The behaviour of the differential Galois group on the generic and special fibres: A Tannakian approach

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Abstract

Let $\mathfrak{o}$ be a complete DVR of fraction field K and algebraically closed residue field k. Let A be an $\mathfrak{o}$-adic domain which is smooth and topologically of finite type. Let $\mathscr{D}$ be the ring of $\mathfrak{o}$-linear differential operators over A and let $\mathscr{M}$ be a $\mathscr{D}$-module which is finitely generated as A-module. Given an $\mathfrak{o}$-point of $\mathrm{Spf}(A)$ we construct using a Tannakian theory of Bruguières-Nori, a faithfully flat $\mathfrak{o}$-group-scheme Π which is analogous–in the sense that its category of dualizable representations is equivalent to a category of $\mathscr{D}$-modules–to the Tannakian group-scheme (the differential Galois or monodromy group) associated to a $\mathscr{D}$-module over a field. We show that the differential Galois group G of the reduced $\mathscr{D}$-module $\mathscr{M}\otimes k$ is a closed subgroup of $\Pi\otimes k$, which coincides with $(\Pi\otimes k)_{\mathrm{red}}$ when Π is finite, and gives back, in any case, the differential Galois group of $\mathscr{M}\otimes K$ upon tensorisation with K. [1]

1 Introduction

The goal of this work is to understand the phenomenon described in Conjecture 8.5 of [11] through the point of view of Tannakian theory. The latter theory is used not only as a method but as an orientation; one of our goals was to provide a Tannakian interpretation of the linear algebraic group envisaged by Matzat and van der Put and therefore be able to call it a differential Galois group.

Let us describe the problematic loosely in our terminology. Let $\mathfrak{o}$ be a complete DVR of mixed characteristic $(0,p)$ with residue field k and fractions field K. Let A be the ring of power series in one variable with coefficients in $\mathfrak{o}$ and convergence radius at least one. For a given differential module (M,∂)

[1] 1991 Mathematics Subject Classification: Primary 13N10, 14F10. Secondary 14L15, 20C99

over $A \otimes K$, one can assume that there exists a finite A-submodule $\mathscr{M}$ of M with $\mathscr{M} \otimes K = M$ which is invariant under the action of the operators

$$\partial_n := \frac{\partial^n}{n!}.$$

Call $\mathscr{M}$ an integral structure on M. Such $\mathscr{M}$ can be reduced modulo the maximal ideal of $\mathfrak{o}$ giving rise to a $\mathscr{D}_{\mathbb{A}^1_k/k}$-module M_0. One then asks:

how to relate the differential Galois groups of M and M_0?

The answer proposed by [11] is that (1) there should exist a linear algebraic group G over $\mathfrak{o}$ whose generic fibre is the differential Galois group of M and whose special fibre contains the differential Galois group of M_0 as a closed subgroup. Moreover, (2) if the differential Galois group of M were to be finite, the closed embedding would be an isomorphism *after a convenient change of the integral structure* $\mathscr{M}$ (a criticism of this second part is in the Example of section 6.2.3). The solution would have to go through the construction of Picard-Vessiot extensions of A.

In this work, we have chosen to concentrate on the highlighted question and we assume that the module $\mathscr{M}$ is fixed throughout. Our aspirations then were: (a) generalize the above problematic to other $\mathfrak{o}$-adic rings (higher dimensions); (b) use a Tannakian theorem of Bruguières-Nori [2] to construct the group G; (c) interpret the representations of G in terms of $\mathscr{D}$-modules over $\operatorname{Spf} A$; (d) give a different formulation, together with a proof, of part (2) of the conjecture. Note that it is (c) which allows us to call G a differential Galois group.

With the methods we use, (a) poses no problem at all and the generality is present from the beginning. To achieve (b) we shed some light on the tensor structure of the representations category of a faithfully flat group scheme over a PID (section 3). The theorem of Bruguières-Nori constructs a flat group-scheme starting from a certain $\mathfrak{o}$-linear monoidal category and in order to create a meaningful group-scheme we had to use the directions provided by section 3–see definitions 10, 21 and 23. Task (c) comes from a careful understanding of [2] and the construction in part (b) (see Corollary 7 and Theorem 24). To make the correct statement regarding finite differential Galois groups–goal (d)–we were inspired by Example 8.6 in [11] and the liberty that the Theorem of Bruguières-Nori provides. This liberty to which we are referring is the possibility to see the differential Galois groups as *affine and faithfully flat* $\mathfrak{o}$*-group-schemes* in place of linear algebraic groups (a term which is loosely used in [11]) and therefore detect the likelihood of a non-reduced special fibre (see the example at the end of section 6.2.3). The result which deals with (d) is Theorem 28.

Of course, since we are starting with a $\mathscr{D}$-module $\mathscr{M}$ over A, the angle here differs from that in [11] where only the generic fibre is required to be fixed (the problem of a *varying* integral structure will be dealt with in another work [5]). Nevertheless, concentrating on the integral structure and studying its differential Galois theory *per se* is a novelty which does not really appear in the conjecture of loc.cit. (but appears in the tentative solution and in [10]).

Finally we make a few remarks concerning existing ideas around the topic "reduction of the differential Galois group." Firstly we point out that Matzat [10, Thm. 5.4] gave an answer–under certain extra assumptions–to part (1) of the conjecture in [11]. The construction there is entirely different and is "point-theoretic" rather than "group-scheme-theoretic". It has the advantage of being closer to ordinary Galois theory. Secondly, after this work was submitted for publication, Y. André called our attention to a theorem of O. Gabber–in the case of equicharacteristic–which shows that the differential Galois group "can only diminish after specialization." (See the Remark (a) in section 6.2.3.) The proof also works in the mixed characteristic case. This construction has the advantaged of giving, together with our results, a simple description of the searched differential Galois group. But it is important to remark that to deserve the name of differential Galois group one of the two interpretations "Galois theoretic" or "Tannakian" should be present.

Some notations and standard terminology

modules For a commutative ring Λ, the category of finitely generated Λ-modules is denoted by Λ-mod. The category of all Λ-modules is denoted by Λ-Mod. If Λ is instead a co-algebra, then we define analogous notations for the comodules: Λ-comod and Λ-Comod.

Group theory Let $G = \operatorname{Spec} R$ be a group scheme over the commutative ring Λ. A representation of G will be an R-comodule $\rho : M \longrightarrow M \otimes_\Lambda R$ such that M is finitely generated over Λ. A comodule which is not necessarily f.g. over Λ will be called a G-module.

The category of representations of G will be denoted $\mathrm{Rep}_\Lambda(G)$; the category of G-modules will be denoted $\mathrm{Rep}'_\Lambda(G)$. If G is *flat* over Λ, then $\mathrm{Rep}'_\Lambda(G)$ is abelian [6, p. 33]. If Λ is noetherian, the same is true about $\mathrm{Rep}_\Lambda(G)$.

The left regular G-module is denoted by (R, ρ_l). The right regular is denoted by (R, ρ_r).

A representation V of G is faithful when V is projective as a Λ-module and the homomorphism of group schemes $\rho : G \longrightarrow \mathrm{GL}(V)$ is a closed embedding.

If $\alpha : H \longrightarrow G$ is a homomorphism of group schemes, then the functor induced by α between the categories of representations will be denoted by $\mathrm{res}(\alpha)$.

2 The results of Bruguières and Nori

The theory of Tannakian categories associates to each neutral Tannakian category C over a field κ an affine group scheme G over κ whose category of representations is naturally equivalent to C. However, the presence of a ground field is necessary for the theory to work this nicely as one can see, for example, in the crucial condition $\mathrm{End}_C(\mathbf{1}) = \kappa$ (forgotten by Saavedra and explored by Deligne). In this section we will review and adapt [2] where Bruguières, inspired by Nori, proposes a Tannakian theory over a ground ring of global dimension ≤ 2.

Unfortunately, Bruguières wrote up his theory without the usual dualization that passes from a co-commutative Hopf-algebra to the ring of functions of a group-scheme (but as Bruguiȓes points out, the role of the dualization is only justified because we are more used to think of representations of groups than of algebras). So the task of adapting [2] to the more usual context was left to us. Our adaptations are minor and were constantly guided by our interest in stratified sheaves–the reader who wishes to assume that the monoidal categories here are those introduced in section 4 will certainly understand the formalism better. In practical terms, the goal is to construct, starting from a monoidal category C_0 over a noetherian ring of global dimension ≤ 2, a group-scheme whose category of representations is closely related (but not identical!) to C_0.

In sections 2.1 and 2.2 we review the main results and definitions given in [2]. In section 2.3 we give a more classical formulation–group-schemes and representations–to the results previously presented. For the applications to follow (Definition 23 and Theorem 24) the most important result of this section is Corollary 7. It is the closest we could get, under the present assumptions, to the main statement of Tannakian theory over a field [3, Thm. 2.11].

The reader is directed to Ch. 7 of [8] for the concepts concerning monoidal categories. In order to save words, *we shall call a monoidal category what [8] calls a monoidal symmetric category.*

2.1 The monoidal category $\mathbf{P}_\Lambda$ and the the monoid $\mathrm{End}(T)$

Let Λ be a noetherian ring.

Definition 1. *1. The category $\mathbf{P}_\Lambda$ has as objects projective limits $\varprojlim_{i\in I} M_i$, where I is a directed set (small co-filtered category in Maclane's terminology) and each M_i is a finitely generated Λ-module. If $\mathbf{M} = \varprojlim_i M_i$ and*

$\mathbf{N} = \varprojlim_j N_j$ *are two objects of* $\mathbf{P}_\Lambda$*, we have*

$$\mathrm{Hom}(\mathbf{M}, \mathbf{N}) := \varprojlim_j \varinjlim_i \mathrm{Hom}_\Lambda(M_i, N_j).$$

2. *We will say that an arrow* $\varphi : \mathbf{M} \longrightarrow \mathbf{N}$ *is strict if (a)* $I = J$*; (b) we can find a representative* $\varphi_{ij} \in \mathrm{Hom}_\Lambda(M_i, N_j)$ *for* φ *with* $i = j$*.*
3. *The category* $\mathbf{P}^*_\Lambda$ *is the full sub-category of* $\mathbf{P}_\Lambda$ *whose objects consist of limits of projective (locally free)* Λ*-modules.*

Some properties of $\mathbf{P}_\Lambda$: (a) The category $\mathbf{P}_\Lambda$ is Λ-linear, abelian and monoidal with the tensor product being:

$$\mathbf{M} \Box \mathbf{N} = \varprojlim_{(i,j)} M_i \otimes_\Lambda N_j.$$

(b) $\mathbf{P}^*_\Lambda$ is a monoidal sub-category .

(c) The category Λ-mod is canonically embedded in $\mathbf{P}_\Lambda$. This embedding is exact, Λ-linear and monoidal.

The category $\mathbf{P}_\Lambda$ is introduced to support the study of Tannakian reconstruction. In fact, it is in this category where $\mathrm{End}(T)$ lives. Let us be specific. Consider C a category and let $T : C \longrightarrow \Lambda$-mod be a functor. Assume for the moment that the C is finite: there are only finitely many objects in C. Let

$$\varphi = (\varphi_x) \in \prod_{x \in C} \mathrm{End}_\Lambda(Tx).$$

To φ associate

$$\delta(\varphi) \in \prod_{s:x \to y} \mathrm{Hom}_\Lambda(Tx, Ty), \quad \delta_s = T(s) \circ \varphi_x - \varphi_y \circ T(s), \tag{1}$$

where s runs over a finite set $\{s_i\}_i \in \mathrm{Arr}(C)$ such that any Tu is a linear combination of some of the Ts_i. Now *define*

$$\mathrm{End}(T) = \ker \delta.$$

So $\mathrm{End}(T)$ is a finite Λ-algebra and every Tx is naturally an $\mathrm{End}(T)$-module. In fact, we have the commutative diagram

$$\begin{array}{ccc} & & \mathrm{End}(T)\text{-mod} \\ & \overset{\overline{T}}{\nearrow} & \downarrow{\scriptstyle \text{forget}} \\ C & \xrightarrow[T]{} & \Lambda\text{-mod} \end{array} \tag{2}$$

All is very simple in this formulation, but this idea is the heart of Saavedra's theory of Tannakian reconstruction.

For a category C in general, we have to construct $\mathrm{End}(T)$ as an object of $\mathbf{P}_\Lambda$. So let I be the directed set (or a directed pre-order in the terminology of [8, Ch. XI, p. 211]) of all full and finite sub-categories of C. If T_i represents the restriction of T to one of these categories, we have an object

$$\mathrm{End}(T) := \varprojlim_{i\in I} \mathrm{End}(T_i) \in \mathbf{P}_\Lambda.$$

With the monoidal structure $\Box : \mathbf{P}_\Lambda \times \mathbf{P}_\Lambda \longrightarrow \mathbf{P}_\Lambda$ introduced above, we have a *strict* arrow

$$\mathrm{End}(T)\Box\mathrm{End}(T) \longrightarrow \mathrm{End}(T)$$

which turns $\mathrm{End}(T)$ into a monoid (algebra) [8, VII §3, p. 170] of $\mathbf{P}_\Lambda$. Note that each $Tx \in \Lambda\text{-mod} \subset \mathbf{P}_\Lambda$ is now an $\mathrm{End}(T)$-module and every arrow $Ts : Tx \longrightarrow Ty$ is a morphism of $\mathrm{End}(T)$-modules (the terminology should be self-explanatory, but the reader can consult [8, VII §4]). If we let $\mathrm{End}(T)$-mod denote the category of objects in Λ-mod which have a left action of the monoid $\mathrm{End}(T)$ (we regard Λ-mod as a full sub-category of $\mathbf{P}_\Lambda$), we have obtained a factorization of T identical to that in diagram (2).

2.2 The main result in [2]

Definition 2 ([2])**.** *i) A category C is said to be monoidal right exact abelian category over Λ if:*
a) C is abelian and Λ-linear.
b) It has a monoidal structure $\otimes : C \times C \longrightarrow C$ such that $\otimes$ is Λ-bilinear.
c) For every $U \in C$, the functor $?\otimes U$ is right-exact (same statement holds for $U\otimes ?$).
d) If I is the identity object of C, then $\mathrm{End}(I) = \Lambda$.

Theorem 3 ([2], Thm. 3)**.** *Let C_0 be a monoidal category. Let C be a monoidal right exact abelian category over the noetherian ring Λ. Let $\omega_0 : C_0 \longrightarrow \Lambda\text{-mod}$, $\iota : C_0 \longrightarrow C$ and $\omega : C \longrightarrow \Lambda\text{-mod}$ be monoidal functors. Assume that*

1. *Λ has global dimension ≤ 2 (the kernel of a homomorphism between projective modules is also projective).*
2. *For each $x \in C_0$, $\omega_0 x$ is projective over Λ.*
3. *$\omega_0 = \omega \circ \iota$ as monoidal functors.*
4. *ω is Λ-linear, faithful and exact.*

Then

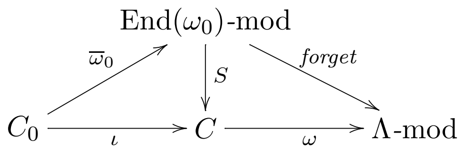
i) End(ω_0) *has a natural structure of co-commutative bi-monoid (bi-algebra) in the category* $\mathbf{P}^*_\Lambda$.

ii) Giving the category End(ω_0)-mod *the monoidal structure arising from the bi-algebra structure of* End(ω_0), *the functor* $\overline{\omega}_0 : C_0 \longrightarrow$ End(ω_0)-mod *is monoidal. Also, the forgetful functor* End(ω_0)-mod $\longrightarrow$ Λ-mod *is monoidal strict.*

iii) There exists a monoidal, exact, Λ*-linear and faithful functor*

$$S : \mathrm{End}(\omega_0)\text{-mod} \longrightarrow C$$

which induces a factorization of ι *making the diagram*



commute up to monoidal isomorphism.

