Introduction to Shape Optimization

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First, a little warning : these notes are not as precise as a research paper or a book. There are probably many imprecisions ; our goal is mostly to give some intuition, some tools, some ideas, and some references. The motivated reader should definitely try to fill the gaps, read references, and correct possible mistakes.

Second, let us give the main general references : most of the content of this class can be found in [53]. See also the classical [12]. For more recent results about the case of spectral functionals, see [35]. For everything related to isoperimetric problem, we refer to [44] (see also the classical [3]). For the topic of quantitative inequalities, we refer to the review paper [32].

1 What is Shape Optimization ? / Content of the class

1.1 Very loose description

We focus on optimization problems of the form

$$\inf\left\{J(\Omega),\ \ \Omega\in\mathcal{S}_{ad}\right\} \tag{1}$$

where $\mathcal{S}_{ad}$ is a class of sets in $\mathbb{R}^d$ and $J:\mathcal{S}_{ad}\to\mathbb{R}$ is called a shape functional.

The questions we will ask ourselves for this kind of problem are the usual questions of optimization. This is very similar to the study of

$$\inf\left\{\int_D L(x,u(x),\nabla u(x))dx, u\in X(D)\right\}$$

which falls into the field of calculus of variations (here $L:D\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ is given and $u:D\to\mathbb{R}$ is the unknown, living in the space $X(D)$). These questions are :

1. Does problem (1) admit a solution, that is to say, is there $\Omega^*\in\mathcal{S}_{ad}$ such that

   $$\forall\Omega\in\mathcal{S}_{ad},\quad J(\Omega)\geq J(\Omega^*)$$

   in which case we may write

   $$J(\Omega^*)=\min\left\{J(\Omega),\ \ \Omega\in\mathcal{S}_{ad}\right\}\quad ?$$

   The method is the usual *direct method of the calculus of variations* that we briefly describe here : We denote $\alpha=\inf\left\{J(\Omega),\ \ \Omega\in\mathcal{S}_{ad}\right\}\in\overline{\mathbb{R}}$. Assuming that $\mathcal{S}_{ad}$ is nonempty, there always exists (Ω_n) a sequence in $\mathcal{S}_{ad}$ such that

   $$J(\Omega_n)\xrightarrow[n\to\infty]{}\alpha.$$

   (a) the first step is to obtain compactness for the sequence (Ω_n). We seek for a metric d that allows to say that there exists σ an extraction and $\Omega_\infty\in\mathcal{S}_{ad}$ such that

   $$\Omega_{\sigma(n)}\xrightarrow[n\to\infty]{d}\Omega_\infty.$$

   (b) to conclude, we need to show that the convergence for d is enough to pass to the limit in $J(\Omega_{\sigma(n)})$. More precisely, it is sufficient to prove

   $$J(\Omega_\infty)\leq\liminf_{n\to\infty}J(\Omega_{\sigma(n)})$$

   which refers to the lower semi-continuity for J.

   These two requirements (compactness for (Ω_n) and continuity for J) are "opposite" (the easier it is to have compactness, the harder it will be for J to be continuous), so it requires a good balance. Depending on J and $\mathcal{S}_{ad}$, there are different topologies one can try to use.

First topic of this course : Existence theory for shape optimization.

2. Can one identify/describe "the" solution(s) ?

   For example, when one hopes that the solution is a (euclidian) ball, one may expect to prove such result. Let us first notice that it is quite "exceptional" to hope for an explicit identification of the solution : in many situations, it may be too ambitious to explicitly identify the minimizer(s), though we may hope to describe them.

   But even then, when you guess an explicit solution like the ball B, it is rare that a direct computation leads to the result (I mean something like "$\forall\Omega \in \mathcal{S}_{ad}$ one can directly prove that $J(\Omega) \geq J(B)$"[1]), so even in this situation the questions under study are interesting.

   So it makes sense to ask : are there symmetries we might expect (cf. class by A. Burchard), can one prove that the solution is a "nice" set ("nice" may mean smooth for example, but it can mean something else), can one get analytical or geometrical informations to help understanding the solutions ?

   Despite symmetrizations, one of the main tool for this step is to go back to the *birth* of calculus of variations with the ideas of Euler and Lagrange, which in modern terms means we want to write optimality conditions on Ω^* solution to (1). For this, we need to perform small deformations of Ω^* and apply differential calculus.

   Nevertheless, this idea is not straightforward to apply as the set of shapes is not a normed vector space.

   Second topic : Shape derivatives and how to write optimality conditions for (1)

   Let us remark that one main difficult in this step is, when we expect it, to prove that the solution of (1) is smooth enough (say Lipschitz) : indeed, in that case, one can for example say that the solution is locally the graph of a function (named $u : D \to \mathbb{R}$), which allows to perform a deformation (move the boundary as the graph of $u + t\varphi$ with φ is compactly supported in D).

   As a consequence, this step will be devided in two topics for this class :

   Third topic : Regularity theory for some examples of (1).

3. Finally, especially when an explicit description of the solutions is unreachable, how can we compute numerically optimal shapes for (1) ? This will be the topic of the class by C. Dapogny and E. Oudet.

To conclude, let us note that an important question that will be driving many ideas in this class is the following :

How can one parametrize shapes with functions ?

Indeed, when we deal with functions, we have many normed vector spaces in hand to work with, which allows to manipulated metrics and differential calculus.

1.2 Examples

Isoperimetric problems : Let us start with the most basic and famous isoperimetric problem. Given Ω in $\mathbb{R}^d$ (measurable), one can define :

$|\Omega| = \mathrm{Vol}(\Omega) = \int_\Omega 1dx$ the Lebesgue measure of Ω, which will be called the volume of Ω.

If Ω is a reasonably smooth shape (say Lipschitz[2]), we can also define

$$P(\Omega) = \int_{\partial\Omega} 1d\sigma = \mathcal{H}^{d-1}(\partial\Omega) \tag{2}$$

the *perimeter*[3] of Ω.

1. It is well-known that such a proof using Fourier series leads to the proof of the isoperimetric inequality in the plane, see [52] for example

2. Classically, a set is said to be Lipschitz its boundary is locally the graph of a Lipschitz function *and* if Ω lies on one side of this graph. This definition can be generalized with $C^{k,\alpha}$ instead of Lipschitz.

3. in fact, if $d = 2$, this is the "usual" perimeter, if $d = 3$ this is often called the surface area of Ω, but we will stick with the common denomination of perimeter in any dimension.

Here $d\sigma$ is a boundary integral, which is well-defined if Ω is Lipschitz. Also, $\mathcal{H}^{d-1}$ refers to the $(d-1)$-dimensional Hausdorff measure, see classical monographs on measure theory (for example [44, Section 1, chapter 3]).

So the famous isoperimetric problem can be formulated as

$$\min\left\{P(\Omega),\ \ \Omega\subset\mathbb{R}^d,\ \ |\Omega|=m\right\} \tag{3}$$

We are for now being unprecise on the definition of $P(\Omega)$ when Ω is not smooth, see Section 2 [4].

Theorem 1.1 *Let m in $(0,+\infty)$. Then*

- *Any ball of volume m is solution of (3). In other words, if Ω is a measurable subset of $\mathbb{R}^d$ with volume m, then*
$$P(\Omega)\geq P(B)$$
where B is any ball of volume m.
- *In fact, any solution to (3) is in some sense equal to a ball.*

This result has a long history, see for example [52] and [32, Section 1]. Many ideas emerged from the work of Steiner in 1838 we provided partial proofs with different symmetrization procedures, but he actually showed a uniqueness result only, meaning that he proved that if Ω is a (reasonably smooth) solution, then it must be a ball. We have to wait the end of the 19^{th}-beginning of the 2020^{th} century for a complete proof of this result [5].

Of course, the story doesn't stop with this result : many variations on this original isoperimetric problem remain interesting and possibly unsolved. Let us quote a few of them :

1. Relative isoperimetric problem : given D a "box" (open bounded set in $\mathbb{R}^d$) describe the solution for
$$\inf\left\{\mathcal{H}^{d-1}(\partial\Omega\cap D),\Omega\subset D,|\Omega|=m\right\}$$
where $m\in(0,|D|)$. For example, if D is a cube ($(0,1)^3$), then up to my knowledge a complete description of the solutions for every values of m is still missing (though it is known that a solution Ω^* exists and that its free boundary $(\partial\Omega^*\cap D)$ is smooth). Note that the solution is not a portion of a ball for some values of m.
2. Gamow liquid drop model : given $\mu>0$, this modelization leads to the problem
$$\inf\left\{P(\Omega)+\mu\int_\Omega\int_\Omega\frac{1}{|x-y|}dxdy,\ \ \Omega\subset\mathbb{R}^3,\ \ |\Omega|=m\right\}.$$
Here there is a competition between two functionals as P is minimized by any ball and $\Omega\mapsto\int_\Omega\int_\Omega\frac{1}{|x-y|}dxdy$ is maximized by a centered ball. It has been proved by Knüpfer-Muratov in [50] that
   - if m is small then the solution is the ball (unique up to translations),
   - if m is large then there is no solution,

   and it is an open problem to prove that only these two situations may occur.
3. Reverse isoperimetric problem : let us look at the following less usual problem : given $D\subset\mathbb{R}^d$ a box, we are interested in reverse estimate for the perimeter by looking at
$$\sup\left\{P(\Omega),\ \Omega\subset D,\ \ |\Omega|=m,\ \ \Omega\text{ is a convex set}\right\}.$$

4. In (2), formula $\int_{\partial\Omega}1d\sigma$ a priori requires some regularity, but the formula $\mathcal{H}^{d-1}(\partial\Omega)$ is well-defined for any set Ω, but as we will see in Section 2, this is not the suitable definition for our purpose.

5. After a quick research, here are the informations I collected (thanks to David Krejčiřík for pointing these references to me) : Burago and Zalgaller write in their book [17, page 90] that for the planar case, the first rigorous proofs are due to Edler (1882), for $d=3$ by Schwarz (1884) and for any dimension by Lusternik (1935) who used an iterated symmetrization technique. Nevertheless, in this last paper, according to Almut Burchard, one argument is missing. It seems indeed that Bandle in [4] attributes the general case to Schmidt (1939). If anyone has better informations, let me know.

There is here a convexity constraint on the admissible shapes. It is easy to see that without such constraint, the problem is ill-posed (the supremum is infinity, cf Exercise 2.2), however, it is not very hard to prove (see Section 2) that there is a solution to this problem if D is bounded and $m \in (0, |D|)$. However the identification of the solutions (depending on m and D) is far from understood. Nevertheless, it is known that if $d = 2$ then the free boundary $\partial\Omega^* \cap D$ is a polygonal line ([39]).

PDE/Spectral problems : Again, let us focus on the most famous example : given Ω an open set in $\mathbb{R}^d$ with finite volume, one can define

$$\lambda_1(\Omega) = \inf\left\{\frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}, u \in H_0^1(\Omega)\right\}$$

where $H_0^1(\Omega) := \overline{C_c^\infty(\Omega)}^{H^1}$ is the closure of smooth and compactly supported functions for the H^1-norm $\|\varphi\|_{H^1} := \sqrt{\|\varphi\|_{L^2}^2 + \|\nabla\varphi\|_{L^2}^2}$. It is classical that this infimum is achieved, and $\lambda_1(\Omega)$ is called the first Dirichlet eigenvalue of the Laplace operator on Ω (see the class by B. Colbois and P. Freitas for more details).

A similar result as Theorem 1.1 states :

Theorem 1.2 (Faber-Krahn) *For any $m > 0$, any ball of volume m is solution of*

$$\min\left\{\lambda_1(\Omega),\ \ \Omega \textit{ open set in } \mathbb{R}^d,\ \ |\Omega| = m\right\} \tag{4}$$

Again, the story is far from ending here :

1. a study of the solutions of

$$\min\left\{\lambda_3(\Omega),\ \ \Omega \text{ open set in } \mathbb{R}^d,\ \ |\Omega| = m\right\}$$

where λ_3 is the third Dirichlet eigenvalue of the Laplace operator for Ω, is still an open problem. In particular, it is conjectured that the disk is a solution if $d = 2$, and it is expected that the ball is not a solution if $d = 3$ (numerical evidence by E. Oudet). Note that it is known that there exists a solution ([19, 36]) but not that it is smooth (see however the most recent regularity results [16, 49] in this direction).

2. in the book [35] one can find many variations on spectral shape optimization problems. Changing boundary conditions, consider more involved operators, or operators defined on $\partial\Omega$ rather than Ω leads to many interesting problems, many of which are open.

Many other examples : Of course, we have only described the most famous type of shape optimization problems, but there are many others. One can for example consider problems that both involve a geometrical aspect (in the spirit of the isoperimetric problem) and a PDE aspect (like the spectral problems mentioned above).

Let us mention that we did focus on problems that are quite academic, but of course there are many applied/industrial problem that enter the framework of shape optimization, and that this is a very active field (with purpose of modelling the construction of mechanical pieces, find ways to adapt to additive manufacturing...). However, the tools required to handle these more applied problems are the similar to the ones we will study in this class, with the clear drawback that the theoretical understanding of applied problems is far less complete than for the academic problems we will mention in this class!

