ON THE NUMBER OF FROBENIUS–TRIVIAL VECTOR BUNDLES ON SPECIFIC CURVES

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Abstract. This note investigates, through direct computational methods, the existence of infinitely many isomorphism classes of stable vector bundles which become trivial after being pulled back by the Frobenius morphism. We obtain examples, in characteristic two, where infinitely many such isomorphism classes exist. In characteristic three, however, the computations show that the aforementioned sets are finite.

1. Introduction

This notes endeavours to investigate through explicit examples a problem in positive characteristic algebraic Geometry first raised by Nori [N82, Conjecture, p. 89] and studied by [MS02], [MS08] and [P07]. We fix an algebraically closed field k of characteristic $p > 0$, and give ourselves a smooth, projective, and connected curve C over k. Letting $\mathrm{Fr} : C \to C$ stand for the absolute Frobenius morphism, we define

(1)

$$S(C,r,t) = \left\{ \begin{array}{c} \text{isomorphism classes of stable vector bundles of rank } r \text{ over } C \\ \text{whose pull–back by the } t\text{–th power of Fr is isomorphic to } \mathcal{O}_C^r \end{array} \right\}.$$

As pointed out by [MS08], understanding $\#S(C,r,t)$ is the key to "base–change" properties for the fundamental group scheme [N82]. More precisely, Mehta and Subramanian introduced the *local fundamental group scheme* [MS08, p. 207]; this is a quotient of Nori's fundamental group scheme which permits us to capture the failure of "good base–change". In [MS08, Theorem, p.208] it is proved that if $S(C,r,t)$ is finite for each (r,t), then the local fundamental group scheme of $C \otimes_k k'$, for any algebraically closed extension k'/k, is obtained from that of C, and conversely. The optimistic view, of course, says that $\#S(C,r,t) < \infty$ for all (r,t). However, building on a precise analysis of the moduli space of bundles of rank two, Pauly [P07] showed that this is unrealistic. He constructed, in characteristic two, a positive dimensional family of isomorphism classes of stable bundles of rank two over a curve C of genus two, whose members are all trivial when pulled back by the

2000 *Mathematics Subject Classification.* 14H60, 14Q05, 16Z05, 16G99.

Key words and phrases. Positive characteristic algebraic Geometry; F–trivial vector bundles; local fundamental group scheme; computations in algebraic Geometry.

fourth power of the Frobenius morphism. More succinctly, Pauly gave an example in which $S(C, 2, 4)$ is infinite.

In what follows, we concentrate on direct computations of $\#S(C, 2, 1)$. Our work produces curves C over $k = \overline{\mathbb{F}}_2$ with $\#S(C, 2, 1) = \infty$, and curves C, now over $\overline{\mathbb{F}}_3$, where $\#S(C, 2, 1) < \infty$. See sections 6 and 7. Of course, the curves over $\overline{\mathbb{F}}_2$ give counterexamples to Nori's conjecture [N82, Conjecture p.89], [MS02, Conjecture(ii), p.144], while those over $\overline{\mathbb{F}}_3$ do not; we have included the latter to show the limits of our methods and hence avoid false hopes.

The method we have chosen has two pillars. The first and most obvious is to use Cartier's theorem [Ka70, Theorem 5.1] on the p–curvature to pass from vector bundles trivialized by the Frobenius morphism to vector bundles with connections. The second, is E. Letzter's procedure [L08] to explicitly detect the occurrence of an infinity of isomorphism classes of semisimple representations of a certain algebra. The algebras intervening, on the other hand, come from the explicit expression for the p–curvature of a connection.

Conventions and notations. Throughout this note, C will stand for a a smooth, connected and projective curve over an algebraically closed field k of characteristic $p > 0$. Its genus is denoted by g, its function field by K. The absolute Frobenius morphisms of C is denoted by $\mathrm{Fr} : C \to C$. By a vector bundle, we understand a locally trivial coherent sheaf of finite rank. A *trivial* vector bundle is a vector bundle which is isomorphic to some $\mathcal{O}^n$.

