# ON THE STRUCTURE OF AFFINE FLAT GROUP SCHEMES OVER DISCRETE VALUATION RINGS, II

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ABSTRACT. In the first part of this work [DHdS17], we studied affine group schemes over a discrete valuation ring by means of Neron blowups. We also showed how to apply these findings to throw light on the group schemes coming from Tannakian categories of $\mathcal{D}$-modules. In the preset work we follow up on this theme. We show that a certain class of affine group schemes of "infinite type", Neron blowups of formal subgroups, are quite typical. We also explain how these group schemes appear naturally in Tannakian categories of $\mathcal{D}$-modules. To conclude we isolate a Tannakian property of affine group schemes which allows one to verify if its coordinate ring is projective over the base ring.

## 1. INTRODUCTION

In this paper we continue the analysis of flat *affine* group schemes over a discrete valuation ring (DVR) $R$ started in [DHdS17]. The results presented here throw light mainly on three fundamental questions, which are:

(**UB**) Ubiquity of "blowups": Assume that $R$ is complete and let $G$ be an affine and flat group scheme over $R$ whose generic fibre $G \otimes K$ is of finite type. Is it the case that $G$ is the blowup of a formal subgroup (see below) of some group scheme *of finite type* over $R$?
(**SA**) "Strict pro-algebraicity": Let $G$ be an affine and flat group scheme over $R$; is it possible to write $G$ as $\varprojlim_i G_i$, where each $G_j \to G_i$ is a *faithfully flat* morphism of flat group schemes *of finite type*?
(**PR**) "Projectivity": Let $G$ be as before. What conditions should the category of representations $\mathrm{Rep}_R(G)$ enjoy so that $R[G]$ is projective as an $R$-module?

In [DHdS17, Section 5] we found a process which produced, starting from an affine flat group scheme of finite type $G$ and a flat closed formal subgroup $\mathfrak{H}$ of the completion $\widehat{G}$, a new group scheme $\mathcal{N}_{\mathfrak{H}}^{\infty}(G)$. (This is called the Neron blowup of $\mathfrak{H}$ in $G$.) From the examples, we realised that many complicated cases of morphisms $G' \to G$ inducing isomorphisms on generic fibres were quite close to these blowups, hence Question (**UB**) immediately imposes. Question (**SA**) is inspired by a fundamental fact from the theory of affine group schemes over a field, see [Wa79, 3.3 and 14.1]. Question (**PR**) arises when studying group schemes obtained from Tannakian categories.

If Neron blowups of closed subgroups on the special fibre [WW80] were one of the main tools of [DHdS17], in the present work we rely much more on the notion of a Neron blowup of a *formal* subgroup scheme mentioned above (for details see [DHdS17, Section 5] as well as Section 2.2). They turn out to play a role in studying all three of the above questions (and in fact immediately answer (**SA**) in the negative.)

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Let us now review the remaining sections of the paper separately. But first, some words concerning terminology. In what follows, "group scheme" shall always mean "*affine* group scheme." For unexplained notations, the reader is directed to the corresponding parts of the text or to the list at the end of this section.

Section 2 exists mainly to establish notations and explain colloquially the operation of blowing up a formal subgroup scheme.

In Section 3 we address Question (**UB**). As Corollary 3.3 argues, the positive answer is not very far from the truth if $R$ is of characteristic $(0,0)$. In fact, the particularity which gives these blowups such a prominent role is not exclusive of characteristic $(0,0)$, and has to do, as visible in Theorem 3.1, with the "constancy" of the centres in the standard sequence (recalled in Section 2.1). This property had already been detected in [DHdS17, Section 6], and used to arrive at a less clear result than Theorem 3.1 below. Section 3 also contains some applications of Theorem 3.1 to unipotent group schemes.

In section 4 we study Tannakian envelopes of abstract groups (Definition 4.1); let us explain our motivations. Group schemes live symbiotically with Tannakian categories, and categories of $\mathcal{D}$-modules consist one of our preferred sources of examples. On the other hand, $\mathcal{D}$-modules are algebraic manifestations of the idea of fundamental group, so that we are naturally led to explore the group schemes associated to abstract groups. After developing the rudiments of the theory (Sections 4.1 and 4.2) and observing that some abstract groups are more docile than others (see Example 4.7), we show that Neron blowups of formal subgroups appear already in relation to the Tannakian envelope of an infinite cyclic group. See Section 4.3. Consequently, these envelopes can easily fail to enjoy "strict pro-algebraicity".

In Section 5 we concentrate on the case $R = \mathbb{C}[\![t]\!]$, so that a purely complex analytic theory becomes available. We go on to show that, under the correct assumptions, the category $\mathcal{D}(X/S)$-**mod** is equivalent to the category of $R$-linear representations of the fundamental group $\pi_1((X\otimes\mathbb{C})^{\mathrm{an}})$, see Theorem 5.10. This allows us to use the constructions of Section 4 to note–see Corollary 5.11–that Neron blowups of formal subgroups do play a role in differential Galois theory over $R$. (This theory is to be understood as the study of the group schemes appearing in the Tannakian category $\mathcal{D}$-**mod**. See [DHdS17, Section 7].)

In Section 6 we address (**PR**) through the concept of *prudence*. The precise definition of the term is given in Definition 6.7, but the idea behind it is quite simple to grasp. Let $\pi$ stand for a uniformizer of $R$; prudence wishes to draw attention to the property "invariance modulo $\pi^n$ for all $n$ implies invariance." As it turns out, the theory of Neron blowups of formal subgroups together with Kaplansky's fundamental result concerning modules over $R$ (Theorem 6.1) allows us to link prudence to *projectivity* (or freeness) of the ring of functions of our group schemes, see Theorem 6.11 for the precise statement. As the reader familiar with formal geometry will immediately recognise, prudence is quite close to Grothendieck's notion of algebraization. As a fruit of this proximity we can offer Theorem 6.20, which answers Question (**PR**) completely for the group schemes in (a certain type of) differential Galois theory.

**Notations, conventions and standard terminology.**

(1) The ring $R$ is a discrete valuation ring with quotient field $K$ and residue field $k$. An uniformizer is denoted by $\pi$, except in Section 5, where we call it $t$ to avoid confusion.
(2) The spectrum of $R$ is denoted by $S$, and the quotient ring $R/(\pi^{n+1})$ is denoted by $R_n$.
(3) The characteristic of $R$ is the ordered pair $(\mathrm{char}.\,K, \mathrm{char}.\,k)$.

(4) If $M$ is an $R$-submodule of $N$, we say that $M$ is **saturated** in $N$ if $N/M$ has no $\pi$-torsion.
(5) Given an object $X$ over $R$ (a scheme, a module, etc), we sometimes find useful to write $X_k$ instead of $X \otimes_R k$, $X_K$ instead of $X \otimes_R K$, etc. If context prevents any misunderstanding, we also employ $X_n$ instead of $X \otimes_R R_n$.
(6) To avoid repetitions, by a **group scheme** over some ring $A$, we understand an **affine group scheme** over $A$.
(7) The category of group schemes over a ring $A$ will be denoted by $(\mathbf{GSch}/A)$; the full subcategory whose objects are $A$-flat will be denoted by $(\mathbf{FGSch}/A)$.
(8) If $V$ is a free $R$-module of finite rank, we write $\mathbf{GL}(V)$ for the general linear group scheme representing $A \mapsto \mathrm{Aut}_A(V \otimes A)$. If $V = R^n$, then $\mathbf{GL}(V) = \mathbf{GL}_n$.
(9) If $G$ is a group scheme over $R$, we let $\mathrm{Rep}_R(G)$ stand for the category of representations of $G$ which are, as $R$-modules, *of finite type* over $R$. (We adopt Jantzen's and Waterhouse's definition of representation. See [Ja03, Part I, 2.7 and 2.8, 29ff] and [Wa79, 3.1-2, 21ff].) The full subcategory of $\mathrm{Rep}_R(G)$ having as objects those underlying a free $R$-module is denoted by $\mathrm{Rep}_R^\circ(G)$.
(10) Given an arrow $\rho : G \to H$ of group schemes over $R$, we let $\rho^\# : \mathrm{Rep}_R(H) \to \mathrm{Rep}_R(G)$ stand for the associated functor.
(11) If $\Gamma$ is an abstract group and $A$ is a commutative ring, we write $A\Gamma$ for the group ring of $\Gamma$ with coefficients in $A$. We let $\mathrm{Rep}_A(\Gamma)$ stand for the category of left $A\Gamma$-modules whose underlying $A$-module is of finite type. The full subcategory of $\mathrm{Rep}_A(\Gamma)$ whose objects are moreover projective $A$-modules shall be denoted by $\mathrm{Rep}_A^\circ(\Gamma)$.
(12) For an affine scheme $X$ over $R$, we call $\mathcal{O}(X)$ the ring of functions of $X$, and denote it by $R[X]$. More generally, if $A$ is any $R$-algebra, we write $A[X]$ to denote $\mathcal{O}(X \otimes_R A)$.
(13) If $G \in (\mathbf{FGSch}/R)$, its right-regular module is obtained by letting $G$ act on $R[G]$ via right translations. It is denoted by $R[G]_{\mathrm{right}}$ below.
(14) For an affine $R$-adic formal scheme $\mathfrak{X}$, we write $R\langle\mathfrak{X}\rangle$ for the ring $\mathcal{O}(\mathfrak{X})$. If $\mathfrak{X}$ is the $\pi$-adic completion (or completion along the closed fibre) of some affine $R$-scheme $X$, we write $R\langle X\rangle$ or $R[X]^\wedge$ to denote $R\langle\mathfrak{X}\rangle$.
(15) A formal group scheme over $R$ is a group object in the category of affine formal $R$-adic schemes.
(16) If $X$ is a ringed space, we let $|X|$ stand for the its underlying topological space.
(17) If $G$ is a group scheme over a ring $A$, and $\mathfrak{a} \subset R[G]$ is the augmentation ideal, we define the conormal module $\omega(G/A)$ as being the $A$-module $\mathfrak{a}/\mathfrak{a}^2$ [DG70, II.4.3.4]. Its dual $A$-module is the Lie algebra $\mathrm{Lie}(G)$ of $G$ (see II.4.3.6 and II.4.4.8 of [DG70]). Similar conventions are in force when dealing with formal affine group schemes.
(18) By a $\otimes$-category or tensor-category we mean a $\otimes$-category ACU in the sense of [Sa72, I.2.4, 38ff], or a tensor category in the sense of [DM82, Definition 1.1, p.105]. Unless mentioned otherwise, all $\otimes$-functors are ACU [Sa72, I.4.2] and all $\otimes$-natural transformation are unital (which are the conventions of [DM82, Section 1]).

## 2. Preliminary material

For the convenience of the reader we briefly recall some thoughts from [DHdS17] and [WW80] which are employed here. On the other hand, we make use of the basic properties of Neron blowups (see [DHdS17, Section 2.1] and [WW80, Section 1]) without the same pedagogy.

