a solution $x_\mu$ to (34) for every $\mu \in (0, \infty)$. First notice that if $g(x_\mu) = a$, then it is admissible for (33), and we easily deduce that $x_0$ is then also a solution to (34). Assume instead that $g(x_\mu) > a$ for every $\mu > 0$ and let us look for a contradiction. To that end, we write the optimality condition for (34), which gives (since the constraint is not saturated)

$$f'(x_\mu) + \mu\, g'(x_\mu) = 0. \tag{36}$$

Assuming some compactness on the set $\{x_\mu\}_{\mu>0}$, $x_\mu$ converges (up to a subsequence) to some $\widetilde{x}_0$ as $\mu \to +\infty$. It is clear, using $f(x_\mu) + \mu(g(x_\mu) - a) \leq f(x_0)$ that $g(\widetilde{x}_0) = a$ (and that $\widetilde{x}_0$ solves (33)). But then (36) leads to a contradiction as $\mu \to \infty$ at least if we (naturally) asssume that $a$ is not a critical value of $g$ (i.e. $g'(\widetilde{x}_0) \neq 0$).

About the second question, we first notice that a necessary condition on $\mu$ so that $x_0$ (solution of (33)) be also solution of (34) is $\mu \geq \Lambda$, where $\Lambda$ is the Lagrange multiplier associated with $x_0$ as defined in (35). Indeed, the optimality condition for (34) is then satisfied at $x_0$ and classically means (Karush-Kuhn-Tucker conditions) that there exists $\gamma_\mu$ such that

$$f'(x_0) + \mu g'(x_0) + \gamma_\mu g'(x_0) = 0, \qquad \text{and } \gamma_\mu \in (-\infty, 0],$$

the sign of $\gamma_\mu$ coming from the fact that the constraint is "$g \geq a$". Therefore, again if $a$ is not a critical value of $g$, this implies: $\Lambda = \mu + \gamma_\mu \leq \mu$.

It is then natural to hope that for any $\mu > \Lambda$, a solution of (33) is also a solution of (34). With some strong convexity assumptions, this is actually the case as proved for example in [6]. Since we are here far from this kind of 'convex' situation, let us refer heuristically to a weaker and local result which will be quite in the spirit of one of the main steps in the proof of Theorem 2.19. More precisely, let us replace (34) by the following 'local' version where $h > 0$ is fixed:

$$\min \left\{ f(x) + \mu\, [g(x) - a]\,,\ g(x) \in [a, a+h] \right\}. \tag{37}$$

Then the previous remarks made for (34) are still valid: if $f, g$ are $C^1$ and that we have some adequate compactness properties, for $\mu$ large enough, $x_0$ is solution of this penalized problem; moreover, this requires $\mu \geq \Lambda$ where $\Lambda$ is defined in (35).

Let us prove that $\mu > \Lambda$ is indeed (generally) sufficient. For this, we assume that $\Lambda$ is the only Lagrange multiplier for the constrained problem (33) or more generally that it is the upper (finite) bound of the Lagrange multipliers (we indeed have to take into account that there may be several solutions to (33), associated to different Lagrange multipliers).

Let $\mu = \Lambda + \varepsilon, \varepsilon > 0$ be fixed. Let $x_h$ be the solution of (37) and assume $g(x_h) > a$ for every $h > 0$. Equation (36) is satisfied as before. Similarly also, using compactness, we may assume that $x_h$ converges to some $\widetilde{x}_0$, which is solution of (33) since the constraints on $x_h$ lead to $g(\widetilde{x}_0) = a$. Therefore, on one hand, passing to the limit in (36), we obtain

$$f'(\widetilde{x}_0) + (\Lambda + \varepsilon) g'(\widetilde{x}_0) = 0,$$

while on the other hand $f'(\widetilde{x}_0) + \lambda g'(\widetilde{x}_0) = 0$ holds for some $\lambda \leq \Lambda$, whence a contradiction (assuming as before that $g'(\widetilde{x}_0) \neq 0$).

### 2.3.3 Penalization of the volume constraint in Problem (10)

Our Problem (10) was proved to be equivalent to the penalized version (11) in Proposition 2.6 (see also (12) for a penalized "domain" version). Actually, we can check that, according to the previous

heuristic analysis, if our problem was differentiable, then the best penalized factor $\mu^*$ would essentially be the Lagrange multiplier $\Lambda$ of the constraint problem. Let us formally make the computation. The Euler-Lagrange equation for Problem (6) inside $D$ is

$$-|\nabla u_{\Omega^*}|^2 + \Lambda = 0 \text{ on } \partial\Omega^* \text{ or also } -\partial_\nu u_{\Omega^*} = \sqrt{\Lambda} \text{ on } \partial\Omega^*.$$

Integrating $-\Delta u_{\Omega^*} = \lambda_a u_{\Omega^*}$ on $\Omega^*$ gives

$$P(\Omega^*)\sqrt{\Lambda} = -\int_{\partial\Omega^*} \partial_\nu u_{\Omega^*} = -\int_{\Omega^*} \Delta u_{\Omega^*} = \lambda_a \int_{\Omega^*} u_{\Omega^*} \leq \lambda_a |\Omega^*|^{1/2}.$$

Inserting now the isoperimetric inequality $P(\Omega^*) \geq C(d)|\Omega^*|^{(d-1)/d}$, we obtain exactly the estimate $\Lambda \leq \mu^*$ where $\mu^*$ is defined at the end of the proof of Proposition 2.6. Actually, the proof of this proposition is nothing but a rigourous justification of the computation we just made here.

Now, the proof of the regularity of the boundary stated in Theorem 2.19 will require to make a "local" penalized version of the type (37). But the heuristic tools described in the previous papragraph are difficult to justify rigourously. Let us list the difficulties we face to apply such strategy to (6) and lead to a complete proof of Proposition 2.23 stated below:

- Compactness and continuity arguments are used several times, including in order to get existence for problems (34) or (37). This requires to obtain $H^1$ bounds, and weak $H^1$ continuity of the functionals.
- We use Euler-Lagrange equations, which requires that functionals are differentiable. As the natural choice of space is $X = H_0^1(D)$ it is important to notice here that the functionals (mainly the constraint) are not classically differentiable. Therefore, at every step of the proof we write the Euler-Lagrange equation in a very weak way, using shape derivatives: in other words, if $u \in H_0^1(D)$ is a minimizer for our problem, we compare $u$ to $u \circ T_t$ where $t$ is a small parameter, and $T_t : D \to D$ is a family of smooth vector fields, close to the Identity in the $C^1$ norm. This gives, in our case: for all $\Phi \in C_0^\infty(D, \mathbb{R}^d)$,

  $$\int_D 2(D\Phi\nabla u, \nabla u) - \int_D |\nabla u|^2 \nabla \cdot \Phi + \lambda_a \int_D u^2 \nabla \cdot \Phi = \Lambda \int_{\Omega_u} \nabla \cdot \Phi$$

  (which is another way to formulate (30)).
- Because of the difficulties stated above, we need to localize the argument. Namely, we fix $B$ a ball centered at a point of $\partial\Omega_{u_0}$ and we prove that if the radius of $B$ is small enough, then we can penalize the constraint if we restrict the test-functions to $H_0^1(B) \subset H_0^1(D)$.
- For our purpose, we also need to study the penalization procedure for sets $\Omega$ such that $|\Omega| < a$ (see (39)). Using the monotonicity of $\lambda_1$, it is easy to see that having a vanishing penalization $\mu = 0$ is valid in that case. However, it is important for the rest of the proof to explain that some positive values of the penalization parameter $\mu$ are also valid, at least when $|\Omega|$ is close to $a$. This rely on the fact, as explained above, that this parameter $\mu$ can be chosen as close to $\Lambda$ (the Lagrange multiplier for problem (6)) as one wants, and on the fact that $\Lambda$ is positive. This fact is highly non-trivial, see Remark 2.24.

**Remark 2.22.** Of course, if we were studying the following penalized problem

$$\min \{\lambda_1(\Omega) + \mu|\Omega|,\ \Omega \subset D\}, \tag{38}$$

we would not be facing the same difficulties. This problem is easier to analyze, and the result of Theorem 2.19 is actually also valid for solutions of (38).

When $D = \mathbb{R}^d$ (or more generally when $D$ is star-shaped), using the homogeneity properties of the functionals $\Omega \mapsto \lambda_1(\Omega), |\Omega|$, one can find an explicit $\mu$ so that problem (6) is equivalent to (38). This is proved in Proposition 3.1 (even for the $\lambda_k$-problem). Therefore, in a sense, the regularity theory is easier in that case (though as noticed before, this case is not very relevant for $D = \mathbb{R}^d$ since we already know that $\Omega^*$ is a ball).

We know state the following theorem, which is a main step in the proof of Theorem 2.19: let $u$ be a solution of (10) and $B_R$ be a ball included in $D$ and centered on $\partial\Omega_u \cap D$. We define

$$\mathcal{F} = \{v \in H_0^1(D), u - v \in H_0^1(B_R)\}.$$

We denote $J(w) = \int_D |\nabla w|^2 - \lambda_a \int_D w^2$ for $w \in H_0^1(D)$. For $h > 0$, we denote by $\mu_-(h)$ the biggest $\mu_- \geq 0$ such that,

$$\forall\, v \in \mathcal{F} \text{ such that } a - h \leq |\Omega_v| \leq a,\ J(u) + \mu_-|\Omega_u| \leq J(v) + \mu_-|\Omega_v|. \tag{39}$$

We also define $\mu_+(h)$ as the smallest $\mu_+ \geq 0$ such that,

$$\forall\, v \in \mathcal{F} \text{ such that } a \leq |\Omega_v| \leq a + h,\ J(u) + \mu_+|\Omega_u| \leq J(v) + \mu_+|\Omega_v|. \tag{40}$$

**Proposition 2.23.** *Let $u, B_R$ and $\mathcal{F}$ as above. Then for $R$ small enough (depending only on $u, a$ and $D$), there exists $\Lambda > 0$ and $h_0 > 0$ such that,*

$$\forall\, h \in (0, h_0),\ 0 < \mu_-(h) \leq \Lambda \leq \mu_+(h) < +\infty,$$

*and, moreover,*

$$\lim_{h\to 0} \mu_+(h) = \lim_{h\to 0} \mu_-(h) = \Lambda. \tag{41}$$

The proof is detailed in [10], and follows the heuristic proof from Paragraph 2.3.2 with the main steps and difficulties described just above.

**Remark 2.24.** As we noticed before, the most difficult statement here (and the most needed one in the next Paragraph) is the fact that $\Lambda > 0$, which implies that $\mu_-(h) > 0$ for $h$ small enough. The proof of this fact can be found in [8, Propositions 6.1 and 6.2] or [10, Proposition 2.6]: the main idea is to prove that if $\Lambda = 0$, then $-\Delta u = \lambda u$ in the whole box $D$, as in a weak sense, $|\nabla u| = \Lambda = 0$ on $\partial\Omega_u$, which implies that the measure $\Delta u$ has no singularity across $\partial\Omega_u$. Once this fact is proven, we easily get a contradiction.

**Remark 2.25.** Note that a related but different approach has been followed in [1] where they deal with the constrained version of (26), namely

$$\min\left\{\int_D |\nabla u|^2,\ \ u \in H^1(D),\ \ |\Omega_u| = a,\ \ \ u = u_0 \text{ on } \partial D\right\}. \tag{42}$$

They introduce the penalized version

$$\min\left\{\int_D |\nabla u|^2 + f_\varepsilon(|\Omega_u|),\ \ u \in H^1(D),\ \ \ u = u_0 \text{ on } \partial D\right\}, \tag{43}$$

where $f_\varepsilon(x) = \frac{1}{\varepsilon}(x - a)$ if $x \geq a$ and $f_\varepsilon(x) = \varepsilon(x - a)$ if $x \leq a$. They prove that the regularity theory can be applied to $u_\varepsilon$, and that for $\varepsilon$ small enough, $u_\varepsilon$ is such that $|\Omega_{u_\varepsilon}| = a$ and therefore $u_\varepsilon$ actually solves (26). Though the results look very similar and are indeed based on similar observations, namely that the main point is to prove the estimate $0 < c \leq \mu_\varepsilon \leq C$ where $|\nabla u_\varepsilon|^2 = \mu_\varepsilon$ is the optimality condition for (43), we notice two main differences with the approach described above:

- first, the strategy in [1] leads to the weaker result that there exists smooth solutions to the constrained problem (42) (namely $u_\varepsilon$ for $\varepsilon$ small enough), while we get here that any solution is smooth,
- next, the authors use in [1] the regularity theory in order to assert that solutions of (43) are actually also solutions to (42) for $\varepsilon$ small enough. For this step, they apply the regularity theory from [3] to $u_\varepsilon$, in order to prove the estimate $0 < c \leq \mu_\varepsilon \leq C$ mentioned before. The approach described here relies on weaker properties and can therefore be applied to a wider class of examples. Several ideas can for instance be applied to the analysis of (2) where $f$ is not a priori assumed to be nonnegative (see [8]). A complete regularity theory is however still to be done for this 'not signed' situation as we particularly discuss in the next Section.

Nevertheless, it is of course interesting to apply the ideas of [1] to (6). This has been done in the literature in [58]. See also [34] where a similar penalization is studied to prove that there exists an open set which solves (6), and [27, Chapter 7] where the same penalization is used as well.

### 2.3.4 Conclusion

Let $u$ be a solution of (10) and $B_R, \mathcal{F}$ as in the previous paragraph. Before giving a sketch of the proof of Theorem 2.19, we need the following Lemma, which gives a rigorous sense to the fact that, at a point $x \in \partial\Omega_u$, the gradient of $u$ is bounded from above and from below.