1.3 One result that will be a consequence of all tools from this class

Let us start with a basic remark from the analysis of functions of one variable. Let $f : \mathbb{R} \to \mathbb{R}$ and assume x_0 is a minimum for f over $\mathbb{R}$. We would like to know, whether the fact that the value of $f(x)$ is close to $f(x_0)$ implies that x is close to x_0 in a quantitative way.

An easy assumption leading locally to the result is, when f is of class C^2, to show that $f''(x_0) > 0$. Indeed, we must have that $f'(x_0) = 0$, and if x is close to x_0, then one can write

$$f(x) = f(x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + o((x - x_0)^2) \geq f(x_0) + \frac{f''(x_0)}{4}(x - x_0)^2$$

in a neighborhood of x_0. This inequality quantifies how much x must be close to x_0 if $f(x)$ is close to $f(x_0)$.

Basically, the question for this paragraph is to wonder if such a result and such a strategy could be adapted to shape optimization problems, for example to the isoperimetric problem (3). The result is the following :

Theorem 1.3 (Quantitative isoperimetric inequality) *There exists $\gamma_d > 0$ a positive constant such that*

$$\forall \Omega \text{ measurable with } |\Omega| \in (0, \infty), \qquad \frac{P(\Omega) - P(B)}{P(B)} \geq \gamma_d d_{L^1}(\Omega, B)^2 \tag{5}$$

where

$$d_{L^1}(\Omega, B) := \min \left\{ \frac{|\Omega \Delta (B + x)|}{|B|}, x \in \mathbb{R}^d \right\}$$

is called the Fraenkel asymmetry, and B is a ball of the same volume as Ω.

There are three proofs for this result : [31, 29, 21], the first one focusing on symmetrization techniques, the second one on optimal transport, and the third one, on the tools of calculus of variations. Therefore, we will discuss only the third proof, and we refer to the review paper [32] for more details in this direction.

To conclude this section with another open problem, note that the knowledge of the optimal constant γ_d is not known, even for $d = 2$ (see [5]).

2 Existence theory

For this topic, let us quote [10] and [35, Chapter 2]. We also mainly follow [53, Chapter 2-3-4].

We are given J a shape functional and $\mathcal{S}_{ad}$ a class of admissible domains, and we would like to prove the existence of Ω_0 such that

$$\Omega_0 \in \mathcal{S}_{ad} \text{ and } J(\Omega_0) = \min \Big\{ J(\Omega), \;\; \Omega \in \mathcal{S}_{ad} \Big\}.$$

One big issue is that there is no "canonical" topology in the set of shapes, so we have to make a choice. As noted in the introduction, this choice should provide both some compactness for shapes and continuity for J.

This question can be crucial : Steiner in 1838 thought he gave a proof of the isoperimetric inequality, but in fact he proved that

If there is a minimizer of J, then it must be the ball

but filling the remaining gap in the proof was not an easy task.

Example 2.1 (Non-existence in shape optimization) Before going into more details, let us present loosely a situation where we do not expect that a shape optimization problem admits a solution : say you have a glass of water, and you have some fixed amount of ice you want to put in the water in order to cool down the temperature of the water as fast as possible. A first remark is that it makes more sense to break the ice into small pieces so that you can spread them in the water. As a consequence, it is understandable that any arrangement of the ice could be improved by breaking the ice even more. It is a situation where we do not expect that there is a solution.

We need some extra geometric condition to hope for existence. For example, maybe limiting the number of ice pieces.

For a similar but more mathematical discussion, see [12, Section 4.2].

Exercise 2.2 For more precise examples of non-existence, show that the following problems have no solution :

$$\sup\Big\{P(\Omega),\quad \Omega\subset D,\quad |\Omega|=m\Big\}$$

$$\sup\Big\{\lambda_1(\Omega),\quad \Omega\subset D,\quad |\Omega|=m\Big\}$$

$$\sup\Big\{P(\Omega),\quad \Omega\subset\mathbb{R}^d,\ \ \Omega \text{ convex },\quad |\Omega|=m\Big\},$$

where D is an bounded open set in $\mathbb{R}^d$.

In the first two problems, does a connectedness assumption on the admissible shapes leads to existence ?

2.1 Hausdorff convergence, existence for geometrical constrained situations

We mentioned in the introduction that a big theoretical question in shape optimization is to "parametrize" shapes with functions, in order to transfer the analytical tools of functional spaces. A first way to do it is to parametrize a shape through its distance function :

Definition 2.3 *Let D be a compact set of $\mathbb{R}^d$ (that will play the role of a "box" containing all considered shapes).*

1. *Given K_1, K_2 two nonempty compact sets in D, we define*

$$d^H(K_1,K_2)=\|d_{K_2}-d_{K_1}\|_{\infty,D} \text{ where } d_K(x):=\inf_{y\in K}|x-y|.$$

2. *Given Ω_1,Ω_2 two open sets in D, we define*

$$d_H(\Omega_1,\Omega_2)=d^H(D\setminus\Omega_1, D\setminus\Omega_2)$$

Those are both called the Hausdorff distance (either among compact sets or among open sets).

Exercise 2.4 Check that these are indeed well-defined metrics on $\mathcal{K}(D)$ the set of non-empty compact sets of D and $\mathcal{O}(D)$ the set of open sets of D respectively. As a consequence, we have a natural notion of Hausdorff convergence in $\mathcal{K}(D)$ and $\mathcal{O}(D)$.

Remark 2.5 — We used a box D for convenience, so that all the involved sets are bounded and all the functions have finite values, but it is possible to define the Hausdorff distance for closed sets in $\mathbb{R}^d$.

— There are other equivalent definitions for these metrics : if K_1, K_2 are compact set in D, one can show

$$\begin{aligned} d^H(K_1,K_2) &= \max\left\{\sup_{x\in K_1} d_{K_2}(x),\ \sup_{x\in K_2} d_{K_1}(x)\right\} \\ &= \inf\Big\{\alpha>0,\ K_2\subset K_1^\alpha,\ \ K_1\subset K_2^\alpha\Big\} \end{aligned}$$

where we defined $K^\alpha=\{x\in\mathbb{R}^d, d(x,K)\le\alpha\}$.

The main interesting property of the Hausdorff metric is summarized in the following result :

Theorem 2.6 *Let D be a compact set in $\mathbb{R}^d$. Then $(\mathcal{K}(D), d^H)$ (and equivalently $(\mathcal{O}(D), d_H)$ are compact metric spaces.*

In other words, if (K_n) is a sequence of non-empty compact sets included in a fixed compact domain D, then there exists φ an extraction and K_∞ a nonempty compact set such that

$$K_{\varphi(n)} \xrightarrow[n\to+\infty]{d^H} K_\infty.$$

Proof. Let (K_n) a sequence of compacts subsets of D. As D is compact itself, the functions (d_{K_n}) are uniformly bounded (by the diameter of D) and they are classically 1-lipschitz. From Ascoli's Theorem, we know that there exists σ an extraction and $f : D \to \mathbb{R}$ such that

$$d_{K_{\sigma(n)}} \xrightarrow[n\to\infty]{\|\cdot\|_\infty} f.$$

It remains to show that there exists K_∞ a nonempty compact set of D such that $f = d_{K_\infty}$. We first notice from the previous convergence that f is nonnegative and 1-lipschitz. We define simply $K_\infty = \{f = 0\}$, which is a closed subset of D (as f is continuous) hence a compact set.

First, from the Lipschitz property and the sign of f, we can write for $x \in D$:

$$\forall y \in K_\infty,\quad f(x) = |f(x) - f(y)| \leq |x - y|$$

so taking the infimum in $y \in K_\infty$ we obtain $f(x) \leq d_{K_\infty}(x)$.

Second, for $x \in D$, for every $n \in \mathbb{N}$, by compactness of $K_{\sigma(n)}$ (and that it is not empty), there exists $x_{\sigma(n)} \in K_{\sigma(n)}$ such that

$$d_{K_{\sigma(n)}}(x) = |x_{\sigma(n)} - x|. \tag{6}$$

We would like to pass to the limit in the previous equality. By compactness of D, there exists σ' an extraction and $x_\infty \in D$ such that $x_{\sigma\circ\sigma'(n)}$ converge to x_∞. So passing to the limit in (6) (with $\sigma \circ \sigma'(n)$ instead of $\sigma(n)$), we obtain

$$f(x) = |x_\infty - x|$$

on one hand, while we also know

$$f(x_\infty) = \lim_n d_{K_{\sigma\circ\sigma'(n)}}(x_\infty) \leq \lim_n |x_{\sigma\circ\sigma'(n)} - x_\infty| = 0$$

which shows that $x_\infty \in K_\infty$ (which is then nonempty). As a consequence, $f(x) = |x_\infty - x| \geq d_{K_\infty}(x)$, which leads to $f = d_{K_\infty}$ and the fact that $K_{\sigma(n)}$ converge in the sense of Hausdorff to K_∞.

The case of open sets (Ω_n) follows by applying the previous result to $(K_n = D \setminus \Omega_n)$. $\square$

So the Hausdorff convergence answered positively (and with minor conditions) to the question of compactness. However, as we will see now, classical shape functionals are not continuous for the Hausdorff convergence :

Exercise 2.7

1. Let (x_n) be a countable dense subset of $[0,1]^2$, and define $\forall n \in \mathbb{N},\quad K_n = \{x_0, x_1, \dots, x_n\}$. Show that K_n converge in the sense of Hausdorff to $[0,1]^2$.
2. Consider
$$\Omega_n = \left\{(x,y) \in (0,1)^2,\quad y < \frac{1}{2} + \frac{\sin(nx)}{4}\right\}$$
and compute its limit in the sense of Hausdorff.
3. (a) Show that the volume is not upper semi-continuous among open sets.
   (b) Show however that the volume is lower semi-continuous among open sets. [6]
4. Show that the perimeter functional [7] is neither upper nor lower semi-continuous for the Hausdorff convergence.
5. Let Ω be an open bounded set.
   (a) Show that if $d \geq 2$ and if $x \in \Omega$, then
   $$\lambda_1(\Omega \setminus \{x\}) = \lambda_1(\Omega).$$
   (b) Show that λ_1 is not continuous for the Hausdorff convergence.

Despite these continuity issues, the Hausdorff convergence is far from useless : indeed, one can retrieve continuity for these shape functionals if one has some extra-geometrical conditions on the considered shapes : for example, let us introduce for $\varepsilon > 0$ the class $\mathcal{O}_\varepsilon$ of open sets satifying the ε-cone condition, that is to say

$$\forall x \in \partial\Omega,\ \exists \nu_x \text{ a unit vector s.t. } \forall y \in \overline{\Omega} \cap B(x,\varepsilon),\ C(y, \nu_x, \varepsilon) \subset \Omega$$

where $C(y, \nu_x, \varepsilon)$ is the cone of vertex y, direction ν_x, radius ε and angle ε. It can be shown (see [53, Th 2.4.7, Rk 2.4.8]) that this is equivalent to being a lipschitz domains with a control of the constants only in terms of ε.

We have for example the following result (see [53, Th 2.4.10 - Th 3.2.13]) :

6. Show that if Ω_n converge to Ω in the sense of Hausdorff, then one has $\mathbb{1}_\Omega \leq \liminf_n \mathbb{1}_{\Omega_n}$ and conclude with Fatou's Lemma.

7. You can use $\mathcal{H}^{d-1}(\partial\Omega)$ or the De Giorgi perimeter introduced in Section 2.2.

Proposition 2.8 *If Ω_n is a sequence of open sets in $\mathcal{O}_\varepsilon$ converging to Ω in the sense of Hausdorff, then*

$$|\Omega_n| \xrightarrow[n\to\infty]{} |\Omega|, \qquad \mathcal{H}^{d-1}(\partial\Omega_n) \xrightarrow[n\to\infty]{} \mathcal{H}^{d-1}(\partial\Omega), \qquad \lambda_1(\Omega_n) \xrightarrow[n\to\infty]{} \lambda_1(\Omega).$$

We will give the first ingredients of the proof for PDE-like functionals in Section 2.3.

As a consequence, the Hausdorff distance allows to show that the following problems have solutions :

$$\max\left\{P(\Omega),\ \Omega \text{ convex} \subset D,\ |\Omega| = m\right\}, \quad \max\left\{\lambda_1(\Omega),\ \Omega \text{ convex} \subset D,\ |\Omega| = m\right\}$$

$$\min\left\{P(\Omega) - \mu\lambda_1(\Omega),\ \Omega \text{ convex} \subset D,\ |\Omega| = m\right\}, \quad \min\left\{\lambda_1(\Omega) - \mu P(\Omega),\ \Omega \text{ convex} \subset D,\ |\Omega| = m\right\}$$

where D is a bounded convex set of $\mathbb{R}^d$, $\mu \geq 0$ and $m \in (0, |D|)$. The analysis of these problems is not an easy task : the first two are reverse isoperimetric and Faber-Krahn problems respectively, and the last ones have two competitive terms. It is only known, for now, that

1. in dimension 2, the solutions to the first two problems have a polygonal free boundary (see [39, 41])
2. in dimension 2, the solution to the third problem is $C^{1,1}$, and the solution to the fourth problem has again a polygonal free boundary (see [40])

2.2 Characteristic functions and Perimeter constraints

Another basic idea here to "parametrize" shapes with functions is to associate to the set Ω (measurable) the function $\mathbb{1}_\Omega$ (which is called the characteristic function of Ω). This allows to define very simply a form of metric/convergence (among sets with finite volume), by using the L^1-norm. Note that in this framework, we are in fact considering shapes up to a set of 0-Lebesgue measure. This might be an issue if the shape functional we consider is not invariant by the choice of representative. For example,

$$\Omega \mapsto \lambda_1(\Omega), \qquad \text{and} \qquad \Omega \mapsto \mathcal{H}^{d-1}(\partial\Omega)$$

are not really adapted to this framework [8].