2.1. **The standard sequences** [DHdS17, Section 2], [WW80, Section 1]**.** Let

$$\rho : G' \longrightarrow G$$

be an arrow of $(\mathbf{FGSch}/R)$ inducing an isomorphism on generic fibres. We then associate to $\rho$ its *standard sequence*:

$$\begin{array}{ccccccc} & & G' & & & & \\ & & \downarrow{\scriptstyle \rho_{n+1}} & & \searrow{\scriptstyle \rho_0 = \rho} & & \\ \cdots & \longrightarrow & G_{n+1} & \xrightarrow[\varphi_n]{} & \cdots & \xrightarrow[\varphi_0]{} & G_0 = G \end{array}$$

as follows. The arrow $\varphi_0 : G_1 \to G_0$ is the Neron blowup of $B_0 := \mathrm{Im}(\rho_0 \otimes k)$, and $\rho_1$ is the morphism obtained by the universal property. Now, if $\rho_n : G' \to G_n$ is defined, then $\varphi_n : G_{n+1} \to G_n$ is the Neron blowup of

$$B_n := \mathrm{Im}(\rho_n \otimes k)$$

and $\rho_{n+1}$ is once more derived from the universal property. In particular, for each $n$, the morphisms $\rho_n \otimes k : G' \otimes k \to B_n$ and $\varphi_n : B_{n+1} \to B_n$ are faithfully flat because of [Wa79, Theorem 14.1]. In [DHdS17, Theorem 2.11], it is proved that the morphism

$$G' \longrightarrow \varprojlim_n G_n$$

obtained from the above commutative diagram is an isomorphism of group schemes.

2.2. **Blowing up formal subgroup schemes** [DHdS17, Section 5]**.** We assume that $R$ is *complete*. Let $G$ be a group scheme over $R$ which is flat and of finite type. Denote by $\widehat{G}$ its $\pi$-adic completion (or completion along the closed fibre): it is a group object in the category of $R$-adic affine formal schemes. Let $\mathfrak{H} \subset \widehat{G}$ be a closed, *flat* formal subgroup scheme and let $I_n \subset R[G]$ be the ideal of $H_n := \mathfrak{H} \otimes R_n$; note that $\pi^{n+1} \in I_n$. We then define the Neron blowup of $\mathfrak{H}$, $\mathcal{N}_{\mathfrak{H}}^{\infty}(G)$, as being the group scheme whose Hopf algebra is

$$\varinjlim_n R[G][\pi^{-n-1} I_n].$$

There is a natural morphism of group schemes $\mathcal{N}_{\mathfrak{H}}^{\infty}(G) \to G$ inducing an isomorphism on generic fibres. One fundamental feature of $\mathcal{N}_{\mathfrak{H}}^{\infty}(G)$ which is worth mentioning here is that $\mathcal{N}_{\mathfrak{H}}^{\infty}(G) \otimes R_n \to G \otimes R_n$ induces an isomorphism between the source and $H_n$ [DHdS17, Corollary 5.11].

If $H \subset G$ is a closed, flat subgroup scheme, we let $\mathcal{N}_H^{\infty}(G)$ stand for the Neron blowup of the (necessarily $R$-flat) closed subgroup $\widehat{H} \subset \widehat{G}$. It is a simple matter to verify that $R[\mathcal{N}_H^{\infty}(G)] = \varinjlim_n R[G][\pi^{-n-1} I]$, where $I$ is the ideal of $H$.

## 3. The ubiquity of blowups of formal subgroups

In this section, we assume that $R$ is *complete*. Let $G' \in (\mathbf{FGSch}/R)$ have a generic fibre $G' \otimes K$ of finite type (over $K$). This being so, there exists some $G \in (\mathbf{FGSch}/R)$, now of finite type over $R$, which is the target of a morphism

$$\rho : G' \longrightarrow G$$

inducing an isomorphism on generic fibres. Then, as explained on Section 2.1, we have the standard sequence of $\rho$:

$$\begin{array}{ccccccc} & & G' & & & & \\ & & \downarrow{\scriptstyle \rho_{n+1}} & \searrow^{\rho_0=\rho} & & & \\ \cdots \longrightarrow & & G_{n+1} & \xrightarrow[\varphi_n]{} & \cdots & \xrightarrow[\varphi_0]{} & G_0 = G. \end{array}$$

To state Theorem 3.1 below, we let

$$\begin{aligned} B_n &= \text{image of } \rho_n \otimes k : G' \otimes k \to G_n \otimes k \\ &= \text{centre of } \varphi_n : G_{n+1} \to G_n. \end{aligned}$$

**Theorem 3.1.** (1) *Assume that for all $n$, the morphisms $\varphi_n : B_{n+1} \to B_n$ are isomorphisms. Then*

(a) *the morphism induced between $\pi$-adic completions $R\langle G\rangle \to R\langle G'\rangle$ is surjective.*

(b) *Let $\mathfrak{H}$ stand for the closed formal subgroup scheme of $\widehat{G}$ cut out by the kernel of $R\langle G\rangle \to R\langle G'\rangle$. (Clearly $\widehat{G}' \simeq \mathfrak{H}$.) Then there exists an isomorphism of group schemes*

$$\sigma : G' \xrightarrow{\sim} \mathcal{N}_{\mathfrak{H}}^{\infty}(G)$$

*rendering*

$$\begin{array}{ccc} G' & \xrightarrow{\sigma} & \mathcal{N}_{\mathfrak{H}}^{\infty}(G) \\ & {\scriptstyle\rho}\searrow & \downarrow \\ & & G \end{array}$$

*commutative.*

(2) *Assume that the characteristic of $k$ is zero. Then, there exists $n_0 \in \mathbf{N}$ such that, for all $n \geq n_0$, the arrows $\varphi_n : B_{n+1} \to B_n$ are isomorphisms.*

The proof shall need

**Lemma 3.2.** *The obvious morphisms $G' \otimes k \to B_n$ induce an isomorphism*

$$G' \otimes k \xrightarrow{\sim} \varprojlim_n B_n.$$

*Proof.* One considers the commutative diagram of $k$-algebras obtained from the construction of the standard sequence

$$\begin{array}{ccccccccc} \cdots \longrightarrow & k[G_{n-1}] & \longrightarrow & k[G_n] & \longrightarrow & k[G_{n+1}] & \longrightarrow \cdots \\ & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \\ \cdots \longrightarrow & k[B_{n-1}] & \hookrightarrow & k[B_n] & \hookrightarrow & k[B_{n+1}] & \longrightarrow \cdots \end{array}$$

(As usual, $\hookrightarrow$ means injection, and $\twoheadrightarrow$ surjection.) It is then clear that

$$\varinjlim_n k[G_n] \longrightarrow \varinjlim_n k[B_n]$$

is an isomorphism. As explained in Section 2.1, the canonical arrow $G' \to \varprojlim G_n$ induces an isomorphism. Since taking tensor products commutes with direct limits [Mat89, Theorem A.1], the result follows. □

*Proof of Theorem 3.1.* (1a) The morphism $G' \otimes k \to G \otimes k$ induces an isomorphism between $G' \otimes k$ and $B_0$ because of the assumption and of Lemma 3.2. Hence, the arrow between $k$-algebras $k[G] \to k[G']$ is surjective. As is well-known (see for example [AM69, 10.23(ii), p. 112]), the induced arrow between completions $R\langle G\rangle \to R\langle G'\rangle$ is then surjective.

(1b) Let $I_n \subset R[G]$ be the ideal of the closed immersion $G' \otimes R_n \to G \otimes R_n \to G$ (in particular $\pi^{n+1} \in I_n$). Writing $\mathcal{N}$ instead of $\mathcal{N}^\infty_{\mathfrak{H}}(G)$, the definition says that

$$R[\mathcal{N}] = \bigcup_n R[G][\pi^{-n-1} I_n].$$

Since, by construction, the morphism of $R$-algebras $R[G] \to R[G']$ sends $I_n$ to the ideal $\pi^{n+1}R[G']$, we conclude that $R[G] \to R[G']$ factors thought $R[\mathcal{N}]$, that is, $\rho : G' \to G$ factors through $\mathcal{N} \to G$. Using that $\mathcal{N} \otimes k \to G \otimes k$ induces an isomorphism between $\mathcal{N} \otimes k$ and $\mathfrak{H} \otimes k$ [DHdS17, Corollary 5.11], we conclude that $G' \otimes k \to \mathcal{N} \otimes k$ is also an isomorphism. Together with the fact that $G' \otimes K \to \mathcal{N} \otimes K$ is an isomorphism, this easily implies that $G' \to \mathcal{N}$ is an isomorphism [WW80, Lemma 1.3].

(2) As already remarked in Section 2.1, the morphisms

$$\varphi_n|_{B_{n+1}} : B_{n+1} \longrightarrow B_n$$

are all faithfully flat. It then follows from [SGA3$_{\text{new}}$, VI$_B$, Proposition 1.2, p.335] that

$$(*) \qquad \dim B_{n+1} = \dim B_n + \dim \operatorname{Ker}(\varphi_n|_{B_{n+1}}).$$

Using the equality $\dim G_n \otimes k = \dim G_n \otimes K$ [SGA3$_{\text{new}}$, VI$_B$, Corollary 4.3, p.358], we conclude that

$$\begin{aligned} \dim B_n &\le \dim G_n \otimes k \\ &= \dim G_n \otimes K \\ &= \dim G \otimes K. \end{aligned}$$

Let $n_0 \in \mathbf{N}$ be such that $\dim B_{n_0}$ is maximal. We then derive from equation $(*)$ above that for all $n \ge n_0$,

$$(**) \qquad \dim \operatorname{Ker}(\varphi_n|_{B_{n+1}}) = 0.$$

Because of the assumption on the characteristic, we know that $B_n$ and $G_n \otimes k$ are *smooth* $k$-schemes, so that $B_n \otimes k \to G_n \otimes k$ is a regular immersion [EGA IV$_4$, 17.12.1, p.85]. Now, as explained in [DHdS17, Section 2.2], the group scheme $\operatorname{Ker}(\varphi_n \otimes k)$ is isomorphic to a vector group $\mathbf{G}_{a,k}^{c_n}$. Consequently, $\operatorname{Ker}(\varphi_n|_{B_{n+1}})$ is isomorphic to a closed subgroup scheme of $\mathbf{G}_{a,k}^{c_n}$; the hypothesis on the characteristic then implies that $\operatorname{Ker}(\varphi_n|_{B_{n+1}})$ is either trivial or positive dimensional (by [DG70, Proposition IV.2.4.1], say). Equation $(**)$ then shows that $\operatorname{Ker}(\varphi_n|_{B_{n+1}}) = \{e\}$ if $n \ge n_0$. We have therefore proved that $\varphi_n|_{B_{n+1}}$ is a closed immersion [SGA3$_{\text{new}}$, VI$_B$, Corollary 1.4.2, p.341], and consequently an isomorphism if $n \ge n_0$. □

**Corollary 3.3.** *If $k$ is of characteristic zero, there exists $n_0 \in \mathbf{N}$ and a formal closed flat subgroup $\mathfrak{H} \subset \widehat{G}_{n_0}$ such that $G' \simeq \mathcal{N}^\infty_{\mathfrak{H}}(G_{n_0})$.* □

We now wish to derive from Theorem 3.1 some consequences concerning unipotent group schemes over $R$ of characteristic $(0,0)$; all hinges on the well-known principle that, in this case, Lie subalgebras are "algebraic." Recall that $U \in (\mathbf{FGSch}/R)$ is called *unipotent* if $U \otimes K$, respectively $U \otimes k$, is a unipotent group scheme over $K$, respectively $k$. (It should be noted that in certain cases it is sufficient to require this condition only for $K$. See for example Theorem 2.11 of [To15].)