□

The functor S obtained in Theorem 3 is also universal in a sense made explicit in [2]. As we shall not find any use to this property, we will not review it here.

2.3 Dualizing: classical Tannakian statements

Let Λ be a noetherian ring.

We now find a different formulation for Theorem 3 which is much more convenient for our purposes. This formulation will be made possible by the construction of a monoidal *contravariant* functor

$$D : \mathbf{P}^*_\Lambda \longrightarrow \Lambda\text{-Mod}.$$

The definition is immediate:

$$D : \varprojlim_i M_i \mapsto \mathrm{Hom}_{\mathbf{P}_\Lambda}(\varprojlim_i M_i, \mathbf{1}) = \varinjlim_i \mathrm{Hom}_\Lambda(M_i, \Lambda).$$

Lemma 4. *The functor D is Λ-linear and monoidal.*

Proof. 1. The category (Λ − Mod) is small co-complete; this means that small co-limits (= direct limits) always exist in Λ − Mod. In [8, Ch. 9 §8; p. 231] we find that $\varinjlim_{(i,j)} M_i \otimes N_j = \varinjlim_i M_i \otimes \varinjlim_j N_j$.

2. If M and N are projective elements of (Λ−mod) it follows that $\mathrm{Hom}_\Lambda(M \otimes N, \Lambda) = \mathrm{Hom}_\Lambda(M, \Lambda) \otimes_\Lambda \mathrm{Hom}_\Lambda(N, \Lambda)$. So the natural transformation

$$D \circ \Box \Rightarrow \otimes_\Lambda \circ D \times D$$

is a natural isomorphism.

□

Hence, D takes monoids (resp. co-monoids) in $\mathbf{P}^*_\Lambda$ to co-monoids (resp. monoids) in Λ-Mod.

Let $\mathbf{A} \in \mathbf{P}^*_\Lambda$ be a monoid. Another important property of D is that it will give an equivalence between the categories $\mathbf{A}$-mod (see [8, VII, §4]) and the category $D\mathbf{A}$-comod. To see this, we construct a pair of functors from $(\mathbf{P}^*_\Lambda)^{opp} \times (\Lambda\text{-mod})^{opp} \times (\Lambda\text{-mod})$ to (Λ-Mod)

$$F_1(\mathbf{X}, M, N) = \mathrm{Hom}_\Lambda(M, N \otimes D(\mathbf{X})); \quad F_2(\mathbf{X}, M, N) = \mathrm{Hom}_{\mathbf{P}_\Lambda}(\mathbf{X} \Box M, N).$$

Let us unravel the definitions (for a Λ-module L we denote the dual module $\mathrm{Hom}_\Lambda(L, \Lambda)$ by L^*):

$$F_1(\mathbf{X}, M, N) = \mathrm{Hom}_\Lambda(M, N \otimes D(X)) \approx \varinjlim_i \mathrm{Hom}_\Lambda(M, N \otimes X_i^*), \text{ as } M \text{ is f.g. over } \Lambda$$

and

$$F_2(\mathbf{X}, M, N) = \varinjlim_i \mathrm{Hom}_\Lambda(X_i \otimes M, N), \text{ by definition.}$$

Since we are taking $\mathbf{X}$ in $\mathbf{P}^*_\Lambda$, we get an isomorphism

$$\Phi_{(\mathbf{X}, M, N)} : F_1(\mathbf{X}, M, N) \xrightarrow{\cong} F_2(\mathbf{X}, M, N).$$

It is obvious that Φ is natural in (M, N); in order to check naturality of Φ we can assume (M, N) fixed and we are left with checking naturality of $\Phi(?, M, N)$, which is straightforward. Therefore, if $M \in \Lambda$-mod is an $\mathbf{A}$-module, we can transport the multiplication map $\mathbf{A} \Box M \longrightarrow M$ to a comodule map $M \longrightarrow M \otimes_\Lambda D\mathbf{A}$. It easy to show that this construction gives an equivalence of categories (below we give a more concrete approach to this minor result):

Lemma 5. *Let $\mathbf{A}$ be a monoid in $\mathbf{P}^*_\Lambda$. Then $D(\mathbf{A})$ is a co-monoid in Λ-Mod (a co-algebra over Λ) and there is a natural equivalence*

$$\eta : \mathbf{A}\text{-mod} \longrightarrow D\mathbf{A}\text{-comod}$$

which is the identity on the underlying category of finite Λ-modules. □

We will now give a more down-to-earth approach to these constructions. This will be greatly simplified if the monoid $\mathbf{A}$ in $\mathbf{P}^*_\Lambda$ is such that

$$\mathbf{A} = \varprojlim_{i\in I} A_i$$

where each A_i is a finite and projective Λ-algebra and the transition maps $\psi_{ij} : A_j \longrightarrow A_i$ are all homomorphisms of Λ-algebras (this is the case for $\mathrm{End}(T)$ and therefore the most interesting). In particular, the category $\mathbf{A}$-mod is just the direct limit of categories

$$\varinjlim_{i\in I} A_i\text{-mod}.$$

In this setting, the above lemma is just an application of a well-known fact, see I-8.6 of [6]. But this assumption is also convenient to study the case where $\mathbf{A}$ has more structure. Assume that $\mathbf{A}$ is bi-monoid in $\mathbf{P}^*_\Lambda$ with co-multiplication $\Delta : \mathbf{A} \longrightarrow \mathbf{A}\Box\mathbf{A}$. Then the category $\mathbf{A}$-mod is endowed with a monoidal structure which makes the forgetful functor

$$\mathbf{A}\text{-mod} \longrightarrow \Lambda\text{-mod}$$

monoidal. The construction is as follows. Let $(M,\alpha),(N,\beta) \in \mathbf{A}$-mod, where $\alpha : \mathbf{A}\Box M \longrightarrow M$ is giving the $\mathbf{A}$-module structure (analogous for β). Define the action as the composition

$$\begin{array}{ccccc} \mathbf{A}\Box(M\Box N) & \xrightarrow{\Delta\Box M\Box N} & \mathbf{A}\Box\mathbf{A}\Box M\Box N & \xrightarrow{\approx} & (\mathbf{A}\Box M)\Box(\mathbf{A}\Box N) \\ & \searrow & & & \downarrow{\scriptstyle \alpha\Box\beta} \\ & & & & M\Box N \end{array}$$

Since D is monoidal, $D\mathbf{A}$ is a bi-monoid in Λ-Mod and the category $D\mathbf{A}$-comod becomes a monoidal category in such a way that the forgetful functor to Λ-mod is monoidal.

Claim: The natural equivalence η is monoidal.

Proof: This is just a matter of unravelling the definitions. Take M, N as above. By definition, there is some $i_0 \in I$ such that M and N are in fact A_{i_0}-modules. So $\eta(M)$ and $\eta(N)$ are in fact $A^*_{i_0}$-comodules. Let $\rho : M \longrightarrow M\otimes A^*_{i_0}$ and $\sigma : N \longrightarrow N \otimes A^*_{i_0}$ be the comodule morphisms. Given $x \in M$, we have an obvious map $\rho_x : A_{i_0} \longrightarrow M$ (analogous notation for N). By definition, Δ is given by a projective system

$$(\Delta_j : A_{i(j)} \longrightarrow A_j \otimes A_j)_{j\in I},\ i(j) \geq j.$$

In particular, we have $\Delta_{i_0} : A_{i_1} \longrightarrow A_{i_0} \otimes A_{i_0}$ for some $i_1 \geq i_0$.

The $\mathbf{A}$-module structure of $M \otimes_\Lambda N$ $(=M\Box N)$ is given by an A_{i_1}-module structure θ characterized in the commutative diagram

$$\begin{array}{ccccc} A_{i_1} \otimes (M \otimes N) & \xrightarrow{\Delta_{i_0}\otimes M \otimes N} & (A_{i_0} \otimes A_{i_0}) \otimes (M \otimes N) & \xrightarrow{\approx} & (A_{i_0} \otimes M) \otimes (A_{i_0} \otimes N) \\ & \searrow^{\theta} & & & \downarrow{\scriptstyle \alpha\otimes\beta} \\ & & & & M \otimes N. \end{array}$$

Hence, the $D(\mathbf{A})$-comodule structure on $\eta(M\Box N)$ is given by

$$\varphi : M \otimes N \longrightarrow M \otimes N \otimes A_{i_1}^*$$

with

$$\varphi_{x\otimes y} : A_{i_1} \longrightarrow M \otimes N, \quad \varphi_{x\otimes y}(a) = \theta(a \otimes x \otimes y),\ x \otimes y \in M \otimes N.$$

If

$$\Delta_{i_0}(a) = \sum_r a_r^{(1)} \otimes a_r^{(2)}, \quad \text{then} \quad \varphi_{x\otimes y}(a) = \sum_r \alpha(a_r^{(1)} \otimes x) \otimes \beta(a_r^{(2)} \otimes y).$$

On the other hand, the comodule structure of $\eta(M) \otimes_{D(\mathbf{A})} \eta(N)$ is obtained by the composition τ defined in the commutative diagram

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\rho\otimes\sigma} & (M \otimes A_{i_0}^*) \otimes (N \otimes A_{i_0}^*) \\ & \searrow^{\tau} & \downarrow \\ & & (M \otimes N) \otimes A_{i_1}^*. \end{array} \tag{3}$$

The vertical arrow is

$$\mathrm{Hom}_\Lambda(A_{i_0}, M)\otimes\mathrm{Hom}_\Lambda(A_{i_0}, N) \longrightarrow \mathrm{Hom}_\Lambda(A_{i_1}, M\otimes N), \quad (f\otimes g) \mapsto (f\otimes g)\circ\Delta_{i_0}.$$

The horizontal arrow of diag. (3) is $x \otimes y \mapsto \rho_x \otimes \sigma_y$. Therefore

$$\tau_{x\otimes y}(a) = \sum_r \rho_x(a_r^{(1)}) \otimes \sigma_y(a_r^{(2)}).$$

By definition, $\rho_x(a^{(1)}) = \alpha(a^{(1)} \otimes x)$ and $\sigma_y(a^{(2)}) = \beta(a^{(2)} \otimes y)$; this proves that $\varphi = \tau$ and hence that η is monoidal. The claim is proved.

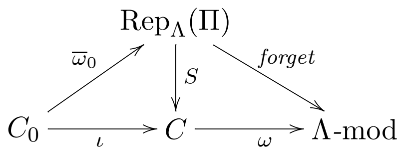
Now we can start rewriting the results of section 2.2 in terms of the material developed in this section. First we have:

Notation: The full sub-category of C whose objects have a dual will be denoted by $C^{\#}$. For this notion of duality the reader is directed to the discussion preceding Proposition 3 in [2] (which follow identically the definition in the case of tensor categories over a field).

Theorem 6. *Adopt the notations and assumptions of Theorem 3. Assume furthermore that every element of C_0 admits a dual in C_0. Then there exits a flat Λ-group-scheme $\Pi = \Pi(C_0, C, \omega)$, a monoidal, exact, Λ-linear and faithful functor*

$$S : \mathrm{Rep}_\Lambda(\Pi) \longrightarrow C$$

and a commutative (up to monoidal isomorphism) diagram



If moreover $\mathrm{Spec}\,\Lambda$ *is connected, then* Π *is faithfully-flat over* Λ.

Proof. The point of duality is to make the bi-monoid $\mathrm{End}(\omega_0)$ into a Hopf-monoid (that is, a bi-monoid with an isomorphic antipode). This is standard and the details will be omitted. Let us prove now the statement about the faithful flatness of

$$\mathscr{O}(\Pi) := D(\mathrm{End}(\omega_0)).$$

Clearly $D(\mathbf{A})$ is flat for any given $\mathbf{A} \in \mathbf{P}^*_\Lambda$. Let $\mathbf{A} = \mathrm{End}(\omega_0)$. We have to show that given $\mathfrak{m} \in \mathrm{Max}(\Lambda)$, $\mathfrak{m}D(\mathbf{A}) \neq D(\mathbf{A})$. Assume otherwise. Letting $\varepsilon : \mathbf{A} \longrightarrow \Lambda$ be the co-identity and Δ the co-multiplication, it follows from the equality $\mathrm{id} = (\mathrm{id}\square\varepsilon) \circ \Delta$ that $\mathrm{id} \in \mathfrak{m}\mathrm{Hom}(\mathbf{A}, \mathbf{A})$. Unraveling the definition, we obtain that, for any $i \in I$, there exists $J(i) \in I$ such that the transition map $\psi_{ij} : A_j \longrightarrow A_i$ satisfies

$$\mathrm{Im}(\psi_{ij}) \subseteq \mathfrak{m}A_i \text{ for every } j \geq J(i).$$

But ψ_{ij} is a homomorphism of Λ-algebras and therefore $1_{A_i} \in \mathfrak{m}A_i$ (we silently exclude the trivial situation); this is in contradiction with the fact that A_i is faithfully flat over Λ (as $\mathrm{Spec}\,\Lambda$ is connected, any finite and projective Λ-module is faithfully flat).

□

The next corollary is, for this work, the most important consequence of the Tannakian theory so far presented. The hypothesis *iv)* and *v)* below will be explored later (see sections 3.2 and 4.3).

Corollary 7. *Assume that*

i) $\iota : C_0 \longrightarrow C$ *is full and faithful (we identify* C_0 *with a full subcategory of* C*).*

ii) Any object of C *which is isomorphic to an object of* C_0 *is an object of* C_0.

iii) C_0 *is an additive* Λ*-linear category and* ι *preserves that structure.*

iv) The category $C^{\#}$ *is stable by sub-objects.*

v) If $\alpha : u \longrightarrow v$ *is an inclusion (monomorphism) with* $v \in C_0$ *and* $\mathrm{coker}(\alpha) \in C^{\#}$*, then* $u \in C_0$*.*

vi) Any $z \in C^{\#}$ *which is a quotient of an object of* C_0 *is in* C_0*.*

Then $\overline{\omega}_0$ *induces an equivalence between* C_0 *and the category* $\mathrm{Rep}^{\#}_{\Lambda}(\Pi)$*.*

Proof. This is essentially contained in [2] but we will give a proof. Since ι is full and faithful, it is easy to see that $\overline{\omega}_0$ is full and faithful. Since every $u \in C_0$ has a dual, so does any $\omega_0(u)$, this means that ω_0 takes values in the full sub-category $\mathrm{Rep}^{\#}_{\Lambda}(\Pi)$.

The hard part is to show essential surjectivity of $\overline{\omega}_0$. It will be a consequence of the following:

Claim: Any $W \in \mathrm{Rep}^{\#}_{\Lambda}(\Pi)$ is a cokernel of an arrow $\overline{\omega}_0(\alpha) : \overline{\omega}_0(x) \longrightarrow \overline{\omega}_0(y)$.

Granted the claim, the rest is just abstract non-sense. As $\omega\mathrm{coker}(\alpha)$ is projective, it is easy to see that $\mathrm{coker}(\alpha)$ admits a dual in C (or check [2, Prp. 3-(2)]). By assumption *vi)*, $\mathrm{coker}(\alpha) \in C_0$. The exact sequence

$$\omega(x) \longrightarrow \omega(y) \longrightarrow \omega(\mathrm{coker}(\alpha)) \longrightarrow 0$$

is just the image sequence of

$$\overline{\omega}_0(x) \longrightarrow \overline{\omega}_0(y) \longrightarrow \overline{\omega}_0(\mathrm{coker}(\alpha)) \longrightarrow 0$$

via the forgetful functor. Hence $\overline{\omega}_0(\mathrm{coker}(\alpha))$ is isomorphic to W in $\mathrm{Rep}_{\mathfrak{o}}(\Pi)$ and we are done.