**Lemma 2.26.** *There exist $C_1, C_2, r_0 > 0$ such that, for $B(x_0, r) \subset B$ with $r \leq r_0$,*

$$\begin{aligned} &\text{if } \frac{1}{r} \fint_{\partial B(x_0,r)} u \geq C_1 \quad \text{then} \quad u > 0 \text{ on } B(x_0, r), \\ &\text{if } \frac{1}{r} \fint_{\partial B(x_0,r)} u \leq C_2 \quad \text{then} \quad u \equiv 0 \text{ on } B(x_0, r/2). \end{aligned} \tag{44}$$

The detailed proof is given in [10], and we describe here the main ingredients. The first part was already proven earlier as we have seen that $u$ is Lipschitz, but one can now give a slightly different proof of this fact, following the arguments from [3]. The idea is to use (40) for a suitable test function, namely $v$ such that $-\Delta v = \lambda_a v$ in $B(x_0, r)$ and $v = u$ on $B(x_0, r)^c$. Again, it is interesting to notice that we only use perturbation from outside, namely $v$ such that $\Omega_u \subset \Omega_v$.

The second part says that the gradient does not degenerate on the free boundary, and as we noticed before, this rely on the fact that $\Lambda > 0$ and that we can penalize the volume constraint for some positive $\mu_-(h)$. Similarly to the previous point, we now use (39) for $v$ defined such that $v = 0$ in $B(x_0, r/2)$, $-\Delta v = \lambda_a u$ on $B(x_0, r) \setminus B(x_0, r/2)$ and $v = u$ on $B(x_0, r)^c$.

With this result, we are in position to give the main steps for the proof of Theorem 2.19.

1. The proof is now, using (44) in Lemma 2.26, the same as in [32] or in [3]: we first show a volume density estimate, namely that that there exists $C_1, C_2$ and $r_0$ such that, for every $B(x_0, r) \subset B$ with $r \leq r_0$,
$$0 < C_1 \leq \frac{|B(x_0, r) \cap \Omega_u|}{|B(x_0, r)|} \leq C_2 < 1,$$
and also an estimate for the measure $(\Delta u + \lambda_a u)$
$$C_1 r^{d-1} \leq (\Delta u + \lambda_a u)(B(x_0, r)) \leq C_2 r^{d-1}.$$
The proof is the same as in [32] with $\lambda_a u$ instead of $f$. It gives directly (using classical Geometric Measure Theory arguments, see section 5.8 in [28]) the first point of Theorem 2.19, namely that $\Omega^*$ has local finite perimeter and $\mathcal{H}^{d-1}((\partial\Omega_u \setminus \partial^*\Omega_u) \cap D) = 0$.

2. For the second point, we see that $\Delta u + \lambda_a u$ is absolutely continous with respect to $\mathcal{H}^{d-1} \lfloor \partial\Omega_u$ which is a Radon measure (using the first point), so we can use Radon-Nikodym's Theorem. To compute the Radon's derivative, we argue as in Theorem 2.13 in [32] or (4.7,5.5) in [3]. The main difference is that here, we have to use (41) in Proposition 2.23 to show that, if $u_0$ denotes a blow-up limit of $u(x_0 + rx)/r$ (when $r$ goes to 0), then $u_0$ is such that,
$$\int_{B(0,1)} |\nabla u_0|^2 + \Lambda |\{u_0 \neq 0\} \cap B(0,1)| \leq \int_{B(0,1)} |\nabla v|^2 + \Lambda |\{v \neq 0\} \cap B(0,1)|,$$
for every $v$ such that $v = u_0$ outside $B(0,1)$. To show this, in [3] or in [32] the authors use only perturbations in $B(x_0, r)$ with $r$ goes to 0, so using (41), we get the same result. We can compute the Radon's derivative and get (in $B$)
$$\Delta u + \lambda_a u = \sqrt{\Lambda} \mathcal{H}^{d-1} \lfloor \partial\Omega_u,$$
which means we have proven the second point in Theorem 2.19.

3. Now, $u$ is a weak-solution in the sense recalled in Paragraph 2.3.1, and with Theorem 2.20 we directly get the regularity of $\partial^*\Omega_u$.

4. If $d = 2$, in order to have the regularity of the whole boundary, we have to show that Theorem 6.6 and Corollary 6.7 in [3] (which are stated for solutions and not weak-solutions) are still true for our

problem. The corollary directly comes from the theorem. So we need to show that, if $d = 2$ and $x_0 \in \partial\Omega_u$, then

$$\lim_{r\to 0} \fint_{B(x_0,r)} \max\{\Lambda - |\nabla u|^2, 0\} = 0. \tag{45}$$

As in [3, Theorem 6.6], the idea is to use (39) with $v = \max\{u - \varepsilon\zeta, 0\}$ and $\zeta \in C_0^\infty(B)^+$, and $h = |0 < u \le \varepsilon\zeta| \le |\{\zeta \neq 0\}|$, and we get

$$\int_{\{0<u<\varepsilon\zeta\}} (\Lambda - |\nabla u|^2) \le \int_{\{u\ge\varepsilon\zeta\}} \varepsilon^2|\nabla\zeta|^2 + (\Lambda - \mu_-(h))h.$$

The only difference now with [3] is the last term. Using Proposition 2.23 again, we see that $(\Lambda - \mu_-(h))h = o(h)$, so we can choose the same kind of $\zeta$ and $\varepsilon$ as in [3] to get (45) (see also Theorem 5.7 in [8] for more details).

□

## 2.4 Remarks and perspectives

### 2.4.1 About the connectedness assumption

In this paragraph, we discuss the hypothesis "$D$ is connected" in Theorem 2.19. We begin with the following example, taken from [9] which proves that the optimal set $\Omega^*$ may be irregular if $D$ is not connected, although $u_{\Omega^*}$ is very regular.

**Example 2.27.** *(from [9])* We choose $D = D_1 \cup D_2$, where $D_1, D_2$ are disjoint disks in $\mathbb{R}^2$ of radius $R_1, R_2$ with $R_1 > R_2$. If $a = \pi R_1^2 + \varepsilon$, then the solution $u$ of (10) coincides with the first eigenfunction of $D_1$ and is identically 0 on $D_2$, and thus $\Omega_u = D_1$ and $|\Omega_u| < a$.
In this case, we can choose any open subset $\omega$ of $D_2$ with $|\omega| = \varepsilon$. Then $\Omega^* := D_1 \cup \omega$ is a solution of (9). Since $\omega$ may be chosen as irregular as one wants, this proves that optimal domains are not regular in general.

However, we are able to prove the following proposition.

**Proposition 2.28** (The non-connected case)**.** *If we suppose that $D$ is not connected and with finite measure, the problem* (10) *still has a solution $u$ which is locally Lipschitz continuous in $D$. If $\omega$ is any open connected component of $D$, we have three cases:*

1. *either $u > 0$ on $\omega$,*
2. *or $u \equiv 0$ on $\omega$,*
3. *or $0 < |\Omega_u \cap \omega| < |\omega|$ : in that case $\Omega_u \subset \omega$ and $\partial\Omega_u$ has the same regularity as stated in Theorem 2.19.*

*If $|\Omega_u| < a$, then only the first two cases can appear.*

**Remark 2.29.** It follows from Proposition 2.28 that we obtain the same regularity as in the connected case. Indeed, in the first two cases, $\partial\Omega_u \cap \omega = \emptyset$.

**Remark 2.30.** To summarize, in all cases, there exists a solution $\Omega^*$ to (9) which is regular in the sense of Theorem 2.19, but there may be some other non regular optimal shape. And if $D$ is connected, any optimal shape is regular.

*Proof of Proposition 2.28.* The existence and the Lipschitz regularity are stated in Theorem 2.15. In particular $\Omega_u$ is open. If $|\Omega_u| < a$, then by Remark 2.5, we are in one the two first situations. Let us now assume that $|\Omega_u| = a$. Then, as proved below:

A) either $|\Omega_u \cap \omega| = |\omega|$: then we are again in the first situation,

B) or $|\Omega_u \cap \omega| < |\omega|$: then $\Omega_u \subset \omega$ and $u$ is solution of (10) with $D$ replaced by $\omega$. In particular, the regularity result of Theorem 2.19 applies (whence Point 3.).

To obtain A) and B), recall first that $u_{|\omega} \in H_0^1(\omega)$: indeed, $u$ is the limit in $H_0^1(D)$ of $u_n \in C_0^\infty(D)$. Then $(u_n)_{|\omega} \in C_0^\infty(\omega)$ and converges in $H^1(\omega)$ to $u_{|\omega}$. Since also $u \in H_0^1(\Omega_u)$, it follows that $u_{|\Omega_u \cap \omega} \in H_0^1(\Omega_u \cap \omega)$. On the other hand, $-\Delta u = \lambda_a u$ on $\Omega_u$ and therefore on $\Omega_u \cap \omega$ so that $\lambda_1(\Omega_u \cap \omega) = \lambda_a$.

In the case A), we remark that, by minimality in (10) (since $|\omega| \leq a$), and by monotonicity of $\lambda_1(\cdot)$

$$\lambda_a = \int_D |\nabla u|^2 \leq \int_D |\nabla u_\omega|^2 = \lambda_1(\omega) \leq \lambda_1(\Omega_u \cap \omega) = \lambda_a.$$

We also have $\lambda_a = \int_{\Omega_u \cap \omega} |\nabla u|^2 / \int_{\Omega_u \cap \omega} u^2 = \int_\omega |\nabla u|^2 / \int_\omega u^2$. Since $\omega$ is connected, $\lambda_1(\omega)$ is simple and therefore $u_\omega = u_{|\omega} / \|u_{|\omega}\|_{L^2(\omega)}$ and $u = u_\omega > 0$ on $\omega$ (see Remark 2.5).

In the case B), if we had $|\Omega_u \cap \omega| < a$, then we could find an open set $\hat{\omega}$ such that

$$\Omega_u \cap \omega \subset \hat{\omega} \subset \omega,\ \lambda_1(\hat{\omega}) < \lambda_1(\Omega_u \cap \omega) = \lambda_a,\ |\hat{\omega}| \leq a,$$

and this would be a contradiction with the minimality of $\lambda_a$. Thus $|\Omega_u \cap \omega| = a \geq |\Omega_u|$, that is $|\Omega_u \cap \omega| = |\Omega_u|$ which implies that $u = 0$ a.e. (and therefore eveywhere) on the complement of $\omega$. Whence $\Omega_u \subset \omega$. □

### 2.4.2 Full regularity and improvement of the estimate of the singular set

It is natural to ask whether the regularity stated in Theorem 2.19 can be improved. As it mainly rely on the use of the theory of Free boundary regularity for problem (29), we state here the different improvement that have been made in the literature since the original paper [3]:

- When concerned with regularity for weak solutions of (29), it is important to notice that regularity does not occur in general in dimension 3 or higher. Indeed, we recall here the construction of [3, Example 2.7]: using the spherical coordinates $(r, \phi, \theta)$ so that $x = r(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$, we search for a function $u$ of the form $rh(\theta)$, harmonic in $B(0,1) \subset \mathbb{R}^3$:

$$\begin{aligned}\Delta u = \tfrac{1}{r^2}\partial_r(r^2\partial_r u) + \tfrac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta\partial_\theta u) + \tfrac{1}{r^2\sin^2\theta}\partial_\phi^2 u = \\ = \tfrac{1}{r\sin\theta}\left[2\sin\theta.h(\theta) + (\sin(\theta).h')'(\theta)\right].\end{aligned}$$

  Adding the condition $h'(\frac{\pi}{2}) = 0$ we obtain the explicit solution

$$h(\theta) = 2 + \cos(\theta)\log\frac{1-\cos\theta}{1+\cos\theta}.$$

  Then, defining $u(x) = rh_+(\theta)$ and $\theta_0$ the unique zero of $h$ in $(0, \frac{\pi}{2})$, we have that $\Omega_u = \{(r,\theta,\phi), \theta_0 < \theta < \pi - \theta_0\}$ is singular. Moreover, it is easy to notice that in the basis $(e_r, e_\theta, e_\phi)$ we have

$$\nabla u = \partial_r u.e_r + \frac{1}{r}\partial_\theta u.e_\theta + \frac{1}{r\sin\theta}\partial_\phi u.e_\varphi = h(\theta)e_r + h'(\theta)e_\theta \text{ in } \Omega_u,$$

  so that

$$|\nabla u| = h'(\theta_0) \text{ on } \partial\Omega_u \setminus \{0\}.$$

  It is easy to see that this function satisfies the shape derivative formulation of the free boundary (30), and is also a weak solution defined in Section 2.3.1. This proves that there exist critical sets for problem (26) in $\mathbb{R}^3$ that are singular. It can be seen that this example is not a minimizer; see also [40].

  Nevertheless, it is proven in [59] by G. Weiss that, for weak solutions (as defined in Section 2.3.1), in the case $f = 0$ and $g$ is a positive constant:

$$\dim_{\mathcal{H}}(\partial\Omega_u \setminus \partial^*\Omega_u) \leq d - 3, \tag{46}$$

  (and in the case $d = 3$ it is known that singularities are isolated points) which recovers Theorem 2.21 and improves the estimate (32). This result is optimal as the previous critical set in $\mathbb{R}^3$ has a 0-dimensional singularity.