Remark 2.9 A first basic compactness property : for every sequence (Ω_n) of shapes, the sequence $(\mathbb{1}_{\Omega_n})$ is bounded in $L^\infty(\mathbb{R}^d)$ which can be seen as the dual of the separable Banach space $L^1(\mathbb{R}^d)$. From Banach-Alaoglu's Theorem [9], there exists σ an extraction and $\chi \in L^\infty(\mathbb{R}^d)$ such that

$$\mathbb{1}_{\Omega_{\sigma(n)}} \rightharpoonup \chi \quad \text{weak}^* - L^\infty$$

which means

$$\forall \varphi \in L^1(\mathbb{R}^d), \qquad \int_{\Omega_{\sigma(n)}} \varphi dx \longrightarrow \int_{\mathbb{R}^d} \chi\varphi dx.$$

However, even though we know that $\chi \in L^\infty(\mathbb{R}^d)$ and that $0 \leq \chi \leq 1$, the function χ may not be a characteristic function in general. This remark is nevertheless far from useless, it leads to the notion of relaxation, see for example [53, Sections 7.2-7.3].

Exercise 2.10 Build a sequence Ω_n such that $\mathbb{1}_{\Omega_n}$ weakly*-L^∞ converge to a function that is not a characteristic function.

In order to "solve" this issue, we need to get a stronger compactness, and it is natural to that end to put also a bound on the "variations" of $\mathbb{1}_{\Omega_n}$. To give meaning to this idea, we use the framework of differentiation in the sense of distributions :

8. This does not mean that it is impossible to use it, see for example [53, Section 4.5] and [26].
9. This result says that if E is a separable Banach space and if (ℓ_n) is a bounded sequence in E', then there exists σ an extration and $\ell_\infty \in E'$ such that

$$\ell_{\sigma(n)} \xrightarrow[n\to\infty]{\text{weak}^*} \ell_\infty, \quad \text{that is to say } \forall x \in E,\ \ell_{\sigma(n)}(x) \xrightarrow[n\to\infty]{} \ell_\infty(x)$$

Definition 2.11 *Let Ω be a measurable set in $\mathbb{R}^d$. We denote*[10]

$$\begin{aligned} P(\Omega) = \|\nabla \mathbb{1}_\Omega\|_{TV} &:= \sup\Big\{\langle \nabla \mathbb{1}_\Omega, \varphi\rangle,\ \varphi \in C^1_c(\mathbb{R}^d,\mathbb{R}^d)\ \ \|\varphi\|_\infty \le 1\Big\} \\ &= \sup\Big\{\int_\Omega \operatorname{div}(\varphi),\ \varphi \in C^1_c(\mathbb{R}^d,\mathbb{R}^d),\ \ \|\varphi\|_\infty \le 1\Big\}. \end{aligned}$$

the perimeter of Ω. In particular we say that Ω is a set of finite perimeter if $P(\Omega) < \infty$.

Remark 2.12 — In other words, Ω is of finite perimeter if and only if $\mathbb{1}_\Omega \in BV(\mathbb{R}^d)$.

— This "new" definition of a perimeter replaces the first guess $\mathcal{H}^{d-1}(\partial\Omega)$ in a convenient way for our purpose. A consequence of the Gauss-Green Theorem if that if Ω is smooth enough (Lipschitz) then we still have $P(\Omega) = \mathcal{H}^{d-1}(\partial\Omega)$:
Indeed, in that case we have

$$\forall \varphi \in C^1_c(\mathbb{R}^d,\mathbb{R}^d),\ \ \int_\Omega \operatorname{div}(\varphi)dx = \int_{\partial\Omega} \varphi\cdot\nu_{\partial\Omega} d\mathcal{H}^{d-1}$$

(where $\nu_{\partial\Omega}$ denotes the outward unit normal vector to $\partial\Omega$), which leads easily to

$$P(\Omega) \le \mathcal{H}^{d-1}(\Omega).$$

To obtain the converse inequality, we need to use $\varphi \in C^1_c(\mathbb{R}^d,\mathbb{R}^d)$ such that its trace on $\partial\Omega$ is close to $\nu_{\partial\Omega}$, see for example [53, Prop 2.3.3].

Continuity properties :

Proposition 2.13 *Let (Ω_n) and Ω measurable domains in $\mathbb{R}^d$.*

$$\mathbb{1}_{\Omega_n} \xrightarrow[n\to\infty]{L^1} \mathbb{1}_\Omega \quad\Longrightarrow\quad \begin{cases} |\Omega| = \lim_n |\Omega_n| \\ P(\Omega) \le \liminf_n P(\Omega_n) \end{cases}$$

Proof. Let us assume $\mathbb{1}_{\Omega_n} \xrightarrow[n\to\infty]{L^1} \mathbb{1}_\Omega$. For the volume, we simply have to write

$$\big||\Omega_n| - |\Omega|\big| \le \int_{\mathbb{R}^d} |\mathbb{1}_{\Omega_n} - 1_\Omega|.$$

Let $\varphi \in C^1_c(\mathbb{R}^d,\mathbb{R}^d)$ such that $\|\varphi\|_\infty \le 1$. Then (from Lebesgue's dominated convergence theorem) we have

$$\int_\Omega \operatorname{div}(\varphi) = \lim_n \int_{\Omega_n} \operatorname{div}(\varphi) \le \liminf P(\Omega_n)$$

and taking the supremum in φ leads to the result. □

Compactness result :

Proposition 2.14 *Let D a bounded open set, and (Ω_n) a sequence of shapes in D such that $(|\Omega_n|)$ and $(P(\Omega_n))$ are bounded. Then there exists σ an extraction and Ω_∞ a set in $\mathbb{R}^d$ such that*

$$\mathbb{1}_{\Omega_{\sigma(n)}} \xrightarrow[n\to\infty]{L^1} \mathbb{1}_{\Omega_\infty} \qquad \textit{and} \qquad \nabla\mathbb{1}_{\Omega_{\sigma(n)}} \xrightarrow[n\to\infty]{weak^*-(C^0)'} \nabla 1_{\Omega_\infty}$$

Sketch of proof (see [53, Th 2.3.11] for more details) This result rely on two seperate steps :

10. The notation $\|\cdot\|_{TV}$ stands for the *total variation*-norm : it is classical that if $T \in (C^\infty_c(\mathbb{R}^d,\mathbb{R}^d))'$ is a (vector-valued) distribution and if

$$\sup\Big\{\langle T,\varphi\rangle,\ \varphi \in C^\infty_c(\mathbb{R}^d,\mathbb{R}^d),\ \|\varphi\|_\infty \le 1\Big\} < +\infty$$

then T can be extended as a continuous linear form on $C^0_{\to 0}(\mathbb{R}^d,\mathbb{R}^d)$ (continuous functions going to 0 at infinity), which we call a (vector-valued) Radon measure (this denomination is justified by Riesz' representation Theorem).

1. the assumptions imply that $(\mathbb{1}_{\Omega_n})$ is bounded in $BV(D)$: we can therefore see both $(\mathbb{1}_{\Omega_n})$ bounded in $L^1(D) \subset (C^0(D))'$ and $(\nabla \mathbb{1}_{\Omega_n})$ bounded in $C^0(D, \mathbb{R}^d)'$. From Banach-Alaoglu's Theorem, there exists $f \in C^0(D)'$ and μ in $C^0(D, \mathbb{R}^d)'$ and σ an extraction such that

$$\mathbb{1}_{\Omega_{\sigma(n)}} \xrightarrow[n\to\infty]{weak*-(C^0)'} f, \qquad \nabla \mathbb{1}_{\Omega_{\sigma(n)}} \xrightarrow[n\to\infty]{weak*-(C^0)'} \mu.$$

As the weak*-C^0 convergence implies the convergence in the sense of distribution and the derivative is continuous in the sense of distribution, we necessarily have $\nabla f = \mu$, so in particular $f \in BV(D)$.

2. As usual it remains to show that f is an chararectistic function. For that one can prove (in the same spirit as Rellich Theorem) that the previous convergence actually implies that $\mathbb{1}_{\Omega_{\sigma(n)}}$ converges to f strongly in $L^1_{loc}(D)$. In particular, up to a subsequence, $\mathbb{1}_{\Omega_{\sigma(n)}}$ converges to f almost everywhere, so the equality $\mathbb{1}_{\Omega_n}(1-\mathbb{1}_{\Omega_n}) = 0$ gives at the limit $f(1-f) = 0$, which concludes the proof.

□

Corollary 2.15 *Let D be a bounded set in $\mathbb{R}^d$ and $m \in (0, |D|)$. Then the problem*

$$\inf\Big\{P(\Omega), \;\; \Omega \subset D, \quad |\Omega| = m\Big\} \tag{7}$$

has a solution.

Proof. A preliminary step (that we leave to the reader) is to prove that the class of admissible sets is not empty. Then one can consider (Ω_n) a minimizing sequence : as $(|\Omega_n|)$ is constant and $P(\Omega_n)$ converges to the infimum, one can apply Proposition 2.14, so there exists σ an extraction and $\Omega_\infty \subset D$ a set of finite perimeter such that $\Omega_{\sigma(n)}$ converges to Ω_∞ in $L^1(D)$. From Proposition 2.13, we deduce that $|\Omega_\infty| = m$ and $P(\Omega_\infty) \le \liminf P(\Omega_{\sigma(n)})$ so Ω_∞ is indeed a solution to the isoperimetric problem. □

Remark 2.16 If D is an open set in $\mathbb{R}^d$, one can consider

$$\begin{aligned} P(\Omega, D) = \|\mathbb{1}_\Omega\|_{TV(D)} &:= \sup\Big\{\langle \nabla \mathbb{1}_\Omega, \varphi\rangle, \; \varphi \in C^1_c(D, \mathbb{R}^d) \;\; \|\varphi\|_\infty \le 1\Big\} \\ &= \sup\Big\{\int_\Omega \operatorname{div}(\varphi), \; \varphi \in C^1_c(D, \mathbb{R}^d), \;\; \|\varphi\|_\infty \le 1\Big\}. \end{aligned}$$

which is the perimeter relative to D, and coincides with $\mathcal{H}^{d-1}(\partial\Omega \cap D)$ when Ω is smooth enough. All the previous results can be adapted to this more general framework.

2.3 Existence for PDE functionals

Preliminary remarks : Think of a situation where the shape functional under study depends on an elliptic PDE over the shape $\Omega \subset D$ (again D is compact), with Dirichlet boundary conditions on $\partial\Omega$ [11]. For example, we have $f \in L^2(D)$ and for each Ω we have $u^f_\Omega (= u_\Omega)$ [12] the unique solution of

$$\left\{ \begin{array}{l} -\Delta u_\Omega = f \text{ in } \Omega \\ u_\Omega \in H^1_0(\Omega) \end{array} \right. \tag{8}$$

Let us first notice that we can easily see $H^1_0(\Omega)$ as a subset of $H^1_0(D)$ by extending functions by 0 outside Ω [13]. Also, the previous equation implies the following bound which is uniform in Ω : from (8) we deduce

$$\|\nabla u_\Omega\|^2_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}\|u_\Omega\|_{L^2(\Omega)} \le \|f\|_{L^2(D)}\|u_\Omega\|_{H^1(D)}.$$

But we also have

$$\|u_\Omega\|^2_{H^1(D)} = \|\nabla u_\Omega\|^2_{L^2(D)} + \|u_\Omega\|^2_{L^2(D)} \le \|\nabla u_\Omega\|^2_{L^2(D)} + \frac{1}{\lambda_1(D)}\|\nabla u_\Omega\|^2_{L^2(D)} = \left(1 + \frac{1}{\lambda_1(D)}\right)\|\nabla u_\Omega\|^2_{L^2(D)}$$

11. The case of other boundary conditions is quite difficult : see for example [13, 14, 15] where a free boundary approach is used to deal with Robin boundary conditions.
12. We will drop the exponent f so that the notations are more readible, we will put f when needed.
13. This requires no regularity ; another way to see this is that $C^\infty_c(\Omega) \subset C^\infty_c(D)$ so by definition $H^1_0(\Omega) \subset H^1_0(D)$.

(this is the Poincaré inequality) so combing the two previous inequalities we obtain

$$\|u_\Omega\|_{H^1(D)} \leq \left(1 + \frac{1}{\lambda_1(D)}\right) \|f\|_{L^2(D)}.$$

This remark provide some compactness. Indeed if (Ω_n) is a sequence of open sets in D, then (u_{Ω_n}) is bounded in $H^1(D)$ so up to a subsequence there exists $u^* \in H^1_0(D)$ such that u_{Ω_n} converges to u^* strongly in $L^2(D)$ and weakly in $H^1_0(D)$.