Let us prepare the terrain: Assume that $R$ has characteristic $(0,0)$ and that $U \in (\mathbf{FGSch}/R)$ is unipotent and of finite type; we also abbreviate $\omega_U = \omega(U/R)$. A most fundamental tool for studying $U$ is the exponential morphism

$$\exp : \operatorname{Lie}(U)_a \longrightarrow U$$

which is explained in terms of functors in [DG70, II.6.3.1, 264ff]. (In loc.cit. the authors work over a base field, but this assumption is unnecessary for the construction.) Endowing

$\mathrm{Lie}(U)_a$ with its BCH multiplication, exp becomes an *isomorphism* of group schemes [To15, Section 1.3]. Since we wish to deal with formal group schemes as well, it is convenient to elaborate on a purely algebraic expression for $\exp^\# : R[U] \to \mathbf{S}(\omega_U)$. (Recall that $\mathrm{Lie}(U)_a = \mathrm{Spec}\,\mathbf{S}(\omega_U)$.)

Write $D : R[U] \to \omega_U$ for the canonical $\varepsilon$-derivation; this gives rise to an $\varepsilon$-derivation, denoted likewise, $D : R[U] \to \mathbf{S}(\omega_U)$ into the symmetric $R$-algebra. For each $n > 0$, we write $D^n : R[U] \to \mathbf{S}(\omega_U)$ for the convolution product of $D$ with itself $n$ times. In terms of Sweedler's notation, this reads

$$D^n(f) = \sum_{(f)} D(f_{(1)}) \cdots D(f_{(n)}), \qquad f \in R[U]. \tag{1}$$

Then, agreeing to fix $D^0(f) = \varepsilon(f) \cdot 1$, we define

$$\eta_t(f) := \sum_{n=0}^{\infty} \frac{D^n}{n!}(f)t^n, \quad f \in R[U].$$

(This is an element of $\mathbf{S}(\omega)[\![t]\!]$.) Leibniz' formula $D^n(\varphi\psi) = \sum_{m=0}^{n} \binom{n}{m} D^m(\varphi)D^{n-m}(\psi)$ proves that $\eta_t$ is a morphism of $R$-algebras, while the expression

$$D^{\ell+m}(f) = \sum_{(f)} D^\ell(f_{(1)})D^m(f_{(2)})$$

allows one to verify that $\eta_t$ satisfies the uniqueness statement concerning the exponential given in [DG70, II.3.6.1, p. 264]. In conclusion,

$$\exp^\# = \sum_{n=0}^{\infty} \frac{D^n}{n!}.$$

We are now in a position to comfortably prove the

**Proposition 3.4.** *We keep the notations and assumptions made just above. Write $\mathfrak{U}$ for the $\pi$-adic completion of $U$ and let $\mathfrak{H} \subset \mathfrak{U}$ be a formal flat closed subgroup scheme. Then, there exists a closed flat subgroup $H \subset U$ inducing $\mathfrak{H}$. Said differently, these formal subgroup schemes are algebraizable.*

*Proof.* We start by remarks on the structure of $\mathfrak{H}$. We note that $\mathfrak{H}$ is connected because the unipotent group scheme $\mathfrak{H} \otimes k$ is, as a scheme, an affine space [DG70, Proposition IV.2.4.1, p.497]. Also, since $\mathfrak{H} \otimes k$ is smooth over $k$, the fibre-by-fibre smoothness criterion [EGA, $\mathrm{IV}_4$, 17.8.2, p.79] proves that $\mathfrak{H} \otimes R_n$ is smooth over $R_n$. We conclude that the conormal module $\omega(\mathfrak{H} \otimes R_n/R_n)$ is free over $R_n$; as the canonical arrow $\omega(\mathfrak{H}/R) \otimes R_n \to \omega(\mathfrak{H} \otimes R_n/R_n)$ is bijective [DG70, I.4.1.6, p.99], $\omega(\mathfrak{H}/R)$ is free over $R$.

The canonical morphism $R[U] \to R\langle U \rangle$ induces a bijection $\omega(U/R) \to \omega(\mathfrak{U}/R)$ so that we can associate to $\omega(\mathfrak{H}/R)$, which is a quotient Lie-coalgebra of $\omega(\mathfrak{U}/R)$, a quotient Lie coalgebra $\omega(\mathfrak{H}/R)$ of $\omega(U/R)$. Let $H \subset U$ be a closed subgroup scheme which corresponds, via exp, to $\omega(\mathfrak{H}/R)$. (In this case $H \simeq \mathrm{Spec}\,\mathbf{S}(\omega(\mathfrak{H}/R))$ and the kernel of $\omega(U/R) \to \omega(H/R)$ is the kernel of $\omega(U/R) \to \omega(\mathfrak{H}/R)$.)

**Claim.** The closed immersion $\widehat{H} \to \mathfrak{U}$ factors as $\widehat{H} \to \mathfrak{H} \to \mathfrak{U}$, where $\widehat{H} \to \mathfrak{H}$ is a closed immersion inducing an isomorphism $\omega(\mathfrak{H}/R) \xrightarrow{\sim} \omega(\widehat{H}/R)$.

We ease notation by putting $\omega_{\mathfrak{H}} = \omega(\mathfrak{H}/R)$, etc. Write $\nu := \mathrm{Ker}(\omega_U \to \omega_{\mathfrak{H}})$ and let $I \subset R\langle U \rangle$ stand for the ideal cutting out $\mathfrak{H}$. Let $\widehat{D}^n$, respectively $\widehat{\exp}^\#$, stand for the prolongation of $D^n$, respectively $\exp^\#$, to $\pi$-adic completions. Then $\widehat{D}(I) \subset \nu$, and using equation (1), we conclude that $\widehat{\exp}^\#$ takes $I$ to the ideal $(\nu) \cdot \mathbf{S}(\omega_U)^\wedge$, which means that

$\mathfrak{H} \to \mathfrak{U}$ factors $\widehat{H} \to \mathfrak{U}$. Since $\widehat{H} \to \mathfrak{U}$ is a closed immersion so is $\widehat{H} \to \mathfrak{H}$; since $\omega_{\mathfrak{U}} \to \omega_{\widehat{H}}$ and $\omega_{\mathfrak{U}} \to \omega_{\mathfrak{H}}$ have the same kernel, the claim is proved.

Using again that $\omega(\mathfrak{H}/R) \otimes R_n \stackrel{\sim}{\to} \omega(\mathfrak{H}_n/R_n)$ and $\omega(H/R) \otimes R_n \stackrel{\sim}{\to} \omega(H_n/R_n)$ [DG70, I.4.1.6, p.99], we conclude that the closed immersion $H_n \to \mathfrak{H}_n$ induces an isomorphism $\omega(\mathfrak{H}_n/R_n) \to \omega(H_n/R_n)$. Now, [DG70, I.4.4.2, p.109] shows that $\widehat{\mathcal{O}}_{H_n,e}$, resp. $\widehat{\mathcal{O}}_{\mathfrak{H}_n,e}$, is isomorphic to the completion of $\mathbf{S}(\omega(H_n/R_n))$, resp. $\mathbf{S}(\omega(\mathfrak{H}_n/R_n))$, with respect to the augmentation ideal, so that we derive an isomorphism $\widehat{\mathcal{O}}_{\mathfrak{H}_n,e} \stackrel{\sim}{\to} \widehat{\mathcal{O}}_{H_n,e}$. This, in turn implies $\mathcal{O}_{\mathfrak{H}_n,e} \stackrel{\sim}{\to} \mathcal{O}_{H_n,e}$. Hence, $H_n \to \mathfrak{H}_n$ is flat at the point $e$, and consequently flat allover [SGA3$_{\rm new}$, VI$_B$, Proposition 1.3]. We can therefore say that the closed immersion $H_n \to \mathfrak{H}_n$ is an open immersion. Because $H_n$ and $\mathfrak{H}_n$ are connected, we conclude that $H_n \to \mathfrak{H}_n$ is an isomorphism which, in turn, implies that $\widehat{H} \to \mathfrak{H}$ is an isomorphism. □

The following corollary shall prove useful in Section 6.

**Corollary 3.5.** *Assume that $R$ is of characteristic $(0,0)$ and let $U \in (\mathbf{FGSch}/R)$ be unipotent and of finite type. Given an arrow*

$$\rho : G \longrightarrow U$$

*of $(\mathbf{FGSch}/R)$ which induces an isomorphism on generic fibres, either $G$ is of finite type, or $R[G]$ contains an $R$-submodule isomorphic to $K$.*

*Proof.* We assume that $G$ is not of finite type over $R$. Let

$$\cdots \longrightarrow U_n \longrightarrow \cdots \longrightarrow U$$

be the standard sequence of $\rho$. It follows from Corollary 3.3 that for some $n_0 \in \mathbf{N}$, $G \simeq \mathcal{N}^{\infty}_{\mathfrak{H}}(U_{n_0})$, where $\mathfrak{H}$ is a closed and flat subgroup of $\widehat{U}_{n_0}$. Since the kernel of $U_{n+1} \otimes k \to U_n \otimes k$ is unipotent [WW80, Theorem 1.5], basic theory [DG70, IV.2.2.3, p.485] tells us that $U_{n_0}$ is unipotent. So Proposition 3.4 can be applied: there exists some flat and closed $H \subset U_{n_0}$ such that $\mathfrak{H} = \widehat{H}$. Now, if $f \in R[U_{n_0}] \setminus \{0\}$ belongs to the ideal cutting out $H$, then, in $R[G]$, $f$ can be *uniquely* divided by any power of $\pi$, so that $\cup_m R\pi^{-m} f$ is isomorphic to $K$. □

## 4. Neron blowups of formal subgroup schemes in the Tannakian theory of "abstract" groups

In Section 3 we saw that blowing up a formal subgroup scheme is a very pertinent operation in the theory of flat group schemes over $R$. We now investigate if such group schemes play a role in the Tannakian theory of abstract groups. Apart from the results of Section 5, our motivation to carry the present study comes from the fact that Tannakian categories of abstract groups produce easily many interesting examples of group schemes.

Section 4.1 contains preparatory material on the Tannakian categories in question. Section 4.2 explains how to mimic, in our setting, the folkloric computation of differential Galois groups from the monodromy representation. Section 4.3 shows that Neron blowups of formal subgroup schemes appear naturally in the Tannakian theory of abstract groups, already in characteristic $(0,0)$. The results here shall be employed in Section 5 to study categories of $\mathcal{D}$-modules.