- In another work of G. Weiss [60], a better estimate is obtained for the singular set of minimizers for (26), and he also gives a strategy to obtain an optimal estimate: indeed the author proves there exists $k^* \in \mathbb{N} \cup \{+\infty\}$ such that if $u$ solves (26), then

$$\dim_{\mathcal{H}}(\partial\Omega_u \setminus \partial^*\Omega_u) \leq d - k^* \tag{47}$$

  (where we understand $\partial\Omega_u \setminus \partial^*\Omega_u = \emptyset$ if $d - k^* < 0$) and he gives a characterization of $k^*$ as the minimal dimension so that there exists a singular homogeneous minimizer. This result rely on a monotonicity formula to prove that blow-ups are homogeneous, and a dimension reduction argument, similar to Federer's strategy in the regularity theory for perimeter minimizers. As it was known from [3] that there is no singularity in dimension 2 (see Theorem 2.21) for weak solutions (and so for minimizers), it is known that $k^* \geq 3$ and therefore Weiss recovers the estimate (46) of weak solutions, and gives a strategy to improve it.

- After Weiss' results, three improvements have been given about the number $k^*$:

  1. in [21] it is proven there is no singular cone in $\mathbb{R}^3$, and therefore $k^* \geq 4$,
  2. in [26] it is proven that there exists a singular cone in $\mathbb{R}^7$, and therefore $k^* \leq 7$.
  3. in [41] it is proven that there is no singular cone in $\mathbb{R}^4$, and therefore $k^* \geq 5$.

  Let us conclude that it is conjectured that $k^* = 7$.

It is natural to expect that the regularity for the free boundary problem (10) is very similar, therefore we propose:

**Open problem 2.31.** *Prove that solutions $\Omega^*$ to problem* (6) *are such that*

$$\partial\Omega^* = \partial^*\Omega^* \quad \text{if } d < 7, \quad \text{and} \quad \dim_{\mathcal{H}}(\partial\Omega^* \setminus \partial^*\Omega^*) \leq d - 7 \quad \text{if } d \geq 7.$$

Of course, this open problem can be decomposed into two different open questions: first show that Weiss' results can be applied to the solutions of (6), and that an estimate like (47) can be obtained with a critical exponent $k^\sharp$ (which may actually be the same as $k^*$), then identify $k^\sharp$.

# 3 Minimization for $\lambda_k$

Here we consider the general minimization problems with $k \geq 1,\ \mu \in (0,\infty),\ 0 < a < |D|$:

$$\min\{\lambda_k(\Omega);\ \Omega \subset D,\ \Omega \text{ quasi} - \text{open } |\Omega| = a\}, \tag{48}$$

$$\min\{\lambda_k(\Omega) + \mu|\Omega|;\ \Omega \subset D,\ \Omega \text{ quasi} - \text{open}\}. \tag{49}$$

Applying the existence results of [27, Chapter 2], we know that (48) and (49) have solutions when $D = \mathbb{R}^d$ (and this is a highly nontrivial result). They are moreover bounded and with finite perimeter. Actually, the existence proof requires to prove some a priori regularity properties for the expected optima (see [27, Chapter 2])).

## 3.1 Penalized is equivalent to constrained in $\mathbb{R}^d$

A main remark is that when $D = \mathbb{R}^d$, then the two problems (48) and (49) are equivalent for a good choice of $\mu$. Obviously, any solution $\Omega^*$ of (49) is a solution of (48) with $a = |\Omega^*|$, and this for all $\mu \in (0,\infty)$. Conversely

**Proposition 3.1.** *Let $D = \mathbb{R}^d$. Then a solution of (48) is solution of (49) with $\mu = 2\lambda_k(\Omega^*)/ad$.*

*Proof.* Let $\Omega^*$ be a solution of (48) and let $\mu := 2\lambda_k(\Omega^*)/ad$. Then for all quasi-open set $\Omega$ with $|\Omega| < +\infty$, we have

$$\lambda_k(\Omega^*) \leq \lambda_k\left((a|\Omega|^{-1})^{1/d}\Omega\right) = (a^{-1}|\Omega|)^{2/d}\lambda_k(\Omega), \tag{50}$$

which implies that for all $t > 0$

$$g(t) := a^{2/d}t^{-2}\lambda_k(\Omega^*) + a^{1+2/d}\mu\, t^d \leq t^{-2}|\Omega|^{2/d}\lambda_k(\Omega) + a^{1+2/d}\mu\, t^d.$$

The minimum of $t \in (0,\infty) \mapsto g(t)$ is reached at $t=1$ so that $g(1) \le t^{-2}|\Omega|^{2/d}\lambda_k(\Omega) + a^{1+2/d}\mu\, t^d$ for all $t \in (0,\infty)$. Choosing $t = (a^{-1}|\Omega|)^{1/d}$ leads to

$$a^{-2/d} g(1) = \lambda_k(\Omega^*) + \mu|\Omega^*| \le \lambda_k(\Omega) + \mu|\Omega|.$$

□

**Remark 3.2.** It is interesting to notice that, if $D$ is star-shaped (say around the origin), that is $\big[\Omega \subset D \to t\Omega \subset D,\ \forall\, t \in [0,1]\big]$, then *a solution $\Omega^*$ of (48) is also a super-solution of (49)* with the same $\mu$ as in Proposition 3.1 that is

$$\lambda_k(\Omega^*) + \mu a \le \lambda_k(\Omega) + \mu|\Omega| \text{ for all } \Omega^* \subset \Omega \subset D. \tag{51}$$

Indeed, we do the same proof as above, using that if $\Omega^* \subset \Omega \subset D$, then $a|\Omega|^{-1} \le 1$ so that $(a|\Omega|^{-1})^{1/d}\Omega \subset D$ and (50) holds. The rest of the proof remains unchanged.

## 3.2 A Lipschitz regularity result for optimal eigenfunctions

The following result is proved in [14], [57].

**Theorem 3.3.** *Let $D = \mathbb{R}^d$ and let $\Omega^*$ be a solution of the minimization problem (48) or (49). Then there exists an eigenfunction associated with $\lambda_k(\Omega^*)$ which is Lipschitz continuous.*

The proof of this result is quite more involved than for the case $k=1$. According to Proposition 2.10, it would be sufficient to prove that one of the eigenfunctions $u = u_{\Omega^*}$ satisfies

$$|\Delta|u||\,(B(x_0,r)) \le C r^{d-1}, \tag{52}$$

around each $x_0$ where $u(x_0) = 0$. This property can actually be proved under the extra (strong) assumption that $\lambda_k(\Omega^*) > \lambda_{k-1}(\Omega^*)$. This is very restrictive, but it is nevertheless a starting point of the proof.

If this strict inequality does not hold, the strategy of [14] is to consider a series of auxiliary approximate shape optimization problems involving the lower $i$-th eigenvalues for $i \le k$. Using extensively that (52) holds even for *super-solutions*, they prove that the approximate state functions are uniformly Lipschitz continuous and converge to one of the $k$-th eigenfunctions, whence the result.

Let us describe a little more the main steps of this proof.

1. A first step in reaching (52) is to try to prove that, for all functions $\varphi \in C_0^\infty(B(x_0,r))$, we have an estimate like we proved in the case $k=1$ (see (24)), namely

   $$|\langle \Delta u + \lambda_k u, \varphi\rangle| \le C\,|B(x_0,r)|^{1/2}\|\nabla\varphi\|_{L^2}. \tag{53}$$

   This will actually be proved only under the extra assumption that $\lambda_k(\Omega^*) > \lambda_{k-1}(\Omega^*)$, see below. A main point for what is coming below is that this estimate will hold for any *super-solution* of the penalized problem (49).

   As we saw in the proof of Theorem 2.15, (53) directly implies (52) when $u \ge 0$ (see Remark 2.16 for more comments on this point). In the general case, (53) only provides an estimate of $|\Delta u^+|(B(x_0,r))$ in terms of $|\Delta u^-|(B(x_0,r))$ and conversely. Therefore we need one more information to bound both of them. This is given by the Monotonicity Lemma 3.4 below.

2. At this point, it is at least possible to prove the continuity of $u$. The proof is rather direct at points $x_0$ where $u(x_0) \ne 0$ (since we formally have $\Delta u + \lambda_a u = 0$ around this $x_0$). Continuity at points $x_0$ of the free boundary where $u(x_0) = 0$ is more involved and uses the estimate (53).

3. As just explained, the lacking information to estimate $|\Delta(u^+ + u^-)|(B(x_0,r))$ may be done around points $x_0$ where $u(x_0) = 0$ thanks to the following celebrated *Monotonicity Lemma* by Caffarelli-Jerison-Kenig [20] which says the following:

**Lemma 3.4.** *Let $U \in H^1(B(0,R))$ and*

$$\Delta U^+ \geq -a, \quad \Delta U^- \geq -a \ \ on \ \ B(0,R) \ for \ some \ a \geq 0.$$

*Set*

$$\Phi(r) := \left( \frac{1}{r^2} \int_{B(0,r)} \frac{|\nabla U^+|^2}{|x|^{d-2}} \right) \left( \frac{1}{r^2} \int_{B(0,r)} \frac{|\nabla U^-|^2}{|x|^{d-2}} \right).$$

*Then,*

- *if $a = 0$, $r \in (0,R) \mapsto \Phi(r)$ is nondecreasing.*
- *in all cases, there exists $C$ such that*

$$\forall r \in (0, R/2), \ \ \Phi(r) \leq C \left[ 1 + \int_{B(0,R)} U^2 \right]. \tag{54}$$

The case $a = 0$ was first proved in the seminal paper [4] by Alt-Caffarelli-Friedman and in this case, we do have an actual monotonicity property. The estimate (54) implied by this monotonicity lemma when $a = 0$ was extended to the non-homogeneous case in [20]. This was proved under a continuity assumption for $U$ which was later dropped by B. Velichkov [57], [56].

This lemma essentially says that for such functions $U$, both $\nabla U^+$ and $\nabla U^-$ cannot be bad at the same time around $x_0$. Thus, if there exists some control on one by the other (and this is given by (53)), then we control both of them. The complete proof of this may be found in [9] (see also in [14], [57]).

4. Now, the question is: how does one prove (53)? The starting point is the 'supersolution property'

$$\lambda_k(\Omega^*) \leq \lambda_k(\Omega^* \cup B(x_0,r)) + \mu \, |B(x_0,r)| \tag{55}$$

implied by (49). In the case of $k = 1$, we apply the optimal property to the optimal eigenfunction $u$ and to the perturbation $u + t\varphi, \varphi \in C_0^\infty(B(x_0,r)$ to obtain

$$\int_D |\nabla u|^2 \leq \int_D |\nabla(u + t\varphi)|^2 + \mu \, |B(x_0,r)|,$$

from which we easily deduce (53) (see the proof of Theorem 2.15 for such an argument). But, for higher eigenfunctions $k \geq 2$, one cannot work so easily with test functions due to the more complex variational functional characterization. Then, two main ideas are used in [14].

5. **Case** $\lambda_k(\Omega^*) > \lambda_{k-1}(\Omega^*)$ : it can be proved (see [14],[57]) that for all $v \in H_0^1(B(x_0,r))$ with $\int |\nabla v|^2 \leq 1$ and for $r \in (0, r_0)$,

$$\lambda_k(\Omega^* \cup B(x_0,r)) \leq \frac{\int |\nabla(u+v)|^2 + (\lambda_{k-1}(\Omega^*) + 1) \int |\nabla v|^2}{\int (u+v)^2 - \int |\nabla v|^2/2}.$$

Plugging this information into (55) gives after a simple computation

$$|\langle \Delta u + \lambda_k(\Omega^*) u, v \rangle| \leq C|B(x_0,r)| + C_k \int |\nabla v|^2, \ C_k = 1 + \lambda_{k-1}(\Omega^*)/2 + \lambda_k(\Omega^*)/4.$$

The choice of $v := |B(x_0,r)|^{1/2} \varphi / \|\nabla \varphi\|_{L^2}$ ($r$ small enough) leads to the estimate (53) and the expected (52) follows as well as Lipschitz continuity.

6. **Case** $\lambda_k(\Omega^*) = \lambda_{k-1}(\Omega^*)$: this is the most difficult case, and also the most frequent since optimal eigenvalues are often multiple. Then the idea of [14] is to consider the problem with $\varepsilon > 0$ small:

$$\min\{(1-\varepsilon)\lambda_k(\Omega) + \varepsilon \lambda_{k-1}(\Omega) + 2\mu \, |\Omega|; \Omega \supset \Omega^*\}, \tag{56}$$

which does have a quasi-open solution (see [27, Chapter 2]). Then, there are two cases (A) and (B):

(A) Suppose there exists a sequence of optimizers $\Omega_{\varepsilon_n}$ with $\lim_{n\to\infty}\varepsilon_n = 0$ and such that $\lambda_k(\Omega_{\varepsilon_n}) > \lambda_{k-1}(\Omega_{\varepsilon_n})$. From (56), using also that $\lambda_{k-1}(\cdot)$ is nonincreasing for the inclusion, we have

$$\lambda_k(\Omega_{\varepsilon_n}) + \frac{2\mu}{1-\varepsilon}|\Omega_{\varepsilon_n}| \le \lambda_k(\Omega) + \frac{2\mu}{1-\varepsilon}|\Omega|,$$

for all $\Omega \supset \Omega_{\varepsilon_n}$. As in the points 4) and 5) above (where we used only the supersolution property of the optimizer), we deduce that $u_n := u_{\Omega_{\varepsilon_n}}$ is Lipschitz continuous and it can be checked that the Lipschitz constant does not depend on $\varepsilon$. At this step, convergence arguments must be used to prove that $\Omega_\varepsilon$ converges in some weak sense, but strong enough so that its limit is solution of the limit problem

$$\min\{\lambda_k(\Omega) + 2\mu|\Omega|; \Omega \supset \Omega^*\}. \tag{57}$$

But it is easily seen that a solution of this problem necessarily coincides with $\Omega^*$. Finally, it can be checked that the limit of the uniformly Lipschitz sequence $u_n$ converges to one of the $k$-th eigenvalue of $\Omega^*$. Whence the statement of Theorem 3.3.