The proof of the claim follows from the material of sections 3 and 4 of [2] (in fact, it is contained in the proof of Prp. 2 of that paper). Let us explain it from basic principles. If we see $\mathrm{Rep}^{\#}_{\Lambda}(\Pi)$ as the category of projective Λ-modules in $\mathrm{End}(\omega_0)$-mod, then any W as above is the cokernel of an arrow $E^b \longrightarrow E^a$, where $E = \mathrm{End}(\omega_0|D)$, $D \subseteq C_0$ is finite and E is seen as an E-module via multiplication on the left. So we only need to show that E lies in the essential image of $\overline{\omega_0}$. We assume for convenience that every projective and finite Λ-module is free.

In section 2 of loc.cit., Bruguières introduces a functor

$$\underline{\mathrm{Hom}}_C(?,?) : \Lambda\text{-mod}^{opp} \times C \longrightarrow C$$

(in the construction C can be replaced by any Λ-linear abelian category) which satisfies:

1. It is Λ-linear and left exact.

2. For M free of rank m, $\underline{\mathrm{Hom}}_C(M,x) = x^{\oplus m}$ (note that these two first properties already tell us how to construct $\underline{\mathrm{Hom}}(?,?)$).
3. $\omega \circ \underline{\mathrm{Hom}}_C(?,?) = \underline{\mathrm{Hom}}_{\Lambda\text{-mod}}(?,?) \circ \mathrm{id} \times \omega$.
4. $\underline{\mathrm{Hom}}_{\Lambda\text{-mod}}(M,N)$ is nothing but $\mathrm{Hom}_\Lambda(M,N)$ seen as a Λ-module in the natural fashion.

Consider the arrow

$$\partial = \partial^0 - \partial^1 : \bigoplus_{u\in D} \underline{\mathrm{Hom}}_C(\omega u, u) \longrightarrow \bigoplus_{s:v\to w\in \mathrm{Arr}(D)} \underline{\mathrm{Hom}}_C(\omega v, w);$$

$$\partial^0_{u,s} = \begin{cases} 0, \text{ if } u \neq v \\ \underline{\mathrm{Hom}}_C(\omega u, ?)(s), \text{ if } u = v \end{cases}$$

$$\partial^1_{u,s} = \begin{cases} 0, \text{ if } u \neq w \\ \underline{\mathrm{Hom}}_C(?, u)(\omega s), \text{ if } u = w \end{cases}$$

As each $\omega(u)$ is free for $u \in C_0$, then $\partial \in \mathrm{Arr}(C_0)$. Because of hypothesis *iv*) and *v*) above $\ker \partial \in C_0$. Now property 4 above gives

$$\omega_0(\partial) = \delta : \bigoplus_{u\in D} \mathrm{Hom}_\Lambda(\omega u, \omega u) \longrightarrow \bigoplus_{s:v\to w} \mathrm{Hom}_\Lambda(\omega v, \omega w),$$

where δ is as in equation (1). Regarding each $\underline{\mathrm{Hom}}_C(\omega u, v)$ as a direct sum of v's (property 2), we see that the $\mathrm{End}(\omega_0)$-module structure of $\omega_0\underline{\mathrm{Hom}}_C(\omega u, v)$ is the structure of $\mathrm{End}(\omega_0|D)$-module of $\mathrm{Hom}_\Lambda(\omega u, \omega v)$ given by composition on ωv: $(\psi_w)\cdot\varphi = \psi_v \circ \varphi$. Hence, $\overline{\omega}_0(\ker\partial) = E$ and this is what we wanted in order to prove the claim.

□

3 Tannakian group theory for faithfully flat group-schemes over a PID

In Tannakian theory over a ground field, there is a specific condition on the category $\mathrm{Rep}(G)$ for a representation V of an affine group-scheme G to be faithful. It asserts that V is faithful if and only if any representation can be obtained as a sub-quotient of a direct sum of objects of the form $V^{\otimes a} \otimes (V^\vee)^{\otimes b}$. Over more general base rings this condition does not hold any longer even for the most basic of examples (see the example in section 3.2). By investigating the latter example, we are led to the understanding which provides the correct definition over a principal ideal domain (Definition 10 and Proposition 12). Here, correct means that with this definition we are able to say when a representation V is faithful by analyzing the relation of the family

$$\left(V^{\otimes a_1} \otimes (V^\vee)^{\otimes b_1}\right) \oplus \cdots \oplus \left(V^{\otimes a_r} \otimes (V^\vee)^{\otimes b_r}\right);\ (a_1, \dots, a_r), (b_1, \dots, b_r) \in \mathbb{N}^r$$

with the rest of the category of representations. Of course, this study is carried in order to say when the group-schemes defined in section 2 are algebraic; in fact, the characterizations obtained here allowed the discovery of the hypothesis in Corollary 7 above.

In order to achieve this cognition, we drew inspiration from the pioneering work of Tannaka–coefficients of representations, section 3.1–rather than from that of Saavedra. This is one of the most pleasant experiences in Mathematics when knowledge that seemed obsolete (at least for these algebraic purposes) comes into use again.

Let $G = \operatorname{Spec} R$ be an affine faithfully flat group scheme over a commutative noetherian base ring Λ. We remind the reader that, by convention, a representation of G is finitely generated over Λ and the R-comodules which are not necessarily finitely generated are called G-modules. We will say that a representation of G is **free** if the underlying Λ-module is free.

3.1 The space of coefficients

Definition 8. *Let (W, ρ) be a free representation of G. Pick a basis $\{w_i\}$ of W and let $\rho(w_j) = \sum_i w_i \otimes \rho_{ij}$. The ρ_{ij} are called the matrix coefficients of ρ for the basis $\{w_i\}$. Define the space of coefficients of W, denoted $C(W)$, as the Λ-sub-module of R generated by the ρ_{ij}.*

Note that $C(W)$ is naturally a G-sub-module of (R, ρ_r).

Lemma 9. *Let $(U, \sigma), (W, \rho)$ be free representations of G.*

i) $C(U \oplus W) = C(U) + C(W)$.

ii) Each element $f \in C(U \otimes W)$ is of the form $\Sigma g_i h_i$, where $g_i \in C(U)$ and $h_i \in C(W)$.

iii) If χ is a character of G, then $C(W \otimes \chi) = \chi \cdot C(W)$; in the second member of the equality we see the character χ as a group-like element of G.

iv) If there is a surjection $U \longrightarrow W$, then $C(W) \subseteq C(U)$.

v) $C(U^\vee)$ is contained in the Λ-module generated by $\delta^{-1} \cdot \{f_1 \cdots f_{d-1};\ f_j \in C(U)\}$, where δ is the group-like element of R corresponding to the determinant representation of U and $d = \operatorname{rk}(U)$.

vi) If W is a G-sub-module of (R, ρ_r), then $W \subseteq C(W)$.

Proof. This is all linear algebra. We only prove v) and vi).

v) Give U a basis $\{u_i\}_{1\leq i\leq d}$ so that $\sigma(u_j) = \sum_i u_i \otimes \sigma_{ij}$. Then $U^\vee$ has a basis $\{u_i^*\}$ such that the comodule map is $u_j^* \mapsto \sum_i u_i^* \otimes \tau_{ji}$, where $(\tau_{ij}) = (\sigma_{ij})^{-1}$. But $\det(\sigma_{ij}) \cdot \tau_{ij} = (-1)^{i+j}\Delta_{ij}$, where Δ_{ij} is a $(d-1)\times(d-1)$-minor of (σ_{ij}).

vi) Let $f_1, \ldots, f_m$ be a basis of W. Let $\varepsilon : R \longrightarrow \Lambda$ be the co-identity. As the composition

$$\varepsilon \otimes \mathrm{id}_R \circ \rho_r : R \longrightarrow R \otimes R \longrightarrow R$$

is just the identity, it follows that $f_j = \sum_i \rho_{ij} \cdot \varepsilon(f_i)$. □

3.2 Closed embeddings and algebraicity

Definition 10 (Special sub-quotients)**.** *Let W be a free representation of G. A sub-representation $W' \subseteq W$ is special if W' and W/W' are free. A free representation U is called a special sub-quotient of W if there is a special sub-representation $W' \subseteq W$ such that U is a (G-equivariant) quotient of W'.* ***Language:*** *We say that an embedding $\alpha : W \longrightarrow W'$ is special if the sub-object* $\operatorname{Im}\alpha$ *is special.*

From now on we assume that Λ is a principal ideal domain (PID).

Some properties: Let M_1, M_2 and M_3 be free representations of G. Then
1) If $M_1 \subseteq M_2$ and $M_2 \subseteq M_3$ are special, then $M_1 \subseteq M_3$ is also special.
2) Take $M_1 \subseteq M_2$ special and let $\varphi : M_3 \longrightarrow M_2$ be surjective. Then $\varphi^{-1}M_1 \subseteq M_3$ is special.
3) If M_1 is a special sub-quotient of M_2 and M_2 is a special sub-quotient of M_3, then M_1 is a special sub-quotient of M_3.
4) If $M_1 \subseteq M_2 \subseteq M_3$, with $M_1 \subseteq M_3$ special, then $M_1 \subseteq M_2$ is special.

Lemma 11. *Let (W, σ) be a free representation of G of rank r. Denote by $\sigma : W \longrightarrow W \otimes R \cong (R, \rho_r)^{\oplus r}$ the natural embedding. Let $U \subseteq R$ be a finitely generated G-sub-module which contains* $\operatorname{Im}(\sigma)$*. Then $\sigma : W \longrightarrow U$ is special.*

Proof. Pick a basis $\{w_i\}_{1\leq i\leq r}$ of W so that the embedding above is $\sigma : w_j \mapsto \sum_i w_i \otimes \sigma_{ij}$. We need to check that $(W \otimes R)/W$ has no torsion. Assume that there is $a \in \Lambda$ such that $a\sum_i w_i \otimes f_i = \sum_j \alpha_j \sigma(w_j) = \sum_{i,j} w_i \otimes \alpha_j\sigma_{ij}$ or, in terms of matrices:

$$(\sigma_{ij}) \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} = a \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}.$$

Because the matrix (σ_{ij}) is invertible, it follows that the each $\alpha_j \in Ra \cap \Lambda = \Lambda a$ (by faithful flatness). It follows that there is no a-torsion in $(W \otimes R)/W$. □

We are now going to use the notion of special sub-quotients in order to obtain a way to decide whether a free representation of G is faithful. Let (V, ρ)

be a free representation of G of rank d. We make some classical constructions in linear algebra concerning V (the PID hypothesis on Λ is not necessary for these). Start with the determinant representation δ: it is a free Λ-module of rank one where G acts via the group-like element δ of R. Define

$$\Theta_V(a,b,1) = \Big(\mathbf{1} \oplus (V^d)^{\otimes 1} \oplus \cdots \oplus (V^d)^{\otimes a}\Big) \otimes (\delta^{-1})^{\otimes b}; \quad \Theta_V(a,b,c) = \Theta(a,b,1)^{\oplus c} \quad (a,b,c \in \mathbb{N}).$$

(We will usually drop the subscript referring to V in the notation). More generally, one defines:

$$V^a_b = \bigoplus_{i=1}^{s} V^{\otimes a_i} \otimes (V^\vee)^{\otimes b_i}, \quad a = (a_1, \dots, a_s); b = (b_1, \dots, b_s) \in \mathbb{N}^s. \qquad (4)$$

There are natural arrows in $\mathrm{Rep}'_\Lambda(G)$

$$\theta_{a,b,c} : \Theta(a,b,c) \longrightarrow (R, \rho_r)^{\oplus c}$$

defined by arrows

$$\theta_{a,b,1} : (V^d)^{\otimes a} \otimes (\delta^{-1})^{\otimes b} \longrightarrow (R, \rho_r)$$

as follows. Take $\{v_i\}$ a basis of V and let $\{v_{ij}\}$ be a basis of V^d where, for fixed i, $\{v_{i1}, \dots, v_{id}\}$ is a basis of V identified with the original basis $\{v_i\}$. Hence the comodule map for V^d is

$$v_{ij} \mapsto \sum_l v_{il} \otimes \rho_{lj}$$

and mapping v_{ij} to ρ_{ij} gives the G-equivariant map $V^d \longrightarrow (R, \rho_r)$. It is obvious to extend this to any tensor power $(V^d)^{\otimes a} \longrightarrow (R, \rho_r)$ because ρ_r(=comultiplication on R) is also a homomorphism of rings. It follows that the map $\theta_{a,0,1}$ sends a pure tensor

$$v_{I_1} \otimes \cdots \otimes v_{I_a} \in (V^d)^{\otimes a}, \quad I \in \{1, \dots, d\} \times \{1, \dots, d\}$$

to the element

$$\rho_{I_1} \cdots \rho_{I_a}.$$

Taking the tensor product $(V^d)^{\otimes a} \otimes (\delta^{-1})^{\otimes b}$ means that we are keeping the basis but changing the coefficients by multiplication by the group-like element δ^{-b}. So we map the pure tensor $v_{I_1} \otimes \cdots \otimes v_{I_a} \in (V^d)^{\otimes a}$ to $\delta^{-b} \cdot \rho_{I_1} \cdots \rho_{I_a}$ instead.

Notation: Let (V, ρ) be a free representation of G. Adopt the following notations:

$$S_V := \bigcup_{a \geq 1} \mathrm{Im}\, \theta_{a,0,1}, \qquad S'_V = \bigcup_{a,b \geq 1} \mathrm{Im}\, \theta_{a,b,1} \qquad (5)$$

These are clearly G-sub-modules of (R, ρ_r). We can now state the most important result of this section.

Proposition 12. *Assume that the free representation (V, ρ) is faithful. Then any free representation (W, σ) is a special sub-quotient of some V_b^a (a and b are multi-indexes and the notation is explained in eq. (4)). Conversely, if any free representation of G is a special sub-quotient of V_b^a, then V is faithful.*

Proof. Let r be the rank of (W, σ). We can embed W equivariantly in $(R, \rho_r)^{\oplus r}$. As the representation V is faithful, we have $R = S'_V$ and by tensoring with some $\delta^{\otimes b}$ we have $W \otimes \delta^{\otimes b} \subseteq S_V'^{\oplus r}$. Hence $W \otimes \delta^{\otimes b}$ is a special sub-object of some $\operatorname{Im} \theta_{a,0,r}$ (by Lemma 11). By the Snake Lemma

$$\operatorname{Im} \theta_{a,0,r} / W \otimes \delta^{\otimes b} \cong \Theta(a, 0, r) / \theta_{a,0,r}^{-1}(W \otimes \delta^{\otimes b})$$

and we see that W is a special sub-quotient of $\Theta(a, b, r)$. It is not hard to see that $\Theta_V(a, b, c)$ is a special sub-quotient of some V_β^α and the first implication of the proposition follows. Let us prove the converse, which will be a consequence of:

Claim: Let $(X, \xi) \subseteq (Y, \eta)$ be free representations and assume that $X \subseteq Y$ is special. Then $C(X) \subseteq C(Y)$.
Proof: Since Y/X is free, any linear functional $\varphi : X \longrightarrow \Lambda$ extends to a functional $\Phi : Y \longrightarrow \Lambda$. Now $C(X)$ is by definition the set of all $\xi_{x,\varphi} := (\varphi \otimes \mathrm{id}_R) \circ \xi(x) = (\Phi \otimes \mathrm{id}_R) \circ \eta(x)$, where Φ is an extension of φ to Y; we are done.