The free associative (and unital) algebra on m generators $X_1, \ldots, X_m$ over k will be denoted by $k\{X_1, \ldots, X_m\}$. An associative algebra on m generators will be a quotient of $k\{X_1, \ldots, X_m\}$ by some two sided ideal. If Λ is an associative k–algebra, $\mathrm{Rep}(\Lambda)$ will stand for the category of representations of Λ on finite dimensional vector spaces.

2. Frobenius trivial vector bundles

For the sake of discussion, we say that a vector bundle E over C is *Frobenius–trivial* if the pull–back Fr^*E is a trivial vector bundle. (The terminology varies in the literature, compare [MS08, p. 207] and [P07, p. 2708].) The set of all Frobenius trivial vector bundles over C will be denoted by $\mathcal{FT}$. The reader can easily verify that each vector bundle in $\mathcal{FT}$ is semi–stable of degree zero. (For standard material on stability and semi–stability, see [S82].) It is folklore that $\mathcal{FT}$, with the obvious kernels and cokernels, is an abelian category; there is a hint of this statement in [MS08, Remark 2, p.208], and a proof can be extracted from [BdS10, Corollary 2.3(iii), p.4]. The latter result can also be used to prove

Lemma 1. *An object E of the abelian category $\mathcal{FT}$ is simple if and only if the vector bundle E is stable.*

Proof. The "if" part is straightforward. We prove the "only if" clause. Assume that E is a simple object of $\mathcal{FT}$ of rank r. Let $\varphi : E \to V$ be a an epimorphism of coherent sheaves, where V is a vector bundle of degree zero and rank $d \leq r$. We want to prove that $d = r$ to guarantee stability [S82, pp. 14–15]. (Note that E is already semi–stable.) In what follows we show that if $f : C' \to C$ is a finite morphism from a projective, smooth and irreducible k–scheme such that $f^*E \cong \mathcal{O}_{C'}^r$, then $f^*V \cong \mathcal{O}_{C'}^d$. This will imply that V is an object of $\mathcal{FT}$. The simplicity of E as an object of $\mathcal{FT}$ will then force the equality $d = r$.

Let G stand for the Grassmann variety $\mathrm{Grass}(r, d)$ and $\mathcal{U}$ for the universal vector bundle on it. See [N05, 5.1.5(2), p.110] for details and notation. The epimorphism $f^*(\varphi) : f^*E \to f^*V$ gives rise to a morphism

$$\gamma : C' \longrightarrow G,$$

such that $\gamma^*\mathcal{U} \cong f^*V$. Now $\det \mathcal{U}$ is very ample on G [N05, 5.1.6, p.112ff], while $\deg f^*V = 0$. Therefore, γ has to be constant, from which we conclude that $f^*V \cong \mathcal{O}_{C'}^d$, i.e. V is an object of $\mathcal{FT}$. □

3. Connections on $\mathcal{O}^n$ and matrices

Let $\mathcal{CT}$ denote the category whose objects are couples (E, ∇) consisting of a *trivial* vector bundle E together with a connection $\nabla : E \to E \otimes_{\mathcal{O}} \Omega^1$, and whose arrows are just horizontal morphisms. (More details on the terminology can be found in [Ka70, §1].) This obviously is an abelian category. We let $\mathcal{CT}^0$ denote the full subcategory of $\mathcal{CT}$ whose objects have trivial p–curvature (for the definition, see [Ka70, §5]). Cartier's theorem [Ka70, Theorem 5.1] has as consequence that for each $(E, \nabla) \in \mathcal{CT}^0$, the sheaf of $\mathcal{O}$–modules $\mathrm{Fr}_*(E^\nabla)$ satisfies $\mathrm{Fr}^*(\mathrm{Fr}_*(E^\nabla)) \simeq E$. Together with the fact that $\mathcal{FT}$ is an abelian category, we obtain that $\mathcal{CT}^0$ is in fact a sub–abelian category of $\mathcal{CT}$. This remark permits us to revisit Cartier's theorem and arrive at:

Corollary 2. *The construction of Cartier* [Ka70, Theorem 5.1] *defines an equivalence of abelian categories* $\mathcal{FT} \xrightarrow{\sim} \mathcal{CT}^0$.