(B) Suppose there exists $\varepsilon_0 \in (0,1)$ such that $\Omega_{\varepsilon_0}$ is a solution of (56) and $\lambda_k(\Omega_{\varepsilon_0}) = \lambda_{k-1}(\Omega_{\varepsilon_0})$. But $\Omega_{\varepsilon_0}$ is then also solution of (57) and therefore coincides with $\Omega^*$. Using that $\lambda_k(\cdot)$ is nonincreasing for the inclusion, we deduce from (56) and $\Omega_{\varepsilon_0} = \Omega^*$ that

$$\lambda_{k-1}(\Omega^*) + 2\mu\varepsilon_0^{-1}|\Omega^*| \le \lambda_{k-1}(\Omega) + 2\mu\varepsilon_0^{-1}|\Omega|,$$

for all $\Omega^* \subset \Omega$. Thus $\Omega^*$ is also a *super-solution* for $\lambda_{k-1}(\cdot) + 2\mu\varepsilon_0^{-1}|\cdot|$. Therefore, one can start again the discussion:

(B1) $\lambda_{k-1}(\Omega^*) > \lambda_{k-2}(\Omega^*)$,

(B2) $\lambda_{k-1}(\Omega^*) = \lambda_{k-2}(\Omega^*)$,

and we repeat the same analysis with adequate auxiliary problems, in the same spirit, and at most a finite number of times. Theorem 3.3 follows.

**Remark 3.5.** It is not known whether all $k$-th eigenfunctions are Lipschitz continuous. As proved in the same paper [14], it is the case when minimizing functions of the eigenvalues which involve all of them like

$$\min\left\{\sum_{j=1}^{p}\lambda_j(\Omega) + \mu|\Omega|;\ \Omega \subset \mathbb{R}^d,\ \Omega \text{ quasi}-\text{open}\right\}. \tag{58}$$

In order to understand this fact (see Corollary 3.8 below), let us mention that the strategy for Theorem 3.3 can be generalized to deal with problems like

$$\min\left\{F(\lambda_{k_1}(\Omega)), \dots, F(\lambda_{k_p}(\Omega)) + \mu\,|\Omega|;\ \Omega \subset \mathbb{R}^d,\ \Omega \text{ quasi}-\text{open}\right\}, \tag{59}$$

where $0 < k_1 < k_2 < \ldots < k_p$ and $F : \mathbb{R}^p \to [0,\infty)$ is locally bi-Lipschitz function, increasing in each variable. Indeed, for such functionals, the following result holds.

**Theorem 3.6.** *(see[14]) Let $\Omega^*$ be a bounded optimal shape of (59) (or even only a bounded* super-solution*). Then there exists a sequence of orthonormal eigenfunctions $u_{k_1}, \dots, u_{k_p}$ corresponding to each of the eigenvalues $\lambda_{k_1}, \dots, \lambda_{k_p}$ which are Lipschitz continuous. Moreover*

- *if $\lambda_{k_j}(\Omega^*) > \lambda_{k_j-1}(\Omega^*)$ for some $j$, then the full eigenspace corresponding to $\lambda_{k_j}(\Omega^*)$ consists of Lipschitz continuous functions;*
- *if $\lambda_{k_j}(\Omega^*) = \lambda_{k_{j-1}}(\Omega^*)$ for some $j$, then there exists at least $k_j - k_{j-1} + 1$ orthonormal eigenfunctions corresponding to $\lambda_{k_j}(\Omega^*)$ which are Lipschitz continuous.*

**Remark 3.7.** Note the difference between $\lambda_{k_j-1}$ and $\lambda_{k_{j-1}}$ in the above theorem.

As a consequence of this theorem, the following holds for any optimal solution of Problem (58).

**Corollary 3.8.** *Let $\Omega^*$ be a solution of (58). Then, all eigenfunctions corresponding to the eigenvalues $\lambda_j(\Omega^*)$, $j = 1, \dots, p$ are Lipschitz continuous on $\mathbb{R}^d$ and $\Omega^*$ is equal a.e. to an open set.*

*Proof.* By the previous theorem, all eigenfunctions corresponding to the eigenvalues $\lambda_j(\Omega^*)$ are Lipschitz continuous. Now let $\Omega^{**} := \cup_{j=1}^p [u_j \neq 0]$ where $u_1, \dots, u_p$ is an orthonormal set of normalized eigenfunctions corresponding respectively to the $\lambda_j(\Omega^*)$, $j = 1, \dots, p$. This set $\Omega^{**}$ is open and $\Omega^{**} \subset \Omega^*$. Moreover, $u_j \in H_0^1([u_j \neq 0])) \subset H_0^1(\Omega^{**})$ and satisfies $\int_{\Omega^*} \nabla u_j \nabla \varphi = \lambda_j(\Omega^*) \int_{\Omega^*} u_j \varphi$ for all $\varphi \in H_0^1(\Omega^*)$ and therefore for all $\varphi \in H_0^1(\Omega^{**})$. Thus, all the $\lambda_j(\Omega^*)$ are also eigenvalues on $\Omega^{**}$, with at least the same multiplicity. Due to the monotonicity for the inclusion, we actually have $\lambda_j(\Omega^{**}) = \lambda_j(\Omega^*)$ for all $j = 1, \dots, p$. Now, using also the optimality of $\Omega^*$, we may write

$$\sum_{j=1}^p \lambda_j(\Omega^*) + \mu|\Omega^*| \leq \sum_{j=1}^p \lambda_j(\Omega^{**}) + \mu|\Omega^{**}| = \sum_{j=1}^p \lambda_j(\Omega^*) + \mu|\Omega^{**}|.$$

Since $|\Omega^{**}| \leq |\Omega^*|$, this implies that equality holds and this proves that $\Omega^*$ is open up to a set of zero Lebesgue measure. □

**Remark 3.9.** As proved in [14], this corollary may be extended in two directions:

- first $\sum_j \lambda_j(\cdot)$ may be replaced by $F\big(\lambda_1(\cdot), \dots, \lambda_p(\cdot)\big)$ where $F : \mathbb{R}^p \to [0, \infty)$ is locally bi-Lipschitz and increasing with respect to each variable;
- then, to the pure constrained problem, namely

$$\min\{F(\lambda_1(\Omega), \dots, \lambda_p(\Omega)); \Omega \text{ quasi} - \text{open }, |\Omega| = 1\}. \tag{60}$$

Let us explain why these extensions hold.

- Extension to $F$ (with the penalized term $\mu|\cdot|$) is done as in Corollary 3.8 by using Theorem 3.6 above.
- Extension to the constrained problem is done by proving that an optimal solution of (60) is a super-solution of the penalized version for some $\mu > 0$ (in the spirit of Proposition 3.1 and Remark 3.2). Indeed, if $\Omega^*$ is an optimal set of (60), and $\Omega$ a quasi-open set with finite measure such that $\Omega^* \subset \Omega$, then letting $t := [|\Omega|/|\Omega^*|]^{1/d} > 1$, we have

$$\begin{cases} F\big(\dots, \lambda_j(\Omega^*), \dots\big) \leq F\big(\dots, \lambda_j(\Omega/t), \dots\big) = F\big(\dots, t^2\lambda_j(\Omega), \dots\big) \\ \leq F\big(\dots, \lambda_j(\Omega), \dots\big) + \|F\|_{Lip}(t^2 - 1)\sum_j \lambda_j(\Omega) \\ \leq F\big(\dots, \lambda_j(\Omega), \dots\big) + \|F\|_{Lip}(t^d - 1)\sum_j \lambda_j(\Omega^*) \\ \leq F\big(\dots, \lambda_j(\Omega), \dots\big) + \mu[|\Omega| - |\Omega^*|], \end{cases}$$

  with $\mu = \|F\|_{Lip}|\Omega^*|^{-1}\sum_j \lambda_j(\Omega^*)$.

**Open problem 3.10.** *Concerning the minimization of $\lambda_k(\Omega), k \geq 2$ as in Problems (48) or (49):*

- *Does there exist an optimal solution which is open?*
- *What about Lipschitz continuity of all $k$-th eigenvalues?*

Partial answers are given in the following paragraph when $k = 2$.

## 3.3 More about $k = 2$

### 3.3.1 An example with singular solutions

We go back to problems (48)-(49) with $k = 2$. First, let us give an example showing that, as for $k = 1$, if the box $D$ is not connected, then a quite different qualitative behavior may occur. We saw (see Corollary 2.17), that the optimal first eigenfunction is nevertheless (locally) Lipschitz continuous and consequently, there is an optimal set which is open, but optimal sets are not all open. For $k = 2$, the situation is even worse since the second eigenfunctions which are optimal for (48) may not be regular. This is seen on the following example.

**Example 3.11.** Let $D := D_1 \cup D_2 \subset \mathbb{R}^2$ where $D_1, D_2$ are *disjoint open disks* of radius respectively $R_1 > 0$ and $R_2 = R_1(1+2\varepsilon)$, $\varepsilon > 0$. Let $a := \pi R_1^2[1+(1+\varepsilon)^2]$. Let $\Omega^*$ be an optimal quasi-open solution of (48) with $k = 2$, namely

$$\lambda_2(\Omega^*) = \min\{\lambda_2(\Omega);\ \Omega \subset D,\ \Omega \text{ quasi} - \text{open},\ |\Omega| = a\}, \quad \Omega^* \subset D,\ |\Omega^*| = a. \tag{61}$$

By monotonicity, $\lambda_2(\Omega^*) \geq \lambda_2(D) = \lambda_1(D_1)$. Actually, equality holds since, by minimality, $\lambda_2(\Omega^*) \leq \lambda_2(D_1 \cup D_3)$ where $D_3$ is the disk of radius $R_1(1+\varepsilon)$ with the same center as $D_2$ (note that $|D_1 \cup D_3| = a$). And $\lambda_2(D_1 \cup D_3) = \lambda_1(D_1)$. Thus, $D_1 \cup D_3$ is also optimal.

Now we may perturb $D_3$ (for instance near its boundary) into an open set $D' \subset D_2$ so that:

a) $|D'| = |D_3|$, which means $|D_1 \cup D'| = a$,

b) $\lambda_1(D_3) \leq \lambda_1(D') < \lambda_1(D_1)$ and therefore $\lambda_2(D_1 \cup D') \leq \lambda_1(D_1) = \lambda_2(\Omega^*)$ ($\leq \lambda_2(D_1 \cup D')$ by optimality);

c) the boundary of $D'$ is irregular.

Then, since $\lambda_2(D_1 \cup D') = \lambda_1(D_1)(= \lambda_2(\Omega^*)), |D_1 \cup D'| = a$, $D_1 \cup D'$ is also a solution of the above problem, but it is not regular (one could even choose $D'$ so that it be only quasi-open and not a.e. equal to an open set).
Now, we can perturb $D'$ into $D''$ so that $|D''| = |D_3|$, and $\lambda_1(D'') = \lambda_1(D_1)$ (for instance by taking off larger and larger circles from $D'$). In this case, $D_1 \cup D''$ is still optimal, but one of its second eigenfunctions (namely, the first eigenfunction of $D''$) is not regular.

Note that, in this situation, it may happen that a solution of (48) is not a solution of (49), no matter the value of $\mu > 0$. For instance, there does not exist any $\mu > 0$ such that $D_1 \cup D_3$ is a solution of (49) although it is solution of (48) as we just saw. Indeed, let $D_4$ be the unit disk with the same center as $D_2$. Then

$$\lambda_2(D_1 \cup D_4) + \mu|D_1 \cup D_4| = \lambda_1(D_1) + \mu[|D_1| + |D_4|] < \lambda_1(D_1) + \mu[|D_1| + |D_3|] = \lambda_2(D_1 \cup D_3) + \mu|D_1 \cup D_3|.$$

The same remark is valid for $D_1 \cup D'$ or $D_1 \cup D''$.

On the other hand, as indicated in Remark 3.2, if $D$ is starshaped, then a solution of (48) is also a super-solution of (49) for some adequate $\mu > 0$. It is very likely that the regularity analysis made to prove Theorem 3.6 would extend from $D = \mathbb{R}^d$ to "good" boxes $D$, locally inside $D$.

### 3.3.2 There are open optimal sets

As a partial answer to the open problems indicated at the end of Section 3.2, let us mention the following result proved in [57, 15] and which uses the regularity result of Theorem 2.15.

**Theorem 3.12.** *Let $D \subset \mathbb{R}^d$ be open, connected and with finite measure. Let $\Omega^*$ be a solution of*

$$\min\{\lambda_2(\Omega) + \mu|\Omega|,\ \Omega \subset D,\ \Omega \text{ quasi} - \text{open}\}. \tag{62}$$

*Then, $\Omega^*$ is a.e. equal to an open set.*

*Proof.* Let us indicate the main steps of the proof. We denote by $u_1, u_2$ a first and a second eigenfunction on $\Omega^*$ with $\int_{\Omega^*} u_1 u_2 = 0, u_1 \geq 0$. We denote $\Omega_1 = [u_1 > 0], \Omega^+ = [u_2 > 0], \Omega^- = [u_2 < 0]$.