Now let $W \subseteq (R, \rho_r)$ be a G-sub-module of finite rank. Choose $U \subseteq V_b^a$ special which has W as a quotient. Then $W \subseteq C(W) \subseteq C(U) \subseteq C(V_b^a) \subseteq S'_V$ (Lemma 9). This shows that $R = S'_V$ and the proposition is proved. ∎

Of course, the special sub-quotients are essential to all this as the following example shows:

Example: Let $\Lambda = \mathbb{Z}_3[\sqrt{-3}]$. It is a DVR with uniformizer $\varpi = \sqrt{-3}$. Let G be the finite etale group scheme $\mathbb{Z}/3\mathbb{Z}$. We write

$$R = \mathscr{O}(G) = \Lambda[x]/(x^3 - x)$$

and let the co-multiplication be

$$\Delta(x) = x \otimes 1 + 1 \otimes x - \frac{3}{2}(x^2 \otimes x + x \otimes x^2).$$

Consider the character (L, θ) of G given by the group-like element

$$\theta = 1 + \frac{\varpi}{2}x - \frac{3}{2}x^2.$$

The dual representation $(L, \theta)^\vee$ is given by the group-like element

$$\theta^{-1} = \theta^2 = 1 - \frac{\varpi}{2}x - \frac{3}{2}x^2.$$

Note that the right-regular representation (R, ρ_r) embeds in $T := \mathbf{1} \oplus L \oplus L^\vee$ since T is naturally a sub-representation of (R, ρ_r) and R/T is killed by 3. In other words, we note that the map

$$R \longrightarrow T, \quad (1, x, x^2) \mapsto 3(1, x, x^2) = (3, \varpi(\theta^2 - \theta), 2 - (\theta + \theta^2))$$

is an equivariant embedding.

Any free representation of finite rank W of G is a sub-representation of some $(R, \rho_r)^{\oplus r}$; it follows that any free representation of G is a sub-representation of $T^{\oplus r}$. But of course, T is not faithful, as the group-scheme $G \otimes \mathbb{F}_3$ has no non-trivial characters (it is unipotent).

3.3 $\mathscr{O}(H) \subseteq \mathscr{O}(G)$

We maintain Λ a PID.

Proposition 13. *Let H be another faithfully flat Λ-group and consider a homomorphism $\alpha : G \longrightarrow H$ which induces an inclusion of rings $\alpha^* : \mathscr{O}(H) \subseteq \mathscr{O}(G) = R$. We have:*

i) the restriction functor

$$\mathrm{res}(\alpha) : \mathrm{Rep}_\Lambda^\#(H) \longrightarrow \mathrm{Rep}_\Lambda^\#(G)$$

is full and faithful (the hypothesis on Λ is not used here).

ii) The full subcategory $\mathrm{Rep}_\Lambda^\#(H)$ is closed under special sub-quotients.

iii) Let $U \in \mathrm{Rep}_\Lambda^\#(H)$ be a faithful representation. The functor $\mathrm{res}(\alpha)$ gives a monoidal equivalence between $\mathrm{Rep}_\Lambda^\#(H)$ and the full subcategory of $\mathrm{Rep}_\Lambda^\#(G)$ whose objects are special sub-quotients of the $\mathrm{res}(\alpha)(U_b^a)$.

Proof. *i*) Clearly $\mathrm{res}(\alpha)$ is faithful and we have to prove that it is full. By duality, we only need to show that for $M \in \mathrm{Rep}_\Lambda^\#(H)$ the natural map $M^H \longrightarrow M^G$ is bijective. Let $\sigma : M \longrightarrow M \otimes \mathscr{O}(H)$ be the comodule map for M and $r(\sigma) : M \longrightarrow M \otimes \mathscr{O}(G)$ the comodule map for $\mathrm{res}(\alpha)(M)$. The proposition is then an application of the Snake Lemma to the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M^H & \longrightarrow & M & \xrightarrow{\sigma - \mathrm{id}_M \otimes 1} & M \otimes \mathscr{O}(H) \\ & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}_M \otimes \alpha^*} \\ 0 & \longrightarrow & M^G & \longrightarrow & M & \xrightarrow{r(\sigma) - \mathrm{id}_M \otimes 1} & M \otimes \mathscr{O}(G). \end{array}$$

ii) We only deal with the case of special sub-representations, the general case is entirely analogous. In this situation, given N a special sub-object of $\mathrm{res}(\alpha)(M)$, there is a basis $\{m_i\}_{1 \le i \le s}$ of M such that $\{m_i\}_{1 \le i \le r}$ is a basis of

N. If σ_{ij} are the matrix coefficients of M, it follows that $\alpha^*(\sigma_{ij}) = 0$ for every $i > r$ and $j \leq r$; as a consequence N is also invariant by the comodule map σ.

iii) Obvious from *i*) and *ii*) and Proposition 12.

□

4 The differential Galois group over $\mathfrak{o}$

In this section we define some of the categories we are interested in: $\mathbf{str}(\mathfrak{X}/\mathfrak{o})$ and $\mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ (see the end of section 4.1) where $\mathfrak{X}$ is an affine formal scheme which is adic and smooth over the complete DVR $\mathfrak{o}$. To ease referencing we have included section 4.1, there we collect some folkloric facts about differential operators over adic formal schemes. In section 4.2 we show how to give $\mathbf{str}(\mathfrak{X}/\mathfrak{o})$ a fibre functor so that the Tannakian theory of Bruguières-Nori (section 2) is put into practice. As we are interested in defining the differential Galois group of $\mathscr{M} \in \mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$–which should be algebraic over the base ring–, we need to look at certain sub-categories of $\mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ whose objects are "dominated" by $\mathscr{M}$ (so that we can copy the Tannakian theory over a field); this is done in section 4.3 where the main definition made, Definition 23, follows the teachings of section 3.2 and gives a crucial algebraicity result, Theorem 24, for the Tannakian group-scheme furnished by Bruguières-Nori.

Let $\mathfrak{o}$ be a complete DVR of residue field k, local parameter $\varpi \neq 0$ and fraction field K. We make no assumptions on the characteristics, but in order to make the theory of differential operators work properly, we need to assume that k is *perfect*. This guarantees that all Tannakian differential Galois group-schemes for stratified sheaves over smooth k-schemes are reduced (see [4, Thm. 11], the hypothesis that k is algebraically closed is not used there).

We let A be a flat $\mathfrak{o}$-adic ring which is topologically of finite presentation (type) over $\mathfrak{o}$:

$$A = \mathfrak{o}\langle T_1, \ldots, T_r\rangle/\mathfrak{a},$$

where $\mathfrak{o}\langle T\rangle$ is the ϖ-adic completion of $\mathfrak{o}[T]$. We also suppose that $\mathfrak{X} := \mathrm{Spf}(A)$ is connected (which is the same as connectedness of $\mathrm{Spec}\, A$) and smooth over $\mathfrak{o}$. Smoothness is to be understood in the following sense. There exists a homomorphism of $\mathfrak{o}$-algebras

$$\psi : \mathfrak{o}\langle x_1, \ldots, x_n\rangle \longrightarrow A$$

such that the induced morphisms $\psi \otimes (\mathfrak{o}/\varpi^{\mu})$ are etale for each $\mu \in \mathbb{N}$. This requirement gives a very simple structure for the ring of $\mathfrak{o}$-linear differential operators $\mathscr{D}_{\mathfrak{X}/\mathfrak{o}}$ on $\mathfrak{X}$ (see section 4.1).

The special fibre of $\mathfrak{X}$ is denoted by X_0 and $\mathscr{O}(X_0) =: A_0$. The generic fibre (seen as a scheme or as a rigid space) is denoted by X and $\mathscr{O}(X) =: A_K$. We assume that $\mathfrak{X}$ has an $\mathfrak{o}$-rational point $\xi : \operatorname{Spf} \mathfrak{o} \longrightarrow \mathfrak{X}$: The induced k-(resp. K-) rational point of X_0 is denoted ξ_0 (resp. ξ_K). This point will give the fibre functor.

4.1 Differential operators

I thank Pierre Berthelot for suggesting the approach on differential operators given here. The suggestion helped us overcome some needless difficulties and allowed us to use documented knowledge to support the results. These are certainly well-known for mathematicians working in Rigid Geometry. We have limited ourselves in making the definitions and pointing out the conclusions which will allow the reader to construct more satisfactory knowledge using any standard text on differential operators (e.g. [1] or EGA IV_4).

Let Λ be a noetherian ring which is complete for the $\mathfrak{a}$-adic topology. Let R be a Λ-adic ring, that is, R is a Λ-algebra which is complete for the $\mathfrak{a}R$-adic topology. We further assume that R is noetherian. Let $\Lambda_\mu = \Lambda/\mathfrak{a}^\mu$ and denote the base change map $\Lambda\text{-Mod} \longrightarrow \Lambda_\mu\text{-Mod}$ by a sub-script $(?)_\mu$.

Recall

Definition 14. *Let $P_{R/\Lambda} = R\hat{\otimes}_\Lambda R$ and let $\mathfrak{I}$ be the kernel of multiplication $P_{R/\Lambda} \longrightarrow R$. The module of principal parts of order $\leq n$ of R over Λ, $P^n_{R/\Lambda}$, is the quotient $P_{R/\Lambda}/\mathfrak{I}^{n+1}$. It can be seen as an R-algebra in two different ways via the homomorphisms $d_1(a) = a \otimes 1$ and $d_2(a) = 1 \otimes a$.*

Assume further that R is topologically of finite type over Λ:

$$R = \Lambda\langle T_1, \dots, T_r\rangle/\mathfrak{b} \quad (\Lambda\langle T\rangle \text{ is the } \mathfrak{a}\text{-adic completion of } \Lambda[T]).$$

Under this condition it is easy to see that $P^n_{R/\Lambda}$ is finite over R.

Proposition 15 (EGA IV_4, 16.4.5, p. 19)**.** *The natural map*

$$P^n_{R/\Lambda} \otimes \Lambda_\mu \longrightarrow P^n_{R_\mu/\Lambda_\mu}$$

is an isomorphism. In particular,

$$P^n_{R/\Lambda} = \varprojlim_{\mu} P^n_{R_\mu/\Lambda_\mu}.$$

For us (and most rigid geometers) a homomorphism of Λ-adic algebras $\varphi : R \longrightarrow S$ is said to be etale if each one of the φ_μ is an etale homomorphism of rings.

Theorem 16. *Let S be another Λ-adic ring which is topologically of finite type and let $\varphi : R \longrightarrow S$ be an etale homomorphism. Then the natural homomorphism*

$$S\hat{\otimes}_R P^n_{R/\Lambda} \longrightarrow P^n_{S/\Lambda}$$

is an isomorphism.

Proof. By definition, $\varphi_\mu : R_\mu \longrightarrow S_\mu$ is an etale homomorphism of noetherian rings: of finite type and formally etale for the discrete topologies. From [14, Thm. 3.2.3, p.39] the natural homomorphism

$$S_\mu \otimes_{R_\mu} P^n_{R_\mu/\Lambda_\mu} \longrightarrow P^n_{S_\mu/\Lambda_\mu}$$

is an isomorphism. Using Proposition 15 the result follows by taking the limit over μ and from the definition of complete tensor product (EGA I (**0**,7.7.1), p. 75). □

Recall that R is said to be Λ-smooth if there is an etale homomorphism

$$\psi : \Lambda\langle T_1, \dots, T_s\rangle \longrightarrow R.$$

Corollary 17. *Assume that R is Λ-smooth and adopt the notation above. If $x_1, \dots, x_s$ are the images of $T_1, \dots, T_s$, then $P^n_{R/\Lambda}$ is free on the basis*

$$\prod_{i=1}^{s}(1\hat{\otimes}x_i - x_i\hat{\otimes}1)^{q_i}, \quad (q_1, \dots, q_s) \in \mathbb{N}^s, \ \sum q_i \leq n.$$

In particular, the ring of Λ-linear differential operators over R is the sub-algebra $\mathscr{D}$ of $\mathrm{End}_\Lambda(R)$ (R acts on the left) freely generated over R by operators

$$\partial_{\mathbf{q}} = \frac{1}{\mathbf{q}!}\frac{\partial^{\mathbf{q}}}{\partial x^{\mathbf{q}}}, \quad \mathbf{q} \in \mathbb{N}^s,$$

where the $\partial_{\mathbf{q}}$ satisfy the usual relations

$$\partial_{\mathbf{q}} x^{\mathbf{r}} = \begin{pmatrix} \mathbf{r} \\ \mathbf{q} \end{pmatrix} x^{\mathbf{r}-\mathbf{q}}; \quad \partial_{\mathbf{q}}\partial_{\mathbf{r}} = \begin{pmatrix} \mathbf{q}+\mathbf{r} \\ \mathbf{q} \end{pmatrix} \partial_{\mathbf{q}+\mathbf{r}}$$

(see also 2.6 of [1]).

□

Remarks: (1) On the proof of Theorem 16: We could not find any reference in EGA supporting the fact that P^n base-changes nicely for etale morphism. On the other hand the corresponding result for $\Omega^1 \cong \mathfrak{I}/\mathfrak{I}^2$ (where $\mathfrak{I}$ is the ideal of the kernel of multiplication $R\hat{\otimes}_\Lambda R \longrightarrow R$) is carefully analyzed in EGA IV_4. As Berthelot points out, if R is known to be Λ-smooth, then Theorem 16 follows from

$$\mathrm{Gr}^\bullet_{\mathfrak{I}}(P_{R/\Lambda}) \cong \mathrm{Symm}^\bullet_R(\mathfrak{I}/\mathfrak{I}^2)$$

and the case of Ω^1.

(2) We will only be interested in *smooth* $\mathfrak{o}$-adic rings; from the previous remark, the use of [14] in Theorem 16 could have been avoided.

The objects of study and important notations:

1. The category $\mathbf{str}(\mathfrak{X}/\mathfrak{o})$ will be called the category of stratified sheaves or modules over $\mathfrak{X}$. The objects are finitely generated A-modules $\mathscr{M}$ endowed with a left action of the ring of differential operators $\mathscr{D}_{\mathfrak{X}/\mathfrak{o}}$

$$\nabla : \mathscr{D} \longrightarrow \mathrm{End}_{\mathfrak{o}}(\mathscr{M})$$

which is compatible with the left-action of A on $\mathscr{D}$ and $\mathscr{M}$. The arrows are homomorphisms of $\mathscr{D}_{\mathfrak{X}/\mathfrak{o}}$-modules and are called horizontal maps. This category is monoidal and the forgetful functor $\mathbf{str}(\mathfrak{X}/\mathfrak{o}) \longrightarrow \mathrm{coh}(\mathfrak{X})$ is monoidal strict [4], [11]. To other formulations of these definitions, the reader should consult [1]–these will be used in section 6.2.1.

2. The category $\mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ will be the full sub-category of $\mathbf{str}(\mathfrak{X}/\mathfrak{o})$ whose objects are flat as A-modules (and hence locally free of constant rank). That the notation is compatible with the notation introduced in section 2.3 will follow from Lemma 19. For $\mathscr{M} \in \mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ we let $\mathscr{M}^{\vee}$ denote its dual.

3. Analogous notations are in force for the special and generic fibres of A: $\mathbf{str}(X_0/k)$ and $\mathbf{str}(X/K)$. The natural monoidal functor

$$\mathbf{str}(\mathfrak{X}/\mathfrak{o}) \longrightarrow \mathbf{str}(X_0/k)$$

will be named r. Note that, since

$$\mathscr{D}_{\mathfrak{X}/\mathfrak{o}} \otimes_{\mathfrak{o}} k = \mathscr{D}_{X_0/k},$$

r admits a section $\mathbf{str}(X_0/k) \longrightarrow \mathbf{str}(\mathfrak{X}/\mathfrak{o})$ which is the identity on the underlying categories of A-modules.