4.1. **Representations of abstract groups.** (In this section we make no supplementary assumption on $R$.)

Let $\Gamma$ be an abstract group. In what follows, we convey some thoughts on the existence and basic properties of a "Tannakian envelope" for $\Gamma$ over $R$, that is, a *flat* affine group scheme $\Pi$ such that $\mathrm{Rep}_R(\Pi) \simeq \mathrm{Rep}_R(\Gamma)$. Of course, contrary to the case of a ground field, flatness is not gratuitous, so that the existence of such an envelope imposes one extra property on $\mathrm{Rep}_R(\Gamma)$ which we briefly explain. Recall from [DH17, Definition 1.2.5] that $\mathrm{Rep}_R(\Gamma)$ is *Tannakian* if every $V \in \mathrm{Rep}_R(\Gamma)$ is the target of an epimorphism $\widetilde{V} \to V$ from an object $\widetilde{V} \in \mathrm{Rep}_R^\circ(\Gamma)$. If $\mathrm{Rep}_R(\Gamma)$ is not Tannakian, it still contains the full Tannakian subcategory $\mathrm{Rep}_R(\Gamma)^{\mathrm{tan}}$ consisting of those objects satisfying the aforementioned condition.

**Definition 4.1.** (1) We say that $\Gamma$ is *Tannakian over $R$* if $\mathrm{Rep}_R(\Gamma)$ is Tannakian in the sense explained above.

(2) The *Tannakian envelope* of $\Gamma$ over $R$ is the flat group scheme constructed from $\mathrm{Rep}_R(\Gamma)^{\mathrm{tan}}$ and the forgetful functor $\mathrm{Rep}_R(\Gamma)^{\mathrm{tan}} \to R$-**mod** by means of the main theorem of Tannakian duality (see [Sa72, II.4.1, p.152] or [DH17, Theorem 1.2.6]). Note that, whether $\Gamma$ is Tannakian or not, its does envelope exist.

**Example 4.2.** Any *finite* abstract group $\Gamma$ is Tannakian. (One uses that the multiplication morphism $R\Gamma \otimes_R V \to V$ is equivariant.)

In order to explore the defining property of a Tannakian group, we use the following terminology.

**Definition 4.3.** Let $V \in \mathrm{Rep}_R(\Gamma)$.

(1) Assume that $\pi V = 0$, that is, $V$ is a $k$-module. An object $\widetilde{V} \in \mathrm{Rep}_R^\circ(\Gamma)$ such that $\widetilde{V}/\pi\widetilde{V} \simeq V$ is called a *lift* of $V$ from $k$ to $R$.

(2) An objet $\widetilde{V} \in \mathrm{Rep}_R^\circ(\Gamma)$ which is the source of an epimorphism $\widetilde{V} \to V$ is called a *weak lift* of $V$.

**Proposition 4.4** (cf. [DH17, Proposition 5.2.2])**.** *Assume that each $V \in \mathrm{Rep}_R(\Gamma)$ which is annihilated by $\pi$ has a weak lift. Then every $E \in \mathrm{Rep}_R(\Gamma)$ has a weak lift.*

*Proof.* Given $M \in \mathrm{Rep}_R(\Gamma)$, we define

$$r(M) = \min\{s \in \mathbf{N} \,:\, \pi^s M_{\mathrm{tors}} = 0\}.$$

We shall proceed by induction on $r(E)$, the case $r(E) = 0$ being trivial. Assume $r(E) = 1$, so that $\pi E$ is torsion-free. Let $q : E \to C$ be the quotient by $\pi E$; since $C$ is annihilated by $\pi$, the hypothesis gives us $\widetilde{C} \in \mathrm{Rep}_R^\circ(\Gamma)$ and an epimorphism $\sigma : \widetilde{C} \to C$. We then have a commutative diagram with exact rows

$$\begin{CD} 0 @>>> \pi E @>>> E @>q>> C @>>> 0 \\ @. @A{\sim}AA @A{\tau}AA \square @A{\sigma}AA @. \\ 0 @>>> \pi E @>>> \widetilde{E} @>>{\chi}> \widetilde{C} @>>> 0, \end{CD}$$

where the rightmost square is cartesian and $\tau$ is surjective. Since $\pi E$ and $\widetilde{C}$ are torsion-free, so is $\widetilde{E}$, and we have found a weak lift of $E$.

Let us now assume that $r(E) > 1$. Let $N = \{e \in E \,:\, \pi e = 0\}$ and denote by $q : E \to C$ the quotient by $N$. It then follows that $\pi^{r(E)-1} C_{\mathrm{tors}} = 0$, so that $r(C) \le r(E) - 1$. By induction there exists $\widetilde{C} \in \mathrm{Rep}_R^\circ(\Gamma)$ and a surjection $\sigma : \widetilde{C} \to C$. We arrive at commutative diagram with exact rows

$$\begin{CD} 0 @>>> N @>>> E @>q>> C @>>> 0 \\ @. @A{\sim}AA @A{\tau}AA \square @A{\sigma}AA @. \\ 0 @>>> N @>>> \widetilde{E} @>>{\chi}> \widetilde{C} @>>> 0, \end{CD}$$

where the rightmost square is cartesian and $\tau$ is surjective. Since $\widetilde{C}$ is torsion-free as an $R$-module, we conclude that $\widetilde{E}_{\rm tors} = N$, so that $r(\widetilde{E}) \le 1$. We can therefore find $\widetilde{E}_1$ and a surjection $\widetilde{E}_1 \to \widetilde{E}$ and consequently a surjection $\widetilde{E}_1 \to E$. □

The simplest cases where any $V \in \mathrm{Rep}_R(\Gamma)$ annihilated by $\pi$ has a lift occurs when $R$ has a coefficient field [Mat89, p.215] (in this case the reduction morphism $\mathrm{GL}_r(R) \to \mathrm{GL}_r(k)$ has a section) or when the group in question is free. Hence:

**Corollary 4.5.** *A free group is Tannakian. If $R$ has a coefficient field, then any abstract group is Tannakian.* □

**Corollary 4.6.** *Let $\Pi$ denote the Tannakian envelope of $\Gamma$ over $R$. Let $\Theta$ denote the Tannakian envelope of $\Gamma$ over $k$. The following claims hold true.*

(1) *There exists a canonical faithfully flat arrow of group schemes $h : \Theta \to \Pi \otimes k$.*
(2) *The morphism $h$ is an isomorphism if and only if $\Gamma$ is Tannakian.*

*Proof.* (1) According to [Ja03, Part I, 10.1], $\mathrm{Rep}_k(\Pi\otimes k)$ can be identified with the full subcategory of $\mathrm{Rep}_R(\Pi)$ consisting of those representations annihilated by $\pi$. Hence, $\mathrm{Rep}_k(\Pi\otimes k)$ can be identified with the full tensor subcategory of $\mathrm{Rep}_R(\Gamma)^{\rm tan}$ of objects which are annihilated by $\pi$. We then derive a fully faithful $\otimes$-functor $\eta : \mathrm{Rep}_k(\Pi \otimes k) \to \mathrm{Rep}_k(\Gamma)$ which, on the level of vector spaces is just the identity. (Note that if $V \in \mathrm{Rep}_k(\Pi \otimes k)$, then $\eta(V)$ always admits a weak lift.) Moreover, $\eta$ is also closed under taking subobjects. Tannakian duality then produces a morphism $h : \Theta \to \Pi\otimes k$ which is, in addition, faithfully flat [DM82, Proposition 2.21, p.139].

(2) Assume now that $h$ is an isomorphism, so that $\eta$ is an equivalence. Then, any $V \in \mathrm{Rep}_k(\Gamma)$ admits a weak lift, so that, by Proposition 4.4, any $E \in \mathrm{Rep}_R(\Gamma)$ admits a weak lift. This means that $\mathrm{Rep}_R(\Gamma)$ is Tannakian. The converse is also simple. □

**Example 4.7** (Non-Tannakian groups)**.** We assume that $R$ is of mixed characteristic $(0, p)$. Let $\Gamma$ be a periodic group of finite exponent (all orders are divisors of a fixed integer). A theorem of Burnside [Dix71, Theorem 2.9, p.40] says that every morphims $\Gamma \to \mathrm{GL}_r(R)$ must then have a *finite image*, so that a morphism $\Gamma \to \mathrm{GL}_r(k)$ which admits a lift must have a finite image. This fact puts heavy restrictions on a Tannakian periodic group of finite exponent. For, if $\Gamma$ is moreover Tannakian, then all representations $\rho : \Gamma \to \mathrm{GL}_r(k)$ must have a finite image. (By the Tannakian property some $\sigma : \Gamma \to \mathrm{GL}_{r+h}(k)$ of the form

$$\begin{bmatrix} * & * \\ * & \rho \end{bmatrix}$$

admits a lift.) Hence, such a periodic group cannot be "linear over $k$". A specific counter-example is then given by the additive group $(k, +)$, if $k$ is infinite.

4.2. **Computing faithfully flat quotients of the Tannakian envelope.** (In this section we make no supplementary assumption on $R$.)

We fix $G \in (\mathbf{FGSch}/R)$ and an abstract group $\Gamma$, whose Tannakian envelope over $R$ is denoted by $\Pi$ (see Definition 4.1). In this section, we shall first explain how to associate to each arrow of abstract groups $\varphi : \Gamma \to G(R)$ a morphism $u_\varphi : \Pi \to G$. Then, we set out to determine under which conditions on $\varphi$, $u_\varphi$ is faithfully flat. We inform the reader that the analogous situation over a base field is folkloric (meaning that some cursory discussions can be found in the literature).

Before starting, let us fix some notations. We write $\omega_\Gamma : \mathrm{Rep}_R(\Gamma)^{\rm tan} \to R\text{-}\mathbf{mod}$ and $\omega_G : \mathrm{Rep}_R(G) \to R\text{-}\mathbf{mod}$ for the forgetful functors. The set $\mathbf{Fun}_*^{\otimes}(\mathrm{Rep}_R(G), \mathrm{Rep}_R(\Gamma)^{\rm tan})$ is formed by all $\otimes$-functors $\eta$ satisfying, as $\otimes$-functors, the equality $\omega_G \circ \eta = \omega_\Gamma$. Given $H \in (\mathbf{FGSch}/R)$, we define $\mathbf{Fun}_*^{\otimes}(\mathrm{Rep}_R(G), \mathrm{Rep}_R(H))$ analogously. Elements in these sets shall be called *pointed* $\otimes$-functors. Finally, if $\gamma \in \Gamma$ and $M \in \mathrm{Rep}_R(\Gamma)$, respectively $g \in G(R)$ and $N \in \mathrm{Rep}_R(G)$, we let $\gamma_M : M \to M$, respectively $g_N$, stand for the action of $\gamma$ on $M$, respectively $g$ on $N$.