Let us first prove that we may choose $u_2$ so that both $\Omega^+, \Omega^-$ are not empty (*this uses that $D$ is connected and may not hold otherwise as seen with Example 3.11*). Assume that $u_2 \geq 0$ on $D$ (that is $\Omega^- = \emptyset$). Then, $\lambda_2(\Omega^*) = \lambda_1(\Omega^+)$ and the relation $\int_{\Omega^*} u_1 u_2 = 0$ implies $u_1 \equiv 0$ on $\Omega^+$. We have $\lambda_1(\Omega_1) = \lambda_1(\Omega^*)$. Let us prove that

$$\lambda_1(\Omega^*) = \lambda_2(\Omega^*). \tag{63}$$

It will follow that we may replace $u_2$ by $u_2 - u_1$ so that the 'new' $\Omega^+$ and $\Omega^-$ are not empty as expected.

Assume by contradiction that $\lambda_1(\Omega^*) < \lambda_2(\Omega^*)$, that is $\lambda_1(\Omega_1) < \lambda_1(\Omega^+)$. Since $D$ is connected and open, we may find $x_0 \in \partial\Omega^+$ and $r > 0$ such that, if $\Omega^r := \Omega^+ \cup B(x_0, r)$, then,

$$\Omega^r \subset D,\ \ |\Omega^r| > |\Omega^+|,\ \lambda_1(\Omega^r) \in \big(\lambda_1(\Omega_1^r), \lambda_1(\Omega^+)\big),$$

where $\Omega_1^r$ is chosen so that $\Omega_1^r \subset \Omega_1 \setminus \Omega^r$ and $|\Omega_1^r \cup \Omega^r| = |\Omega_1 \cup \Omega^+|$. Note that we use the continuity at 0 of $r \in [0, r_0) \mapsto (\lambda_1(\Omega^r), \lambda_1(\Omega_1^r))$. We then have

$$\lambda_2(\Omega_1^r \cup \Omega^r) + \mu|\Omega_1^r \cup \Omega^r| = \lambda_1(\Omega^r) + \mu|\Omega_1^r \cup \Omega^r| < \lambda_2(\Omega^*) + \mu|\Omega^*|,$$

which is a contradiction with the minimality of $\Omega^*$. Whence (63).

Thus we have $\Omega^+, \Omega^-$ not empty and

$$\lambda_2(\Omega^*) = \lambda_1(\Omega^+) = \lambda_1(\Omega^-) = \lambda_2(\Omega^+ \cup \Omega^-),\ |\Omega^+ \cup \Omega^-| = |\Omega^*|,$$

the last identity coming from the minimality: $\lambda_2(\Omega^*) + \mu|\Omega^*| \leq \lambda_2(\Omega^+ \cup \Omega^-) + \mu|\Omega^+ \cup \Omega^-|$.

Now we remark that $\Omega^+$ (resp. $\Omega^-$), are *subsolutions* of (62). Indeed, for $\omega \subset \Omega^+$, we may write

$$\lambda_2(\Omega^*) + \mu|\Omega^*| \leq \lambda_2(\omega \cup \Omega^-) + \mu|\omega \cup \Omega^-|.$$

This is the same as

$$\lambda_1(\Omega^+) + \mu[|\Omega^+| + |\Omega^-|] \leq \lambda_1(\omega) + \mu[|\omega| + |\Omega^-|],$$

or

$$\lambda_1(\Omega^+) + \mu[\Omega^+| \leq \lambda_1(\omega) + \mu|\omega|,$$

whence the subsolution property. The same holds for $\Omega^-$. It follows from a (nontrivial) result in [57, 15] that there exist two *open* sets $D^+, D^- \subset D$ such that

$$\Omega^+ \subset D,\ \Omega^- \subset D,\ D^+ \cap \Omega^- = \emptyset,\ D^- \cap \Omega^+ = \emptyset.$$

Actually, this result relies on the fact that subsolutions of (62) are also subsolutions for the torsion energy as explained in [27, Chapter 2,(1.7) and Paragraph 1.4.1]. Let us show that $\Omega^+$ is solution of the following problem

$$\min\{\lambda_1(\Omega); \Omega \subset D^+,\ \Omega \text{ quasi} - \text{open},\ |\Omega| = |\Omega^+|\}. \tag{64}$$

Assume by contradiction that there exists $\Omega \subset D^+$ such that $\lambda_1(\Omega) < \lambda_1(\Omega^+)[= \lambda_1(\Omega^-) = \lambda_2(\Omega^*)], |\Omega| = |\Omega^+|$. Then we argue as above by introducing $\Omega_r^- := \Omega^- \cup B(x_0, r)$ for some $x_0 \in \partial\Omega^-, r > 0$ such that

$$\Omega_r^- \subset D,\ |\Omega_r^-| > |\Omega^-|,\ \lambda_1(\Omega_r^-) \in \big(\lambda_1(\Omega^r), \lambda_1(\Omega^-)\big),$$

where $\Omega^r$ is chosen so that $\Omega^r \subset \Omega \setminus \Omega_r^-$ and $|\Omega_r^- \cup \Omega^r| = |\Omega \cup \Omega^-|$. We then have

$$\lambda_2(\Omega_r^- \cup \Omega^r) + \mu|\Omega_r^- \cup \Omega^r| = \lambda_1(\Omega_r^-) + \mu|\Omega_r^- \cup \Omega^r| < \lambda_2(\Omega^*) + \mu|\Omega^*|,$$

which is a contradiction with the minimality of $\Omega^*$. Thus $\Omega^+$ is solution of (64).

Since $D^+$ is open, we may apply Theorem 2.15 which says that $u_2$ is locally Lipschitz continuous on $D^+$. It follows that $\Omega^+ = [u_2 > 0]$ is open. Similarly, $\Omega^-$ is open so that $\Omega^+ \cup \Omega^-$ is an open optimal set with the same measure as $\Omega^*$. □

The same question can be asked for the constrained problem:

**Open problem 3.13.** *Are the minimal shapes of*

$$\min\{\lambda_2(\Omega); \Omega \subset D,\ \Omega \text{ quasi} - \text{open},\ |\Omega| = a\}$$

*open subsets of $D$ ?*

# 4 Singularities due to the box or the convexity constraint

In this section, we study the regularity up to the boundary of the box $D$ for the problem

$$\min\{\lambda_1(\Omega),\ \ \Omega \subset D, \Omega \text{ quasi} - \text{open}, |\Omega| = a\}, \tag{65}$$

where $D$ is a *smooth* open set of $\mathbb{R}^2$. If $\Omega^*$ solves (65), we expect the contact between $\partial\Omega^*$ and $\partial D$ to be a bit smooth (see below), but as we will see, the smoothness is in general limited.

In order to insist on the fact that this appearance of a (mild) singularity is not only due to the box, we also show that a similar behavior applies to the solutions of the following problem:

$$\min \{\lambda_2(\Omega), \Omega \text{ open and convex}, |\Omega| = a\}. \tag{66}$$

This problem is peculiar because of the convexity constraint: indeed, if we drop this constraint, it is well-known that the solution of this problem is any disjoint union of two balls of volume $a/2$ (see Figure 1), which is clearly not convex, therefore Problem (66) is interesting on its own. It has been studied in [36] where it is proven, in particular, that the optimal shape is not a stadium (convex hull of two tangent balls). They also obtain a rough description of the optimal shape, under the a priori assumption that the shape is $C^{1,1}$, and has a simple geometry. But as shown in this Section, the a priori $C^{1,1}$-regularity assumption is too optimistic (see also Remark 4.3 for a related comment).

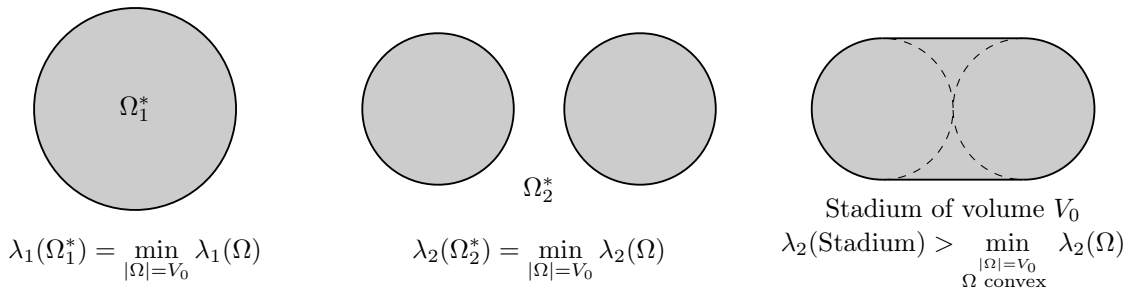


Figure 1: Minimization of the first two eigenvalues under volume constraint

The previous sections were concerned with the regularity of the pieces of the free boundary $\partial\Omega^*$ where it is expected to give sense to the optimality condition $|\nabla u|^2 = \Lambda$ where $\Lambda \in (0, \infty)$ is a Lagrange multiplier and $u$ is the eigenfunction associated to the eigenvalue under study. Here we focus here on the global regularity of the boundary $\partial\Omega^*$. We divide this section in 3 paragraphs. First we describe the situation and the possible regularity that we can expect from the optimality condition, seen as an partially overdetermined problem. Then we focus on a particular case of (65) where $D$ is a strip. Finally we give some partial answers for (66). As the conclusions are only partial, we describe open problems that would be interesting to fully investigate.

## 4.1 Regularity for partially overdetermined problem

For both of these problems, (65) or (66), if $\Omega^*$ is an optimal shape, the boundary $\partial\Omega^*$ can be decomposed into two subsets, namely the free boundary $\Gamma_1 \subset \partial\Omega^*$ where one can write an optimal condition for optimality, and the set $\Gamma_2 \subset \partial\Omega^* \setminus \Gamma_1$ of saturation of the constraint. More precisely,

- for problem (65), $\Gamma_1 = \partial\Omega^* \cap D$; we have seen that this boundary is locally smooth, and that $|\nabla u|^2 = \Lambda$ on $\Gamma_1$, where $u$ is the normalized first eigenfunction of $\Omega^*$ and $\Lambda \in (0, \infty)$ is a Lagrange multiplier for the volume constraint. The set $\Gamma_2$ is equal to $\partial\Omega^* \cap \partial D$. If this set is empty (or reduced to one point), then by Serrin's result on overdetermined boundary problems (see [53]), the set $\Omega^*$ must be a ball (which is the unconstrained minimizer, so the box is irrelevant). When this set is not empty (which is the case when $D$ does not contain any ball of volume $a$), it means that the constraint $\Omega \subset D$ is active.

- for problem (66), one can define

$$\Gamma_1 := \{x \in \partial\Omega^* \ / \ \exists r > 0 \text{ such that } B_r(x) \cap \Omega^* \text{ is strictly convex}\} \tag{67}$$

  (where we understand an open set $\omega$ to be 'strictly convex' if $\forall (x, y) \in \overline{\omega}$ with $x \neq y, \forall t \in (0, 1), tx + (1-t)y \in \omega$). The set $\Gamma_1$ is a relatively open subset of $\partial\Omega^*$. It will improperly be called the strictly convex parts of the boundary. This set can be understood as the part of $\partial\Omega^*$ where the curvature is positive, though one has to be careful to give sense to that, as the curvature is defined in a weak

sense, and is a priori only a measure, see Section 4.3 for more details. We can define $\Gamma_2$ as the union of all closed nontrivial segments which are included in $\partial\Omega^*$. This set represents the part of $\partial\Omega^*$ where the convexity constraint is saturated (vanishing curvature). It is a priori difficult to understand the structure of the sets $\Gamma_1$ and $\Gamma_2$: in particular, with the previous definitions, it is not true in general that $\Gamma_1 \cup \Gamma_2 = \partial\Omega^*$. We notice that it is possible to construct a $C^{1,1}$ convex domain such that $\Gamma_1 = \emptyset$ and $\Gamma_2$ is strictly contained (and dense) in $\partial\Omega^*$. Such singular convex set can be obtained by taking the epigraph of $f : [0,1] \to \mathbb{R}$ such that $f'' = \mathbb{1}_K$ (where $f''$ is understood in the sense of distributions) and $K$ is a compact set with positive measure and empty interior. Nevertheless, if one assume that $\Gamma_2$ is made of a finite number of segments, then $\partial\Omega^* = \Gamma_1 \cup \Gamma_2$.

Let us focus here on the regularity of a point which is at the intersection of $\overline{\Gamma_1}$ and $\Gamma_2$. In both cases, the situation is the following:

- on the side $\Gamma_1$, one has the overdetermined equation $|\nabla u(x)|^2 = \Lambda$, where either $u$ is the first eigenfunction, or the second eigenfunction of $\Omega^*$ (depending on whether $\Omega^*$ solves (65) or (66)). This fact is not so easy to prove in the case (66), as a smooth deformation of a strictly convex set does not necessarily remain convex, see Section 4.3.
- on the side $\Gamma_2$, we have an information about the geometry, *up to the intersection point*: either this part is flat (for solutions of (66)) or it is smooth (for solutions of (65), assuming the box $D$ is smooth).

Notice first that it is likely to expect the contact to be $C^1$: indeed, assume that the boundary is piecewise smooth around $x_0$ and that there is a (convex and non-flat) corner at $x_0$. Then it is classical that $\nabla u(x)$ goes to 0 when $x \to x_0$ in $\overline{\Omega}$, but this would contradict the fact that $|\nabla u(x)|^2 = \Lambda$ on $\Gamma_1$. This proof is not completely valid as we do not know, even applying the results of the previous sections, that $\partial\Omega^*$ is piecewise smooth. However, it implies that we expect the optimal shape to be at least $C^1$. For problem (66), we will give a proper proof of this fact in Section 4.3; for problem (65), it seems this is not proved anywhere yet.

In the following result, knowing that the contact is $C^1$, we prove higher regularity, and analyze the possible singularity near such a point.