We fix a positive integer n. Let $d : \mathcal{O}^n \to \mathcal{O}^n \otimes \Omega^1_C$ be the obvious connection on the trivial vector bundle $\mathcal{O}^n$. If ∇ is any other connection, the difference $\nabla - d : \mathcal{O}^n \to \mathcal{O}^n \otimes \Omega^1$ is $\mathcal{O}$–linear. Conversely, for any $\mathbf{A} \in \mathrm{Hom}_{\mathcal{O}}\left(\mathcal{O}^n, \mathcal{O}^n \otimes \Omega^1\right)$, $d + \mathbf{A}$ is a connection. We can therefore identify the space of all connections on $\mathcal{O}^n$ with the vector space $\mathrm{Hom}_{\mathcal{O}}\left(\mathcal{O}^n, \mathcal{O}^n \otimes \Omega^1\right)$. Letting M_n stand for the vector space of all $n \times n$ matrices, from the previous considerations it ensues that the set of all connections on $\mathcal{O}^n$ is, non–canonically, in bijection with $M_n^{\times g}$. Explicitly, we fix a basis $\theta_0, \dots, \theta_{g-1}$ of $H^0(\Omega^1_C)$ to obtain a bijection

$$\begin{array}{rcl} M_n^{\times g} & \to & \{\text{connections on } \mathcal{O}^n\}, \\ \mathbf{A} = (A_0, \dots, A_{g-1}) & \mapsto & d_{\mathbf{A}} = d + \sum_0^{g-1} A_\kappa \otimes \theta_\kappa. \end{array}$$

We now take into consideration the horizontal homomorphisms. Let m be another positive integer and let $\gamma : k^m \to k^n$ be a linear map. It is straightforward to verify that $\gamma : \mathcal{O}^m \to \mathcal{O}^n$ induces a horizontal homomorphism $(\mathcal{O}^m, d_{\mathbf{A}}) \to (\mathcal{O}^n, d_{\mathbf{B}})$ if and only if $\gamma A_\kappa = B_\kappa \gamma$ for all κ. In other words, the map $\mathbf{A} \mapsto d_{\mathbf{A}}$ *defines an equivalence* from the category of representations of the free associative algebra on g generators $\mathrm{Rep}(k\{X_0, \dots, X_{g-1}\})$ to $\mathcal{CT}$.

4. On the p–curvature

Let $\mathbf{A} = (A_0, \dots, A_{g-1}) \in M_n^{\times g}$. We now consider the p–curvature [Ka70, §5] of a connection $d_{\mathbf{A}}$ on $\mathcal{O}^n$. This is a morphism of sheaves of abelian groups

$$\Psi_{\mathbf{A}} : \mathcal{D}er(C) \to \mathcal{E}nd_{\mathcal{O}}(\mathcal{O}^n)$$

which satisfies

$$\Psi_{\mathbf{A}}(a \cdot \partial) = a^p \cdot \Psi(\partial) \tag{2}$$

for sections a of $\mathcal{O}$ and ∂ of $\mathcal{D}er(C)$ over any given open of C. In other words, $\Psi_{\mathbf{A}}$ is an $\mathcal{O}$–linear morphism from $\mathcal{D}er(C)$ to $\mathrm{Fr}_*\mathcal{E}nd_{\mathcal{O}}(\mathcal{O}^n)$. Let U be a dense open subset of C where Ω^1_C is freely generated by dx, $x \in \mathcal{O}(U)$. If $\frac{d}{dx}$ stands for the canonical section of $\mathcal{D}er(C)$ over U obtained from dx, then

$$\begin{aligned} \Psi_{\mathbf{A}}\left(\frac{d}{dx}\right) &= \left(\frac{d}{dx} + \sum_{\kappa=0}^{g-1} A_\kappa \cdot \left\langle \theta_\kappa|_U, \frac{d}{dx}\right\rangle\right)^p \\ &= \left(\frac{d}{dx} + T\right)^p, \end{aligned} \tag{3}$$