Let $\varphi : \Gamma \to G(R)$ be a morphism of abstract groups. For each $M \in \mathrm{Rep}_R(G)$ and each $\gamma \in \Gamma$, we have an $R$-linear automorphism $\varphi(\gamma)_M$. In this way, we obtain a pointed $\otimes$-functor

$$\varphi^\natural \colon \mathrm{Rep}_R(G) \longrightarrow \mathrm{Rep}_R(\Gamma)^{\mathrm{tan}}$$

verifying the equations

$$\gamma_{\varphi^\natural(M)} = \varphi(\gamma)_M \text{ as elements of } \mathrm{Aut}_R(M),$$

for all $\gamma \in \Gamma$ and $M \in \mathrm{Rep}_R(G)$.

Conversely, let $T : \mathrm{Rep}_R(G) \to \mathrm{Rep}_R(\Gamma)^{\mathrm{tan}}$ be a pointed $\otimes$-functor. If $\gamma \in \Gamma$ is fixed, and $M \in \mathrm{Rep}_R(G)$ is given, then $\gamma_{TM}$ is an $R$-linear automorphism of $M$. It is a simple matter to show that the family $\{\gamma_{TM} : M \in \mathrm{Rep}_R(G)\}$ defines a $\otimes$-automorphism of $\omega_G$, that is, an element of $\mathrm{Aut}^\otimes(\omega_G)$. (It is equally simple to note that $\gamma \mapsto \{\gamma_{TM} : M \in \mathrm{Rep}_R(G)\}$ is in fact an arrow of groups.) Since the canonical arrow $G(R) \to \mathrm{Aut}^\otimes(\omega_G)$ is bijective [Sa72, II.2.5.4, p.133], we arrive at a morphism of *groups*

$$T^\flat : \Gamma \longrightarrow G(R)$$

verifying the equations

$$[T^\flat(\gamma)]_M = \gamma_{TM} \text{ as elements of } \mathrm{Aut}_R(M),$$

for all $\gamma \in \Gamma$ and $M \in \mathrm{Rep}_R(G)$.

**Proposition 4.8.** *The arrows $\varphi \mapsto \varphi^\natural$ and $T \mapsto T^\flat$ are inverse bijections between* $\mathrm{Hom}(\Gamma, G(R))$ *and* $\mathbf{Fun}_*^\otimes(\mathrm{Rep}_R(G), \mathrm{Rep}_R(\Gamma)^{\mathrm{tan}})$. *In addition, the construction $(-)^\natural$ is functorial in the sense that, for each $\rho : G \to H$ in* $(\mathbf{FGSch}/R)$*, the diagram*

$$\begin{array}{ccc} \mathrm{Hom}(\Gamma, G(R)) & \xrightarrow{(-)^\natural} & \mathbf{Fun}_*^\otimes(\mathrm{Rep}_R(G), \mathrm{Rep}_R(\Gamma)^{\mathrm{tan}}) \\ {\scriptstyle \rho(R)\circ(-)}\downarrow & & \downarrow{\scriptstyle (-)\circ\rho^\#} \\ \mathrm{Hom}(\Gamma, H(R)) & \xrightarrow[(-)^\natural]{} & \mathbf{Fun}_*^\otimes(\mathrm{Rep}_R(H), \mathrm{Rep}_R(\Gamma)^{\mathrm{tan}}) \end{array}$$

*commutes.*

*Proof.* One immediately sees that $[(\varphi^\natural)^\flat(\gamma)]_M = \varphi(\gamma)_M$ for any $M \in \mathrm{Rep}_R(G)$. Since the natural arrow $G(R) \to \mathrm{Aut}^\otimes(\omega_G)$ is bijective, we conclude that $(\varphi^\natural)^\flat(\gamma) = \varphi(\gamma)$. To prove that $(T^\flat)^\natural : \mathrm{Rep}_R(G) \to \mathrm{Rep}_R(\Gamma)^{\mathrm{tan}}$ coincides with $T$, we need to show that for each $\gamma \in \Gamma$ and each $M \in \mathrm{Rep}_R(G)$, the $R$-linear automorphisms $\gamma_{(T^\flat)^\natural(M)}$ and $\gamma_{TM}$ are one and the same. Now, $\gamma_{(T^\flat)^\natural(M)}$ is just $[T^\flat(\gamma)]_M$, which is the only element of $G(R)$ inducing the automorphism $\gamma_{TM}$. The verification of the last property is also very simple and we omit it. □

Given $\gamma \in \Gamma$, let $u(\gamma) \in \Pi(R) = \mathrm{Aut}^\otimes(\omega_\Gamma)$ be defined by

$$u(\gamma)_M = \gamma_M \text{ as elements of } \mathrm{Aut}_R(M).$$

In this way, we construct a morphism of groups

$$u : \Gamma \longrightarrow \Pi(R)$$

and then a pointed $\otimes$-functor $u^\natural : \mathrm{Rep}_R(\Pi) \to \mathrm{Rep}_R(\Gamma)^{\tan}$. Composing the canonical equivalence $\mathrm{Rep}_R(\Gamma)^{\tan} \to \mathrm{Rep}_R(\Pi)$ with $u^\natural$, we obtain precisely the identity, so that $u^\natural$ is also a $\otimes$-equivalence. Using Proposition 4.8 and the bijection

$$\mathrm{Hom}(G, H) \longrightarrow \mathbf{Fun}_*^{\otimes}(\mathrm{Rep}_R(H), \mathrm{Rep}_R(G)), \quad \rho \longmapsto \rho^{\#},$$

explained in [Sa72, II.3.3.1, p.148], we arrive at:

**Corollary 4.9.** *The function*

$$\mathrm{Hom}(\Pi, G) \longrightarrow \mathrm{Hom}(\Gamma, G(R)), \quad \rho \longmapsto \rho(R) \circ u$$

*is a bijection.* □

**Remark 4.10.** Corollary 4.9 gives a different way to think about the Tannakian envelope; needless to say this is a common viewpoint in algebraic Topology, see [ABC+, Appendix A].

Using Corollary 4.9, we can easily detect when a morphism $\Gamma \to G(R)$ presents $G$ as a faithfully flat quotient of $\Pi$.

**Definition 4.11.** Let $G$ be a flat group scheme over $R$, and $\varphi : \Gamma \to G(R)$ a homomorphism of abstract groups. Write $\varphi(\gamma)_k : \mathrm{Spec}\, k \to G_k$, respectively $\varphi(\gamma)_K : \mathrm{Spec}\, K \to G_K$, for the closed immersion associated to $\varphi(\gamma) : \mathrm{Spec}\, R \to G$ by base-change. We say that $\varphi$ is *dense* if $\{\varphi(\gamma)_k\}_{\gamma\in\Gamma}$, respectively $\{\varphi(\gamma)_K\}_{\gamma\in\Gamma}$, is schematically dense in $G_k$ [EGA, $\mathrm{IV}_3$, 11.10.2, p.171], respectively in $G_K$.

**Remark 4.12.** The somewhat complicated definition given above is due to the usual discrepancy between topological and schematic properties. Needless to say, if in the above definition $G_k$ and $G_K$ are reduced, then the condition simply means that the image of $\Gamma$ in $|G_k|$ and in $|G_K|$ is dense [EGA, $\mathrm{IV}_3$, 11.10.4]. But in the case of non-reduced schemes one has to proceed with caution as a simple example shows (take $\Gamma = \{\pm 1\}$, $R = \mathbf{Z}_2$ and $G = \mu_{2,R}$). The economically oriented reader will also find useful to know that requiring the family $\{\varphi(\gamma)\}_{\gamma\in\Gamma}$ to be schematically dense does not imply, in general, the above condition (same example as before).

**Proposition 4.13.** *The following claims about the Tannakian envelope* $\Pi$ *of* $\Gamma$ *are true.*

(1) *The morphism* $u : \Gamma \to \Pi(R)$ *is dense.*
(2) *Both fibres of* $\Pi$ *are reduced.*
(3) *Let* $G \in (\mathbf{FGSch}/R)$ *and let* $\varphi : \Gamma \to G(R)$ *be a dense morphism. Then, the morphism* $u_\varphi : \Pi \to G$ *associated to* $\varphi$ *by means of Corollary 4.9 is faithfully flat.*

*Proof.* (1) and (2). Let $B$ be the smallest $K$-subgroup scheme of $\Pi_K$ bounding ("majorant" in French) the family $\{u(\gamma)_K\}_{\gamma\in\Gamma}$ [$\mathrm{SGA3_{new}}$, $\mathrm{VI}_B$, Proposition 7.1, p.384]. Let $i : \overline{B} \to \Pi$ be the schematic closure of $B$ in $\Pi$ [EGA, $\mathrm{IV}_2$, 2.8, 33ff]. Note that, for each $\gamma \in \Gamma$, the arrow $u(\gamma) : \mathrm{Spec}\, R \to \Pi$ factors as

$$(*) \qquad u(\gamma) = i(R) \circ \psi(\gamma),$$

where $\psi(\gamma) : \mathrm{Spec}\, R \to \overline{B}$ is an $R$-point of $\overline{B}$. In addition, since $i(R) : \overline{B}(R) \to \Pi(R)$ is a monomorphism, $\psi : \Gamma \to \overline{B}(R)$ is a morphism of *groups*. By Corollary 4.9, there exists a unique morphism $u_\psi : \Pi \to \overline{B}$ such that

$$(**) \qquad \psi(\gamma) = u_\psi(R) \circ u(\gamma).$$

From $(*)$ and $(**)$ we deduce that $u(\gamma) = (i \circ u_\psi)(R) \circ u(\gamma)$, and hence $\mathrm{id}_\Pi = i \circ u_\psi$ (due to Corollary 4.9). Therefore $i$ must be an isomorphism. Using [$\mathrm{SGA3_{new}}$, $\mathrm{VI}_B$, Proposition 7.1(ii)], we conclude that $\Pi_K$ is reduced, and that the set of points associated to $\{u(\gamma)_K\}_{\gamma\in\Gamma}$ is dense in it. In this case, it is obvious that $\{u(\gamma)_K\}_{\gamma\in\Gamma}$ is also schematically dense [EGA, $\mathrm{IV}_3$, 11.10.4].

In the same spirit, let $j : H \to \Pi_k$ be the smallest $k$-subgroup bounding $\{u(\gamma)_k\}_{\gamma\in\Gamma}$ and let $\zeta : \Pi' \to \Pi$ be the Neron blowup of $\Pi$ at $H$. By the definition of a Neron blowup, for each $\gamma \in \Gamma$ there exists $u'(\gamma) : \operatorname{Spec} R \to \Pi'$ such that

$$(\dagger) \qquad u(\gamma) = \zeta(R) \circ u'(\gamma).$$

Since $\zeta(R) : \Pi'(R) \to \Pi(R)$ is a monomorphism of groups, we conclude that $u' : \Gamma \to \Pi'(R)$ is a homomorphism. Corollary 4.9 provides $v : \Pi \to \Pi'$ such that

$$(\ddagger) \qquad u'(\gamma) = v(R) \circ u(\gamma).$$

As a consequence of (†) and (‡), we have $u(\gamma) = (\zeta v)(R) \circ u(\gamma)$. Another application of Corollary 4.9 leads to $\zeta v = \mathrm{id}_\Pi$. Together with the fundamental criterion [Wa79, 14.1, Theorem], this shows that $\zeta_k : \Pi'_k \to \Pi_k$ is faithfully flat. Consequently $H = \Pi_k$. As in the previous paragraph, these findings show that $\Pi_k$ is reduced and that $\{u(\gamma)_k\}_{\gamma\in\Gamma}$ is schematically dense.