**Proposition 4.1** ([46]). *Let $\Omega$ be an open bounded set of $\mathbb{R}^2$, $x_0 \in \partial\Omega$, $\gamma_1 \subset \partial\Omega$ and $\gamma_2 \subset \partial\Omega$ two relatively open connected sets, such that*

- $\overline{\gamma_1} \cap \overline{\gamma_2} = \{x_0\}$
- $\overline{\gamma_1 \cup \gamma_2}$ *is $C^1$, and $\overline{\gamma_2}$ is $C^\infty$.*

*We assume there exists $u \in C^2(\Omega) \cap C^1(\Omega \cup \gamma_1) \cap L^\infty(\Omega)$ satisfying*

$$\begin{cases} -\Delta u &= f(u) & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \\ |\nabla u|^2 &= \Lambda > 0 & \text{on } \gamma_1, \end{cases} \tag{68}$$

*where $f : \mathbb{R} \to \mathbb{R}$ is a $C^\infty$ function, and $f(u) \geq 0$ in a neighborhood of $x_0$. Then,*

- *either $\overline{\gamma_1 \cup \gamma_2}$ is $C^\infty$,*
- *or there exists $k \in \mathbb{N}^*$ such that $\overline{\gamma_1 \cup \gamma_2}$ is $C^{k,\frac{1}{2}}$ and $\forall \varepsilon > 0, \overline{\gamma_1 \cup \gamma_2}$ is not $C^{k,\frac{1}{2}+\varepsilon}$.*

**Sketch of proof:** thanks to the fact that we are in a two-dimensional framework, we use the conformal map of the set $\Omega^*$ (or only a neighborhood of $x_0$ in $\overline{\Omega^*}$) in order to understand its regularity, namely there is a biholomorphic map $\phi : \mathbb{H} \to \Omega^*$ where $\mathbb{H} = \{z \in \mathbb{C}, Im(z) < 0\}$, which is such that $\phi(0) = x_0$, $\phi^{-1}(\gamma_1) \subset \mathbb{R}_-$, $\phi^{-1}(\gamma_2) \subset \mathbb{R}_+$. Then, the regularity of $\partial\Omega$ and $\gamma_2$ can be seen as an information on the regularity of the trace of $Arg(\phi')$, respectively on $\partial\mathbb{H}$ or $\mathbb{R}_+$ (as it is a parametrization of the angle of the tangent vector to $\partial\Omega^*$, see for example [51]), and the regularity of $|\nabla u|$ on $\gamma_1$ can be seen as an information on the regularity of the trace of $\log(|\phi'|)$ on $\mathbb{R}_-$ (which is seen transporting (68) on $\mathbb{H}$ and studying the regularity of $u \circ \phi$). As these two functions, $\log(|\phi'|)$ and $Arg(\phi')$, are harmonic and conjugated to each other, these informations can be seen as a mixed boundary value problem, and the analysis of singularities for such problems leads to the fact that either $Arg(\phi')$ is smooth on $\overline{\mathbb{H}}$, or its behavior near 0 is of the form $r^{k+1/2}\cos(k\varphi/2)$ for some $k \in \mathbb{N}^*$, where $(r, \varphi)$ are radial coordinates in $\mathbb{H}$, see [31]. This leads to the result, as the regularity of $Arg(\phi')$ on $\overline{\mathbb{H}}$ near 0 implies the regularity of $\partial\Omega$ in a neighborhood of $x_0$. $\square$

## 4.2 Minimization of $\lambda_1$ in a strip

We focus here on the particular case where $D = \mathbb{R} \times (-M, M)$ is a strip in $\mathbb{R}^2$. Making good use of the symmetries of this box, we are able to give a complete description of the regularity of optimal shapes for (65), though it is expected that a similar behavior happens for more general boxes, see the end of this Section for open problems.

**Proposition 4.2.** *Let $a > 0$ and $D = \mathbb{R} \times (-M, M)$ for some $M > 0$. Let $\Omega^* \subset \mathbb{R}^2$ be a solution of* (65). *We assume that the contact between $\partial\Omega^*$ and $\partial D$ is tangential. Then*

- *either $\Omega^*$ is a disk,*
- *or* $\left[ \partial\Omega^* \in C^{1,\frac{1}{2}}, \text{ and } \forall \varepsilon > 0, \partial\Omega^* \notin C^{1,\frac{1}{2}+\varepsilon} \right]$.

**Remark 4.3.** This result is a bit surprising if we have in mind the behavior of solutions to the constrained isoperimetric problem

$$\min \{P(\Omega), \ \ \Omega \subset D, \ \ |\Omega| = a\}.$$

Indeed, in that case the contact between $\Omega^*$ optimal and the boundary of the box is expected to be $C^{1,1}$. In dimension 2 for example, this fact is easy to understand as the optimal shape is made of pieces of arc of circles, touching the boundary tangentially (it is easy to see, for example writing optimality conditions near the contact point, that having a non-flat angle of contact is not optimal). For a more general result, see [55].

**Sketch of proof:** (see also [45, 46])
1) It is well known that the solution of (65) is the ball of volume $a$, if this one is admissible (included in $D$). If such a ball does not exist, we already saw that any optimal shape $\Omega^*$ should touch the boundary of the box on a nontrivial set. Since the cylindrical box $D = \mathbb{R} \times (-M, M)$ has two orthogonal symmetry axes, one can prove using two Steiner symmetrization that $\Omega^*$ also has two axes of symmetry and is vertically and horizontally convex (see for example [29] for more details), and therefore the free boundary $\Gamma_1 = \partial\Omega^* \cap D$ necessarily has exactly two connected components, and the remaining boundary $\Gamma_2 = \partial\Omega^* \cap \partial D$ is the union of two segments.

2) Thus, as we assume that the contact is tangential, we know that the full boundary is $C^1$, and applying Proposition 4.1 around one "corner" (a point of $\overline{\Gamma_1} \cap \Gamma_2$), we get that $\partial\Omega^*$ is $C^{1,\frac{1}{2}}$ or at least $C^{2,\frac{1}{2}}$ (by symmetry the regularity at each corner is the same).

3) From there, we assume the contact is $C^2$ and seek for a contradiction: to that end we adapt an argument from [36]. The idea is that from the regularity of $\Omega^*$, we get that $u$ is $C^2$ on $\overline{\Omega^*}$, and from this information, we will obtain a contradiction by studying the nodal sets of $\partial_x u$, showing that one of them (denoted $\omega$) is such that $\partial_x u$ is a first eigenfunction for the Dirichlet-Laplacian on $\omega$, which is indeed a contradiction with the strict monotonicity of the eigenvalues, as $\lambda_1(\Omega^*) < \lambda_2(\Omega^*) = \lambda_1(\omega)$ while $\omega \subset \Omega^*$. As we know that $-\Delta\partial_x u = \lambda_2(\Omega^*)\partial_x u$ on $\Omega^*$, one only needs to check that such $\omega$ can be chosen such that $\partial_x u = 0$ on $\partial\omega$.

So, to find $\omega$, we show that from the $C^2$ regularity of $u$, derivating tangentially $|\nabla u|^2$ on $\Gamma_1 = \partial\Omega^* \cap D$ gives $\partial_{xy} u(x_0) = 0$ where $x_0$ is (say) the upper left "corner" of the optimal shape. We also know $\partial_x u(x_0) = 0$ (as $u = 0$ on the upper segment), so from the strong maximum principle, $x_0$ belongs to the closure of both the sets $[\partial_x u > 0]$ and $[\partial_x u < 0]$. Let us denote $\omega$ one connected component of $[\partial_x u < 0]$ that has points in a neighborhood of $x_0$. Then from the symmetries of $\Omega^*$, $u$ is even, so $\omega$ is on the left of the vertical axe of symmetry of $\Omega^*$. As we easily check that $\partial_x u \geq 0$ on the left part of the free boundary and that $\partial_x u = 0$ on $\partial D \cap \partial\Omega^*$, we easily conclude that $\partial_x u = 0$ on $\partial\omega$, which completes the contradiction and the proof. □

We conclude this section with the following open problem:

**Open problem 4.4.** *Concerning Problem* (65) *where $D$ is a smooth open set in $\mathbb{R}^2$, can we prove that any optimal shape $\Omega^*$ is globally $C^{1,\frac{1}{2}}$?*

Of course, a similar question in higher dimension can be asked, but the regularity is already limited by the possible singularities of the free boundary, so it makes more sense to obtain a more satisfying regularity theory of the free boundary first, as it is done in $\mathbb{R}^2$ so far, see Section 2.4.2.

## 4.3 Minimization of $\lambda_2$ with convexity constraint

We address here the question of the regularity of an optimal shape $\Omega^*$ for problem (66).

**Theorem 4.5.** *Let $a > 0$ and let $\Omega^* \subset \mathbb{R}^2$ be a solution of the minimization problem* (66)*, that is to say an optimal convex set of given area for the second Dirichlet-Laplacian eigenvalue.*
*We assume:*

$$\Omega^* \text{ contains at most a finite number of segments in its boundary.} \tag{69}$$

*Then*

$$\Omega^* \text{ is } C^{1,\frac{1}{2}}, \text{ and } \forall\, \varepsilon > 0,\ \Omega^* \text{ is not } C^{1,\frac{1}{2}+\varepsilon}. \tag{70}$$

**Remark 4.6. About assumption** (69)**:** as we noticed before (see (67)) the boundary of a convex shape contains two specific subsets, $\Gamma_1$ the strictly convex part, and $\Gamma_2$ the flat parts, but in general, even if the set $\Omega^*$ is a bit regular, these pieces of the boundary can have a highly non-trivial structure. We know though that $\Gamma_2$ is not empty, and even contains at least two segments (see the proof below). We notice that it is announced in [36, 35] that $\Gamma_2$ is made exactly of two segments (and then that these segments are parallel), but it seems the proof is not complete. This explains the geometric assumption, which does not appear in [36, 35], but is implicitly used in these papers. About the strictly convex part, even if $\Omega^*$ is assumed to be a bit smooth (say $C^1$, as we have a proof of this fact, see below), as far as we know it is not even clear that $\Gamma_1$, defined as in (67), is nonempty. With assumption (69), everything becomes more simple, and one can focus on the singularities at junction points between a flat part and a strictly convex part.

**Remark 4.7.** As noticed below, the first step in the proof of this result is to prove that $\Omega^*$ is $C^1$. This fact is actually very general, namely the result in [11] states that any optimal shape for

$$\min\{F(\lambda_1(\Omega), \dots, \lambda_k(\Omega)) + \mu|\Omega|, \quad \Omega \subset D, \ \ \Omega \text{ open and convex}\}$$

(where $F : \mathbb{R}^k \to \mathbb{R}$ is Lipschitz continuous, $\mu \in (0, \infty)$ and $D$ is open) is $C^1$. Compared to the results in Section 3, this one is much easier since the convexity guarantees some a priori regularity for any optimal shape $\Omega^*$. The difficulty though, is to deal with the convexity constraint to go farther and reach $C^1$, see also Section 5.

**Sketch of proof:** 1) As we noticed in the beginning of Section 4.1, we want to apply Proposition 4.1, and to that end, one needs first to prove an a priori $C^1$-regularity of $\Omega^*$. This can be done with an argument taken from [11] (see Remark 4.7), that we briefly reproduce here (in $\mathbb{R}^2$ for simplicity, but this argument is valid in any dimension): we notice in particular that this argument does not use (69). Because of the a priori convexity, proving $C^1$ regularity is equivalent to proving that $\partial\Omega^*$ has no corner. By contradiction, if $\Omega^*$ had such corner, cutting this corner at a size $\varepsilon$ into a set $\Omega^*_\varepsilon$ would lead to $\lambda_2(\Omega^*_\varepsilon) = \lambda_2(\Omega^*) + o(\varepsilon^2)$ (this relies on the fact that, in a weak sense $\nabla u$ goes to 0 at a convex corner, see [11] for more details) while $|\Omega^*| - |\Omega^*_\varepsilon| \geq c\varepsilon^2$ for some $c$ (depending on the angle of the corner). But by a classical scaling argument (see Section 3.1), $\Omega^*$ minimizes (among convex domains) $\lambda_2(\Omega) + \alpha|\Omega|$ for a suitable $\alpha > 0$, while $\Omega^*_\varepsilon$ has a lower energy than $\Omega^*$, because of the previous estimates, which is a contradiction.

2) The next step to prepare the application of Proposition 4.1, is to write the optimality condition. To that end, one can prove that the second eigenvalue of $\Omega^*$ is simple so that it is shape differentiable. Then, on the strictly convex part, one can write $|\nabla u|^2 = \Lambda$, by focusing on smooth deformations supported on $\Gamma_1$, and taking the convex hull, see [36] for more details.

We are also interested in the existence of a segment in the boundary: this fact is easy as $u$ has a nodal line that hits the boundary of $\Omega^*$ at exactly two points (from a result of Melas-Alessandrini, see [50, 2]), where $\nabla u$ must vanish, which is incompatible with the optimality condition just proven before.

3) We then apply Proposition 4.1 and conclude that either the shape is $C^{1,1/2}$ and not $C^{1,1/2+\varepsilon}$, or at least $C^{2,1/2}$.