4.2 Fibre functor

Proposition 18. *The category* $\mathbf{str}(\mathfrak{X}/\mathfrak{o})$ *is an abelian and* $\mathfrak{o}$*-linear. The functor* $\omega = \xi^* : \mathbf{str}(\mathfrak{X}/\mathfrak{o}) \longrightarrow (\mathfrak{o} - \mathrm{mod})$ *is faithful and exact.*

The proof will make use of the following lemma and its corollary.

Lemma 19. *The category* $\mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ *is the full subcategory of* $\mathbf{str}(\mathfrak{X}/\mathfrak{o})$ *whose objects have no* ϖ*-torsion when regarded as* A*-modules.*

Proof. Let $(\mathscr{M}, \nabla)$ be an object of $\mathbf{str}(\mathfrak{X}/\mathfrak{o})$. By the local flatness criterion [9, Thm. 22.3, p. 174], $\mathscr{M}$ is flat as an A-module if and only if $\mathrm{Tor}_1^A(\mathscr{M}, A_0) = 0$ and $\mathscr{M}/\varpi\mathscr{M} =: M_0$ is flat over A_0. Now M_0 is a $\mathscr{D}_{X_0/k}$-module over the algebraic smooth k-scheme X_0 and hence is flat (locally free). The condition on Tor is precisely the absence of ϖ-torsion. □

Corollary 20. *The category* $\mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ *is stable under sub-objects: if* $\mathscr{N} \subseteq \mathscr{M}$ *is an inclusion in* $\mathbf{str}(\mathfrak{X}/\mathfrak{o})$ *with* $\mathscr{M} \in \mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$*, then* $\mathscr{N}$ *belongs to* $\mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$.

Proof of Prop. 18. The first assertion is obvious. We now prove that ω is left exact and hence exact. We can assume that $\ker\xi = (x_1, \ldots, x_n)$. Consider an exact sequence

$$0 \longrightarrow \mathscr{M}' \longrightarrow \mathscr{M} \longrightarrow \mathscr{M}'' \longrightarrow 0$$

in $\mathbf{str}(\mathfrak{X}/\mathfrak{o})$. We will show that $0 \longrightarrow \omega\mathscr{M}' \longrightarrow \omega\mathscr{M} \longrightarrow \omega\mathscr{M}'' \longrightarrow 0$ is exact by proving that $\mathrm{Tor}_1^A(\mathscr{M}'', A/(x_1, \ldots, x_n)) = 0$. So we set our goal in proving that $\mathrm{Tor}_1^A(\mathscr{M}, A/(x_1, \ldots, x_n))$ vanishes for any object $\mathscr{M} \in \mathbf{str}(\mathfrak{X}/\mathfrak{o})$. If $\mathscr{M}$ is in $\mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ then we are done. In general we have an exact sequence in $\mathbf{str}(\mathfrak{X}/\mathfrak{o})$

$$0 \longrightarrow \mathscr{M}_\tau \longrightarrow \mathscr{M} \longrightarrow \mathscr{M}_f \longrightarrow 0 \tag{6}$$

where $\mathscr{M}_\tau$ is the sub-object of $\mathscr{M}$ given by the elements killed by a power of ϖ. By Lemma 19, $\mathscr{M}_f$ is in $\mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ and hence $\mathrm{Tor}_i^A(\mathscr{M}, N) = \mathrm{Tor}_i^A(\mathscr{M}_\tau, N)$ for all $i > 0$. We assume now that $\mathscr{M}$ is killed by ϖ^m. Let $\mathscr{M}_i := \ker(\varpi^i : \mathscr{M} \longrightarrow \mathscr{M})$. We have a filtration of $\mathscr{M}$ by sub-objects

$$0 \subseteq \mathscr{M}_1 \subseteq \cdots \subseteq \mathscr{M}_m = \mathscr{M} \tag{7}$$

and the factor modules $\mathscr{M}_{i+1}/\mathscr{M}_i$ are A_0-modules. Hence, by general homological algebra, we will succeed in showing that $\mathrm{Tor}_1^A(\mathscr{M}, A/(x_1, \ldots, x_n)) = 0$ for all $\mathscr{M} \in \mathbf{str}(\mathfrak{X}/\mathfrak{o})$ if we can show that $\mathrm{Tor}_1^A(\mathscr{M}, A/(x_1, \ldots, x_n)) = 0$ for all $\mathscr{M} \in \mathbf{str}(\mathfrak{X}/\mathfrak{o})$ which are *killed by* ϖ. So let $\mathscr{M} \in \mathbf{str}(\mathfrak{X}/\mathfrak{o})$ be such that $\varpi\mathscr{M} = 0$. Let $T_j = \mathrm{Tor}_1^A(\mathscr{M}, A/(x_1, \ldots, x_j))$. We will use induction on j. Using the exact sequence

$$0 \longrightarrow A \xrightarrow{x_1} A \longrightarrow A/(x_1) \longrightarrow 0$$

we see that T_1 is the kernel of multiplication by x_1 on $\mathscr{M}$. This last group is certainly 0 as the $\mathscr{D}_{X_0/k}$-module $\mathscr{M}$ has no $(x_1 + \varpi A)$-torsion. The first inductive step is taken and we assume that $T_{j-1} = 0$. Let $A_\nu = A/(x_1, \ldots, x_\nu)$. We have an exact sequence of A-modules $0 \longrightarrow A_{j-1} \xrightarrow{x_j} A_{j-1} \longrightarrow A_j \longrightarrow 0$ whose associate long exact sequence gives an exact sequence

$$0 \longrightarrow T_j \longrightarrow A_{j-1} \otimes_A \mathscr{M} \xrightarrow{x_j} A_{j-1} \otimes_A \mathscr{M}.$$

Again, we see that $T_j = \ker(A_{j-1} \otimes_A \mathscr{M} \xrightarrow{x_j} A_{j-1} \otimes_A \mathscr{M})$. But $A_{j-1} \otimes_A \mathscr{M}$ has a natural structure of $\mathscr{D}_{Y/k}$-module, where Y is the smooth algebraic k-scheme

$$\mathrm{Spec}\, A/(\varpi, x_1, \ldots, x_{j-1}),$$

and therefore has no x_j-torsion. Hence $T_j = 0$ and we have just proved that ω is exact.

To show that ω is faithful, we only need to prove that if $\omega\mathscr{M} = 0$ for some $\mathscr{M} \in \mathbf{str}(\mathfrak{X}/\mathfrak{o})$, then $\mathscr{M} = 0$. This is certainly true if $\mathscr{M}$ is in $\mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$, as in this case $\mathscr{M} \subset \mathscr{M} \otimes K$. In general, using the sequence in eq. (6) it follows that $\mathscr{M} = \mathscr{M}_\tau$ and $\mathscr{M}$ is killed by ϖ^m. If $m = 1$, then $\mathscr{M}$ is a $\mathscr{D}_{X_0/k}$-module whose fibre at the k-point associated to ξ is zero: it follows that $\mathscr{M} = 0$. The case where $m > 1$ is treated using the filtration in eq. (7).

□

4.3 Special sub-quotients

Definition 21. *1. An embedding $\mathscr{V}' \subseteq \mathscr{V}$ in $\mathbf{str}(\mathfrak{X}/\mathfrak{o})$ will be called special if $\mathscr{V}, \mathscr{V}'$ and the cokernel $\mathscr{V}/\mathscr{V}'$ are in $\mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$. We will say that $\mathscr{V}'' \in \mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ is a special sub-quotient of $\mathscr{V}$ if there is a special embedding $\mathscr{V}' \subseteq \mathscr{V}$ and a quotient map $\mathscr{V}' \longrightarrow \mathscr{V}''$.*

2. *Given a family of objects $\mathscr{F}$ in $\mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ possessing $\mathbf{1}$ and stable by direct sums, duals and tensor products, we define the category $\langle \mathscr{F} \rangle^s$ as the full sub-category of $\mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ having as objects the special subquotients of objects in $\mathscr{F}$.*

3. *Given $\mathscr{M} \in \mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ and $a, b \in \mathbb{N}$, write*

$$\mathscr{M}_b^a = \mathscr{M}^{\otimes a} \otimes (\mathscr{M}^\vee)^{\otimes b}.$$

For multi-indexes $a, b \in \mathbb{N}^s$, define

$$\mathscr{M}_b^a = \mathscr{M}_{b_1}^{a_1} \oplus \cdots \oplus \mathscr{M}_{b_s}^{a_s}.$$

If the family $\mathscr{F}$ is given by the $\mathscr{M}_b^a$ where a, b are multi-indexes, then the category $\langle \mathscr{F} \rangle^s$ is denoted by $\langle \mathscr{M} \rangle_\otimes^s$.

One of the purposes of the above definition is

Lemma 22. *Let $\mathscr{F}$ be a family of objects as in Definition 21 above. Then $\langle \mathscr{F} \rangle^s$ is stable by direct sums, tensor products, duals and special sub-quotients.*

Proof. Given the rigidity of the definitions this result is quite straightforward. It is obvious that $\langle \mathscr{F} \rangle^s$ is stable by direct sums. Equally obvious is the statement about tensor products, since for $\mathscr{E} \in \mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ the functor $? \otimes \mathscr{E}$ is exact.

We now show that $\langle \mathscr{F} \rangle^s$ is stable by taking duals. Start by observing that an exact sequence

$$0 \longrightarrow \mathscr{E}'' \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}' \longrightarrow 0$$

in $\mathbf{str}(\mathfrak{X}/\mathfrak{o})$ with $\mathscr{E}, \mathscr{E}', \mathscr{E}'' \in \mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ gives rise to an exact sequence

$$0 \longrightarrow (\mathscr{E}')^{\vee} \longrightarrow \mathscr{E}^{\vee} \longrightarrow (\mathscr{E}'')^{\vee} \longrightarrow 0.$$

Let $\mathscr{E}' \subseteq \mathscr{E}$ be special and let $\alpha : \mathscr{E}' \longrightarrow \mathscr{E}''$ be an epimorphism. Assume that $\mathscr{E} \in \mathscr{F}$. The embedding $\mathscr{E}' \subseteq \mathscr{E}$ gives a quotient map $\mathscr{E}^{\vee} \longrightarrow (\mathscr{E}')^{\vee}$ and the quotient map α gives an embedding $(\mathscr{E}'')^{\vee} \subseteq (\mathscr{E}')^{\vee}$. Let $\mathscr{V} \subseteq \mathscr{E}^{\vee}$ be the pre-image of $(\mathscr{E}'')^{\vee}$. The Snake Lemma shows that $\mathscr{E}^{\vee}/\mathscr{V} \cong (\mathscr{E}')^{\vee}/(\mathscr{E}'')^{\vee}$. Because $(\mathscr{E}')^{\vee}/(\mathscr{E}'')^{\vee} \cong \ker(\alpha)^{\vee}$ and $\ker(\alpha) \in \mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ (Cor. 20), it follows that $\mathscr{V}$ is a special sub-object of $\mathscr{E}^{\vee}$ which maps epimorphically to $(\mathscr{E}'')^{\vee}$. Since $\mathscr{E}^{\vee} \in \mathscr{F}$, we obtain that $\langle \mathscr{F} \rangle^{s}$ is stable by duality.

The proof that $\langle \mathscr{F} \rangle^{s}$ is stable by special sub-quotients is just more of the same formalism based on Lemma 19 and Corollary 20 and we omit it. □

Putting Corollary 20, Lemma 22 and Proposition 18 together, we obtain that the category $\langle \mathscr{F} \rangle^{s}$ is under the conditions of Theorem 6 and Corollary 7. Hence, we can make the following definition and take note of the working properties which follow.

Definition 23 (the differential Galois group over $\mathfrak{o}$)**.** *Let $\mathscr{F}$ be a family of objects in $\mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ which is stable under direct sums, tensor products and duals. Also, assume that $\mathbf{1} \in \mathscr{F}$. The differential Galois group of $\mathscr{F}$ at ω is defined as the faithfully flat $\mathfrak{o}$-group-scheme*

$$\Pi(\mathscr{F}, \omega) = \Pi(\langle \mathscr{F} \rangle^{s}, \mathbf{str}(\mathfrak{X}/\mathfrak{o}), \omega)$$

given by Theorem 6. If $\langle \mathscr{F} \rangle^{s} = \langle \mathscr{M} \rangle^{s}_{\otimes}$, we write $\Pi(\mathscr{F}, \omega) = \Pi(\mathscr{M}, \omega)$.

4.3.1 Working properties of the differential Galois group

Almost all important results to follow are based on properties arising from the commutative diagram

$$\begin{array}{ccccc} & & \mathrm{Rep}_{\mathfrak{o}}(\Pi(\mathscr{M}, \omega)) & & \\ & \overset{\overline{\omega}_0}{\nearrow} & \downarrow S & \overset{\text{forget}}{\searrow} & \\ \langle \mathscr{M} \rangle^{s}_{\otimes} & \longrightarrow & \mathbf{str}(\mathfrak{X}/\mathfrak{o}) & \underset{\omega}{\longrightarrow} & (\mathfrak{o} - \mathrm{mod}) \end{array} \tag{8}$$

where S is monoidal, exact, $\mathfrak{o}$-linear and faithful; $\overline{\omega}_0$ is monoidal, $\mathfrak{o}$-linear and fully-faithful.

As we are allowed to use Corollary 7, the first statement of the important theorem below follows. From Proposition 12 we obtain the second statement.

Theorem 24. *The monoidal functor $\overline{\omega}_0 : \langle \mathscr{M} \rangle^{s}_{\otimes} \longrightarrow \mathrm{Rep}^{\#}_{\mathfrak{o}}(\Pi)$ is an equivalence and the representation $\overline{\omega}_0(\mathscr{M})$ is faithful. In particular Π is algebraic.*

4.3.2 The universal torsor or PV-ring

As the definition of the universal torsor or Picard-Vessiot ring given here is *tentative* and will only be needed in the case where $\Pi(\mathscr{M},\omega)$ is finite (in which case a more responsible discussion is available, see Prp. 34) we will be informal in what follows.

Let (R,ρ_l) be the left-regular representation of $\Pi=\Pi(\mathscr{M},\omega)$. It is a monoid (an algebra) in the category $\mathrm{Ind}(\mathrm{Rep}^{\#}_{\mathfrak{o}}(\Pi))$ (all facts used here concerning categories of inductive limits can be found in SGA 4, exposé I, §8). By means of the functor S, we obtain an A-algebra $B_{\mathscr{M}}$. This A-algebra is faithfully flat as the quotient $B_{\mathscr{M}}/A$ is flat. By transport of structure, $B_{\mathscr{M}}$ is a $\mathscr{D}_{\mathfrak{X}/\mathfrak{o}}$-module and multiplication $B_{\mathscr{M}}\otimes_A B_{\mathscr{M}}\longrightarrow B_{\mathscr{M}}$ as well as identity $A\longrightarrow B_{\mathscr{M}}$ are homomorphism of $\mathscr{D}_{\mathfrak{X}/\mathfrak{o}}$-modules. Furthermore, $\mathrm{Spec}\,B_{\mathscr{M}}\longrightarrow\mathrm{Spec}\,A$ is a Π-torsor. From the formalism of Nori [4, 2.3.2(d), p. 701] it follows that $\mathscr{M}\otimes_A B_{\mathscr{M}}\cong B_{\mathscr{M}}^{\oplus r}$ as $\mathscr{D}_{\mathfrak{X}/\mathfrak{o}}$-modules.

It is unclear to us whether this is the "correct" object to be regarded since $B_{\mathscr{M}}$ is in general no longer in the category of $\mathfrak{o}$-adic algebras.