(3) Because $\varphi(\gamma) = u_\varphi \circ u(\gamma)$ for each $\gamma$, it follows that the arrows $K[G] \to K[\Pi]$ and $k[G] \to k[\Pi]$ derived from $u_\varphi$ are injective. The result then follows by applying [DH17, Theorem 4.1.1]. □

**Remark 4.14.** Let $\varphi : \Gamma \to \Gamma'$ be a homomorphism of abstract groups. Then there exists, by the universal property, a unique group scheme homomorphism $\Phi : \Pi \to \Pi'$ rendering the following diagram commutative

$$\begin{array}{ccc} \Gamma & \xrightarrow{\varphi} & \Gamma' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ \Pi & \xrightarrow[\Phi]{} & \Pi'. \end{array}$$

4.3. **The Tannakian envelope of an infinite cyclic group.** In this section we assume $R = k[\![\pi]\!]$, with $k$ of characteristic zero, and we give ourselves an infinite cyclic group $\Gamma$ generated by $\gamma$. In the terminology of Definition 4.1, this is certainly a Tannakian group (apply Corollary 4.5, if anything); let $\Pi$ stand for its Tannakian envelope over $R$.

We wish to show that, despite the very simple nature of $\Gamma$, its envelope *does* allow rather "large" faithfully flat quotients. A precise statement is Proposition 4.15 below.

Let $G = \mathbf{G}_{a,R} \times \mathbf{G}_{m,R}$ have ring of functions $R[G] = R[x, y^\pm]$. In the $\pi$-adic completion $R\langle G\rangle$ of $R[G]$, we single out the element

$$\Phi = y - \exp(\pi x) \in R\langle G\rangle.$$

Obviously, the closed formal subscheme $\mathfrak{H}$ cut out by the ideal $(\Phi) \subset R\langle G\rangle$ is a formal subgroup, isomorphic to the $\pi$-adic completion of $\mathbf{G}_{a,R}$; let $\mathcal{N}$ stand for the Neron blowup of $\mathfrak{H}$ (see Section 2.2). By definition,

$$R[\mathcal{N}] = \varinjlim_n R[G][\pi^{-n-1}\Phi_n],$$

where

$$\Phi_n = y - \left(1 + \pi x + \frac{\pi^2}{2!}x^2 + \cdots + \frac{\pi^n}{n!}x^n\right).$$

We now introduce the "evaluation" at the point $P = (1, e^\pi) \in G(R)$:

$$\mathrm{ev}_P : R[G] \longrightarrow R, \qquad \begin{array}{rcl} x & \longmapsto & 1 \\ y & \longmapsto & e^\pi. \end{array}$$

Since $\mathrm{ev}_P(\Phi_n) \equiv 0 \mod \pi^{n+1}$, $P$ is a point of $\mathcal{N}$. Let $\varphi : \Gamma \to \mathcal{N}(R)$ send $\gamma$ to $P$. According to Corollary 4.9, there exists a morphism $u_\varphi : \Pi \to \mathcal{N}$ and a commutative diagram

$$\begin{array}{ccc} \Pi(R) & \xrightarrow{u_\varphi(R)} & \mathcal{N}(R) \\ {\scriptstyle u}\uparrow & \nearrow{\scriptstyle \varphi} & \\ \Gamma. & & \end{array}$$

**Proposition 4.15.** *The morphism $u_\varphi : \Pi \to \mathcal{N}$ is faithfully flat.*

*Proof.* According to [DHdS17, Corollary 5.11], $\mathcal{N} \otimes k$ is just $\mathfrak{H} \otimes k \simeq \mathbf{G}_{a,k}$; as the image of $P$ in $\mathcal{N}(k)$ is not the neutral element, the assumption on the characteristic proves that the subgroup it generates is schematically dense. On the generic fibre, the Zariski closure $C$ of the subgroup generated by $P$ must be of dimension at least two since both restrictions $C \to \mathbf{G}_{m,K}$ and $C \to \mathbf{G}_{a,K}$ are faithfully flat (the standard facts employed here are in [Wa79], see the Corollary on p.65 and the Theorem on the bottom of p.114). This implies that $C = G_K$ and Proposition 4.13 then finishes the proof. □

**Corollary 4.16.** *The group scheme $\Pi$ is not strictly pro-algebraic (in the sense of Question (**SA**) from the Introduction). In fact, the Tannakian enveloppe $\Pi^\star$ of any abstract group $\Gamma^\star$ admitting $\Gamma$ as a quotient fails to be strictly pro-algebraic.*

*Proof.* We assume that $\Pi$ is strictly pro-algebraic. It is therefore possible to find a $\mathcal{G} \in (\mathbf{FGSch}/R)$ of *finite type* and a *faithfully flat* arrow $q : \Pi \to \mathcal{G}$ factoring $u_\varphi$:

$$\begin{array}{ccccc} & & \mathcal{G} & & \\ & {\scriptstyle q}\nearrow & & \searrow{\scriptstyle \chi} & \\ \Pi & & \xrightarrow[u_\varphi]{} & & \mathcal{N}. \end{array}$$

This is because $K[\mathcal{N}]$ is generated by $\{x, y, y^{-1}\}$ and $R[\mathcal{G}] \subset R[\Pi]$ is saturated. Note that, in this case, $\chi$ is faithfully flat. Together with Exercise 7.9 on p. 53 of [Mat89], this implies that $R[\mathcal{N}]$ is a noetherian ring, something which is certainly false. (To verify this last claim, one proceeds as follows. Let $\mathfrak{a}$ be the augmentation ideal of $R[\mathcal{N}]$, and let $\Psi_n = \pi^{-n-1}\Phi_n$. Then $\pi\Psi_{n+1} \equiv \Psi_n \mod \mathfrak{a}^2$ for all $n \geq 1$, so that $\mathfrak{a}/\mathfrak{a}^2$ is either not finitely generated over $R = R[\mathcal{N}]/\mathfrak{a}$, or $y - 1 - \pi x \in \mathfrak{a}^2$. But this latter condition is impossible.)

To verify the second statement, we observe that if $\psi : \Gamma^\star \to \Gamma$ is an epimorphism, then the induced arrow $\Psi : \Pi^\star \to \Pi$ is faithfully flat due to [DH17, Proposition 3.2.1(ii)]. The exact same proof as above now proves the second statement. □

## 5. Neron blowups of formal subgroup schemes in differential Galois theory

In this section, we set out to investigate whether blowups of closed formal subgroup schemes, as in Section 2.2, appear in differential Galois theory over $R$. (This theory is to be understood as the one explained in [DHdS17, Section 7].) Our strategy is to restrict attention to the discrete valuation ring $\mathbb{C}[\![t]\!]$. We shall see that through a "Riemann-Hilbert correspondence", which is Theorem 5.10 below, we are able to transplant the results of Section 4.3 concerning the category of representations of an abstract group to swiftly arrive at a conclusion: see Corollary 5.11. The proof of Theorem 5.10 is routine and employs Serre's GAGA, Grothendieck's Existence Theorem in formal Geometry (GFGA), and Deligne's dictionary [Del70, I.2].

5.1. **Preliminaries on $\mathcal{D}$-modules.** We now review some useful notions concerning $\mathcal{D}$-modules. These are employed to establish an algebraization result, Proposition 5.3, further ahead.

Let $T$ be a noetherian scheme and $Y$ a smooth $T$-scheme; we write $\mathcal{D}(Y/T)$ for the ring of differential operators described in [EGA, $\mathrm{IV}_4$, 16.8] and [BO78, §2]. Following the notation of [DHdS17, §7], we let $\mathcal{D}(Y/T)$-**mod** stand for the category of (sheaves of) $\mathcal{D}(Y/T)$-modules on $Y$ whose underlying $\mathcal{O}_Y$-module is coherent. If $T$ is not a $\mathbb{Q}$-scheme, then the ring $\mathcal{D}(Y/T)$ is usually not finitely generated over $\mathcal{O}_Y$, so that a fundamental tool in dealing with $\mathcal{D}$-modules is Grothendieck's notion of stratification.

Let $\mathcal{P}^{\nu}_{Y/T}$ stand for the sheaf of principal parts of order $\nu$ of $Y \to T$ [EGA, $\mathrm{IV}_4$, 16.3.1, p.14]. The scheme $(|Y|, \mathcal{P}^{\nu}_{Y/T})$ [EGA, $\mathrm{IV}_4$, 16.1], usually named the *scheme of principal parts of order* $\nu$, is denoted by $P^{\nu}_{Y/T}$. It comes with two projections $p_0, p_1 : P^{\nu}_{Y/T} \to Y$ and a closed immersion $\Delta : Y \to P^{\nu}_{Y/T}$, and we follow the convention of EGA fixing the first projection as the structural one, while the other is supplementary.

Given $\mathcal{E} \in \mathcal{D}(Y/T)$-**mod**, for any $\nu \in \mathbf{N}$ we can associate an isomorphism of $\mathcal{O}_{P^{\nu}_{Y/T}}$-modules

$$\theta(\nu) : p_1^*(\mathcal{E}) \xrightarrow{\sim} p_0^*(\mathcal{E})$$

which, when pulled-back along $\Delta$ gives back the identity. These are are called Taylor isomorphisms of order $\nu$. Clearly, using the obvious closed immersions $P^{\nu}_{Y/T} \to P^{\nu+1}_{Y/T}$, we obtain the notion of a compatible family of Taylor isomorphisms.

These considerations prompt the definition of a pseudo-stratification of a coherent module $\mathcal{F}$: It is a *compatible* family of isomorphisms

$$\sigma(\nu) : p_1^*(\mathcal{F}) \xrightarrow{\sim} p_0^*(\mathcal{F}), \quad \nu = 0, \ldots,$$

which induce the identity when pulled-back along the diagonal. The pseudo-stratification $\{\sigma(\nu)\}_{\nu \in \mathbf{N}}$ is a stratification if the isomorphisms satisfy, in addition, a cocycle condition [BO78, Definition 2.10]. Coherent modules endowed with stratifications constitute a category, denoted by $\mathbf{str}(Y/T)$. The main point of these definitions is that the Taylor isomorphisms define an equivalence of categories $\mathcal{D}(Y/T)$-**mod** $\to \mathbf{str}(Y/T)$ which preserves all the underlying coherent modules and morphisms [BO78, 2.11].