where $T := \sum_{\kappa=0}^{g-1} A_\kappa \left\langle \theta_\kappa|_U, \frac{d}{dx}\right\rangle$ (this is an endomorphism of $\mathcal{O}_U^n$) and $\langle \bullet, \bullet \rangle : \Omega^1_C \times \mathcal{D}er(C) \to \mathcal{O}$ is the obvious pairing. Formula (2) together with the identity $\frac{d}{dx} \circ M - M \circ \frac{d}{dx} = \frac{dM}{dx}$ allow us to obtain the following expressions for $\Psi_{\mathbf{A}}\left(\frac{d}{dx}\right)$ (for more of these formulas, see [O08, 283ff])

$$p = 2, \qquad T^2 + \frac{dT}{dx} \tag{4}$$

$$p = 3, \qquad T^3 + \frac{dT}{dx} \cdot T + 2T \cdot \frac{dT}{dx} + \frac{d^2T}{dx^2}. \tag{5}$$

.

The vanishing of $\Psi_{\mathbf{A}}\left(\frac{d}{dx}\right)$ as an endomorphism of $\mathcal{O}^n|_U$ is equivalent with the vanishing of $\Psi_{\mathbf{A}}$; this follows from formula (2) and the fact that $\mathrm{Fr}_*\mathcal{E}nd_{\mathcal{O}}(\mathcal{O}^n)$

is a vector bundle, since $\mathrm{Fr}_*\mathcal{O}$ is likewise. A convenient choice of U will then give us relations $B_0, \cdots, B_t$ which the matrices A_κ have to fulfil in order to have $\Psi_{\mathbf{A}}\left(\frac{d}{dx}\right) = 0$. These relations have coefficients in k, so that we obtain a certain associative algebra over k whose representation theory is closely related to $\mathcal{FT}$. To render this algebra conspicuous is the task of the examples.

Proposition 3. *There exists an associative algebra Λ_C over k on g generators and an equivalence of abelian categories $\mathrm{Rep}(\Lambda_C) \xrightarrow{\sim} \mathcal{FT}$. In particular, a Frobenius–trivial vector bundle is stable (respectively direct sum of stable bundles) if and only if the representation of Λ_C corresponding to it is simple (respectively, semi–simple).* □

Since there are only finitely many isomorphism classes of line bundles over C whose Frobenius pull–back is trivial, Proposition 3 has the following consequence.

Corollary 4. *The set $S(C, 2, 1)$ defined in eq. (1) is infinite if and only if there are infinitely many isomorphism classes of semi–simple representations of rank two of Λ_C.*

5. Letzter's algorithm

Building on work of Artin, Procesi, Shirshov, and Belov, Letzter constructs in [L08] an algorithm which permits one to determine if for a given finitely generated associative k–algebra the number of semi–simple representations up to isomorphism is finite or not.

We recall the constructions [L08, §3]. Let $\Lambda = k\{X_1, \dots, X_s\}/(B_1, \dots, B_t)$ be a finitely generated associative k–algebra. Let n be a fixed positive integer and u a variable. Let R be the algebra generated over the field $k(u)$ by the variables $z_{ij}(\ell)$, where $1 \le i, j \le n$ and $1 \le \ell \le s$. We let $\mathbf{z}(\ell)$ stand for the matrix whose (i, j)th entry is $z_{ij}(\ell)$. We then construct the following sets

$$\begin{aligned}
\mathbf{RelEntries} &:= \text{entries of the matrices } B_1(\mathbf{z}(1), \dots, \mathbf{z}(s)), \dots, B_t(\mathbf{z}(1), \dots, \mathbf{z}(s)),\\
\mathbf{RelIdeal} &:= \text{ideal of } R \text{ generated by the set } \mathbf{RelEntries},\\
\mathbf{Monomials}_n &:= \text{set of products having the form } \mathbf{z}(i_1)\cdots\mathbf{z}(i_q), \text{ with } q \le n,\\
\mathbf{CharCoeff}_n &:= \text{characteristic invariants of the matrices in } \mathbf{Monomials}_n.
\end{aligned}$$

If, for some $\chi \in \mathbf{CharCoeff}_n$,

$$1 \notin (u - \chi, \mathbf{RelIdeal}),$$

it follows that χ is **not algebraic modulo RelIdeal** [L08, 2.12, p.3932]. In this case, the set of isomorphism classes of semisimple representations over k which are of rank n is infinite.