5 Relation between the differential Galois group over $\mathfrak{o}$ and over K

Let $\mathscr{M}\in\mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ and let $M\in\mathbf{str}(X/K)$ be the stratified module induced by $\mathscr{M}$ on the generic fibre X (for our purposes, X can be seen as an analytic space or–by the coherence Theorems of Kiehl–as the spectrum $\mathrm{Spec}\,A\otimes K$). Our goal is to compare the differential Galois groups

$\Pi(\mathscr{M})=\Pi(\langle\mathscr{M}\rangle^s_{\otimes},\mathbf{str}(\mathfrak{X}/\mathfrak{o}),\omega)$ (obtained by Bruguières-Nori theory, section 4.3)

and

$\Pi(M)=\Pi(\langle M\rangle_{\otimes},\mathbf{str}(X/K),\omega\otimes K)$ (obtained by classical Tannakian theory).

This is considerably easier than the comparison result between $\Pi(\mathscr{M})$ and $\Pi(\mathscr{M}\otimes k)$ in section 6 below; there is a much stronger relation between $\langle\mathscr{M}\rangle^s_{\otimes}$ and $\langle M\rangle_{\otimes}$:

Theorem 25. *Let $\gamma:\mathbf{str}(\mathfrak{X}/\mathfrak{o})\longrightarrow\mathbf{str}(X/K)$ be the obvious functor. Then*
i) The restricted functor $\gamma|\langle\mathscr{M}\rangle^s_{\otimes}$ takes values in $\langle M\rangle_{\otimes}$.
ii) Given $N\in\langle M\rangle_{\otimes}$, there exists $\mathscr{N}\in\langle\mathscr{M}\rangle^s_{\otimes}$ with $\gamma(\mathscr{N})=N$.
iii) For $\mathscr{N}_1,\mathscr{N}_2\in\langle\mathscr{M}\rangle^s_{\otimes}$, the natural K-linear homomorphism

$$\mathrm{Hom}(\mathscr{N}_1,\mathscr{N}_2)\otimes_{\mathfrak{o}}K\longrightarrow\mathrm{Hom}(\gamma\mathscr{N}_1,\gamma\mathscr{N}_2)$$

is bijective.

Using the flatness of the $\mathfrak{o}$-algebra K and the fact that the construction of Bruguières-Nori in section 2 is just the usual construction when Λ is a field, we obtain:

Corollary 26. *There exists an isomorphism of K-group-schemes*

$$\Pi(M) \longrightarrow \Pi(\mathscr{M}) \otimes K$$

making the diagram of categories

$$\begin{array}{ccccc} \langle \mathscr{M} \rangle_{\otimes}^{s} & & \xrightarrow{\quad\quad\quad\quad \gamma \quad\quad\quad\quad} & & \langle M \rangle_{\otimes} \\ \Big\downarrow{\scriptstyle \varpi_0} & & & & \Big\downarrow{\scriptstyle \omega\otimes K} \\ \mathrm{Rep}_{\mathfrak{o}}(\Pi(\mathscr{M})) & \xrightarrow[\otimes K]{} & \mathrm{Rep}_K(\Pi(\mathscr{M}) \otimes K) & \longrightarrow & \mathrm{Rep}_K(\Pi(M)). \end{array}$$

commute.

After [11] the method of proof of Theorem 25 is certainly folklore. It consists of looking for Banach norms on each stratified module $(N, \nabla) \in \langle M \rangle_{\otimes}$ in such a way that

$$\|\nabla\left(\partial_{\mathbf{q}}\right)\| \leq 1, \quad \mathbf{q} \in \mathbb{N}^n \quad \text{(see Corollary 17 for notation).}$$

Recall that $\mathfrak{o}\langle T_1, \ldots, T_r \rangle$ has a $\mathfrak{o}$-Banach algebra norm $\|\cdot\| : \mathfrak{o}\langle T \rangle : \longrightarrow \{0\} \cup q^{\mathbb{N}}$ (where $\|\varpi\| = q < 1$) such that the topology defined by $\|\cdot\|$ coincides with the ϖ-adic topology. In fact,

$$\varpi^m \mathfrak{o}\langle T \rangle = \{f \in \mathfrak{o}\langle T \rangle : \ \|f\| \leq q^m\} \qquad (m \in \mathbb{N}). \tag{9}$$

Every ideal of $\mathfrak{o}\langle T \rangle$ is closed and we can endow

$$A = \mathfrak{o}\langle T \rangle / \mathfrak{a}$$

with a $\mathfrak{o}$-Banach algebra norm $\|\cdot\|_A$ for which the analogue of eq. (9) is valid. In particular, every finitely generated A-module $\mathscr{E}$ has a structure of A-Banach module $\|\cdot\|_{\mathscr{E}} : \mathscr{E} \longrightarrow \{0\} \cup q^{\mathbb{N}}$ such that

$$\varpi^m \mathscr{E} = \{e \in \mathscr{E} : \ \|e\|_{\mathscr{E}} \leq q^m\}. \tag{10}$$

Let A_K be the affinoid K-algebra $A \otimes_{\mathfrak{o}} K$. For any A-Banach module without torsion $(\mathscr{E}, \|\cdot\|_{\mathscr{E}})$ we induce a structure of A_K-Banach module on $\mathscr{E} \otimes K$ which satisfies (using eq. (10))

$$\mathscr{E} = \{e \in \mathscr{E} \otimes K : \ \|e\| \leq 1\}. \tag{11}$$

Proof of Thm. 25. *i*) is quite obvious due to the $\mathfrak{o}$-flatness of K.

ii) We note that γ is a monoidal functor, therefore $\gamma(\mathscr{M}_b^a) \cong M_b^a$ (notation of definition 21). Put $\mathscr{M}_b^a = \mathscr{E}$ and $M_b^a = E$. Let $N \subseteq E$ be an inclusion in the category $\mathbf{str}(X/K)$. Define the A-module $\mathscr{N} = \mathscr{E} \cap N = \{e \in N : \|e\| \leq 1\}$, the norm $\|\cdot\|$ being the one constructed above. Because A is noetherian and $\mathscr{E}$ is finitely generated, $\mathscr{N}$ is finitely generated over A. Equation (11) shows that $\mathscr{N} \otimes K = N$. As the action of $\mathscr{D}_{\mathfrak{X}/\mathfrak{o}}$ ($\subset \mathscr{D}_{X/K}$) on E leaves N and $\mathscr{E}$ invariant, we see that $\mathscr{N} \in \mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ (use Corollary 20). Note that, by construction, $\mathscr{E}/\mathscr{N}$ has no ϖ-torsion; an application of Lemma 19 shows that $\mathscr{N} \subseteq \mathscr{E}$ is special. We have proved that any sub-object of M_b^a is in the essential image of γ. That any quotient of N is also in the essential image of γ follows easily from the reasoning above and the fact that for $\mathscr{N}' \subseteq \mathscr{N}$ special, the inclusion $\mathscr{N}' \subseteq \mathscr{E}$ is also special.

iii) Using linear algebra we are left with the verification of the tautological fact: For each $(\mathscr{E}, \nabla) \in \mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$, the natural map

$$\mathscr{E}^{\nabla} \otimes_{\mathfrak{o}} K \longrightarrow (\gamma\mathscr{E})^{\nabla}$$

(the superscript ∇ denotes horizontal sections) is bijective. □

6 Relation between the differential Galois group over $\mathfrak{o}$ and the differential Galois group of the reduced module

This section contains the most important results of this work (Theorem 27 and Theorem 28). They relate the differential Galois group of $\mathscr{M} \in \mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ over $\mathfrak{o}$ with the differential Galois group of $\mathscr{M} \otimes k \in \mathbf{str}(X_0/k)$. The inspirations for the statement of the latter theorems are, of course, Conjecture 8.5 and Example 8.6 in [11]. Understanding the latter is an important step in making the correct definitions (see the example in section 6.2.3 and also the negative example in section 3.2).

The proof of Theorem 27 is stimulated by the principle that if the $\mathscr{D}$-modules over $\mathfrak{X}$ correspond to representations of a group-scheme, then the $\mathscr{D}$-modules over X_0 should correspond to the representations of the mod ϖ group and the natural map $\mathbf{str}(X_0/k) \longrightarrow \mathbf{str}(\mathfrak{X}/\mathfrak{o})$ should be seen as the restriction of scalars.

The method of proof used in Theorem 28 (section 6.2.3) is based on Nori's theory of inversion of the Tannakian construction applied to $\mathscr{D}$-modules (see Lemma 34 and [4, 2.3.2]), Grothendieck's remarkable equivalence and the well-understood principle that differential equations with finite differential Galois groups are etale coverings (section 6.2.1).

As we already mentioned in the general introduction, the relation between the angle pursued here and that of [11] is quite strong with one main technical difference: we see the "integral" (over A) module $\mathscr{M}$ as part of the initial data. As the theorems and the example at the end of section 6.2.3 show, this assumption has its advantages.

We fix $\mathscr{M} \in \mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ and let $\Pi = \Pi(\mathscr{M}, \mathbf{str}(\mathfrak{X}/\mathfrak{o}), \omega)$ as in Definition 23. We will use the notations introduced in diagram (8). Also, let $M_0 \in \mathbf{str}(X_0/k)$ be the reduction of $\mathscr{M}$. Denote the differential Galois group of M_0 in the category $\mathbf{str}(X_0/k)$ by $\Pi(M_0)$ (the fibre functor is induced by the k-rational point ξ_0) and put $\Pi \otimes_{\mathfrak{o}} k = \Pi_0$. We want to obtain relations between Π and $\Pi(M_0)$: in section 6.1 we show that $\Pi(M_0)$ is a closed subgroup-scheme of Π_0; in section 6.2 we have Theorem 28 which has as a consequence the production of a class of examples where the closed embedding $\Pi(M_0) \subseteq \Pi_0$ is an isomorphism.

6.1 A closed embedding

Our goal here is to realize $\Pi(M_0)$ as a closed subgroup scheme of Π_0 (Theorem 27 below). This will be achieved, via Tannakian theory, by constructing an exact monoidal functor

$$\nu : \mathrm{Rep}_k(\Pi_0) \longrightarrow \mathrm{Rep}_k(\Pi(M_0))$$

which is the identity on the underlying category of vector-spaces and which has a faithful representation of $\Pi(M_0)$ in its essential image. The construction will use the notion of restriction of scalars in group theory.

Digression: Restriction of scalars [6, Ch. I, 10.1, p. 162] Let G be a flat group-scheme over $\mathfrak{o}$ and write $G_0 = G \otimes_{\mathfrak{o}} k$. The restriction of scalars is a functor

$$\sigma_G : \mathrm{Rep}_k(G_0) \longrightarrow \mathrm{Rep}_{\mathfrak{o}}(G)$$

which is the identity on the underlying categories of $\mathfrak{o}$-modules. It is defined as follows. For V a representation of G_0, let $\rho : V \longrightarrow V \otimes_k \mathscr{O}(G_0)$ be the comodule structure. Since the co-domain of ρ is $V \otimes_k (k \otimes_{\mathfrak{o}} \mathscr{O}(G)) = V \otimes_{\mathfrak{o}} \mathscr{O}(G)$, we have immediately defined on the $\mathfrak{o}$-module V a structure of $\mathscr{O}(G)$-comodule and this is the functor σ. Here are some properties:

1. Let V be a free representation of G, then $\sigma(V \otimes k)$ is the representation $\mathrm{coker}(V \xrightarrow{\varpi} V)$.
2. Let $b : \mathrm{Rep}_{\mathfrak{o}}(G) \longrightarrow \mathrm{Rep}_k(G_0)$ be the base change functor. Then b is a left adjoint of σ and $b \circ \sigma \cong \mathrm{id}$.

3. Let H be another $\mathfrak{o}$-group-scheme and let $f : G \longrightarrow H$ be a homomorphism. Then $\sigma_G \circ \mathrm{res}(f \otimes k) = \mathrm{res}(f) \circ \sigma_H$.
4. There is a natural isomorphism of functors $\otimes_{\mathrm{Rep}_{\mathfrak{o}}(G)} \circ (\sigma \times \sigma) \Rightarrow \sigma \circ \otimes_{\mathrm{Rep}_k(G_0)}$ (because k is a quotient of $\mathfrak{o}$). Of course, this means that σ is a monoidal functor *without* the property of mapping $\mathbf{1}_{\mathrm{Rep}_k(G_0)}$ to $\mathbf{1}_{\mathrm{Rep}_{\mathfrak{o}}(G)}$.

End of digression

Define

$$\nu : \mathrm{Rep}_k(\Pi_0) \longrightarrow \mathbf{str}(X_0/k)$$

as the composition (to repeat: S below is defined by the diagram (8))

$$\mathrm{Rep}_k(\Pi_0) \xrightarrow{\sigma} \mathrm{Rep}_{\mathfrak{o}}(\Pi) \xrightarrow{S} \mathbf{str}(\mathfrak{X}/\mathfrak{o}) \xrightarrow{r} \mathbf{str}(X_0/k).$$

Since r takes $\mathrm{coker}(\mathbf{1} \xrightarrow{\varpi} \mathbf{1})$ to $\mathscr{O}_{X_0}$ with the trivial stratification, it follows that ν is monoidal functor. Note that ν is also exact, since $S \circ \sigma$ takes values in the category of $\mathscr{D}_{\mathfrak{X}/\mathfrak{o}}$-modules which are killed by ϖ.

Theorem 27. *The functor ν above has range in the category $\langle M_0 \rangle_\otimes$ and induces a closed embedding of $\Pi(M_0)$ into $\Pi_0 = \Pi(\mathscr{M}) \otimes_{\mathfrak{o}} k$.*

Proof. The main point is that the representation $U = \overline{\omega}_0(\mathscr{M})$ of Π is faithful (Thm. 24), consequently $U_0 := U \otimes k$ is a faithful representation of Π_0. Given $V \in \mathrm{Rep}_k(\Pi_0)$, there are representations $V'' \subseteq V' \subseteq (U_0)^a_b$ with $V = V'/V''$. Since ν is exact, $\nu(V)$ is a sub-quotient of $\nu((U_0)^a_b) = \nu(U_0)^a_b$. But

$$\sigma(U_0) = \mathrm{coker}(U \xrightarrow{\varpi} U)$$

and then

$$S \circ \sigma(U_0) = \mathrm{coker}(S(U) \xrightarrow{\varpi} S(U)) = \mathrm{coker}(\mathscr{M} \xrightarrow{\varpi} \mathscr{M}).$$

It follows that $\nu(U_0) \cong M_0$. We have proved that $\nu(V)$ is a sub-quotient of some $(M_0)^a_b$ and this shows that ν has range in $\langle M_0 \rangle_\otimes$. It is immediate to see that ν preserves the fibre functors, hence it induces a monoidal functor $\mathrm{Rep}_k(\Pi_0) \longrightarrow \mathrm{Rep}_k(\Pi(M_0))$ which is the identity on the underlying vector spaces. By standard Tannakian theory $\mathrm{Rep}_k(\Pi_0) \longrightarrow \mathrm{Rep}_k(\Pi(M_0))$ comes from a homomorphism of group-schemes; that this homomorphism is a closed embedding is a consequence of the classical criterion presented in [3, 2.21, p. 139]. □

6.2 Finite differential Galois group over $\mathfrak{o}$

Take k algebraically closed. We keep the notations of section 6.1 and assume that

$$\Pi \text{ is finite.}$$

Under this hypothesis we want to be able to say something more about the closed embedding

$$\beta : \Pi(M_0) \longrightarrow \Pi_0$$

constructed in Theorem 27.

Theorem 28. *Under the above assumptions and notations. The closed embedding β induces an isomorphism between $\Pi(M_0)$ and the etale group scheme of connected components of Π_0.*

Let us give some explanation about the group-scheme of connected components of Π and of Π_0 in order to clarify the statement of the above theorem.