But now, having continuity for the PDE (8) would require that $u^* = u^f_\Omega$ for some limit domain Ω. A few remarks :

1. If indeed there is Ω such that $u^* = u_\Omega$, then the convergence of u_{Ω_n} to u_Ω is in fact strong in H^1. This fact follows from the formula
$$\|\nabla u_{\Omega_n}\|^2_{L^2} = \int f u_{\Omega_n}$$
which passes to the limit (here the L^2 convergence of u_{Ω_n} suffices [14]) and shows $\|u_{\Omega_n}\|_{H^1} \xrightarrow[n\to\infty]{} \|u_\Omega\|_{H^1}$, and it is a classical lemma about weak convergence that if the norm is preserved, then the convergence is in fact strong.
2. If we assume that there exists an open set $\Omega \subset D$ such that
$$\forall K \Subset \Omega, \;\; \exists n_0, \; \forall n \geq n_0, \;\; K \subset \Omega_n, \tag{9}$$
we have
$$-\Delta u^* = f \quad \text{in the sense of distribution in } \Omega$$
Indeed if $\varphi \in C^\infty_c(\Omega)$, then its support is compactly supported in Ω so the previous condition allows to have for $n \geq n_0$, $\varphi \in C^\infty_c(\Omega_n)$ so
$$\forall n \geq n_0, \quad \int_D \nabla u_{\Omega_n} \varphi = \int_{\Omega_n} \nabla u_{\Omega_n} \varphi = \int_{\Omega_n} f\varphi = \int_D f\varphi$$
which passes to the limit and gives the result.
3. The remaining question, if we are in the situation of the previous remark, is to wonder whether $u^* \in H^1_0(\Omega)$. This is not the case in general.

Exercise 2.17 Show that condition (9) is true if Ω_n converges to Ω in the sense of Hausdorff.

We are finally led to the following definition :

Definition 2.18 *Let D be an open bounded set. We say that a sequence of open sets (Ω_n) (included in D) γ-converges to an open set $\Omega \subset D$ if for every $f \in L^2(D)$,*

$$u^f_{\Omega_n} \xrightarrow[n\to\infty]{H^1} u^f_\Omega$$

where $(u^f_{\Omega_n})$ and u^f_Ω are solutions of (8) in their respective domains.

Remark 2.19 — It may look strange/strong that this convergence refers to the right-hand side $f \in L^2(D)$. In fact, it doesn't, in particular if it is valid for the particular function $f = 1$, then it is still true for any function $f \in L^2(D)$, see [53, Th 3.2.5].

— If Ω_n γ-converges to Ω, then $\lambda_k(\Omega_n)$ converges to $\lambda_k(\Omega)$ for every $k \in \mathbb{N}^*$. This is due to the fact that γ-convergence implies a strong convergence (in $\mathcal{L}(L^2(D))$ of the resolvant operators, see [53, Lemma 4.7.3].

Example 2.20 (Cioranescu-Murat [22]) Here is a famous example to feel what can "go wrong" for the γ-convergence we are hoping for.

From Exercise 2.7 (questions 1 and 5b) we understand that by making "holes of the size of a point" in a domain Ω, we can have a piece of Ω (or all of it) that disappears for the Hausdorff convergence, but "remains here" for the Dirichlet problem, which means the Dirichlet energy is unmodified. We could argue

14. But if we had taken $f \in H^{-1}(D)$, which is more natural, we would have used the weak H^1 convergence

that this perturbation is a very specific situation. We will nevertheless discuss this idea in a more involved way :

Consider (to simplify) the square $\Omega = (0,1)^2$, and for $n \in \mathbb{N}^*$, $x_{ij} = \left(\frac{i}{n}, \frac{j}{n}\right)$. We then define :

$$\Omega_n = \Omega \setminus \bigcup_{i,j} B(x_{ij}, r_n)$$

where $r_n > 0$ is small enough so that the holes do not intersect each other. The Hausdorff limit of this set is always the empty set.

This example is specific enough so that an explicit computation of the γ-limit of (Ω_n) can be computed. In [22], the authors showed :

1. if the holes are very small (much smaller [15] than e^{-n^2}) then

$$u_{\Omega_n} \xrightarrow[n\to+\infty]{H^1} u_\Omega, \quad \text{ i.e. } \quad \Omega_n \xrightarrow[n\to+\infty]{\gamma} \Omega$$

   The situation is the same as when the holes had radius 0.
2. if the holes are "slightly big" (much bigger [16] that e^{-n^2}), then on the contrary, everything vanishes at the limit :

$$u_{\Omega_n} \xrightarrow[n\to+\infty]{H^1} 0, \quad \text{ i.e. } \quad \Omega_n \xrightarrow[n\to+\infty]{\gamma} \emptyset$$

3. if the holes have an intermediate regime, namely if $r_n = e^{-\alpha n^2}$, then

$$u_{\Omega_n} \xrightarrow[n\to+\infty]{H^1} u^*$$

   where u^* is the solution to

$$\begin{cases} -\Delta u^* + \frac{2\pi}{\alpha} u^* = f \text{ in } \Omega \\ u^* \in H^1_0(\Omega) \end{cases}$$

   Here is a new "strange" term appearing in the PDE at the limit ; so (Ω_n) has no γ-limit. This phenomenon has been generalized to every situation with the notion of capacitary measure by Dal Maso-Mosco [23].

Cases where Hausdorff convergence implies γ-convergence : Let us first start with an easy situation :

Exercise 2.21 Show that when (Ω_n) is a nondecrasing sequence of open sets, then it γ-converges to $\Omega := \cup_n \Omega_n$.

As we mentioned in Section 2.1, if we had some regularity assumptions on the domains, then we can retrieve the continuity properties for the Hausdorff convergence : let us quote two results whose proof can be found in [53].

Theorem 2.22 (Chesnais [20]) *Let $\Omega_n \subset D$ is a sequence of open sets in $\mathcal{O}_\varepsilon$. If Ω_n converges to Ω in the sense of Hausdorff, then it γ-converges to Ω.*

To prove this result, one can use the method from the introduction : only the third step remains, which means to prove that $u^* \in H^1_0(\Omega)$, which relies on the following property

$$H^1_0(\Omega) = \{u \in H^1(\mathbb{R}^d), \ \ u = 0 \text{ almost everywhere on } \Omega^c\}$$

which is not true in general, but is true for Lipschitz sets (see [53, Prop 3.2.16].

15. the precise condition is

$$\frac{\ln(r_n)}{n^2} \xrightarrow[n\to\infty]{} -\infty.$$

16. The precise condition is

$$\frac{\ln(r_n)}{n^2} \xrightarrow[n\to\infty]{} 0.$$

Theorem 2.23 (Sverak [57]) *Assume $d = 2$ and $\Omega_n \subset D$ is a sequence of open sets such that the number of connected component of $D \setminus \Omega_n$ is bounded uniformly in n. If Ω_n converges to Ω in the sense of Hausdorff, then it γ-converges to Ω.*

This result is more involved than the previous one : it actually relies on a generalization of Theorem 2.22 where the cone condition is replaced by a (weaker) capacity condition.

Example of the Dirichlet energy : [17]
Say we would like to focus on the model problem (where as usual D is a bounded open set and $m \in (0, |D|)$)

$$\inf\Big\{E_f(\Omega),\quad \Omega \text{ open subset of } D,\quad |\Omega| = m\Big\}, \tag{10}$$

where we defined for each Ω an open set the Dirichlet energy $E_f(Om)$ associated to the right-hand side f :

$$E_f(\Omega) = \inf\left\{\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega fu,\ u \in H^1_0(\Omega)\right\}. \tag{11}$$

It is well-known that this infimum is achieved by a unique u_Ω solution to (8).

The previous paragraph informs us that using the Hausdorff metric is not sufficient to conclude. Also, the L^1-convergence of characteristic functions seems not really adapted, as E_f is not invariant if one changes Ω with a set of 0-Lebesgue measure.

Let us show, nevertheless, an existence result for (10). Indeed, we can take advantage of two facts (to simplify the situation, think of f as being a positive function in which case the state function u_Ω is positive in Ω). First, E_f is itself defined as a minimum over $H^1_0(\Omega)$ that can be seen as a subset of $H^1_0(D)$, and second, given u_Ω solution to (8), seen as a function of $H^1_0(D)$, one can retrieve Ω as the set of positivity for u_Ω. This allows to deal with problem (10) with the point of view of free boundary problems :

Theorem 2.24 *Let D be an open bounded set in $\mathbb{R}^d$. Then there exists $u \in H^1_0(D)$ solution of*

$$\min\left\{\frac{1}{2}\int_D |\nabla v|^2 - \int_D fv,\quad v \in H^1_0(D),\quad |\{v \neq 0\}| \leq m\right\}. \tag{12}$$

The set $\Omega^* = \{u \neq 0\}$ is such that

$$\forall\Omega \text{ open subset of } D \text{ such that } |\Omega| \leq m,\quad E_f(\Omega^*) \leq E_f(\Omega).$$

Sketch of proof : Let (u_n) a minimizing sequence for (12), and denote α the value of the infimum. We can prove with similar argument as before that it is bounded in $H^1(D)$: more precisely, as it is a minimizing sequence, there exists $M \in \mathbb{R}$ such that

$$\forall n \in \mathbb{N},\ \ \frac{1}{2}\|\nabla u_n\|^2_{L^2} - \|f\|_{L^2}\|u_n\|_{L^2} \leq \frac{1}{2}\int_D |\nabla u_n|^2 - \int_D fu_n \leq M$$

but with the Poincaré inequality in $H^1_0(D)$, there exists $c > 0$ independant of n such that

$$\frac{1}{2}\|\nabla u_n\|^2_{L^2} - \|f\|_{L^2}\|u_n\|_{L^2} \geq c\|u_n\|^2_{H^1} - \|f\|_{L^2}\|u_n\|_{H^1}$$

so necessarily $(\|u_n\|_{H^1})$ is bounded.

So from usual compactness properties, there exists $u^* \in H^1(D)$ such that (up to a subsequence) u_n converges to u^* strongly in $L^2(D)$ and almost everywhere, and ∇u_n converges weakly-L^2 to ∇u^*.

The lower-semincontinuity of the norm for the weak convergence shows that

$$\frac{1}{2}\int_D |\nabla u^*|^2 - \int_D fu^* \leq \alpha.$$

The convergence almost everywhere shows that

$$\mathbb{1}_{\{u^* \neq 0\}} \leq \liminf \mathbb{1}_{\{u_n \neq 0\}} \quad a.e.$$

so with Fatou's lemma, $|\{u^* \neq 0\}| \leq m$, so u^* is indeed a solution to (12).

Let us now consider Ω an open set of D such that $|\Omega| \leq m$.

17. I will work in these notes mainly with E_f instead of λ_1, for example. In fact, most of the ideas I develop here can be adapted to the case of λ_1, with a minor cost on computations. The case of λ_k might however be sometimes more involved.

First, we notice that if we restrict (12) to functions $v \in H^1_0(\Omega^*)$ (which are admissible functions), then u^* solves (11). Also, if $v \in H^1_0(\Omega)$ then $\{v \neq 0\} \subset \Omega$ so $|\{v \neq 0\}| \leq m$ and we can write :

$$\forall v \in H^1_0(\Omega), \quad E_f(\Omega^*) = \frac{1}{2}\int_D |\nabla u^*|^2 - \int_D f u^* \leq \frac{1}{2}\int_D |\nabla v|^2 - \int_D f v$$

so taking the infimum in v, we obtain

$$E_f(\Omega^*) \leq E_f(\Omega).$$

□

Warning : the previous proof is a bit misleading, because there is a catch. We considered the set $\{u^* \neq 0\}$ for $u^* \in H^1(D)$, but this set is a priori not an open set (it is well-known that an H^1 function may not be continuous if $d \geq 2$), which means $E_f(\Omega^*)$ was not *a priori* well-defined. We were also a bit quick when saying $u^* \in H^1_0(\Omega^*)$. Actually, the set Ω^* that we built here is a *quasi-open* set, which by definition is the level set of an H^1-function. To get a better understanding of this would require some extra-knowledge about capacity and quasi-continuous representative of H^1-functions, leading to the fact that (11) can be defined for quasi-open sets once we realize we can give a new definition of the set $H^1_0(\Omega)$ as

$$H^1_0(\Omega) = \{u \in H^1(\mathbb{R}^d), \quad \tilde{u} = 0 \text{ quasi-everywhere on } \Omega^c\}$$

where quasi-everywhere means everywhere except on a set of 0-capacity, and $\tilde{u}$ is the quasi-continuous representative of u. I refer to [53, Section 3.3] for much more details.