6. Examples in characteristic 2

We assume that $k = \overline{\mathbb{F}}_2$ and that $K = k(x, y)$, where x is transcendental and $y^2 + y = f \in k(x)$. In other words, $K/k(x)$ is an Artin–Schreier extension. The place of $k(x)$ corresponding to 0 (resp. ∞) will be denoted by P_0 (resp. P_∞) in all examples below. The divisor of a place P will be denoted by $[P]$.

Example 5. We take $f \in k[x]$ of degree five. By [S09, Proposition 3.7.8, p.127], we know that $g = 2$, that the place P_∞ of $k(x)$ is below a single place Q_∞ of K and that the ramification index is 2. Moreover, from [S09, Proposition 3.7.8(c), p.127] and [S09, Corollary 3.4.7, p.94], we obtain that the divisor of poles of x in K is $2 \cdot [Q_\infty]$, while the divisor of dx in K is $2 \cdot (-2) \cdot [Q_\infty] + 6 \cdot [Q_\infty] = 2 \cdot [Q_\infty]$. Hence, $dx, x \cdot dx$ is a basis for the regular differentials. Then, using the notation introduced in section 3, we have $T = A_0 + A_1 \cdot x$, and

$$\begin{aligned} T^2 + \frac{dT}{dx} &= A_0^2 + (A_0A_1 + A_1A_0) \cdot x + A_1^2 \cdot x^2 + A_1 \\ &= B_0 + B_1 \cdot x + B_2 \cdot x^2. \end{aligned}$$

Therefore, the category of connections on trivial vector bundles with vanishing p–curvature over C is equivalent to the category of representations of the free algebra Λ_C on two generators subjected to the relations $B_0 = B_1 = B_2 = 0$. Using §5 and Macaulay 2 [M2], we conclude that there are only finitely many isomorphism classes of semisimple representations of Λ_C of rank two. (The command lines we have written to arrive at this conclusion are similar to those in Example 6.) From Corollary 4, $\#S(C, 2, 1) < \infty$.

Example 6. Let $f = x^3 + x^{-3}$. From [S09, Proposition 3.7.8, p.127], the only places of $k(x)$ which ramify in K are P_0 and P_∞. In both cases, the ramification index is 2, the genus of K is $\frac{1}{2} \cdot (-2 + 8) = 3$. Let Q_0, respectively Q_∞, denote the *only* place of K above P_0, respectively P_∞. The divisor of x in K is $2 \cdot ([Q_0] - [Q_\infty])$. The divisors of dx in K is, according to [S09, Proposition 3.7.8, p.127] and [S09, Corollary 3.4.7, p.94], $-2 \cdot 2 \cdot [Q_\infty] + 4 \cdot [Q_0] + 4 \cdot [Q_\infty] = 4 \cdot [Q_0]$. Hence the differentials $dx, x^{-1} \cdot dx$ and $x^{-2} \cdot dx$ form a basis of $H^0(\Omega^1_C)$. Keeping with the notation introduced in section 3, we have $T = A_0 + A_1 \cdot x^{-1} + A_2 \cdot x^{-2}$. Therefore,

$$\begin{aligned} \Psi_{\mathbf{A}}\left(\frac{d}{dx}\right) &= T^2 + \frac{dT}{dx} \\ &= A_0^2 + (A_0A_1 + A_1A_0) \cdot x^{-1} + (A_0A_2 + A_1^2 + A_2A_0) \cdot x^{-2} + \\ &\quad +(A_1A_2 + A_2A_1) \cdot x^{-3} + A_2^2 \cdot x^{-4} + A_1 \cdot x^{-2} \\ &:= B_0 + B_1 \cdot x^{-1} + B_2 \cdot x^{-2} + B_3 \cdot x^{-3} + B_4 \cdot x^{-4}. \end{aligned}$$

Thus, the category of connections with vanishing p–curvature is equivalent to the category of representations of the free algebra on three generators subjected to the relations $B_0 = \cdots = B_4 = 0$. We use Macaulay 2 [M2] to implement Letzter's algorithm. Define

```
R=frac(ZZ/2[u])[a_1..a_4,b_1..b_4,c_1..c_4];
```

and the matrices

```
A_0=matrix{{a_1,a_2},{a_3,a_4}};
A_1=matrix{{b_1,b_2},{b_3,b_4}};
A_2=matrix{{c_1,c_2},{c_3,c_4}};
```