4)Then assuming that $\Omega^*$ is $C^{2,1/2}$, a similar argument (but slightly more involved) as in the previous section, leads to a contradiction. Though this argument is written in [36, Theorem 10] with the idea that the optimal shape contains only two parallel segments in its boundary while it is still an open problem (see Remark 4.6), we notice that their proof is still valid up to minor modifications, and reproduce roughly the argument here. One knows there exists one segment on the boundary, touching the nodal line of $u$: we denote it $\Sigma$ and we choose it at the $x$-axis. Then we focus on $\partial_x u$, as it solves $-\Delta\partial_x u = \lambda_2(\Omega^*)\partial_x u$, and vanishes on $\Sigma$ and we seek for connected component of $\{\partial_x u \neq 0\}$ so that $\partial_x u$ vanishes on their

boundary. To that end, we write an optimality condition on $\Sigma = [a, b]$: this is not classical, as clearly for most deformations supported in $\Sigma$, the convexity is not preserved. But with a convex hull method, as explained in [36, Theorem 7], one can write $|\nabla u|^2(x, 0) = \Lambda + w''(x)$, where $w \geq 0$ has triple roots at $a$ and $b$, and $(x, 0)$ parametrizes $\Sigma$. This implies that $\partial_{xy} u$ vanishes at least three times inside $\Sigma$. But as $|\nabla u|^2 = \Lambda$ on the strictly convex part, we also know that $\partial_{xy} u$ vanishes at $a$ and $b$, *thanks to the $C^2$-regularity of $u$ on $\overline{\Omega^*}$*. From there we know that in a neighborhood of $\Sigma$, there are at least 6 connected components of $\{\partial_x u \neq 0\}$ (be careful that globally, some of these "local" connected components may be part of the same "global" connected components of $\{\partial_x u \neq 0\}$). Studying where the nodal lines can end (which is restricted to $\Sigma \cup \{N\} \cup \Sigma_{/\!/}$, with $N$ the point of intersection of the nodal line of $u$ which is not in $\Sigma$ (we recall that the nodal line of $u$ touches the boundary exactly at two points) and $\Sigma_{/\!/}$ the parallel segment of $\Sigma$ in $\partial\Omega^*$, which may be restricted to one point), we deduce that $\partial_x u$ has at least two nodal domains (whose union is denoted $\omega$) so that it vanishes on their boundary. But then, $\lambda_2(\omega) = \lambda_2(\Omega^*)$, which contradicts the strict monotonicity of $\lambda_2$. □

As we mentioned before, the set $\Omega^*$ is poorly understood, from scratch, and informations on its geometry helps to get informations on its regularity. Therefore, we propose the following open questions, which are supported by numerical evidence:

**Open problem 4.8.** *Prove that $\Omega^*$, solution of* (66)*, satisfies:*

- *$\partial\Omega^*$ contains a finite number of segments.*
- *$\Omega^*$ has two orthogonal axes of symmetry.*

# 5 Polygons as optimal shapes

In the previous section, we saw that mild singularities can happen in shape optimization involving eigenvalues. In this section, we will see that we can obtain much stronger singularities. We will provide examples where, in dimension 2, optimal shapes are actually polygons.

The results we are going to describe here can be found in the papers [47, 48, 49], see also [13, 33]. The general framework is to study optimization problems under convexity constraint, in the spirit of (66):

$$\min \left\{ J(\Omega), \ \ \Omega \text{ is an open convex domain in } \mathbb{R}^d \right\}.$$

The first remark is that it is much easier to obtain existence for such problems, compare to similar optimization problems without convexity constraint. Indeed, the convexity provides much stronger compactness properties, and allows to investigate unusual optimization, like maximizing the perimeter, or maximizing eigenvalues. The remaining difficult is usually to avoid that minimizing sequences have a diameter going to $\infty$, or that they collapse (converges to something flat). To avoid these behaviors and enforce existence, we will focus on the following constraints, though our strategy can be applied to more general situations:

$$\min \left\{ J(\Omega), \ \ \Omega \text{ is an open convex domain such that } B(0, a) \subset \Omega \subset B(0, b) \right\}, \tag{71}$$

where $0 < a < b < \infty$.

## 5.1 General result about the minimization of a weakly concave functional

A first general result, stated in dimension 2, asserts that for a wide class of shape functionals, the optimal sets, under convexity constraint, happens to be polygons. In order to describe this result, we recall the following classical parametrization of 2-dimensional convex domains with polar coordinates $(r, \theta) \in [0, \infty) \times \mathbb{T}$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$:

$$\Omega_u := \left\{ (r, \theta) \in [0, \infty) \times \mathbb{R} \ ; \ r < \frac{1}{u(\theta)} \right\}, \tag{72}$$

where $u$ is a positive and $2\pi$-periodic function and is called the gauge function of $\Omega_u$. A simple computation shows that the curvature of $\Omega_u$ is

$$\kappa_{\partial\Omega_u} = \frac{u'' + u}{\left(1 + \left(\frac{u'}{u}\right)^2\right)^{3/2}}. \tag{73}$$

This implies that $\Omega_u$ is convex if and only if $u'' + u \geq 0$, which has to be understood in the sense of $H^{-1}(\mathbb{T})$ if $u$ is not $C^2$. More precisely, if $u \in H^1(\mathbb{T})$ then $u'' + u \geq 0$ if and only if

$$\forall\, v \in H^1(\mathbb{T}) \quad \text{with } v \geq 0, \quad \int_{\mathbb{T}} (uv - u'v')\, d\theta \geq 0.$$

Throughout this section, any function defined on $\mathbb{T}$ is considered as the restriction to $\mathbb{T}$ of a $2\pi$-periodic function on $\mathbb{R}$, with the same regularity. Moreover, it is clear that $\Omega_u$ and $u$ share the same regularity.

With this parametrization, considering $j(u) = J(\Omega_u)$, Problem (71) is equivalent to

$$\min\{j(u),\ u'' + u \geq 0,\ u \in \mathcal{U}_{ad}\}, \quad \text{where } \mathcal{U}_{ad} = \{u \in W^{1,\infty}(\mathbb{T}),\ 1/u \in [a,b]\}. \tag{74}$$

Then we have the following result proven in [48, Theorem 3].

**Theorem 5.1.** *Let $u_0 > 0$ be a solution for (74) and $\mathbb{T}_{in} := \left\{\theta \in \mathbb{T}, a < \dfrac{1}{u_0(\theta)} < b\right\}$. Assume $j : W^{1,\infty}(\mathbb{T}) \to \mathbb{R}$ is $C^2$ and that there exist $s \in [0,1)$, $\alpha > 0$, $\beta \in \mathbb{R}$ such that, for any $v \in W^{1,\infty}(\mathbb{T})$, we have*

$$j''(u_0)(v,v) \leq -\alpha \|v'\|^2_{L^2(\mathbb{T})} + \beta \|v\|^2_{H^s(\mathbb{T})}. \tag{75}$$

*Then*

$$u_0'' + u_0 \text{ is a finite sum of Dirac masses in } \mathbb{T}_{in}.$$

In this statement, the assumption made on $j$ can be seen as a weak concavity property. It implies that, if $v$ has a small support around some point $x_0$ (like $v(x) = v_0(x_0 + \sigma x)$ with $\sigma$ large), then $j''(u_0)(v,v) < 0$. With this remark, the conclusion of the statement appears natural, as it says that minimizers are locally a sum of Dirac masses, while Dirac masses can be seen as extremal points among nonnegative measures, and it is a general fact that minimizers of concave functional are expected to be extremal.

Geometrically, this result is a tool to extract sufficient conditions on the functional $J$ so that solutions of (71) are polygons: indeed, formulae (73) implies that $\Omega_u$ is polygonal if and only if $u'' + u$ is a sum of Dirac masses.

As it is not related to eigenvalues, we do not describe the proof of this result. Let us notice though that it is highly inspired by the paper [44] which deals with the Newton's problem of minimal resistance, one of the oldest shape optimization problems with a convexity constraint.

We conclude noticing that a similar result can be obtained if one add to (74) a constraint of the form $m(u) = 0$ where $m : W^{1,\infty}(\mathbb{T}) \to \mathbb{R}^d$ is $C^2$ and such that $m'(u_0)$ is onto and $\|m''(u_0)(v,v)\| \leq \beta' \|v\|^2_{H^s(\mathbb{T})}$, for some $\beta' \in \mathbb{R}$ and $s \in [0,1)$, see [48, Theorem 4]. We will use this fact for volume constrained problem.

## 5.2 Examples

In order to introduce the list of examples we are interested in, let us recall the reverse isoperimetric inequality, which in the framework of convex geometry is due to Ball (see [5]), and can be stated as the fact that the optimization problem

$$\max\left\{\min_{T \in GL_d(\mathbb{R})} \left\{\frac{P(T(\Omega))}{|T(\Omega)|^{\frac{d-1}{d}}}\right\}, \Omega \text{ open bounded convex and centrally symmetric set of } \mathbb{R}^d\right\} \tag{76}$$

is solved by the unit cube. This can be understood as the maximization of the isoperimetric ratio among centrally symmetric convex bodies, where shapes are understood up to linear inversible transformations. In [47], some shape optimization problem of a similar behavior were introduced, namely:

$$\min\{\mu|\Omega| - P(\Omega),\ \ \Omega \text{ is an open convex domain such that } B(0,a) \subset \Omega \subset B(0,b)\}, \tag{77}$$

whose solutions are trivial for $\mu \in \{0, +\infty\}$, and for which it is expected to obtain interesting optimal shapes for $\mu \in (0,\infty)$. Theorem 5.1 applies for such problem, and if $\Omega^*$ solves (77), then it is polygonal inside $B(0,b) \setminus \overline{B(0,a)}$. To go further, a full description, for any parameters $(a,b,\mu)$ has been achieved in [7], where they show in particular that for $\mu \in (\frac{1}{2b}, \frac{2}{a})$, the optimal shape is actually a full polygon (which means that the contact between $\Omega^*$ and the boundary of the annulus is a finite set of points).

In this section, we will study similar problems, involving the first eigenvalue of the Dirichlet-Laplacian, which can be listed in two sets of examples:

- the shape functional involves again a maximization of the perimeter: in that case, the regularity/singularity of the optimal shape is driven by the perimeter which is the leading term, see Examples 5.2 and 5.3,
- the shape functional does not contain a perimeter term, but involves a maximization of the first eigenvalue. In that case, we only have partial results, though there are a few indications that the behavior should be the same as in the previous items, see Example 5.4.

**Example 5.2** (Negative perimeter penalization)**.** One can study

$$\min\{F(|\Omega|, \lambda_1(\Omega)) - P(\Omega)\ ;\ \Omega \text{ convex}, B(0,a) \subset \Omega \subset B(0,b)\} \tag{78}$$

where $F : (0,+\infty) \times (0,+\infty) \to \mathbb{R}$ is $C^2$. Then any optimal shape $\Omega^*$ is such that each connected component of the free boundary $\partial\Omega^* \setminus (\partial B(0,a) \cup \partial B(0,b))$ is polygonal.

This relies on Theorem 5.1, and the following properties of the second order derivatives of the perimeter, the volume, and the eigenvalue: denoting $p(u) = P(\Omega_u), a(u) = |\Omega_u|, \ell(u) = \lambda_1(\Omega_u)$, we have

$$|a''(u)(v,v)| \le \beta_1 \|v\|^2_{L^2(\mathbb{T})}, \tag{79}$$

$$p''(u)(v,v) \ge \alpha |v'|^2_{L^2(\mathbb{T})} - \beta_2 \|v\|^2_{L^2(\mathbb{T})}, \tag{80}$$

$$|\ell''(u)(v,v)| \le \beta_1 \|v\|^2_{H^{1/2}}(\mathbb{T}). \tag{81}$$

The first two estimates are easily obtained by direct computations, while the third one is much more involved as one has to prove it for rather irregular domains, namely only convex domains. In a smooth setting though, it is easy to get such estimates, using the classical formula for the second order shape derivative of $\lambda_1$, see (92). In [48], we deal with general convex domains in dimension 2, and obtain a weaker version of (81) (sufficient for our purpose) where $1/2$ is replaced by $1/2+\varepsilon$ where $\varepsilon > 0$. A new approach is introduced in [49] which leads to $\varepsilon = 0$ (it is written there for energy functionals, but the same method can be adapted to the eigenvalue case), and the approach in [49] works in any dimension.

**Example 5.3** (Volume constraint and negative perimeter penalization)**.** We can also consider a similar problem with a volume constraint:

$$\min\{J(\Omega) := F(\lambda_1(\Omega)) - P(\Omega)\ ;\ \Omega \text{ convex in } \mathbb{R}^2,\ |\Omega| = a\} \tag{82}$$

where $a \in (0,+\infty)$. Again, any optimal shape of (82) is a polygon, using a volume constraint version of Theorem 5.1 (see the remarks following its statement).

Note that, studying minimizing sequences converging to a segment, one may prove that for a large class of functionals $F$, there exists an optimal shape (for example making good use of the estimate $\lambda_1(\Omega) \ge \frac{\pi^2}{4\mathrm{Diam}(\Omega)^2}$ if $F(x) \ge cx^{1/2}$ at $x \to \infty$ for some $c$ large enough, one can prove that there exists a solution to (82)). Thus considering, for $\mu \in (0,\infty)$ the problem

$$\min\{J(\Omega) := \mu\lambda_1(\Omega) - P(\Omega)\ ;\ \Omega \text{ convex in } \mathbb{R}^2,\ |\Omega| = a\}$$

where there is a competition between minimizing the eigenvalue and maximizing the perimeter, our previous statement asserts that the regularity of the solution is driven by the perimeter term, which means that solutions are polygons.