Corollary 2.25 *There exists Ω^* a quasi-open subset of D solution to*

$$\min\left\{E_f(\Omega), \ \ \Omega \textit{ quasi-open subset of } D, \ |\Omega| = m\right\} \tag{13}$$

Proof. We apply Theorem 2.24, providing a set Ω^*. There are only two differences : first we showed optimality for Ω open, but actually the same proof works for Ω quasi-open. Second, we might have $|\Omega^*| < m$. In that case, we just increase Ω^* in any way to that the volume increases up to m. Because of the monotonicity of E_f, the increased set has a lower energy so it is indeed a solution to (13)

□

A similar approach can be used to prove the existence of a minimizer for

$$\min\left\{\lambda_1(\Omega), \quad \Omega \text{ quasi-open subset of } D, \ \ |\Omega| = m\right\} \tag{14}$$

(see [18]) and more generally

$$\min\left\{F(\lambda_1(\Omega), \cdots, \lambda_k(\Omega), \quad \Omega \text{ quasi-open subset of } D, \ \ |\Omega| = m\right\} \tag{15}$$

where $k \in \mathbb{N}^*$ and $F : \mathbb{R}^k \to \mathbb{R}$ is non-decreasing in each variable and lower-semicontinuous, see [35, Theorem 2.1].

Let us conclude with a famous result :

Theorem 2.26 (Buttazzo-Dal Maso [19]) *Let D be a bounded open set and J a shape functional defined on quasi-open sets of D. We assume*

1. *J is lower semi-continuous for the γ-convergence,*
2. *J is nonincreasing for the inclusion.*

Then for every $m \in (0, |D|)$, there exists a solution to

$$\min\left\{J(\Omega), \ \ \Omega \textit{ quasi-open } \subset D, \ |\Omega| = m\right\}.$$

Example 2.27 This result generalizes the example we dealt with using the free boundary approach, as it is classical that E_f and λ_k satisfy both hypotheses from Theorem 2.26.

Concluding remarks on the unbounded case : A natural question in the study of (15) is whether we can consider the most natural case where $D = \mathbb{R}^d$. This is a difficult question that was solved 20 years after Buttazzo-Dal Maso's result, namely in 2012 independantly (and with different methods) by Bucur in [11] and Mazzoleni-Pratelli in [47] (see also [53, Section 4.8]). I won't give any details, but I will only say that this existence result is achieved (in both proofs) by combining a usual compactness strategy with a (weak) regularity result. So it is worth noticing that the tools from this Section and Section 4 are sometimes working together.

3 Shape derivatives

In the previous section, we gave several ways to parametrize shapes with functions. In this section, we need to find a way to parametrize deformations of shapes. A first basic remark is that we could hope that the parametrization from the previous section would also work. Think for example about $\Omega \mapsto \mathbb{1}_\Omega$: thinking of a perturbation of a shape could be to consider $\mathbb{1}_\Omega + \mathbb{1}_\omega$ where ω is small in L^1, which means it has small volume [18]. Unfortunately, this way of deforming shapes is way too general as most shape functionals will not have a nice expansion with respect to this kind of perturbation (except for very specific functionals like the volume).

So the new balance we are facing is to parametrize *a lot* of perturbations, having in mind that we would like the shape functional to have a nice expansion for those perturbations. Notice that there are, as a consequence, very different ways of thinking of expansions : in particular we will not discuss the notion of topological derivative in this document (see for example [55]), we will only focus on the idea to perturbe "smoothly" the boundary of a fixed shape, in the spirit of Euler and Lagrange perturbations in the calculus of variations.

3.1 Definition

Definition 3.1 *Let J a shape functional* [19] *and Ω_0 a shape in $\mathbb{R}^d$. We call first and second order shape derivative of J at Ω_0, the first and second usual (Fréchet) differential at 0 of*

$$\begin{array}{rccc} \mathcal{J}_{\Omega_0} : & W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ & \theta & \longmapsto & J((Id+\theta)(\Omega_0)) \end{array}$$

when it exists. We also denote

$$J'(\Omega_0) := \mathcal{J}'_{\Omega_0}(0) \qquad \text{and } J''(\Omega_0) := \mathcal{J}''_{\Omega_0}(0).$$

Remark 3.2 — The space $W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)$ is endowed with its usual norm $\|\theta\|_{W^{1,\infty}} := \|\theta\|_{L^\infty} + \|D\theta\|_{L^\infty}$. Let us recall that this is the space of Lipschitz functions from $\mathbb{R}^d$ to $\mathbb{R}^d$, and $D\theta \in L^\infty(\mathbb{R}^d, M_d(\mathbb{R}))$ is defined in the sense of distributions. Note that it also coincide with the classical differential which is well-defined almost everywhere (Rademacher Theorem).

- If $\|\theta\|_{1,\infty} < 1$, then $(\mathrm{Id}+\theta)$ is a $W^{1,\infty}$-diffeomorphism, which means it is Lipschitz, invertible [20], and $(\mathrm{Id}+\theta)^{-1}$ is also $W^{1,\infty}$. As a consequence $(\mathrm{Id}+\theta)(\Omega_0)$ can be seen as a "small" deformation of Ω_0, in particular it does not change the topology of Ω_0.
- The space $W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)$ may be replaced by smaller spaces (with their usual norms).
- From the definitions, we have that $J'(\Omega_0)$ and $J''(\Omega_0)$ are respectively linear and bilinear continuous forms on $W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)$.

Example 3.3 — [Graphs] Assume Ω_0 is locally the sub-graph of $u : D \to (-\alpha, \alpha)$, and $v : D \to \mathbb{R}$ is a Lipschitz function with compact support in D. Let $\rho : \mathbb{R}^d \to [0,1]$ be a Lipschitz function such

18. We could also consider it is small in BV, but this doesn't really help
19. We are being unprecise about the domain of J, but of course it should be well-defined at every shape involved in this definition.
20. This is a consequence of the fixed point Theorem for contractions

that $\rho = 1$ on $D' \times (-\alpha+\varepsilon, \alpha-\varepsilon)$ where $\varepsilon > 0$ is small and D' contains the support of v, and $\rho = 0$ outside of $D \times (-\alpha, \alpha)$. Then one can define

$$\forall x = (x', x_d) \in \mathbb{R}^d, \quad \theta(x) = \begin{cases} \rho(x)v(x')e_d \text{ if } x \in D \times (-\alpha, \alpha) \\ 0 \text{ otherwise.} \end{cases}$$

Then $\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ and $(\mathrm{Id}+\theta)(\Omega_0)$ is the subgraph of $u+v$ over D, and is equal to Ω_0 outside of $D \times (-\alpha, \alpha)$.

— [Nearly spherical sets] We denote, for $u : \mathbb{S}^{d-1} \to (-1, \infty)$ Lipschitz :

$$B_u := \{(r, \varphi) \in \mathbb{R}_+ \times \mathbb{S}^{d-1}, \ \ r < 1 + u(\varphi)\} \tag{16}$$

(where (r, φ) are the spherical coordinates) which is called a nearly spherical set. Again, one can see this set as $(\mathrm{Id}+\theta)(B)$ for $\theta(r, \varphi) = u(\varphi)\rho(r)$ where $\rho : (0, +\infty) \to [0, 1]$ is smooth and such that $\rho = 0$ is $(0, \varepsilon)$ and $\rho = 1$ on $(2\varepsilon, +\infty)$.

3.2 First order examples

In this section, we will study our 3 main examples (volume, perimeter, Dirichlet energy). In particular we will show that they have first and second order shape derivatives (actually derivatives of any order), and compute the first order one.

Volume : The idea is to use a change of variable (which is valid for $W^{1,\infty}$-diffeomorphisms). Let $\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ such that $\|\theta\|_{1,\infty} < 1$. Then one has

$$|(\mathrm{Id} + \theta)(\Omega_0)| = \int_{(\mathrm{Id}+\theta)(\Omega_0)} 1dx = \int_{\Omega_0} 1 \det(\mathrm{Id} + D\theta)dy.$$

Proposition 3.4 *Let Ω_0 be a measurable set. Then*

$$\begin{array}{ccc} W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ \theta & \longmapsto & |(Id+\theta)(\Omega_0)| \end{array}$$

is C^∞ in a neighborhood of 0, and

$$\forall \xi \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d), \quad \mathrm{Vol}'(\Omega_0)(\xi) = \int_{\Omega_0} \mathrm{div}(\xi)dx.$$

If Ω_0 is of class C^1 then one also has

$$\forall \xi \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d), \quad \mathrm{Vol}'(\Omega_0)(\xi) = \int_{\partial\Omega_0} \xi \cdot \nu_{\partial\Omega_0} d\sigma,$$

where $\nu_{\partial\Omega_0}$ denotes the unit outward normal vector to $\partial\Omega_0$.

Remark 3.5 The last part of the result can be generalized to sets of finite perimeter. Indeed, if Ω is such that $P(\Omega) < \infty$, then by Riesz's Theorem, there exists a (vector-valued) Radon measure μ_Ω such that

$$\forall \varphi \in C^1_c(\mathbb{R}^d, \mathbb{R}^d), \quad \int_\Omega \mathrm{div}(\varphi) = \int_{\mathbb{R}^d} \varphi \cdot d\mu_\Omega.$$

From classical measure theory (Lebesgue-Besicovitch differentiation Theorem), we have the existence of $\partial^*\Omega \subset \partial\Omega$ (called the reduced boundary) and $\nu_{\partial\Omega} : \partial^*\Omega \to \mathbb{S}^1$ (the generalized outward unit normal) such that

$$\forall \varphi \in C^1_c(\mathbb{R}^d, \mathbb{R}^d), \quad \int_\Omega \mathrm{div}(\varphi) = \int_{\partial^*\Omega} \varphi \cdot \nu_{\partial^*\Omega} d|\mu_\Omega|.$$

The measure $|\mu_\Omega|$ is the (real-valued) "total variation measure" of μ_Ω. Finally, the De Giorgi's structure Theorem asserts that $|\mu_\Omega| = \mathcal{H}^{d-1}_{\lfloor\partial^*\Omega}$, which finally leads to the following generalized Gauss-Green Theorem :

$$\forall \varphi \in C^1_c(\mathbb{R}^d, \mathbb{R}^d), \quad \int_\Omega \mathrm{div}(\varphi) = \int_{\partial^*\Omega} \varphi \cdot \nu_{\partial^*\Omega} d\mathcal{H}^{d-1}.$$

See for example [44, Chapter 15] for more details.

Proof. For the regularity, we look at the following composition :

$$\begin{array}{ccccccc} W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d) & \longrightarrow & L^\infty(\mathbb{R}^d, M_d(\mathbb{R})) & \longrightarrow & L^\infty(\mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ \theta & \longmapsto & \mathrm{Id}+D\theta & \longmapsto & \det(\mathrm{Id}+D\theta) & \longmapsto & \int_{\Omega_0}\det(\mathrm{Id}+D\theta) \end{array}$$

The first and third one are continuous and linear (hence C^∞), and for the second one it easily follows from the fact that det is a polynomial in its coefficient and L^∞ is an algebra.

For the computation, we recall that if $H \in M_d(\mathbb{R})$ is small, then

$$\det(\mathrm{Id}+H) = \det(\mathrm{Id}) + (\det)'(\mathrm{Id}).H + O(\|H\|^2) = 1 + \mathrm{Tr}(H) + O(\|H\|^2)$$

so

$$\begin{aligned} \forall \xi \text{ small in } W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d),\ \int_{\Omega_0}\det(\mathrm{Id}+D\xi) &= \int_{\Omega_0}\left(1+\mathrm{Tr}(D\xi)+O(\|D\xi\|^2_{L^\infty})\right) \\ &= |\Omega_0| + \int_{\Omega_0}\mathrm{div}(\xi) + O(\|\xi\|^2_{W^{1,\infty}}) \end{aligned}$$

hence the result as $\mathrm{Tr}(D\xi) = \mathrm{div}(\xi)$. □

Remark 3.6 More generally, if one considers $I_f : \Omega \mapsto \int_\Omega f$, we have to study

$$\theta \mapsto \int_{\Omega_0} f\circ(\mathrm{Id}+\theta)\det(\mathrm{Id}+D\theta)dx$$

which requires the differentiability of $\theta \in W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d) \mapsto f\circ(\mathrm{Id}+\theta) \in L^1$.

If $f \in W^{1,1}(\mathbb{R}^d)$ then this function is indeed C^1 in a neighborhood of 0 (see [53, Lemma 5.3.3]), and therefore I_f has a shape derivatives at Ω_0. Let us formally [21] make the computation :

$$f\circ(\mathrm{Id}+\xi)(x) = f(x) + \nabla f(x)\cdot\xi(x) + o(\xi(x)) \tag{17}$$

leads to

$$\begin{aligned} \int_{(\mathrm{Id}+\xi)(\Omega_0)} f &= \int_{\Omega_0}(f+\nabla f\cdot\xi)(1+\mathrm{Tr}(D\xi)) + o(\|\xi\|_{W^{1,\infty}}) \\ &= \int_{\Omega_0} f + \int_{\Omega_0}(\nabla f\cdot\xi + \mathrm{div}(\xi)f) + o(\|\xi\|_{W^{1,\infty}}) \\ &= \int_{\Omega_0} f + \int_{\Omega_0}\mathrm{div}(f\xi) + o(\|\xi\|_{W^{1,\infty}}). \end{aligned}$$

So we obtain

$$\forall \xi \in W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d),\ \ I_f'(\Omega_0)(\xi) = \int_{\Omega_0}\mathrm{div}(f\xi)dx$$

and again if Ω_0 is C^1 then one has

$$\forall \xi \in W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d),\ \ I_f'(\Omega_0)(\xi) = \int_{\partial\Omega_0} f(\xi\cdot\nu_{\partial\Omega_0})d\sigma$$

Perimeter : in a similar way from the case of the volume, we have :

Proposition 3.7 *Let Ω_0 a set of finite perimeter. Then*

$$\theta \in W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d) \mapsto P((Id+\theta)(\Omega_0))$$

is C^∞ in a neighborhood of 0. Moreover

$$\forall \xi \in W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d),\ \ P'(\Omega_0)(\xi) = \int_{\Gamma_0}\mathrm{div}_{\Gamma_0}(\xi)d\sigma$$

where $\Gamma_0 = \partial^*\Omega_0$ is the reduced boundary and div_{Γ_0} is the tangential divergence (defined in (18)).