The relations are therefore

```
B_0=A_0^2;
B_1=A_0*A_1+A_1*A_0;
B_2=A_0*A_2+A_2*A_0+A_1^2+A_1;
B_3=A_1*A_2+A_2*A_1;
B_4=A_2^2;
```

so that the ideal of relations is

```
RelIdeal=
minors(1,B_0)+minors(1,B_1)+minors(1,B_2)+minors(1,B_3)+minors(1,B_4);
```

Then, a direct calculation shows that the $1 \notin (u - \chi, \mathbf{RelIdeal})$ if and only if $\chi = \mathrm{trace}(A_0 A_2)$, $\mathrm{trace}(A_2 A_0)$, $\det(A_1)$, $\det(A_1^2)$. It follows that there are infinitely many isomorphism classes of semisimple representations of dimension 2, so, by Corollary 4, $\#S(C, 2, 1) = \infty$.

Example 7. If we take $f = x^{-5} + x^3$, then [S09, Proposition 3.7.8, p. 127] and [S09, Corollary 3.4.7, p. 94] show that C is of genus 4, and $dx, x^{-1} \cdot dx, x^{-2} \cdot dx, x^{-3} \cdot dx$ is a basis of $H^0(\Omega^1)$. A computation using Macaulay 2 similar to that in Example 6 lets us conclude that there are infinitely many isomorphism classes of semisimple representations of rank 2 of Λ_C. Hence $\#S(C, 2, 1) = \infty$.

7. The case of characteristic 3

We assume that $k = \overline{\mathbb{F}}_3$ and that $K = k(x, y)$, where x is transcendental.

7.1. **Hyperelliptic curves.** Let $y^2 = f$, where $f \in k[x]$ is square free. Then, according to [S09, Example 3.7.6, p.125]

$$g = \begin{cases} (\deg(f) - 1)/2, & \text{if } \deg(f) \text{ is odd} \\ (\deg(f) - 2)/2, & \text{if } \deg(f) \text{ is even.} \end{cases}$$

Moreover, applying [S09, Proposition 3.7.3, p.122] and [S09, Corollary 3.4.7, p.94], the rational differential forms $y^{-1} \cdot dx, xy^{-1} \cdot dx, \ldots, x^{g-1}y^{-1} \cdot dx$ are regular and form a basis of $H^0(\Omega^1_C)$. Also, we have

$$\frac{dy}{dx} = \frac{f'}{2y}, \qquad \frac{d}{dx}(y^{-1}) = -\frac{f'}{2y^3}, \qquad \frac{d^2}{dx^2}(y^{-1}) = -\frac{f''}{2y^3}.$$

Hence, if we write

$$\begin{aligned} T &= T_0 \cdot y^{-1}, \\ &= (A_0 + A_1 \cdot x + \dots) \cdot y^{-1}, \end{aligned}$$

it follows that $\frac{dT}{dx} = T_1 \cdot y^{-3}$ and $\frac{d^2T}{dx^2} = T_2 \cdot y^{-3}$, where

$$\begin{aligned} T_1 &= \left(f' \cdot T_0 + f \cdot \frac{dT_0}{dx} \right), \\ T_2 &= \frac{dT_1}{dx}. \end{aligned}$$

Consequently,

$$\Psi_{\mathbf{A}} \left(\frac{d}{dx} \right) = y^{-3} \cdot \left(T_0^3 + T_2 + [T_1, T_0] \cdot y^{-1} \right).$$

This means that Λ_C is the quotient of $k\{A_0, \dots, A_{g-1}\}$ by the relations $T_0^3 + T_2 = [T_1, T_0] = 0$.

