Fundamental group-schemes in positive characteristic

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Introduction – The topic of this talk is the study the Tannakian group-scheme associated to the category of stratified sheaves (D-modules) on a smooth scheme over an algebraically closed field of positive characteristic. The details and further references are in [3]. This work is based on [4], [6] and [7].

Let k be an algebraically closed field of positive characteristic p. Let X be a smooth k-scheme. The absolute Frobenius morphism of X will be denoted by F. Let $\mathscr{D}_X$ be the $\mathscr{O}_X$-algebra of k-linear differential operators on X as defined in EGA IV_4, 16.7. Another reference for differential operators is [1, Ch. 2].

Definition 1. *The category of stratified sheaves* $\mathbf{str}(X)$ *is the category whose:*

Objects are $(\mathscr{E}, \nabla)$ *with* $\mathscr{E}$ *a coherent* $\mathscr{O}_X$*-module and* $\nabla : \mathscr{D}_X \longrightarrow \mathscr{E}nd_k(\mathscr{E})$ *a homomorphism of* $\mathscr{O}_X$*-algebras.*

Arrows are *homomorphism of* $\mathscr{D}_X$*-modules.*

Remark: Our terminology follows that introduced by Grothendieck in [5].

The category $\mathbf{str}(X)$ is abelian, k-linear and tensor. A stratified sheaf is a generalization of an integrable connection. In fact, if Θ_X denotes the sheaf of tangent vectors of X/k (k-linear derivations), then $\Theta_X \subseteq \mathscr{D}_X$ and the condition that ∇ above is a homomorphism of $\mathscr{O}_X$-algebras implies that $\nabla|\Theta_X$ is an integrable connection on $\mathscr{E}$. In characteristic zero these notions are equivalent (see [1, Ch. 2]) but in positive characteristic a stratification is a much stronger condition, as the following lemma shows:

Lemma 2 ([1])**.** *If* $(\mathscr{E}, \nabla)$ *is a stratified sheaf, then* $\mathscr{E}$ *is locally free.*

Note that the $k[x]$-module $k[x]/(x^p)$ is endowed with an integrable connection but is certainly not locally free.

A more convenient description of the category $\mathbf{str}(X)$ can be obtained by iterating Cartier's result on the p-curvature and Frobenius pull-backs. First a definition.

Definition 3. *The category of* F*-divided sheaves* $\mathbf{Fdiv}(X)$ *is the category whose:*

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Objects are *sequences of coherent* $\mathscr{O}_X$*-modules* $\{\mathscr{E}_i\}_{i\in\mathbb{N}}$ *and isomorphisms of* $\mathscr{O}_X$*-modules* $\sigma_i : F^*\mathscr{E}_{i+1} \longrightarrow \mathscr{E}_i$.

Arrows are *projective systems;* $\{\alpha_i\} : \{\mathscr{E}_i, \sigma_i\} \longrightarrow \{\mathscr{F}_i, \tau_i\}$ *with* α_i $\mathscr{O}_X$*-linear and* $\tau_i \circ F^*(\alpha_{i+1}) = \alpha_i \circ \sigma_i$.

By flatness of Frobenius, it is easy to see that $\mathbf{Fdiv}(X)$ has a structure of abelian k-linear category. There is also an obvious tensor product. If $\{\mathscr{E}_i\}$ is an object of $\mathbf{Fdiv}(X)$, then $\mathscr{E}_0$ is locally free. Using this (and the above hinted result of Cartier) we have:

Theorem 4 (N. Katz, [4])**.** *The categories* $\mathbf{Fdiv}(X)$ *and* $\mathbf{str}(X)$ *are equivalent* k*-linear tensor categories.*

Since all the sheaves in $\mathbf{str}(X)$ and $\mathbf{Fdiv}(X)$ are locally free, given $x_0 \in X(k)$ we obtain the following definition.

Definition 5. *The fundamental group-scheme of* X *at the point* x_0*,* $\Pi_X = \Pi(X, x_0)$*, is the Tannakian group-scheme associated to* $\mathbf{Fdiv}(X)$ *via the fibre functor* x_0^*.

For the construction of the Tannakian group-scheme see [2, Thm. 2.11].

Structural properties of Π_X – By using a method of Nori which interprets exact tensor functors $\mathscr{L} : \mathrm{Rep}_k(G) \longrightarrow \mathrm{coh}(X)$ we can show:

Theorem 6. *The Frobenius homomorphism* $F : \Pi_X \longrightarrow \Pi_X$ *is an isomorphism.*

Corollary 7. *i) Any quotient of* Π_X *is reduced.*
ii) Any pro-finite quotient of Π_X *is pro-etale.*

The same method of Nori allows us to show that the group of connected components of $\pi_0(\Pi_X)$ is none other than the etale fundamental group seen as a constant group-scheme. This is another manifestation of the property "differential equations with finite monodromy are etale coverings". This is silently used in the next result.

Theorem 8. *Assume that* X *is proper over* k*. Then the largest unipotent quotient of* Π_X*,* Π_X^{uni}*, coincides with the largest unipotent quotient of the etale fundamental group.*

The reader should notice that this is quite particular to positive characteristic as one can see from the example of an elliptic curve over $\mathbb{C}$. The above theorem will enable us to derive more precise information about Π_X from known information on the etale fundamental group π_1^{et}.

THE CASE OF AN ABELIAN VARIETY – Using the results above we come to a good description of Π_X in the case where X is an abelian variety.

Theorem 9. *There is a natural isomorphism*

$$\Pi_X \xrightarrow{\cong} T_p(X) \times \mathrm{Diag}(P),$$

where $T_p(X)$ is the p-adic Tate module (of k-rational points) and $\mathrm{Diag}(P)$ *is the diagonal group-scheme [8, 2.2] whose character group is*

$$P = \varinjlim \left(\cdots \xrightarrow{[p]} \mathrm{Pic}^0 \xrightarrow{[p]} \mathrm{Pic}^0 \xrightarrow{[p]} \cdots \right).$$

References

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