If moreover Ω_0 is smooth enough ($C^{1,1}$) then

$$\forall \xi \in W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d),\ \ P'(\Omega_0)(\xi) = \int_{\Gamma_0}\mathcal{H}_{\Gamma_0}(\xi\cdot\nu_{\Gamma_0})d\sigma$$

where $\mathcal{H}_{\Gamma_0}$ denotes the mean curvature [22] over Γ_0

21. This is formal first because we wrote an expansion of f in the classical sense (meaning we assumed f to be C^1) but more importantly because $o(\xi(x))$ in (17) depends on x so we cannot integrate it without more justification. See [53, Lemma 5.3.3] for a proper proof.
22. We define the mean curvature as the sum of principal curvatures.

Sketch of proof : The proof is very similar than for the volume, except that we use two formulas from differential geometry (we assume here some regularity for Ω_0 in order to simplify the presentation) :

1. a change of variable between two manifolds, allowing to write
$$\int_{\partial(\mathrm{Id}+\theta)(\Omega_0)} 1d\sigma = \int_{\partial\Omega_0} 1\mathrm{Jac}_{\partial\Omega_0}(\mathrm{Id}+\theta)d\sigma$$
where $\mathrm{Jac}_{\partial\Omega_0}(T) := \|{}^t(DT)^{-1}\cdot\nu_{\partial\Omega_0}\||\det(DT)|$ is called the tangential Jacobian. The expansion becomes [23] :
$$\begin{aligned}\mathrm{Jac}_{\partial\Omega_0}(\mathrm{Id}+\xi) &= |{}^t(\mathrm{Id}+D\theta)^{-1}\cdot\nu_{\partial\Omega_0}|\det(\mathrm{Id}+D\xi) = \|(\mathrm{Id}-{}^tD\xi)\cdot\nu_{\partial\Omega_0}\|(1+\mathrm{div}(\xi))+o(\|\xi\|_{W^{1,\infty}})\\ &= (1-({}^tD\xi\cdot\nu_{\partial\Omega_0})\cdot\nu_{\partial\Omega_0})(1+\mathrm{div}(\xi))+o(\|\xi\|_{W^{1,\infty}}) = 1+(\mathrm{div}(\xi)-(D\xi\cdot\nu_{\partial\Omega_0})\cdot\nu_{\partial\Omega_0})+o(\|\xi\|_{W^{1,\infty}})\end{aligned}$$
so we are naturally led to define
$$\mathrm{div}_{\partial\Omega_0}(\xi) = \mathrm{div}(\xi)-(D\xi\cdot\nu_{\partial\Omega_0})\cdot\nu_{\partial\Omega_0} \tag{18}$$
2. To obtain the last formula in Proposition 3.7, we use an integration by part formula on M a $(d-1)$-submanifold of $\mathbb{R}^d$ with boundary ∂M :
$$\int_M \mathrm{div}_M(\theta)d\sigma = \int_{\partial M}\theta\cdot n_{\partial M} + \int_M \mathcal{H}_M\theta\cdot\nu_M$$
where n_M is the outward unit co-normal of ∂M, ν_M is the normal vector to M and $\mathcal{H}_M$ is the mean curvature of M. In our case $M=\partial\Omega_0$ and $\partial M=\emptyset$.

□

PDE Example : Let now focus on our example of the Dirichlet energy E_f.

Proposition 3.8 *Let Ω_0 be an open set and $k\in\mathbb{N}^*$. If $f\in H^k(\mathbb{R}^d)$ then $\theta\in W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)\mapsto E_f((Id+\theta)(\Omega_0))$ is C^k in a neighborhood of 0, and*
$$\forall\xi\in W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d),\quad E_f'(\Omega_0)(\xi) = -\frac{1}{2}\int_{\Omega_0} fu'$$
where u' is the unique solution to $\begin{cases} -\Delta u' = 0 \ \text{in } \Omega_0\\ u'+\nabla u_{\Omega_0}\cdot\nu_{\partial\Omega_0}\in H_0^1(\Omega)\end{cases}$

with u_Ω solution to (11). If moreover Ω_0 is smooth enough (Lipschitz) we have
$$\forall\xi\in W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d),\quad E_f'(\Omega_0)(\xi) = -\frac{1}{2}\int_{\partial\Omega_0}(\partial_\nu u_{\Omega_0})^2(\xi\cdot\nu_{\partial\Omega_0})d\sigma.$$

Remark 3.9 In a very similar way, we can compute the first order shape derivative of λ_1 and obtain, if Ω_0 is Lipschitz :
$$\forall\xi\in W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d),\quad \lambda_1'(\Omega_0)(\xi) = -\int_{\partial\Omega_0}(\partial_\nu u_{\Omega_0})^2(\xi\cdot\nu_{\partial\Omega_0})d\sigma.$$
where u_Ω is the first Dirichlet eigenfunction of Ω_0.

Sketch of computation : We start with an instructive but formal computation of the derivative : in order to simply the presentation, let us fix $\xi\in W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)$ and compute a directional derivative. For $t\in\mathbb{R}$ small we denote $\Omega_t=(\mathrm{Id}+t\xi)(\Omega)$ and $u_t=u_{\Omega_t}$. We will differentiate with respect to t (without justifying any existence of derivatives just yet, see below), and we denote u' the derivative at 0 of $t\mapsto u_t$. There are 3 steps :

1. the PDE : we have for every t,
$$-\Delta u_t = f \text{ in } \Omega_t.$$
At least formally, we see that differentiating with t and then taking $t=0$ we get
$$-\Delta u' = 0 \text{ in } \Omega_0.$$

23. We use the classical expansions :
$$(\mathrm{Id}+A)^{-1} = \mathrm{Id}-A+o(A),\qquad \|a+b\| = \|a\|+\frac{a\cdot b}{\|b\|}+o(\|b\|)$$
and also $({}^tA\cdot x)\cdot x = (A\cdot x)\cdot x$.

2. the boundary condition for u : we have

$$\forall t \text{ small}, \forall x \in \partial\Omega_0, \;\; u_t(x + t\xi(x)) = 0$$

which leads to, by differentiation with respect to t to

$$\forall x \in \partial\Omega_0, \;\; u'(x) + \nabla u(x) \cdot \xi(x) = 0.$$

Knowing that $\nabla u = (\partial_\nu u)\nu$ on $\partial\Omega_0$, this can be rewritten $u' = -(\partial_\nu u)(\xi \cdot \nu) = 0$ on $\partial\Omega_0$.

3. Finally, we want to differentiate the energy : from the PDE it is classical that we have the following simplification :

$$E_f(\Omega_t) = \frac{1}{2}\int_{\Omega_t} |\nabla u_t|^2 - \int_{\Omega_t} f u_t = -\frac{1}{2}\int_{\Omega_t} f u_t.$$

We see that contrary to the case of the volume, we want to differentiate an integral over a moving set of a function also depending on t. This is the purpose of the famous Hadamard formula that we can retrieve formally here (see [53, Th 5.2.2] for a proof) : given $t \mapsto g_t$ also assumed to be differentiable, we have

$$\begin{aligned} \int_{\Omega_t} g_t dx &= \int_{\Omega_0} g_t \circ (\mathrm{Id} + t\xi)\det(\mathrm{Id} + t\xi) = \int_{\Omega_0} (g_0 + t g_0')(x + t\xi(x))(1 + t\mathrm{div}(\xi))dx + o(t) \\ &= \int_{\Omega_0} g_0 + t\int_{\Omega_0} \Big[(\nabla g_0 \cdot \xi + g_0') + g_0 \mathrm{div}(\xi)\Big]dx + o(t) = \int_{\Omega} g_0 + t\int_{\Omega_0} \Big[g_0' + \mathrm{div}(g_0\xi)\Big]dx + o(t) \end{aligned}$$

So applying this formula with $g_t = f u_t$ we obtain

$$E_f'(\Omega)(\xi) = -\frac{1}{2}\int_{\Omega_0} \Big[f u' + \mathrm{div}(f u \xi)\Big]dx.$$

Assuming we can integrate by part, we obtain

$$\int_{\Omega_0} \mathrm{div}(f u\xi)dx = \int_{\partial\Omega_0} f u \xi d\sigma = 0$$

because of the boundary condition. This leads to the first expected formula. Now, using the previous the PDE of u and u' and the boundary condition for u', we obtain the second formula :

$$-\frac{1}{2}\int_{\Omega_0} f u' = \frac{1}{2}\int_{\Omega_0} (\Delta u)u' = \frac{1}{2}\int_{\Omega_0} u(\underbrace{\Delta u'}_{=0}) + \frac{1}{2}\int_{\partial\Omega_0} (\partial_\nu u)u' - (\partial_\nu u')u d\sigma = -\frac{1}{2}\int_{\partial\Omega_0} (\partial_\nu u)^2 \xi \cdot \nu_{\partial\Omega_0}.$$

Sketch of proof : We now give some ideas allowing to justifiy the previous computations.

1. First, we want to prove that E_f is shape differentiable (or even C^∞). There several methos : in [27, Chapter 9] for example they use the fact that E_f is itself a minimum, and in [56, Section 2.28] they show with the definition the regularity of $t \mapsto u_t$. As we are interested in higher regularity, we directly focus on another interesting method using the implicit function's Theorem.
First, we go back to Fréchet differential, and for $\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ small, we denote $\Omega_\theta = (\mathrm{Id} + \theta)(\Omega_0)$ and $u_\theta = u_{\Omega_\theta}$. Notice that the function $\theta \mapsto u_\theta$ is not so convenient (the function u_θ is not defined on a fixed set) and it does not usually have good differentiability properties (see [53, Section 5.3.3]). So it is more convenient to consider $\theta \mapsto \widehat{u}_\theta := u_\theta \circ (\mathrm{Id} + \theta) \in H_0^1(\Omega)$. Indeed we have

$$\forall \varphi \in H_0^1(\Omega_\theta), \;\; \int_{\Omega_\theta} \nabla u_\theta \cdot \nabla \varphi = \int_{\Omega_\theta} f u_\theta,$$

and we would like to formulate this equation on the fixed domain Ω_0. If $\varphi \in H_0^1(\Omega)$ then $\varphi \circ (\mathrm{Id} + \theta)^{-1} \in H_0^1(\Omega_\theta)$ so we can use as a test function in the previous equation, and after a change of variable, it leads to :

$$\forall \varphi \in H_0^1(\Omega_0), \;\; \int_{\Omega} \Big[(\nabla u_\theta) \cdot \nabla(\varphi \circ (\mathrm{Id} + \theta)^{-1})\Big] \circ (\mathrm{Id} + \theta)\det(\mathrm{Id} + D\theta) = \int_{\Omega_0} f \circ (\mathrm{Id} + \theta)\varphi \det(\mathrm{Id} + D\theta)$$

which leads to

$$\forall \varphi \in H_0^1(\Omega_0), \int_{\Omega} A(\theta)\nabla \widehat{u}_\theta \cdot \nabla \varphi = \int_{\Omega_0} f \circ (\mathrm{Id} + \theta)\varphi \det(\mathrm{Id} + D\theta)$$

where

$$A(\theta) = (\mathrm{Id} + (D\theta)^{-1})(\mathrm{Id} + {}^t(D\theta)^{-1})\det(\mathrm{Id} + D\theta).$$

So we have to consider

$$\begin{array}{cccc} F: & W^{1,\infty} \times H_0^1(\Omega_0) & \longrightarrow & H^{-1}(\Omega) \\ & (\theta, \varphi) & \longmapsto & -\mathrm{div}(A(\theta)\nabla\varphi - f \circ (\mathrm{Id} + \theta)\det(\mathrm{Id} + D\theta) \end{array}$$

and apply the implicit theorem to F near $F(0, u_{\Omega_0}) = 0$, see [53, Theorem 5.3.2] to show that the map $\theta \in W^{1,\infty} \longmapsto \widehat{u}_\theta \in H_0^1(\Omega)$ is C^k near 0 if $f \in H^k(\mathbb{R}^d)$.

2. With the previous differentiability, we just notice that

$$E_f(\Omega_\theta) = -\frac{1}{2}\int_{\Omega_0} f \circ (\mathrm{Id} + \theta)\widehat{u}_\theta \det(\mathrm{Id} + D\theta)$$

and compute the derivative from there.