Example 8. Let f be an irreducible monic polynomial of degree 5 over $\mathbb{F}_3$. A list of such polynomials can be found in [LN86, p.380]. For each one of these, the associated algebra Λ_C has only finitely many isomorphism classes of semisimple representations of rank 2. The commands for Macaulay 2 [M2] we have used to arrive at the ideal of relations **RelIdeal** in the case $f = x^5 + 2x^4 + 2x^3 + x^2 + 2$ are tailored for generalisation. They are as follows.

```
S=frac(ZZ/3[u])[a_1..a_4,b_1..b_4];
R=S[x];
f=(matrix{{1,2,2,1,0,2}}*matrix{{x^5},{x^4},{x^3},{x^2},{x^1},{x^0}})_(0,0);
f'=diff(x,f);
A_0=sub(matrix{{a_1,a_2},{a_3,a_4}},R);
A_1=sub(matrix{{b_1,b_2},{b_3,b_4}},R);
T_0=A_0+A_1*x;
T_1=f'*T_0+f*diff(x,T_0);
T_2=diff(x,T_1);
RelIdeal=minors(1,(coefficients(T_0^3+T_2)_(0,0))#1)+
minors(1,(coefficients(T_0^3+T_2)_(0,1))#1)+
minors(1,(coefficients(T_0^3+T_2)_(1,0))#1)+
minors(1,(coefficients(T_0^3+T_2)_(1,1))#1)+
minors(1,(coefficients((T_1*T_0-T_0*T_1)_(0,0)))#1)+
minors(1,(coefficients((T_1*T_0-T_0*T_1)_(0,1)))#1)+
minors(1,(coefficients((T_1*T_0-T_0*T_1)_(1,0)))#1)+
minors(1,(coefficients((T_1*T_0-T_0*T_1)_(1,1)))#1).
```

The same procedure can be made for polynomials of degree six (in which case the genus is also two). We have tested with all irreducible monic polynomials of degree six over $\mathbb{F}_3$ (there are 116 of those) and obtained in all cases the same result as for polynomials of degree five. For genus 3, that is, polynomials of degree 7, computations with Macaulay 2 become too long.

Acknowledgments. Thanks are due to C. Pauly and V. Mehta for the interest they took in this work.

References

[BdS10] I. Biswas and J. P. dos Santos, Vector bundles trivialized by proper morphisms and the fundamental group scheme. Journal of the Inst. of Math. Jussieu (2011) 10(2), 225–234. doi:10.1017/S1474748010000071.

[M2] D. Grayson and M. Stillman, Macaulay2, a software system for research in algebraic geometry. Available at `http://www.math.uiuc.edu/Macaulay2/`

[Ka70] N. M. Katz, Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. Inst. Hautes Études Sci. Publ. Math. No. 39 (1970), 175–232.

[L08] E. S. Letzter, Detecting infinitely many semisimple representations in a fixed finite dimension. Journal of Algebra **320** (2008) 3926 – 3934.

[LN86] R. Lidl and H. Niederreiter, Introduction to finite fields and their applications. Cambridge University Press, Cambridge, 1986.

[MS08] V. B. Mehta and S. Subramanian, Some remarks on the local fundamental group scheme. Proc.Indian Acad. Sci. (Math. Sci.) Vol. 118, No. 2, May 2008, pp. 207–211.

[MS02] V. B. Mehta and S. Subramanian, On the fundamental group scheme. Invent. Math. 148 (2002), no. 1, 143 – 150.

[N05] N. Nitsure, Construction of Hilbert and Quot schemes, *in* Fundamental algebraic geometry. Mathematical Surveys and Monographs, 123. American Mathematical Society, Providence, RI, 2005.

[N82] M. V. Nori, The fundamental group scheme, Proc. Indian Acad. Sci. (Math. Sci.) 91 (1982) 73–122.

[O08] B. Osserman, Frobenius unstable bundles and p–curvature. Trans. Amer. Math. Soc. 360 (2008), no. 1, 273–305.

[P07] C. Pauly, A smooth counterexample to Nori's conjecture on the fundamental group scheme. Proc. Amer. Math. Soc. 135 (2007), no. 9, 2707–2711.

[S82] C. S. Seshadri, Fibrés vectoriels sur les courbes algébriques. Notes written by J.–M. Drezet. Astérisque, 96. Société Mathématique de France, Paris, 1982.

[S09] H. Stichtenoth, Algebraic function fields and codes. Second edition. Graduate Texts in Mathematics, 254. Springer-Verlag, Berlin, 2009.

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