Although stratifications are a more intricate version of the natural concept of $\mathcal{D}$-module, their functoriality properties are most welcome. From a commutative diagram of schemes

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & Y \\ \downarrow & & \downarrow \\ U & \longrightarrow & T, \end{array}$$

we obtain another such diagram

$$\begin{array}{ccc} P^{\nu}_Z & \xrightarrow{P^{\nu}(\varphi)} & P^{\nu}_Y \\ {\scriptstyle p_{i,Z}}\downarrow & & \downarrow{\scriptstyle p_{i,Y}} \\ Z & \xrightarrow[\varphi]{} & Y \\ \downarrow & & \downarrow \\ U & \longrightarrow & T, \end{array}$$

where $P^\nu(\varphi)$ is the morphism defined by $|\varphi| : |Z| \to |Y|$ and the arrow of $\mathcal{O}_Z$-algebras (abusively denoted by the same symbol)

$$P^\nu(\varphi) : \mathcal{O}_Z \underset{\mathcal{O}_Y}{\otimes} \mathcal{P}^\nu_{Y/T} \longrightarrow \mathcal{P}^\nu_{Z/U},$$

which is explained on [EGA, IV$_4$, 16.4.3, 17ff]. Consequently, we have an isomorphism of functors

$$P^\nu(\varphi)^* \circ p^*_{i,Y} \simeq p^*_{i,Z} \circ \varphi^*,$$

which allows us to prolong $\varphi^* : \mathbf{Coh}(Y) \to \mathbf{Coh}(Z)$ into

$$\varphi^\# : \mathbf{str}(Y/T) \longrightarrow \mathbf{str}(Z/U), \quad (\mathcal{E}, \{\theta(\nu)\}) \longmapsto (\varphi^*\mathcal{E}, \{\varphi^\#\theta(\nu)\});$$

just define $\varphi^\#\theta(\nu)$ by decreeing that

$$\begin{array}{ccc} P^\nu(\varphi)^* p^*_{1,Y}(\mathcal{E}) & \xrightarrow{P^\nu(\varphi)^*\theta(\nu)} & P^\nu(\varphi)^* p^*_{0,Y}(\mathcal{E}) \\ \sim\downarrow & & \downarrow\sim \\ p^*_{1,Z}\varphi^*(\mathcal{E}) & \xrightarrow[\varphi^\#\theta(\nu)]{} & p^*_{0,Z}\varphi^*(\mathcal{E}) \end{array}$$

commutes. Using the equivalence $\mathcal{D}(Y/T)$-$\mathbf{mod} \simeq \mathbf{str}(Y/T)$ mentioned above, we can also prolong $\varphi^*$ to a functor $\varphi^\# : \mathcal{D}(Y/T)$-$\mathbf{mod} \to \mathcal{D}(Z/U)$-$\mathbf{mod}$.

**Remark 5.1.** Similar notions hold for the case where $g : Y \to T$ is replaced by a smooth morphism of *complex analytic spaces* [Del70, I.2.22]. This being so, we adopt similar notations.

5.2. **An algebraization result.** In this section, we assume that $R$ is complete. Let

$$f : X \longrightarrow S$$

be a smooth morphism (recall that $S = \operatorname{Spec} R$). For every $n \in \mathbf{N}$, we obtain cartesian commutative diagrams

$$\begin{array}{ccc} X_n & \xrightarrow{u_n} & X_{n+1} \\ \downarrow & & \downarrow \\ S_n & \longrightarrow & S_{n+1} \end{array}$$

and

$$\begin{array}{ccc} X_n & \xrightarrow{v_n} & X \\ \downarrow & & \downarrow \\ S_n & \longrightarrow & S, \end{array}$$

where the vertical arrows are the evident closed embeddings. In this case, the arrows

(2) $$P^\nu(v_n) : \mathcal{O}_{X_n} \underset{\mathcal{O}_X}{\otimes} \mathcal{P}^\nu_{X/S} \longrightarrow \mathcal{P}^\nu_{X_n/S_n}$$

are isomorphisms [EGA, IV$_4$, 16.4.5]. Hence, the *dual* arrows

$$\mathcal{D}(X_n/S_n) \longrightarrow \mathcal{O}_{X_n} \underset{\mathcal{O}_X}{\otimes} \mathcal{D}(X/S)$$

are isomorphisms of $\mathcal{O}_{X_n}$-modules (here we used that the modules of principal parts are locally free [BO78, Proposition 2.6]). This particular situation allows us to construct an arrow of $\mathcal{O}_X$-modules

(3) $$\rho_n : \mathcal{D}(X/S) \longrightarrow \mathcal{D}(X_n/S_n).$$

A moment's thought proves that for any differential operator (on some open of $X$) $\partial : \mathcal{O}_V \to \mathcal{O}_V$, $\rho_n(\partial) : \mathcal{O}_{V_n} \to \mathcal{O}_{V_n}$ is just the reduction of $\partial$ modulo $\pi^{n+1}$, so that $\rho_n$ is a morphisms of $v_n^{-1}\mathcal{O}_X$-algebras. Moreover, $\rho_n$ allows us to identify $\mathcal{D}(X/S)/(\pi^{n+1})$ and $\mathcal{D}(X_n/S_n)$ in such a way that the functor $v_n^\# : \mathcal{D}(X/S)$-**mod** $\to \mathcal{D}(X_n/S_n)$-**mod** is the obvious one.

**Definition 5.2.** Denote by $\mathbf{Coh}^\wedge(X)$ the category constructed as follows. Objects are sequences $(\mathcal{E}_n, \alpha_n)$, where $\mathcal{E}_n \in \mathbf{Coh}(X_n)$, and $\alpha_n : u_n^*\mathcal{E}_{n+1} \to \mathcal{E}_n$ an isomorphism. Morphisms between $(\mathcal{E}_n, \alpha_n)$ and $(\mathcal{F}_n, \beta_n)$ are families $\varphi_n : \mathcal{E}_n \to \mathcal{F}_n$ fulfilling $\varphi_n \circ \alpha_n = \beta_n \circ u_n^*(\varphi_n)$, see [Il05, 8.1.4]. (Needless to say, this is the category of coherent modules on the formal completion.) We let $\mathcal{D}(X/S)$-**mod**$^\wedge$ be constructed analogously from $\mathcal{D}(X/S)$-**mod** and the functors $u_n^\#$.

Let

$$\mathbf{Coh}(X) \xrightarrow{\Phi} \mathbf{Coh}(X)^\wedge \quad \text{and} \quad \mathcal{D}(X/S)\text{-}\mathbf{mod} \xrightarrow{\mathrm{D}\Phi} \mathcal{D}(X/S)\text{-}\mathbf{mod}^\wedge$$

be the obvious functors. Grothendieck's existence theorem [Il05, Theorem 8.4.2], henceforth called GFGA, says that *if $f$ is proper*, then $\Phi$ is an equivalence of $R$-linear $\otimes$-categories. Using the notion of stratification, we have:

**Proposition 5.3.** *Assume that $f : X \to S$ is proper. Then,* $\mathrm{D}\Phi$ *is an equivalence.*

*Proof.* For the sake of clarity, we choose to avoid the obvious proof, so that the number of verifications relying on canonical identifications is reduced. In taking this approach, we shall require Lemma 5.4 below; its proof is simple and we leave it to the reader.

*Essential surjectivity:* Due to GFGA, we need to prove the following

**Claim.** Let $\mathcal{E} \in \mathbf{Coh}(X)$, and assume that (i) each $\mathcal{E}_n := \mathcal{O}_{X_n} \otimes \mathcal{E}$ has the structure of a $\mathcal{D}(X_n/S_n)$-module, and (ii) the canonical isomorphisms $\alpha_n : u_n^*\mathcal{E}_{n+1} \to \mathcal{E}_n$ preserve these structures. Then $\mathcal{E}$ carries a $\mathcal{D}(X/S)$-module structure, and $\mathrm{D}\Phi(\mathcal{E}) = (\mathcal{E}_n, \alpha_n)$.

Let $\nu \in \mathbf{N}$ be fixed, and for each $n \in \mathbf{N}$, write

$$\theta(\nu)_n : p_{1,X_n}^*(\mathcal{E}_n) \longrightarrow p_{0,X_n}^*(\mathcal{E}_n)$$

for the Taylor isomorphism of order $\nu$ associated to the $\mathcal{D}(X_n/S_n)$-module $\mathcal{E}_n$. The fact that $\alpha_n$ is an isomorphism of stratified modules implies, after the necessary identifications, that

$$P^\nu(u_n)^*(\theta(\nu)_{n+1}) = \theta(\nu)_n, \qquad \text{for all } n. \tag{4}$$

By the isomorphism (2), we can apply GFGA to the proper $S$-scheme $P_X^\nu$ to obtain an isomorphism of $\mathcal{O}_{P_X^\nu}$-modules

$$\theta(\nu) : p_{1,X}^*(\mathcal{E}) \longrightarrow p_{0,X}^*(\mathcal{E})$$

such that

$$P^\nu(v_n)^*(\theta(\nu)) = \theta(\nu)_n \quad \text{for all } n.$$

Let $V \subset X$ be an open subset constructed from $\mathcal{E}$ as in Lemma 5.4 below, and let $\partial \in \mathcal{D}(X/S)(V)$ be of order $\le \nu$. Let $e \in \mathcal{E}(V)$. We define

$$\nabla_\nu(\partial)(e) := (\mathrm{id} \otimes \partial)\,[\theta(\nu)(1 \otimes e)]\,,$$

where we remind the reader that $p_{0,X}^*(\mathcal{E}) = \mathcal{E} \otimes \mathcal{P}_X^\nu$ and $p_{1,X}^*(\mathcal{E}) = \mathcal{P}_X^\nu \otimes \mathcal{E}$ as sheaves on $|X|$. At this point, $\nabla_\nu(\partial)(e)$ is just a name for a section of $\mathcal{E}(V)$. We have $\mathcal{O}(V)$-linear maps $q_n : \mathcal{E}(V) \to \mathcal{E}_n(V \cap X_0)$ and $\rho_n$ (as in eq. (3)) and it is clear from the definition and eq. (4) that

$$q_n[\nabla_\nu(\partial)(e)] = \rho_n(\partial) \cdot q_n(e), \qquad \text{for each } n. \tag{5}$$

As $(q_n) : \mathcal{E}(V) \to \varprojlim_n \mathcal{E}_n(V \cap X_0)$ is injective (Lemma 5.4-(b)), we can say:

(1) if $\nu \le \nu'$, then $\nabla_\nu(\partial)(e) = \nabla_{\nu'}(\partial)(e)$.
(2) If $\partial'$ is such that $\partial\partial'$ has order $\le \nu$, then $\nabla_\nu(\partial\partial')(e) = \nabla_\nu(\partial)[\nabla_\nu(\partial')(e)]$.

This means that $\mathcal{E}(V)$ now carries the structure of a $\mathcal{D}(X/S)(V)$-module. Because $V$ is affine, this endows $\mathcal{E}|_V$ with a structure of $\mathcal{D}(V/S)$-module. Furthermore, by definition, the Taylor isomorphism of level $\nu$ associated to this structure is simply the restriction of $\theta(\nu)$ to $V$. Hence, covering $X$ by open subsets as these, $\mathcal{E}$ becomes a $\mathcal{D}$-module for which the Taylor isomorphism of order $\nu$ is simply $\theta(\nu)$. By construction (see eq. (4)) we then conclude that $\mathrm{D}\Phi(\mathcal{E}) = (\mathcal{E}_n, \alpha_n)$.