3.3 Structure Theorem and optimality conditions (first order)

In the examples of the previous sections, we noticed, at least when Ω_0 is smooth, that the first order shape derivatives can be written

$$\int_{\partial\Omega_0} g_{\partial\Omega_0}(\xi \cdot \nu_{\partial\Omega_0}) d\sigma.$$

While a priori $J'(\Omega_0)$ is a continuous linear form on $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, we noticed that it depends only on $\xi_{|\partial\Omega_0} \cdot \nu_{\partial\Omega_0}$. This fact is actually valid for any shape functional :

Theorem 3.10 *Assume Ω_0 is C^2 and $\theta \mapsto J((Id + \theta)(\Omega_0))$ is differentiable at 0. Then there exists $\ell_1 : C^1(\partial\Omega) \to \mathbb{R}$ a continuous linear form such that*

$$\forall \xi \in C^1 \cap W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d), \ \ J'(\Omega)(\xi) = \ell_1(\xi_{|\partial\Omega_0} \cdot \nu_{\partial\Omega_0}).$$

This result helps to take into account the invariances in $\theta \mapsto J((\mathrm{Id} + \theta)(\Omega))$:

— if ξ moves only points inside or outside Ω_0, then for every t small one has $J((\mathrm{Id} + t\xi)(\Omega_0)) = J(\Omega_0)$ so by differentiating with respect to t one gets $J'(\Omega_0)(\xi) = 0$,

— more generally, if ξ is tangent on $\partial\Omega_0$, then it "turns" Ω_0 and at least at first order, it doesn't move the shape Ω_0, which explains why $J'(\Omega_0)(\xi) = 0$.

Remark 3.11 — The linear form ℓ_1 (or alternatively $g_{\partial\Omega_0}$ when it exists) are sometimes called the shape gradient of J at Ω_0.

— See [42] for a version of this result when Ω_0 is only assumed to be of finite perimeter.

Sketch of proof : The proof is made of two steps :

1. Let ξ such that $\xi \cdot \nu_{\partial\Omega_0} = 0$ on $\partial\Omega_0$. We introduce the flow $(\gamma_t)_{t\in\mathbb{R}}$ of ξ (which is well-defined as ξ is globally lipschiz), i.e. such that

$$\begin{cases} \forall (t,x), \ \ \frac{\partial}{\partial t}\gamma_t(x) = \xi(\gamma_t(x)) \\ \gamma_0(x) = x \end{cases}.$$

Then by usual invariance result of sets by an ODE (Nagumo's type condition), we have $\gamma_t(\Omega) = \Omega$ for all t. As a consequence

$$\forall t \in \mathbb{R}, \ \ \frac{d}{dt} J(\gamma_t(\Omega)) = 0 (= \mathcal{J}_{\Omega_0}(\gamma_t - \mathrm{Id}))$$

which leads to

$$\forall t \in \mathbb{R}, \ \ \mathcal{J}_{\Omega_0}(\gamma_t - \mathrm{Id}) \left(\frac{\partial}{\partial t}\gamma_t\right) = 0$$

and the value at $t = 0$ indeed gives

$$J'(\Omega_0)(\xi) = 0.$$

2. The second step is more of an algebraic step : we introduce

$$\begin{array}{rccc} \Phi : & C^1 \cap W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) & \longrightarrow & C^1(\partial\Omega) \\ & \xi & \longmapsto & \xi_{|\partial\Omega_0} \cdot \nu_{\partial\Omega_0} \end{array} \tag{19}$$

In the previous step, we proved that $J'(\Omega_0)$ vanishes on $\mathrm{Ker}(\Phi)$, so we can use the universal property of quotient spaces both to $J'(\Omega_0)$ and to Φ. In the case of Φ, this builds a isomorphism between $C^1 \cap W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)/\mathrm{Ker}(\Phi)$ and $\mathrm{Im}(\Phi)$ and it remains to show that $\mathrm{Im}(\Phi) = C^1(\partial\Omega)$, which is a consequence of the regularity assumptions and an extension procedure.

□

Optimality conditions :

Proposition 3.12 *Let D be an open set in $\mathbb{R}^d$.*

1. *Let Ω^* a solution of*

$$\min\left\{P(\Omega), \ \ |\Omega| = m, \ \Omega \subset D\right\}$$

and assume $\partial\Omega^* \cap D$ is of class C^2. Then there exists $\lambda \in \mathbb{R}$ such that

$$\mathcal{H}_{\partial\Omega_0} = \lambda \text{ on } \partial\Omega^* \cap D$$

where $\mathcal{H}_{\partial\Omega_0}$ is the mean curvature of $\partial\Omega^*$.

2. Let Ω^* a solution of

$$\min\left\{\lambda_1(\Omega),\quad |\Omega| = m,\ \Omega\subset D\right\}$$

and assume $\partial\Omega^*\cap D$ is of class C^2. Then there exists $\lambda\in\mathbb{R}$ such that

$$|\nabla u_{\Omega^*}|^2 = \lambda \text{ on } \partial\Omega^*\cap D$$

where u_{Ω^*} is the first Dirichlet eigenfunction of Ω^*.

This is an easy consequence of the Lagrange multiplier Theorem combined with the computations from Proposition 3.7 and Remark 3.9 (and the fact that every function in $C^1(\partial)$ can be written as $\xi\cdot\nu_{\partial\Omega^*}$ for some $\xi\in C^1\cap W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)$ (surjectivity of Φ defined in (19)).

Remark 3.13 — In the field of Riemannian geometry, people are therefore looking for "CMC surfaces" for "constant mean curvature" surfaces.

— In the case of λ_1, this gives a relation between the Faber-Krahn Theorem 1.2 and a result by Serrin asserting that if Ω is a smooth open set and $u\in C^2(\overline{\Omega})$ is such that $u>0$ in Ω and

$$\begin{cases} -\Delta u &=& \lambda u & \text{in } \Omega\\ u &=& 0 & \text{on } \partial\Omega\\ \partial_\nu u &=& c & \text{on } \partial\Omega\end{cases}$$

where $c\in\mathbb{R}$, then Ω must be a ball. Actually, if we know that there exists a smooth minimizer for (4) (which is not an easy task...), then we can apply Serrin's result to prove Faber-Krahn's inequality [24].

3.4 Remarks on second order shape derivatives

Structure Theorem (second order) : There is a structure result about second order shape derivatives, see [51, 24]. The results says that if Ω_0 is C^3 and J admits a second order shape derivative, then there exists ℓ_2 a continuous bilinear form on $C^2(\partial\Omega_0)$ such that

$$J''(\Omega_0)(\xi,\xi) = \ell_2(\xi\cdot\nu,\xi\cdot\nu) + \ell_1(Z_\xi) \tag{20}$$

where $Z_\xi = B(\xi_\tau,\xi_\tau) - 2\nabla_\tau(\xi\cdot\nu)\cdot\xi_\tau$ with $B = D_\tau\nu$ the second fondamental form of $\partial\Omega$, ∇_τ denoting the tangential gradient [25] on $\partial\Omega_0$ and $\xi_\tau = \xi - (\xi\cdot\nu)\nu$ the tangential part of ξ, and ℓ_1 comes from Theorem 3.10.

This has important consequences :

— it is enough to compute $J''(\Omega_0)$ for normal deformations, and (20) result will allow to retrieve the general formula,

— if we restrict to normal deformations $\xi = \varphi\nu$ (smoothly extended to $\mathbb{R}^d$), then $J''(\Omega_0)$ turns out to be a bilinear form in φ.

Example 3.14 (see for example [24] for more details) Here are the second order shape derivative of volume and perimeter : given Ω_0 an smooth set in $\mathbb{R}^d$, thanks to the structure Theorem, it is enough to give the formula assuming $\xi = \varphi\nu$ with $\varphi\in C^\infty(\partial\Omega_0)$ (more precisely, any smooth extension of $\varphi\nu$ to $\mathbb{R}^d$) :

$$\mathrm{Vol}''(\Omega_0)(\xi,\xi) = \int_{\partial\Omega_0}\mathcal{H}\varphi^2,\quad P''(\Omega_0)(\xi,\xi) = \int_{\partial\Omega_0}|\nabla_\tau\varphi^2| + \left[\mathcal{H}^2 - \|B\|^2\right]\varphi^2$$

where $\mathcal{H}$ is the curvature and $\|B\|^2$ is the sum of the squares of the principal curvatures of $\partial\Omega_0$.

Assume now that $\Omega_0 = B_1$ is the ball of radius 1 : then

$$\mathrm{Vol}''(B_1)(\xi,\xi) = (d-1)\int_{\partial B_1}\varphi^2,\quad P''(B_1)(\xi,\xi) = \int_{\partial B_1}|\nabla_\tau\varphi^2| + (d-1)(d-2)\varphi^2.$$

24. This is definitely not the fastest proof of Faber-Krahn's inequality ! Also, as far as I know this method only works in dimension $d\le 5$ using [43].

25. The tangential gradient is $\nabla_\tau g = \nabla g - (\nabla g\cdot\nu)\nu$.

As a "test" case for understanding these tools, let us check that the ball satisfies the first and second order optimality condition for the isoperimetric problem. In order to have the solution as the ball of radius 1, we fix the volume constraint as ω_d the volume of the unit ball. First, we have the first order optimality condition

$$P'(B_1)(\xi) = \int_{\partial B_1} (d-1)\xi \cdot \nu = (d-1)\mathrm{Vol}'(B_1)(\xi).$$

The number $(d-1)$ is the Lagrange multiplier from Proposition 3.12. As a consequence, the second order optimality condition must be

$$\forall \xi \text{ such that } \mathrm{Vol}'(B_1)(\xi) = 0, \quad (P - (d-1)\mathrm{Vol})''(B_1)(\xi, \xi) \geq 0.$$

Because of the first order optimality condition, the formula for $(P - (d-1)\mathrm{Vol})''(B_1)$ only requires the term ℓ_2 (associated to $P - (d-1)\mathrm{Vol}$) from (20). So this time, even though we don't assume ξ to be normal on the boundary, the second order optimality condition only depends on $\varphi = \xi \cdot \nu$. So this condition can be rewritten :

$$\forall \varphi \in C^1(\partial B_1) \text{ such that } \int_{\partial B_1} \varphi = 0, \quad \int_{\partial B_1} |\nabla_\tau \varphi|^2 - (d-1)\int_{\partial B_1} \varphi^2 \geq 0.$$

This inequality is indeed valid and refers to the fact that

$$\lambda_1(\Delta_\tau, \mathbb{S}^{d-1}) = d-1.$$

We finally notice an interesting fact : these quadratic forms can be given in a diagonalized form using the spherical harmonics $(Y^{k,l})_{k\in\mathbb{N}, 1\leq l\leq d_k}$ which is an orthonormal basis of $L^2(\partial B_1)$. In this basis, we can write

$$\varphi = \sum_{k,l} \alpha_{k,l}(\varphi) Y^{k,l} \text{ on } \partial B_1$$

and in these coordinates it can be proven that :

$$\mathrm{Vol}''(B_1)(\xi,\xi) = (d-1)\sum_{k,l}\alpha_{k,l}(\varphi)^2, \quad P''(B_1)(\xi,\xi) = \sum_{k,l}\Big[k^2 + (d-2)k + (d-1)(d-2)\Big]\alpha_{k,l}(\varphi)^2.$$

We can check again the nonnegativity of the second order derivative in a more "concrete" way : :

1. the condition $\mathrm{Vol}'(B_1)(\xi) = 0$ only means $\int_{\partial B_1} \varphi = 0$, i.e. $\alpha_{0,0}(\varphi) = 0$.
2. we then observe, $\forall \xi$ s.t. $\int_{\partial B_1} \varphi = 0$:

$$(P - (d-1)\mathrm{Vol})''(B_1)(\xi,\xi) = \sum_{k\geq 1, l}\Big[\underbrace{k^2 + (d-2)k + (d-1)(d-2) - (d-1)^2}_{=(k-1)(k+d-1)}\Big]\alpha_{k,l}(\varphi)^2, \qquad (21)$$

which is indeed nonnegative.

4 Regularity Theory (just a glimpse)

As we have seen in Section 2, while studying a problem like

$$\min\Big\{J(\Omega), \quad |\Omega| = m\Big\}$$

(without strong geometrical constraints), the strategy to prove existence requires to use a large class of domains (sets of finite perimeter, measurable sets, quasi-open sets), so we obtain *a priori* the existence of a domain with very poor *a priori* regularity. Nevertheless, in many situations, we expect the optimal shape to be rather "nice/smooth".

In order to be convinced by this last assertion, it is usually not so hard to check that assuming that Ω^* is "a bit" smooth (say Lipschitz or a bit more), one can in fact prove that Ω^* is in fact C^∞ or even analytic. This step is available because one can write and use the Euler-Lagrange equation of the problem by using the tools from Section 3.

As a consequence, the main difficulty is in fact to obtain "a bit" of regularity "from scratch" (i.e. only knowing that Ω is measurable or quasi-open). This step is particularly difficult, especially because one can not easily write optimality conditions, and we need to use optimality in a more involved way to prove regularity.