The group of connected components of Π, which will be denoted by $\Pi^{(c)}$ (the cumbersome notation here is to avoid the notational disaster $\pi_0(\Pi)$) is an *etale* group-scheme over $\mathfrak{o}$ which is the co-domain of a homomorphism

$$q : \Pi \longrightarrow \Pi^{(c)}$$

coming from an inclusion of rings $\mathscr{O}(\Pi^{(c)}) \subseteq \mathscr{O}(\Pi)$. This inclusion is obtained by lifting idempotents from $\mathscr{O}(\Pi) \otimes k$ to $\mathscr{O}(\Pi)$ [13, Prp. on p. 43]. Hence, if

$$q_0 : \Pi_0 \longrightarrow \Pi_0^{(c)}$$

defines the group-scheme of connected components of Π_0, then $q_0 = q \otimes_{\mathfrak{o}} k$ (in particular $\mathscr{O}(\Pi^{(c)} \otimes k) \subseteq \mathscr{O}(\Pi \otimes k)$).

Lemma 29. *The embedding $\beta : \Pi(M_0) \longrightarrow \Pi_0$ is still a closed embedding when composed with q_0.*

Proof. Let $\Pi_{0,\mathrm{red}}$ be the closed sub-scheme of Π_0 defined by the ideal of nilpotent elements. It still defines a closed sub-group-scheme of Π_0 such that $q_0|\Pi_{0,\mathrm{red}}$ is an isomorphism [15, 6.8, p. 52]. Because $\Pi(M_0)$ is reduced ([11] or [4, Cor. 12]) and finite, it is etale. Therefore β gives a closed embedding of $\Pi(M_0)$ into $\Pi_{0,\mathrm{red}}$ and when composed with the isomorphism $q_0|\Pi_{0,\mathrm{red}}$ we still have a closed embedding.

□

Let

$$\alpha : \Pi(M_0) \longrightarrow \Pi_0^{(c)}$$

denote this new closed embedding obtained in the above lemma. The proof that α is an isomorphism (which is the contention of Theorem 28) will use the material of sections 6.2.1 and 6.2.2. It will be given in section 6.2.3.

6.2.1 Etale coverings and monoids in $\mathbf{str}(\mathfrak{X}/\mathfrak{o})$

As the reader can guess, the proof of Theorem 28 will rely on Grothendieck's "remarkable equivalence" (EGA IV_4 18.1.2, p. 110) and on the well-understood principle that "etale coverings are stratified sheaves with etale differential Galois group". To fill this last principle with mathematical content, we will study the relation between finite and etale A-algebras and $\mathbf{str}(\mathfrak{X}/\mathfrak{o})$ (summarized in Prp. 31 below).

To understand the following definition, the reader is reminded to consult [8, VII §3, p. 170] for terminology concerning monoids. It is also important to bear in mind that an etale morphism of $\mathfrak{o}$-adic formal schemes $f : \mathfrak{Y} \longrightarrow \mathfrak{S}$ is given by a compatible system of etale morphisms $f_\mu : Y_\mu \longrightarrow S_\mu$, where the sub-script $(\cdot)_\mu$ indicates that we are taking the scheme defined by the ideal ϖ^μ (EGA I, 10.6.11, p. 192).

Definition 30. *Let* $\mathbf{Etstr}(\mathfrak{X}/\mathfrak{o})$ *denote the full sub-category of commutative monoids in* $\mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ *whose objects have the property that the underlying* $\mathscr{O}_{\mathfrak{X}}$*-algebra defines an etale covering of* $\mathfrak{X}$*. Objects of* $\mathbf{Etstr}(\mathfrak{X}/\mathfrak{o})$ *are called etale stratified algebras over* $\mathfrak{X}$*. The analogous definition for* X_0 *is in force:* $\mathbf{Etstr}(X_0/k)$*. Objects of* $\mathbf{Etstr}(X_0/k)$ *are called etale stratified algebras over* X_0*.*

We will also use the notations $\mathbf{Et}(\mathfrak{X})$ (resp. $\mathbf{Et}(X_0)$) to denote the **opposite** category of etale coverings of $\mathfrak{X}$ (resp. X_0). That is, $\mathbf{Et}(\mathfrak{X})$ is the category of finite and etale A-algebras.

Proposition 31. *Consider the commutative diagram of categories*

$$\begin{array}{ccc} \mathbf{Etstr}(\mathfrak{X}/\mathfrak{o}) & \longrightarrow & \mathbf{Et}(\mathfrak{X}) \\ {\scriptstyle r}\big\downarrow & & \big\downarrow \\ \mathbf{Etstr}(X_0/k) & \longrightarrow & \mathbf{Et}(X_0), \end{array}$$

where the horizontal arrows are just the forgetful functors and the vertical are the restrictions to the special fibre. Then each arrow is a monoidal equivalence of categories.

Proof. That the functor $\mathbf{Et}(\mathfrak{X}) \longrightarrow \mathbf{Et}(X_0)$ is an equivalence is Grothendieck's celebrated "remarkable equivalence" (EGA IV_4 18.1.2, p. 110). Therefore, the proposition will follow if we can show that the forgetful functors above induce equivalences. This is an immediate application of Theorem 16:

Claim: Let $\varphi : A \longrightarrow S$ be a finite and etale morphism of $\mathfrak{o}$-adic algebras.

Then the A-module S has a stratification $\{\lambda_n\}_n$ induced by the commutative triangle

$$\begin{array}{ccc} P_A^n \otimes_A S & \xrightarrow{\lambda_n} & S \otimes_A P_A^n \\ & \searrow^{\cong} & \downarrow{\cong} \\ & & P_S^n. \end{array}$$

Under this stratification, the natural arrows $\varphi : A \longrightarrow S$ and mult. : $S \otimes_A S \longrightarrow S$ are horizontal. Furthermore, the stratification $\{\lambda_n\}$ is unique satisfying these conditions.

Proof: It is easy to see that φ is horizontal. Let us take care of multiplication. The arrows λ_n are, by definition, homomorphisms of P_A^n-modules. The stratification of $S \otimes_A S$ is obtained via the canonical identifications $(P^n \otimes S) \otimes_{P^n} (P^n \otimes S) = P^n \otimes (S \otimes_A S)$, $(S \otimes P^n) \otimes_{P^n} (S \otimes P^n) = (S \otimes_A S) \otimes P^n$. So, mult. : $S \otimes_A S \longrightarrow S$ is horizontal is an only if λ_n is a homomorphism of rings. As the isomorphism of Theorem 16 is an isomorphism of P_A^n-algebras, we are done.

Now we consider uniqueness. So let $\varepsilon_n : P^n \otimes S \longrightarrow S \otimes P^n$ define a stratification of S which allows φ and mult. to be horizontal. From what we saw before, this means that ε_n is a homomorphism of P^n-algebras and we want to show that $\varepsilon_n = \lambda_n$ for every n. This will be achieved if we show that $\varepsilon_n \otimes (\mathfrak{o}/\varpi^\mu) = \lambda_n \otimes (\mathfrak{o}/\varpi^\mu)$ for every μ. In view of Proposition 15 above, we can assume that φ is formally etale (replace A, S and etc by $A \otimes (\mathfrak{o}/\varpi^\mu)$, $S \otimes (\mathfrak{o}/\varpi^\mu)$ and etc). By hypothesis, $\varepsilon_0 = \lambda_0 = \text{id}$ and we will show by induction that $\{\varepsilon_n\} = \{\lambda_n\}$; assume that we have $\varepsilon_n = \lambda_n$.

Since φ is formally etale, the same is true for $P^n \longrightarrow P^n \otimes S$ and $P^n \longrightarrow S \otimes P^n$. Let $\pi_{n+1} : P^{n+1} \longrightarrow P^n$ be the canonical projection; note that $\ker(\pi_{n+1})$ is nilpotent, so by flatness of φ, both $\ker(S \otimes \pi_{n+1})$ and $\ker(\pi_{n+1} \otimes S)$ are nilpotent. The definition of stratification requires that $(S \otimes \pi_{n+1}) \circ \square_{n+1} = \square_n \circ (\pi_{n+1} \otimes S)$, where $\square = \lambda$ or ε. Hence we have the commutative diagram

$$\begin{array}{ccc} P^{n+1} \otimes S & \xrightarrow{\varepsilon_n \circ (\pi_{n+1} \otimes S) = \lambda_n \circ (\pi_{n+1} \otimes S)} & S \otimes P^n \\ \uparrow & \overset{?}{\searrow} & \uparrow{\scriptstyle S \otimes \pi_{n+1}} \\ P^{n+1} & \longrightarrow & S \otimes P^{n+1} \end{array}$$

where the dotted arrow ? can be filled either with ε_{n+1} or λ_{n+1}. As $P^n \longrightarrow P^n \otimes S$ is formally etale, we must have $\varepsilon_{n+1} = \lambda_{n+1}$.

$\square$

A particularly pleasant consequence of Proposition 31 is:

Corollary 32. *Any automorphism of $\mathscr{P} \in \mathbf{Et}(\mathfrak{X})$ commutes with the action of the differential operators.*

In this spirit, the following notation is quite useful:
Notation: Let G be an abstract group. The sub-category of $\mathbf{Etstr}(\mathfrak{X}/\mathfrak{o})$ corresponding to those elements in $\mathbf{Et}(\mathfrak{X})$ whose spectrum is a G-torsor over $\mathfrak{X}$ will be denoted by

$$\mathbf{Etstr}(\mathfrak{X}/\mathfrak{o}; G).$$

Using Proposition 31, the above category is none other than the category of G-torsors over $\mathfrak{X}$.

6.2.2 Associated sheaves

The material here is all standard and folkloric. We use this section mainly to fix notations needed in section 6.2.3 below.

Let G be a finite abstract group. Let $\mathscr{P} \in \mathbf{Etstr}(\mathfrak{X}/\mathfrak{o}; G)$ (see the end of section 6.2.1 for notation). Define the functor

$$L_{\mathscr{P}} : \mathrm{Rep}_{\mathfrak{o}}(G) \longrightarrow \mathbf{str}(\mathfrak{X}/\mathfrak{o}), \qquad L_{\mathscr{P}}(V) = (\mathscr{P} \otimes_{\mathfrak{o}} V)^G. \tag{12}$$

Of course, $L_{\mathscr{P}}$ composed with the forgetful functor $\mathbf{str}(\mathfrak{X}/\mathfrak{o}) \longrightarrow \mathrm{coh}(\mathfrak{X})$ is just the associated sheaf construction of [6, I 5.8, p. 84]. It is easy to see that $L_{\mathscr{P}}$ is monoidal, exact and faithful (exactness and faithfulness follow automatically from the standard case, see loc. cit). The following easy lemma will be used later.

Lemma 33. *i) Let $\mathscr{P} \in \mathbf{Etstr}(\mathfrak{X}/\mathfrak{o}; G)$ and $\mathscr{Q} \in \mathbf{Etstr}(\mathfrak{X}/\mathfrak{o}; H)$. Assume that there is a pair (α, Φ) where $\alpha : H \longrightarrow G$ is a homomorphism and $\Phi : \mathscr{P} \longrightarrow \mathscr{Q}$ is an arrow of $\mathbf{Etstr}(\mathfrak{X}/\mathfrak{o})$ such that $\Phi \circ \alpha(h) = h \circ \Phi$ for all $h \in H$. Then there is a natural isomorphism $\varphi : L_{\mathscr{P}} \Rightarrow L_{\mathscr{Q}} \circ \mathrm{res}(\alpha)$ of monoidal functors induced by the pair (α, Φ).*

ii) The composition of functors

$$\mathrm{Rep}_k(G) \xrightarrow{\textit{restrc. of scalars}} \mathrm{Rep}_{\mathfrak{o}}(G) \xrightarrow{L_{\mathscr{P}}} \mathbf{str}(\mathfrak{X}/\mathfrak{o}) \xrightarrow{r} \mathbf{str}(X_0/k)$$

is naturally isomorphic to the functor L_{P_0}, where $P_0 \in \mathbf{Etstr}(X_0/k)$ is induced by $\mathscr{P}$.

Proof. *i)* Let $V \in \mathrm{Rep}_{\mathfrak{o}}(G)$. Using Φ we obtain an arrow in $\mathbf{str}(\mathfrak{X}/\mathfrak{o})$

$$\mathscr{P} \otimes_{\mathfrak{o}} V \xrightarrow{\Phi \otimes V} \mathscr{Q} \otimes_{\mathfrak{o}} V.$$

From the compatibility $\Phi \circ \alpha(h) = h \circ \Phi$ we obtain another arrow in $\mathbf{str}(\mathfrak{X}/\mathfrak{o})$

$$\varphi_V : (\mathscr{P} \otimes V)^G \longrightarrow (\mathscr{Q} \otimes V)^H$$

where H acts on V via α. It is immediate to see that φ is natural and monoidal. From the theory of torsors, we know that φ_V is an isomorphism of coherent sheaves and *i*) is proved. The proof of *ii*) is even easier and we shall omit it.

□

Language: We shall call a morphism $\Phi : \mathscr{P} \longrightarrow \mathscr{Q}$ as in *i*) above a map *covering* the homomorphism α.

6.2.3 Proof of Theorem 28

We will keep the notations fixed in sections 6.1 and 6.2. We will also abbreviate $\Pi(M_0)$ to G_0. As we are assuming that k is algebraically closed, any etale group-scheme over $\mathfrak{o}$ or k will be constant; we will identify the group schemes G_0, $\Pi_0^{(c)}$ and $\Pi^{(c)}$ with the constant groups they induce.

If $V \in \mathrm{Rep}_{\mathfrak{o}}^{\#}(\Pi^{(c)})$ is *faithful*, then $V_0 = V \otimes k$ is a faithful representation of $\Pi^{(c)} \otimes k = \Pi_0^{(c)}$ and, consequently,

$$\mathrm{res}(\alpha)(V_0)$$

is a faithful representation of $G_0 = \Pi(M_0)$ (α is defined by Lemma 29). From Proposition 13, the functor $\mathrm{res}(q)$ induces an equivalence between $\mathrm{Rep}_{\mathfrak{o}}^{\#}(\Pi^{(c)})$ and the full sub-category $\langle \mathrm{res}(q)(V)\rangle_{\otimes}^{s}$ of $\mathrm{Rep}_{\mathfrak{o}}^{\#}(\Pi)$. By Theorem 24 there exists $\mathscr{V} \in \langle \mathscr{M}\rangle_{\otimes}^{s}$ such that $\overline{\omega}_0(\mathscr{V}) = \mathrm{res}(q)(V)$; if this choice is fixed, then

$$\overline{\omega}_0|\langle \mathscr{V}\rangle_{\otimes}^{s} : \langle \mathscr{V}\rangle_{\otimes}^{s} \longrightarrow \langle \mathrm{res}(q)(V)\rangle_{\otimes}^{s} = \mathrm{Rep}_{\mathfrak{o}}^{\#}(\Pi^{(c)}) \text{ is an equivalence.}$$

Write

$$T := S \circ \mathrm{res}(q).$$

The main idea behind Theorem 28 is that if $\alpha : G_0 \longrightarrow \Pi_0^{(c)}$ fails to be an isomorphism, then Proposition 31 will produce a certain factorization of T; this factorization will then lead to a contradiction. For the next proposition we use notation from sections 6.2.2 and 6.2.1. The reader is asked to bear in mind the definition of ν given in section 6.1.