**Example 5.4** (Reverse Faber-Krahn inequality)**.** In [12], motived by the question of adapting the classical Mahler inequality, replacing the area by the first Dirichlet eigenvalue, the authors were naturally led to question the reverse Faber-Krahn inequality in the same spirit as in (76), namely: is the cube solution of

$$\max\left\{\min_{T\in GL_d(\mathbb{R})}\left\{\lambda_1(T(\Omega))|T(\Omega)|^{\frac{2}{d}}\right\}, \Omega \text{ open bounded convex and centrally symmetric set of } \mathbb{R}^d\right\}? \tag{83}$$

This question, even if $d = 2$, is certainly very difficult. As in (77), it motivates the following optimization problems:

$$\max\left\{\mu|\Omega| + \lambda_1(\Omega),\ \ \Omega \text{ open convex domain with } B(0,a) \subset \Omega \subset B(0,b)\right\}, \tag{84}$$

or

$$\max\left\{\lambda_1(\Omega),\ \ \Omega \text{ open convex domain with } \ \Omega \subset B(0,b) \text{ and } |\Omega| = a\right\}, \tag{85}$$

where again $\mu \in \mathbb{R}_+$.

Contrary to the previous examples, the leading term for the geometry of optimal shapes is no longer the perimeter, but the first Dirichlet eigenvalue. Thus, in order to apply Theorem 5.1, we would need a convexity property of the functional $\lambda_1$ in the spirit of (80). More precisely, we wonder whether

$$\ell''(u)(v,v) \geq \alpha |v|^2_{H^{1/2}(\mathbb{T})} - \beta_2 \|v\|^2_{L^2(\mathbb{T})}, \tag{86}$$

where $\ell(u) = \lambda_1(\Omega_u)$, $\Omega_u$ is the set whose gauge function is $u$ and is only assumed to be convex and $|\cdot|_{H^{1/2}(\mathbb{T}}$ is the $H^{1/2}$-semi-norm. Such result would imply that any optimal shape for the previous problems (84), (85) is locally polygonal inside $B(0,b) \setminus \overline{B(0,a)}$ or $B(0,b)$ respectively. Unfortunately, even if (81) was obtained in full generality for convex sets, we are only able to obtain (86), when assuming that the deformation $v$ is supported on a set where $u$ is smooth enough. Therefore, one obtains as a weaker result that if $\Omega^*$ is an optimal shape and $\gamma \subset \partial\Omega^*$, then $\gamma$ cannot, at the same time, be smooth and have a strictly positive curvature. We generalize this result in higher dimension in the next paragraph.

Another interesting result from [12] is that, in the class of convex axisymmetric octagons having vertices at the points $(\pm l, 0)$ and $(0, \pm l)$, the square is a solution of

$$\max\{\lambda_1(\Omega)|\Omega|\}.$$

These results suggest the following open problems:

**Open problem 5.5.**

- *Prove that solutions to* (84) *and* (85) *are polygonal inside the box constraints.*
- *Prove that the square is solution of* (83).

## 5.3 Remarks on the higher dimensional case

In the multi-dimensional case, convexity constraint in shape optimization is much less understood, though there are some results in this direction, see [13, 33] and the work of T. Lachand-Robert, see for example [43, 44, 22]. We describe in this section some results from [49], which can be applied to Examples from Section 5.2.

We can use again the parametrization of convex bodies with their gauge function and obtain a result of the type of Theorem 5.1, but whose conclusion will not allow to prove that optimal shapes are polyhedra.

If $d \geq 2$, and $u : \mathbb{S}^{d-1} \to (0,\infty)$ is given, $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d,\ |x| = 1\}$, we can consider

$$\Omega_u := \left\{ (r,\theta) \in [0,\infty) \times \mathbb{S}^{d-1}, \quad r < \frac{1}{u(\theta)} \right\}. \tag{87}$$

The function $u$ is again called the gauge function of $\Omega_u$. The set $\Omega_u$ is convex if and only if the 1-homogeneous extension of $u$, denoted by the same letter and given by $u(x) = |x|u(x/|x|)$, is convex in $\mathbb{R}^d$ (in this section, we will refer to this property by saying that $u : \mathbb{S}^{d-1} \to \mathbb{R}$ is convex), see [52, Section 1.7] for example. In this way, we describe every bounded convex open set containing the origin. Throughout this section, the regularity of any function defined on $\mathbb{S}^{d-1}$ is seen as the regularity on $\mathbb{R}^d \setminus \{0\}$ of its 1-homogeneous extension, and it is classical that it is equivalent to the regularity of the set $\Omega_u$ itself.

With this parametrization, considering $j(u) = J(\Omega_u)$, problem (71) is equivalent to

$$\min\left\{ j(u),\ u : \mathbb{S}^{d-1} \to (0,\infty) \text{ convex }, \ a \leq 1/u \leq b \right\}. \tag{88}$$

Then in the same spirit as Theorem 5.1, we can prove the following where we denote $|v|^2_{H^1(\mathbb{S}^{d-1})} = \int_{\mathbb{S}^{d-1}} |\nabla_\tau v|^2 d\theta$, $\nabla_\tau =$ tangential gradient on $\mathbb{S}^{d-1}$.

**Theorem 5.6.** *Let $u_0 > 0$ be a solution for* (88) *Assume $j : W^{1,\infty}(\mathbb{S}^{d-1}) \to \mathbb{R}$ is $C^2$ and that there exist $s \in [0,1)$, $\alpha > 0$, $\beta, \gamma \in \mathbb{R}$ such that, for any $v \in W^{1,\infty}(\mathbb{S}^{d-1})$, we have*

$$j''(u_0)(v,v) \leq -\alpha |v|^2_{H^1(\mathbb{S}^{d-1})} + \beta \|v\|^2_{H^s(\mathbb{S}^{d-1})}. \tag{89}$$

*Then the set*

$$T_{u_0} = \{v \in W^{1,\infty}(\mathbb{S}^{d-1}) / \exists \varepsilon > 0, \forall |t| < \varepsilon,\ u_0 + tv \text{ is convex and such that } a \leq u_0 + tv \leq b\}, \tag{90}$$

*is a linear vector space of finite dimension.*

This theorem is a generalization of Theorem 5.1; nevertheless, in dimension 3 or higher, it is not true that $[dim\,(T_{u_0}) < +\infty]$ implies that $\Omega_{u_0}$ is a polyhedra.

**Example 5.7** (Negative perimeter penalization)**.** This result applies to

$$\min\{F(|\Omega|, \lambda_1(\Omega)) - P(\Omega)\ ;\ \Omega \text{ convex}, B(0,a) \subset \Omega \subset B(0,b)\}$$

as explained in [49] (adapting the computation done for the Dirichlet energy to the case of the first eigenvalue). Using the fact that the set defined in (90) is of finite dimension, we easily deduce that if $\omega$ is a $C^2$ relatively open subset of $\partial\Omega^* \cap \{x, a < |x| < b\}$, then the Gauss curvature of $\Omega^*$ vanishes on $\omega$ (otherwise $C_0^\infty(\omega) \subset T_{u_0}$ which is a contradiction with the finite dimension property).

As when $d = 2$, we cannot obtain as good results for Reverse Faber-Krahn type problems. However, in the spirit of [13, Theorem 4.5], we can prove the following;

**Proposition 5.8.** *Let $\Omega^*$ be respectively a solution of* (84) *or* (85) *in $\mathbb{R}^d$. If $(\partial\Omega^* \cap B(0,b)) \setminus \overline{B(0,a)}$ (respectively $\partial\Omega^* \cap B(0,b)$) contains a relatively open set $\omega$ of class $C^2$, then its Gauss curvature vanishes on $\omega$.*

Though this result is new, the computations, and the observation that the second shape derivative of $\lambda_1$ always has a sign, which is the main ingredient in the proof below, can be found in [54].

**Sketch of proof:** We focus first on the case where $\Omega^*$ solves (84). Let us assume that the Gauss curvature of $\omega$ is positive at one point $x_0$, and is therefore (by $C^2$ assumption) greater than some $\alpha > 0$ in a neighborhood $\hat{\omega} \subset \omega$ of $x_0$. Then if $\varphi \in C_c^\infty(\hat{\omega})$, the set $\Omega_t = (Id + t\varphi\nu)(\Omega^*)$ (where $\nu$ is the normal vector to $\partial\Omega^*$, well defined on the support of $\varphi$) is admissible in the sense that it is still convex and satisfies the box constraint. Therefore the optimality conditions are

$$\frac{d}{dt}\lambda_1(\Omega_t)_{|t=0} = -\mu\frac{d}{dt}\mathrm{Vol}(\Omega_t)_{|t=0}, \quad \frac{d^2}{dt^2}\left[\lambda_1 + \mu\mathrm{Vol}\right](\Omega_t)_{|t=0} \le 0. \tag{91}$$

On the other hand, from classical formula for first and second order derivative (see for example [38]), denoting by $u$ the first eigenfunction of $\Omega^*$, we have

$$\frac{d}{dt}\mathrm{Vol}(\Omega_t)_{|t=0} = \int_\omega \varphi, \quad \frac{d}{dt}\lambda_1(\Omega_t)_{|t=0} = -\int_\omega (\partial_\nu u)^2\varphi,$$

$$\frac{d^2}{dt^2}\left[\lambda_1 + \mu\mathrm{Vol}\right](\Omega_t)_{|t=0} = \int_\omega \left[2V_\varphi\partial_\nu V_\varphi + H\left((\partial_\nu u)^2 + \mu\right)\varphi^2\right], \tag{92}$$

where $V_\varphi$ solves

$$\begin{cases} -\Delta V_\varphi &= \lambda_1(\Omega^*)V_\varphi - u\displaystyle\int_\omega (\partial_\nu u)^2\varphi \text{ in } \Omega^*, \\ V_\varphi &= -\varphi\partial_\nu u \text{ on } \partial\Omega^* \quad \text{ and } \int_{\Omega^*} uV_\varphi = 0. \end{cases}$$

By the first relation of (91) and the above formula, we have $\mu = (\partial_\nu u)^2$ on $\hat{\omega}$. Since $H \ge 0$, it follows

$$\begin{cases} \frac{d^2}{dt^2}\left[\lambda_1 + \mu\mathrm{Vol}\right](\Omega_t)_{|t=0} &\ge 2\int_\omega V_\varphi\partial_\nu V_\varphi = 2\int_{\Omega^*}\left(|\nabla V_\varphi|^2 - \lambda_1(\Omega^*)V_\varphi^2\right), \\ &\ge \alpha|\varphi|^2_{H^{1/2}(\omega)} - \beta\|\varphi\|^2_{L^2(\omega)}, \end{cases} \tag{93}$$

for some $\alpha, \beta > 0$, where the last inequality is obtained as follows: first, we use (recall that $V_\varphi = \sqrt{\mu}\,\varphi$ on $\hat{\omega}$)

$$\mu\|\varphi\|^2_{H^{1/2}(\partial\hat{\omega})} = \|V_\varphi\|^2_{H^{1/2}(\partial\Omega^*)} \le C\|V_\varphi\|^2_{H^1(\Omega^*)} = C\left[\int_{\Omega^*} |\nabla V_\varphi|^2 + V_\varphi^2\right]. \tag{94}$$

Then the $L^2$-norm of $V_\varphi$ may be estimated from above by introducing the solution $\psi$ of

$$\psi \in H_0^1(\Omega^*), \quad -\Delta\psi - \lambda_1(\Omega^*)\psi = V_\varphi \text{ in } \Omega^*, \quad \int_{\Omega^*} \psi\, u = 0.$$

This solution exists since $\int_{\Omega^*} u\,V = 0$. Moreover,

$$[\lambda_2(\Omega^*) - \lambda_1(\Omega^*)]\int_{\Omega^*}\psi^2 \le \int_{\Omega^*}|\nabla\psi|^2 - \lambda_1(\Omega^*)\psi^2 = \int_{\Omega^*} V_\varphi\psi,$$

so that $\|\psi\|_{L^2(\omega)} \leq C\|V_\varphi\|_{L^2(\omega)}$ for some $C > 0$ (we use $\lambda_2(\Omega^*) - \lambda_1(\Omega^*) > 0$ since $\Omega^*$ is convex). Multiplying the equation in $\psi$ by $V_\varphi$ gives

$$\int_{\Omega^*} V_\varphi^2 = -\int_{\partial\Omega^*} V_\varphi \partial_\nu \psi = -\sqrt{\mu}\int_\omega \varphi \partial_\nu \psi \leq \sqrt{\mu}\,\|\varphi\|_{L^2(\hat{\omega})}\|\partial_\nu\psi\|_{L^2(\hat{\omega})}. \tag{95}$$

Now, the equation in $\psi$ implies that

$$\|\Delta\psi\|_{L^2(\Omega^*)} \leq \lambda_1(\Omega^*)\|\psi\|_{L^2(\Omega^*)} + \|V_\varphi\|_{L^2(\Omega^*)} \leq C\|V_\varphi\|_{L^2(\Omega^*)}.$$

And near $\omega$, the $H^2$-norm of $\psi$ is controlled by the $L^2$-norms of $\Delta\psi$ and $\psi$ so that $\|\partial_\nu\psi\|_{L^2(\hat{\omega})} \leq C\|V_\varphi\|_{L^2(\Omega^*)}$. Finally, going back to (95) leads to $\|V_\varphi\|_{L^2(\Omega^*)} \leq C\|\varphi\|_{L^2(\hat{\omega})}$.

This, together with (94), proves the estimate (93). Since it is valid for any $\varphi \in C_c^\infty(\hat{\omega})$, this contradicts the second part of (91) and ends the proof in the case of (84).

For the case where $\Omega^*$ solves (85), a similar proof is valid, $\mu$ is the Lagrange multiplier for the volume constraint, and one only have to restrict to deformations $\varphi$ such that $\int_\omega \varphi = 0$ (to preserve the volume constraint at the first order), but the same computations still leads to a contradiction. □

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