4.1 Mild regularity implies lots of regularity

1. For a solution Ω^* of an isoperimetric problem like 7, assume that $\partial\Omega^*$ is locally inside D the graph of a $C^{1,\alpha}$-function u defined on a smooth set $\omega \subset \mathbb{R}^{d-1}$. Then it is possible to prove (∇' denotes the gradient in $\mathbb{R}^{d-1}$) :
$$\forall \varphi \in C_c^1(\omega), \quad \int_\omega \frac{\nabla' u}{\sqrt{1+|\nabla' u|^2}} \cdot \nabla'\varphi dx = \lambda \int_\omega \varphi.$$
This condition is just a weak version of the optimality condition $\mathcal{H}_{\partial\Omega_0} = \lambda$ from Proposition 3.12. It can be rewritten (in a weak sense) as
$$-\mathrm{div}\left(\frac{\nabla' u}{\sqrt{1+|\nabla' u|^2}}\right) = \lambda \text{ in } \omega.$$
This equation is naturally called a mean curvature equation and falls into the framework of nonlinear elliptic PDE, and classical results shows that u must be analytic. See for example [44, Chapter 27] for more details.
2. Similarly if Ω^* is solution to a Faber-Krahn type problem like (14) and if we assume that locally $\omega \subset \partial\Omega^* \cap D$ is the graph of a $C^{1,\alpha}$-function (and as usual we also require Ω^* to lie on one side of the graph), then u_{Ω^*} is $C^{1,\alpha}$ up to this part of the boundary and we can write
$$|\nabla u_\Omega| = c \text{ on } \omega$$
and from the Hopf principle, we can even say that c is strictly positive. The regularity for this type of overdetermined PDE has been studied in [37] where it is shown that again ω is then analytic.

4.2 Quasi-minimizers of the perimeter

The question of regularity for solutions of isoperimetric problems has been a topic of research for most of the second half of the 20$^{\text{th}}$ century, and is still active nowadays. We refer to [44] for an extensive study. Even nowadays there some new developments on Plateau problems or generalized isoperimetric problems.

A flexible notion has emerged from this work :

Definition 4.1 *Let D be an open set and Ω^* be a set of finite perimeter in $\mathbb{R}^d$. We say that Ω^* is a local quasi-minimizer of the perimeter in D if there exists $C \in \mathbb{R}$, $\alpha \in (d-1, d]$ and $r_0 > 0$ such that for every ball $B_r(x) \subset \mathbb{R}^d$ we have*
$$P(\Omega^*) \leq P(\Omega) + Cr^\alpha, \qquad \textit{for every } \Omega \textit{ such that } \Omega\Delta\Omega^* \subset B_r \cap D.$$

Remark 4.2 — There are many variants of this notion (and also several names : sometimes the word "quasi" is replaced by "almost"). In [44], the author localizes the definition (which is more general) and replaces the rest Cr^α by the term $C|\Omega\Delta\Omega'|$ (which is a bit weaker, but applies to many interesting examples).

— It is important to notice that the term r^α is a order strictly less that r^{d-1}, which is the expected order of the perimeter term. So it is natural to hope that a solution of a isoperimetric-like problem is a quasi-minimizer.

The following result is a very deep and difficult result :

Theorem 4.3 ([58]) *Suppose that Ω^* is a local quasi-minimizer of the perimeter in D for some $\alpha \in (d-1, d]$. Then*

1. *the set $\partial^*\Omega^* \cap D$ is locally the graph of a $C^{1,\frac{\alpha-d+1}{2}}$-function, where $\partial^*\Omega^*$ is the reduced boundary introduced in Remark 3.5,*
2. *the singular set $(\partial\Omega \setminus \partial^*\Omega) \cap D$ has dimension at most $d-8$ (and is therefore empty if $d \leq 7$.*

Remark 4.4 — About the Hölder exponent $\frac{\alpha-d+1}{2}$: it makes sense that the higher α is, the better the result is. It is worth noticing though that the best regularity obtained here is $C^{1,\frac{1}{2}}$ when $\alpha = d$.

— Note that we have to precise what is $\partial\Omega$ in the previous statement : indeed this result deals with sets defined almost everywhere while $\partial\Omega$ depends on the representative a.e.. It is in fact classical that if Ω is measurable, then one can choose a representative of Ω so that $\partial\Omega$ is minimal, and for this representative, we have

$$\partial\Omega = \{x \in \mathbb{R}^d, \ \ \forall r > 0, \ \ 0 < |\Omega \cap B_r(x)| < |B_r(x)|\}$$

see [44, Prop 12.19]. Also, when Ω is of finite perimeter, we also have, for this representative, $\partial\Omega = \overline{\partial^*\Omega}$.

— This result is difficult to prove, and we are not going to try to sum up its proof. Nevertheless, it is worth mentioning why this estimate of the singular set cannot be improved in general : one step in the strategy is to study blow-ups near a point x_0 of the boundary, namely limits of

$$\frac{(\Omega - x_0)}{r_n}$$

when r_n is a sequence going to 0. It is proved that this must be a cone that minimizes the perimeter in $\mathbb{R}^d$ (note that the volume constraint "disappeared" at the limit [26]. It has been proved by Simons that such cone must be a halfplane if $d \geq 7$ (which in turn implies the regularity of $\partial\Omega^*$ near x_0). However, in $\mathbb{R}^8$, the following cone

$$\Big\{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4, \ \ |x| < |y|\Big\}$$

is a singular minimizing cone in $\mathbb{R}^8$.

Corollary 4.5 *Let D be an open set in $\mathbb{R}^d$ and $m \in (0, |D|)$. Then the solutions of*

$$\min\Big\{P(\Omega), \ \ \Omega \subset D, \ \ |\Omega| = m\Big\}$$

are smooth in the sense of Theorem 4.3.

Idea of the proof : Let Ω^* a solution. We have to prove that Ω^* is a quasi-minimizer of the perimeter and then one can apply Theorem 4.3 : to that end we can prove that the volume constraint can (locally) be penalized. More precisely, it can be shown (see [44, Example 21.3] or [26, Lemma 4.5]) that there exists $\mu \geq 0$ and $r_0 > 0$ such that Ω^* solves, for every $x \in D$

$$\min\Big\{P(\Omega) + \mu\big||\Omega| - \mu\big|, \ \ \Omega \text{ such that } \Omega\Delta\Omega^* \subset B_{r_0}(x)\Big\}. \tag{22}$$

From this property is it easy to prove that Ω^* is a quasi-minimizer of the perimeter with $\alpha = d$. □

Remark 4.6 In [26] we show that similar regularity properties applies to the following problems that "mix" perimeter and PDE in their energy :

$$\min\Big\{P(\Omega) + E_f(\Omega), \ \ \Omega \subset D, \ \ |\Omega| = m\Big\}, \quad \min\Big\{P(\Omega) + F(\lambda_1(\Omega), \ldots, \lambda_k(\Omega)), \ \ \Omega \subset D, \ \ |\Omega| = m\Big\}$$

(where F is increasing in each variable, and Lipschitz). We also deal with the existence issue : it is interesting to notice that we use the existence tools from Section 2.2, while I mentioned that this look unadapted to the framework of PDE (which depends a priori on the almost-everywhere representative) : in fact we replace these PDE functionals by other ones that are adapted to this framework, prove existence in this other framework and because we prove regularity of optimal shapes in this new context we can conclude that our shapes are actually also solution to the original problem.

26. Since this cones are not bounded I should precise what it means to minimize the perimeter : a set Ω^* is called a perimeter minimizer in $\mathbb{R}^d$ if for all $D \Subset \mathbb{R}^d$ bounded and every Ω such that $\Omega\Delta\Omega' \Subset D$ we have $P(\Omega^*, D) \leq P(\Omega, D)$ (where $P(\Omega, D)$ is defined in Remark 2.16)

4.3 Remarks on the case of PDE

In a similar fashion as in the previous paragraph, we would like to study the regularity properties for

$$\min\Big\{E_f(\Omega),\ \ \Omega\subset D,\ \ |\Omega|=m\Big\},\qquad \min\Big\{\lambda_k(\Omega),\ \ \Omega\subset D,\ \ |\Omega|=m\Big\}$$

or more generally

$$\min\Big\{F(\lambda_1(\Omega),\dots,\lambda_k(\Omega)),\ \ \Omega\subset D,\ \ |\Omega|=m\Big\}.$$

For the problem with E_f, this is due to Briançon [8], and it has been adapted to λ_1 by Briançon and the author in [9] : this relied heavily on the work by Alt and Caffarelli [2] who studied regularity of free boundary problems like the one we encountered in Theorem 2.24. Then further improvment about the regularity have been obtained for these model functionals, see [54, 43].

For the last improvments, see [48, 38, 49].

5 Sketch of proof of Theorem 1.3

As planned, let us check the steps of one of the proofs of Theorem 1.3 using most of the tools from this class. The general strategy we will refer to is due to Cicalese-Leonardi in [21] though we will follow a slightly simplified version from [1]. Note that this general strategy to obtain a quantitative isoperimetric inequality has then been applied to various examples, some which couldn't be achieved by any other strategy so far : see in particular [25, 7, 6, 28, 34, 33, 45, 46] (this list is not exhaustive!). See also the review paper [32] we heavily use for the following lines.

To fix the ideas, B denote the unit ball.

1. First, as a preliminary step, we notice that it is easier to get rid of the measure constraint : in the spirit of (22) (in the proof of Corollary 4.5) one can prove [27] that if $\mu>d$, B is solution to

$$\min\Big\{P(\Omega)+\mu\Big||\Omega|-|B|\Big|,\ \ \Omega\subset\mathbb{R}^d\Big\}.$$

   For the rest of the proof we fix μ a constant stricly larger than d.

2. A second preliminary step is actually due to Fuglede in [30] : he shows that there is a quantitative inequality for domains that are nearly spherical (see Example 3.3). The framework we introduced in Section 3.4 allows to provide a different way to approach the computations by Fuglede, even though in spirit the difficulties are similar : indeed from (21), we notice that

$$\forall\xi \text{ such that } \int_{\partial B}\varphi=0 \text{ and } \int_{\partial B}x\varphi(x)dx=0,$$
$$(P-(d-1)\mathrm{Vol})''(B_1)(\xi,\xi)=\sum_{k\geq 2,l}\Big[(k-1)(k+d-1)\Big]\alpha_{k,l}(\varphi)^2\geq c\|\varphi\|^2_{H^1(\partial B)},$$

   for some $c>0$. In fact, it is natural that we have to assume $\int_{\partial B}x\varphi(x)dx=0$ because the functional is invariant by translation. With this, we can hope that a Taylor expansion leads to

$$\forall u\in W^{1,\infty}(\partial B) \text{ such that } \|u\|_{W^{1,\infty}(\partial B)}<\varepsilon,\quad P(B_u)-P(B)\geq\frac{c}{4}\|u\|^2_{H^1(\partial B)},\tag{23}$$

   where B_u is the nearly spherical sets defined in (16). This fact is actually not so easy as P is shape differentiable in the space $W^{1,\infty}$ while the coercivity norm is in a weaker H^1-norm. It is nevertheless possible to prove that the third order term in the Taylor expansion can still be controlled by the coercive term. See [24] for more details about this step, especially with other shape functionals.

3. We are now in position to start the actual proof of (5) : we argue by contradiction and assume (5) is not true. So there exists $\delta_0>0$, (Ω_n) such that (with $c>0$ coming from (23)

$$\forall n\in\mathbb{N}, |\Omega_n|=|B|,\quad P(\Omega_n)-P(B)\leq\frac{c}{8}d_{L^1}(\Omega_n,B)^2,\ \text{and}\quad P(\Omega_n)\xrightarrow[n\to\infty]{}P(B).\tag{24}$$

27. We should notice here that since this property is now global, this is much easier to prove than (22) in the proof of Corollary 4.5.

(a) First, one can prove that Ω_n can be modified so that we have the extra property that the sequence (Ω_n) remains in B_{R_0} for some R_0. This can be done by s surgery argument removing the pieces of (Ω_n) "going to infinity" (see [32, Lemma 4.2]).

(b) Second, because of compactness and continuity properties (Proposition 2.13 and 2.14), it is easy to check that Ω_n converges to B in L^1.

(c) Let us introduce $\widetilde{\Omega_n}$ solution of

$$\min\left\{P(\Omega)+\left|d_{L^1}(\Omega,B)-d_{L^1}(\Omega_n,B)\right|+\mu\Big||\Omega|-|B|\Big|,\Omega\subset\Omega_{R_0}\right\}.$$

It is easy to prove that these sets exists, again using the tools from Section 2.2.

This is the regularizing procedure of the sequence (Ω_n), called "selection principle" in [21] : the point is to replace Ω_n by the sequence $(\widetilde{\Omega_n})$ satisfying again properties like (24) but being smoother.

(d) Indeed, with the first preliminary step one can see that $(\widetilde{\Omega_n})$ converges again to the ball (in L^1), but this time they are also quasi-minimizer of the perimeter with the same constants (C,α) from Definition 4.1. Further developments of the regularity theory of quasi-minimizers ([32, Theorem 5.2]) shows that in this situation, (Ω_n) must be $C^{1,\alpha}$ perturbation of the ball, that is to say, nearly spherical sets.

But this contradicts (23) and concludes the proof.

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