Proposition 34. *The functor T is naturally isomorphic as a monoidal functor to*

$$L_{\mathscr{P}} : \mathrm{Rep}_{\mathfrak{o}}(\Pi^{(c)}) \longrightarrow \mathbf{str}(\mathfrak{X}/\mathfrak{o}),$$

where $\mathscr{P} = T(\mathscr{O}(\Pi^{(c)}), \rho_l)$ is seen as an object of $\mathbf{Etstr}(\mathfrak{X}/\mathfrak{o}, \Pi^{(c)})$. The functor $\nu \circ \mathrm{res}(q_0) : \mathrm{Rep}_k(\Pi_0^{(c)}) \longrightarrow \mathbf{str}(X_0/k)$ is naturally isomorphic to the functor L_{P_0}, where P_0 is the object of $\mathbf{Etstr}(X_0/k)$ induced by $\mathscr{P}$.

Proof. The second assertion in the statement follows from the first, property 3. (see section 6.1) of the restriction of scalars functor and from Lemma 33. The first assertion follows from a technique of M. Nori; for the case of a ground *field* this is explained in [4, 2.3.2] or in [12, §2], but the proof there applies to the present case. We will indicate (i) why $\mathscr{P} \in \mathbf{Etstr}(\mathfrak{X}/\mathfrak{o}, \Pi^{(c)})$ and (ii) which point of the proof in [4, 2.3.2] needs further attention.
(i) Let $R := \mathscr{O}(\Pi^{(c)})$. Then the co-multiplication

$$\Delta : (R, \rho_l) \longrightarrow (R, \rho_l) \otimes_{\mathfrak{o}} (R, \mathrm{id}_R \otimes 1)$$

becomes an arrow in $\mathrm{Rep}_{\mathfrak{o}}(\Pi^{(c)})$ and hence we have an arrow $T(\Delta) : \mathscr{P} \longrightarrow \mathscr{P} \otimes_{\mathfrak{o}} R$ (the co-domain has the direct sum stratification) in $\mathbf{str}(\mathfrak{X}/\mathfrak{o})$. We also have an isomorphism $\mathscr{P} \otimes \mathscr{P} \xrightarrow{\cong} \mathscr{P} \otimes_{\mathfrak{o}} R$, which is induced by the obvious isomorphism in $\mathrm{Rep}_{\mathfrak{o}}(\Pi^{(c)})$

$$(R, \rho_l) \otimes (R, \rho_l) \xrightarrow{\cong} (R, \rho_l) \otimes (R, \mathrm{id}_R \otimes 1).$$

As $\mathscr{P}$ is faithfully flat as an $\mathscr{O}_{\mathfrak{X}}$-module, it follows that $\mathscr{P}$ has the desired properties.
(ii) Let $(V, \rho) \in \mathrm{Rep}_{\mathfrak{o}}(\Pi^{(c)})$. In loc.cit. we find the construction of $\Pi^{(c)}$-equivariant maps

$$\tau_1 : (V, \rho) \otimes (R, \rho_l) \longrightarrow (V, \rho) \otimes (R, \rho_l) \otimes (R, \mathrm{id}_R \otimes 1)$$
$$\tau_2 : (V, \mathrm{id}_V \otimes 1) \otimes (R, \rho_l) \longrightarrow (V, \mathrm{id}_V \otimes 1) \otimes (R, \rho_l) \otimes (R, \mathrm{id}_R \otimes 1)$$

where

- τ_1 is the comodule map for the representation $(V, \mathrm{id}_V \otimes 1) \otimes (R, \rho_r)$ (and hence just $(V, \rho) \otimes \Delta$) and
- τ_2 is the comodule map for the representation $(V, \rho) \otimes (R, \rho_r)$.

Nori shows that there exists a natural $\Pi^{(c)}$-equivariant isomorphism

$$\theta_V : \mathrm{domain}(\tau_1) \longrightarrow \mathrm{domain}(\tau_2)$$

with the property that $\tau_2 \circ \theta_V = (\theta_V \otimes \mathrm{id}_R) \circ \tau_1$. Applying the functor T, we obtain a commutative diagram in $\mathbf{str}(\mathfrak{X}/\mathfrak{o})$

$$\begin{array}{ccc} T(V) \otimes \mathscr{P} & \xrightarrow{T(\tau_1) - \mathrm{id} \otimes 1} & T(V) \otimes \mathscr{P} \otimes R \\ \downarrow & & \downarrow \\ V \otimes \mathscr{P} & \xrightarrow{T(\tau_2) - \mathrm{id} \otimes 1} & V \otimes \mathscr{P} \otimes R \end{array}$$

where the vertical arrows are isomorphisms (note that in loc.cit. there is a misprint: the term "$-\mathrm{id} \otimes 1$" is missing). The kernel of the lower horizontal arrow is, by definition, $L_{\mathscr{P}}(V)$. The kernel of the upper horizontal arrow will be $T(V)$ since by the claim below the sequence

$$0 \longrightarrow V \xrightarrow{V\otimes 1} V \otimes R \xrightarrow{V\otimes\Delta - V\otimes(\mathrm{id}_R\otimes 1)} V \otimes R \otimes R$$

is exact.
Claim:
coker$(\Delta - \mathrm{id}_R \otimes 1)$ is flat over $\mathfrak{o}$.

Let Γ be the abstract group defining $\Pi^{(c)}$, so that R is the ring of functions $\Gamma \longrightarrow \mathfrak{o}$. Assume that there is $f \in R \otimes R$ such that $\varpi f \in \mathrm{Im}(\Delta - \mathrm{id}_R \otimes 1)$. Then

$$\varpi f(x,y) = g(xy) - g(x), \ \forall x, y \in \Gamma$$

and some $g : \Gamma \longrightarrow \mathfrak{o}$. Making $x = e_\Gamma$, it follows that $\varpi f(e,y) = g(y) - g(e)$ and hence $\gamma(y) := g(y) - g(e) \in (\varpi)$. Since

$$\gamma(xy) - \gamma(x) = (g(xy) - g(e)) - (g(x) - g(e)) = g(xy) - g(x) = \varpi f(x,y)$$

it follows that $f \in \mathrm{Im}(\Delta - \mathrm{id} \otimes 1)$; this proves the claim. $\square$

Let $Q_0 \in \mathbf{Etstr}(X_0/k; G_0)$ have the property that

$$L_{Q_0} : \mathrm{Rep}_k(G_0) \longrightarrow \langle M_0 \rangle_\otimes$$

is a tensor equivalence. If P_0 is as in the above proposition, then we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}_k(\Pi_0^{(c)}) & \xrightarrow{L_{P_0}} & \langle M_0 \rangle_\otimes \\ & \searrow^{\mathrm{res}(\alpha)} & \uparrow L_{Q_0} \\ & & \mathrm{Rep}_k(G_0) \end{array}$$

since, by construction, $\xi_0^* \circ L_{P_0}$ (ξ_0 is the k-rational point of X_0) is naturally isomorphic to the forgetful functor. Such commutative diagram determines a homomorphism $\theta_0 : P_0 \longrightarrow Q_0$ in $\mathbf{Etstr}(X_0/k)$ which covers $\alpha : G_0 \longrightarrow \Pi_0^{(c)}$. From Proposition 31, we obtain a lifting $\theta : \mathscr{P} \longrightarrow \mathscr{Q}$ in $\mathbf{Etstr}(\mathfrak{X}/\mathfrak{o})$ covering α. From Lemma 33(i) we obtain a factorization

$$\begin{array}{ccc} \mathrm{Rep}_{\mathfrak{o}}(\Pi^{(c)}) & \xrightarrow{L_{\mathscr{P}} = T} & \mathbf{str}(\mathfrak{X}/\mathfrak{o}) \\ & \searrow^{\mathrm{res}(\alpha)} & \uparrow L_{\mathscr{Q}} \\ & & \mathrm{Rep}_{\mathfrak{o}}(G). \end{array}$$

(G is the constant group scheme over $\mathfrak{o}$ lifting G_0.)

From the lemma below, such a factorization does not exit unless α is an isomorphism, **therefore we have proved Theorem 28**.

Lemma 35. *Let $\gamma : \Gamma \longrightarrow \Pi^{(c)}$ be a closed embedding of flat $\mathfrak{o}$-group-schemes which is not an isomorphism. Then the functor $T : \mathrm{Rep}_{\mathfrak{o}}(\Pi^{(c)}) \longrightarrow \mathbf{str}(\mathfrak{X}/\mathfrak{o})$ cannot be factored into $T = T' \circ \mathrm{res}(\gamma)$, where $T' : \mathrm{Rep}_{\mathfrak{o}}(\Gamma) \longrightarrow \mathbf{str}(\mathfrak{X}/\mathfrak{o})$ is exact, $\mathfrak{o}$-linear and monoidal and $\omega \circ T'$ is naturally equivalent to the forgetful functor.*

Proof. The reader is asked to bear in mind that we have obtained an *equivalence* $\overline{\omega}_0|\langle \mathscr{V} \rangle_{\otimes}^s : \langle \mathscr{V} \rangle_{\otimes}^s \longrightarrow \mathrm{Rep}_{\mathfrak{o}}^{\#}(\Pi^{(c)})$ such that $\overline{\omega}_0(\mathscr{V}) = V$, with V a faithful representation of $\Pi^{(c)}$.

Because γ is a closed embedding which is not an isomorphism, there exists a pair (W', W) of finite and free $\mathfrak{o}$-modules such that:
1) W' is a proper sub-module of W.
2) W has a structure of $\Pi^{(c)}$-module such that W' **is not stable** under $\Pi^{(c)}$ but is stable under Γ.

Let $\mathscr{W} \in \langle \mathscr{V} \rangle_{\otimes}^s$ be such that $\overline{\omega}_0(\mathscr{W}) = W$ in $\mathrm{Rep}_{\mathfrak{o}}^{\#}(\Pi^{(c)})$. Because γ is a closed embedding we have $\langle \mathrm{res}(\gamma)(V) \rangle_{\otimes}^s = \mathrm{Rep}_{\mathfrak{o}}^{\#}(\Gamma)$ (Proposition 12); as a consequence the functor $T'|\mathrm{Rep}_{\mathfrak{o}}^{\#}(\Gamma)$ takes values in $\langle \mathscr{V} \rangle_{\otimes}^s$. So the image of the inclusion $W' \subset W$ by T' is an arrow $T'(W') \longrightarrow \mathscr{W}$ of $\langle \mathscr{V} \rangle_{\otimes}^s$. Applying $\overline{\omega}_0$ to the latter, we get a structure of $\Pi^{(c)}$-module on the $\mathfrak{o}$-submodule W' of W ($\omega \circ T'$ is the forgetful functor) which is $\Pi^{(c)}$-equivariant (W has the structure of $\Pi^{(c)}$-module with which we have started). This is a contradiction and the lemma is proved. □

Example: The Dwork equation Let $\mathfrak{o} = W[\varpi]$, where W is the ring of Witt vectors over $\overline{\mathbb{F}}_p$ and ϖ is the uniformizer satisfying the equation $\varpi^{p-1} = -p$. Take $A = \mathfrak{o}\langle x \rangle$. Then $\mathscr{D}_{\mathfrak{X}/\mathfrak{o}}$ is the sub-algebra of $\mathrm{End}_{\mathfrak{o}}(A)$ generated by the operators

$$\partial_n x^m = \begin{pmatrix} m \\ n \end{pmatrix} x^{m-n}.$$

Define $(\mathscr{L}, \nabla) \in \mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ by $\mathscr{L} = A \cdot \mathbf{e}$ (free of rank one) and $\nabla(\partial_n)\mathbf{e} = \frac{\varpi^n}{n!}\mathbf{e}$ (usually one should put a sign). As is well-known, there exists $f \in \mathfrak{o}[[x]]$ such that $\nabla(\partial_n)(f\mathbf{e}) = 0$ for all $n > 0$; the power series f is an analytic function on the "open" disk $D(0, 1^-)$ and $g := f^p \in \mathfrak{o}\langle x \rangle$. In particular,

$$\mathscr{L}^{\otimes p} \cong \mathbf{1}.$$

Let $\Pi = \Pi(\mathscr{L})$. It is well-known (and easy to see) that the differential Galois group on the generic fibre is the etale group scheme $\mathbb{Z}/p\mathbb{Z}$. From Corollary 26, $\Pi = \Pi(\mathscr{L})$ is of rank p over $\mathfrak{o}$. Since Π has a faithful character $\overline{\omega}_0\mathscr{L}$ (Theorem 24) which is of order p, we have $\Pi \cong \mu_{p,\mathfrak{o}}$. Therefore, the fact that the stratified module over the affine line in positive characteristic induced by $\mathscr{L}$ becomes trivial is explained–in accordance with Theorem 28–by the non-reduced nature of the group $\Pi \otimes k$.

The Dwork equation also helps us understand why the second part of Conjecture 8.5 in [11] cannot be unconditionally true (at least in the setting presented here). For if $\mathscr{N} \in \mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$ is such that $\mathscr{N} \otimes K \cong \mathscr{L} \otimes K$, then $\mathscr{N}$ has rank one and the least difficult part of the "differential Abhyankar conejcture" [11, Thm. 7.2(1)] asserts that the differential Galois group of $\mathscr{N} \otimes k$ is trivial as well (this group is a subgroup of $\mathbb{G}_{m,k}$). A substitute for the conjecture together with a proof will appear in [5].

Final remarks: (Made after submission.) (a) Let $\mathscr{M} \in \mathbf{str}^{\#}(\mathfrak{X}/\mathfrak{o})$. Then we have a canonical closed embedding $\Pi(\mathscr{M}) \longrightarrow \mathrm{GL}(M)$, where $M = \omega\mathscr{M} = \xi^*\mathscr{M}$. Let $I \subseteq \mathscr{O}(\mathrm{GL}(M))$ be the ideal defining this closed embedding. Since $\mathscr{O}(\mathrm{GL}(M))/I$ is flat over $\mathfrak{o}$, it follows that I is the contraction of its extension to $\mathscr{O}(\mathrm{GL}(M)) \otimes K$. As the induced closed embedding $\Pi(\mathscr{M}) \otimes K \longrightarrow \mathrm{GL}(M) \otimes K$ is just the natural closed embedding of K-group-schemes $\Pi(\mathscr{M} \otimes K) \longrightarrow \mathrm{GL}(M \otimes K)$, it follows that $\Pi(\mathscr{M})$ is the **closure** of $\Pi(\mathscr{M} \otimes K)$ inside the $\mathfrak{o}$-group-scheme $\mathrm{GL}(M)$. In this setting, Theorem 27 above was obtained by O. Gabber and published in [7, Thm. 2.4.1, p. 39]. (In loc.cit the base DVR is of characteristic zero, but the proof can be translated to our situation.) More precisely, Katz proves the following: Let I_{gen} be the ideal of the closed embedding $\Pi(\mathscr{M} \otimes K) \longrightarrow \mathrm{GL}(M \otimes K)$. Then the ideal $I_0 \subseteq \mathscr{O}(\mathrm{GL}(M \otimes k))$ induced by $I_{\mathrm{gen}} \cap \mathscr{O}(\mathrm{GL}(M))$ is contained in the ideal of the closed embedding $\Pi(\mathscr{M} \otimes k) \longrightarrow \mathrm{GL}(M \otimes k)$.

(b) It has been brought to our attention the existence of [16] which deals with "Tannakian" categories over valuation rings. Although a few points in the present work could be facilitated by the results and ideas there, the central Corollary 7 is missing (as it is missing in [2]).

Acknowledgments: I thank B. Matzat for helping me understand better the relation between this work and the conjecture in [11]. I thank P. Berthelot, A. Bruguières and M. van der Put for some interesting remarks. Finally, I wish to thank Nick Shepherd-Barron, resp. Y. André for calling my attention to [2], resp. the above mentioned work of Gabber. During this research I benefited from a Arithmetic Algebraic Network (Marie-Curie) grant and the financial support is greatly appreciated.

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