*Fully faithfulness:* As before, by GFGA, it is sufficient to show

**Claim.** Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{D}(X/S)$-modules, and let $\phi : \mathcal{E} \to \mathcal{F}$ be an $\mathcal{O}_X$-linear arrow such that $\phi_n : v_n^*\mathcal{E} \to v_n^*\mathcal{F}$ is $\mathcal{D}(X_n/S_n)$-linear for all $n$. Then $\phi$ is a morphism of $\mathcal{D}(X/S)$-modules.

Let $V$ be constructed from $\mathcal{F}$ as in Lemma 5.4(a); we show that $\phi|_V$ is $\mathcal{D}(V/S)$-linear. Let $e \in \mathcal{E}(V)$, $\partial \in \mathcal{D}(X/S)(V)$, and write $r_n$ for the natural $\mathcal{O}(V)$-linear morphisms $\mathcal{F}(V) \to v_n^*\mathcal{F}(V \cap X_0)$. It follows easily that $r_n(\partial \cdot \phi(e)) = r_n(\phi(\partial \cdot e))$, so that injectivity of $(r_n) : \mathcal{F}(V) \to \varprojlim v_n^*(\mathcal{F})(V \cap X_0)$ assures that $\phi|_V$ is $\mathcal{D}(V/S)$-linear. Since $X$ can be covered by open subsets like $V$ (Lemma 5.4), we are done. □

**Lemma 5.4.** *Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module.*

(a) *Denote by $\xi_1, \ldots, \xi_r$ the associated points of $\mathcal{M}$* [EGA, IV$_2$, 3.1] *lying on the generic fibre $X_K$. Then, every point $x \in X_k$ has an open affine coordinate neighbourhood $V$ such that if $\xi_j \in V$, then $\overline{\{\xi_j\}} \cap V_k \neq \varnothing$.*
(b) *If $V$ is as before, then $\mathcal{M}(V)$ is $\pi$-adically separated.*
(c) *The scheme $X$ can be covered by finitely many open subsets as $V$ above.* □

5.3. **Local systems and representations of the topological fundamental group.** Let $\Lambda$ be a local $\mathbb{C}$-algebra which, as a complex vector space, is finite dimensional. (In this case, the residue field of $\Lambda$ is $\mathbb{C}$.) Writing $T$ for the analytic space associated to $\Lambda$ [SC, Exposé 9, 2.8], we give ourselves a smooth morphism $g : Y \to T$ of *connected* complex analytic spaces [SC, Exposé 13, §3].

Let us now bring to stage the category $\mathbf{LS}(Y/T)$ of relative local systems [Del70, I.2.22]. These are $g^{-1}(\mathcal{O}_T)$-modules $\mathcal{F}$ on $Y$ enjoying the ensuing property: for every $y \in Y$, there exists an open $V \ni y$ such that the $g^{-1}(\mathcal{O}_T)$-module $\mathcal{F}|V$ belongs to $g^{-1}\mathbf{Coh}(T)$. Under these conditions, any $\mathcal{F} \in \mathbf{LS}(Y/T)$ is a *locally constant* sheaf of $\Lambda$-modules whose values on connected open sets are of finite type. Conversely, any such locally constant sheaf of $\Lambda$-modules satisfies the defining property of $\mathbf{LS}(Y/T)$. Hence, the following arguments constitute a well-known exercise, see Exercise F of Ch. 1 and Exercise F of Ch. 6 in [Sp66]. (Here one should recall that $|Y|$ is the topological space of a manifold.)

Let $y_0 \in |Y|$ be a point and $\Gamma$ the fundamental group of $|Y|$ based at it. Given a locally constant sheaf of $\Lambda$-modules $\mathcal{E}$ on $Y$, we can endow its stalk $\mathcal{E}_{y_0}$ with a left action of $\Gamma$. This construction then allows the definition of a functor

$$\mathbb{M}_{y_0} : \left\{ \begin{array}{c} \text{locally constant sheaves} \\ \text{of } \Lambda\text{-modules whose} \\ \text{stalks are of finite type} \end{array} \right\} \longrightarrow \mathrm{Rep}_\Lambda(\Gamma).$$

**Proposition 5.5.** *The functor $\mathbb{M}_{y_0}$ defines a $\Lambda$-linear $\otimes$-equivalence between $\mathbf{LS}(Y/T)$ and $\mathrm{Rep}_\Lambda(\Gamma)$.* □

5.4. **$\mathcal{D}$-modules on the analytification of an algebraic $\mathbb{C}$-scheme.** Let $\Lambda$ be a local $\mathbb{C}$-algebra which, as a complex vector space, is finite dimensional and write $T = \operatorname{Spec}\Lambda$. We give ourselves a smooth morphism of *algebraic $\mathbb{C}$-schemes* $g : Y \to T$, and set out to explain briefly the equivalence of $\mathcal{D}$-modules on $Y$ and on its analytification (see Proposition 5.6 below).

Following [SGA1, XII, §1], we introduce the smooth morphism of complex analytic space $g^{\mathrm{an}} : Y^{\mathrm{an}} \to T^{\mathrm{an}}$, and the $\otimes$-functor

$$(-)^{\mathrm{an}} : \mathbf{Coh}(Y) \longrightarrow \mathbf{Coh}(Y^{\mathrm{an}}).$$

Using the same notations as in Section 5.1, we remark that $(-)^{\mathrm{an}}$ takes $\mathcal{P}^{\nu}_{Y/T}$ to $\mathcal{P}^{\nu}_{Y^{\mathrm{an}}/T^{\mathrm{an}}}$ (this can be extracted from [SGA1, Exposé XII, §1]) so that, employing the notion of stratifications, we have another "analytification" functor:

$$(-)^{\mathrm{an}} : \mathcal{D}(Y/T)\text{-}\mathbf{mod} \longrightarrow \mathcal{D}(Y^{\mathrm{an}}/T^{\mathrm{an}})\text{-}\mathbf{mod}.$$

(This can certainly be constructed in a more elementary way.) In $g$ is in addition *proper*, Serre's GAGA [SGA1, XII, Theorem 4.4] applied to the proper $\mathbb{C}$-scheme $P^{\nu}_{Y/T}$ implies, by the same technique used in proving Proposition 5.3, that

**Proposition 5.6.** *The functor $(-)^{\mathrm{an}} : \mathcal{D}(Y/T)\text{-}\mathbf{mod} \to \mathcal{D}(Y^{\mathrm{an}}/T^{\mathrm{an}})\text{-}\mathbf{mod}$ is an equivalence of $\Lambda$-linear $\otimes$-categories.* □

To complement this Proposition, we note the following. Let $y$ be a $T$-point of $Y$ sending $|T|$ to $y_0$. If $y^{\mathrm{an}}$ is the associated $T^{\mathrm{an}}$-point of $Y^{\mathrm{an}}$, then it sends $|T^{\mathrm{an}}|$ to $y_0$, and the natural morphism $\mathcal{E}_{y_0} \to (\mathcal{E}^{\mathrm{an}})_{y_0}$ defines an isomorphism $y^*\mathcal{E} \to y^{\mathrm{an}*}\mathcal{E}^{\mathrm{an}}$. This produces a natural isomorphism $y^* \stackrel{\sim}{\to} y^{\mathrm{an}*} \circ (-)^{\mathrm{an}}$.

5.5. **$\mathcal{D}$-modules and relative local systems.** Notations now are those of Section 5.3, so that we are concerned only with complex analytic spaces.

For each $\mathcal{E} \in \mathcal{D}(Y/T)\text{-}\mathbf{mod}$, we consider the sheaf of $\Lambda$-modules ("solutions", "horizontal sections")

$$\mathbf{Sol}(\mathcal{E})(U) := \left\{ \begin{array}{c} \text{sections } s \in \mathcal{E}(U) \text{ annihilated by the} \\ \text{augmentation ideal of } \mathcal{D}(Y/T)(U) \end{array} \right\}.$$

Now, recalling that $\mathcal{D}(Y/T)$-modules are exactly the same thing as integrable connections [BO78, Theorem 2.15], Deligne shows in [Del70, I.2.23] that:

**Theorem 5.7.** *For each $\mathcal{E} \in \mathcal{D}(Y/T)\text{-}\mathbf{mod}$, the sheaf of $\Lambda$-modules $\mathbf{Sol}(\mathcal{E})$ is a relative local system and the morphism of $\mathcal{O}_Y$-modules*

$$\mathcal{O}_Y \otimes_{\Lambda} \mathbf{Sol}(\mathcal{E}) \longrightarrow \mathcal{E}$$

*is an isomorphism. In addition, the functor*

$$\mathbf{Sol} : \mathcal{D}(Y/T)\text{-}\mathbf{mod} \longrightarrow \mathbf{LS}(Y/T)$$

*is a $\otimes$-equivalence of $\Lambda$-linear categories.*

Putting together Proposition 5.5 and Theorem 5.7, we have:

**Corollary 5.8.** *The composite functor*

$$\mathcal{D}(Y/T)\text{-}\mathbf{mod} \xrightarrow{\mathbf{Sol}} \mathbf{LS}(Y/T) \xrightarrow{\mathbb{M}_{y_0}} \mathrm{Rep}_{\Lambda}(\Gamma)$$

*defines a $\otimes$-equivalence of $\Lambda$-linear categories.* □

To complement this corollary, we make the following observations. Let $y : T \to Y$ be a section to the structural morphism taking $|T|$ to $y_0$. Then the natural morphism $\mathbf{Sol}(\mathcal{E})_{y_0} \otimes_{\Lambda} \mathcal{O}_{y_0} \to \mathcal{E}_{y_0}$ is an isomorphism (by Theorem 5.7), so that the natural arrow $\mathbf{Sol}(\mathcal{E})_{y_0} \to \mathcal{E}_{y_0} \to y^*(\mathcal{E})$ is an isomorphism. Hence, if $\omega : \mathrm{Rep}_{\Lambda}(\Gamma) \to \Lambda\text{-}\mathbf{mod}$ stands for the forgetful functor, it follows that $\omega \circ \mathbb{M}_{y_0} \circ \mathbf{Sol} \stackrel{\sim}{\to} y^*$ as $\otimes$-functors.

5.6. **The category** $\mathrm{Rep}_R(\Gamma)^\wedge$**.** In this section we shall assume that $R$ is complete and denote by $t$ a uniformizer. Given an abstract group $\Gamma$, we introduce the category $\mathrm{Rep}_R(\Gamma)^\wedge$ following the pattern of Definition 5.2. Its objects are sequences $(E_n, \sigma_n)_{n\in\mathbf{N}}$ where $E_n$ is a left $R_n\Gamma$-module underlying a finite $R_n$-module, and

$$\sigma_n : R_n \otimes_{R_{n+1}} E_{n+1} \longrightarrow E_n$$

is an isomorphism of $R_n\Gamma$-modules. Term by term tensor product (of $R_n$-modules) defines on $\mathrm{Rep}_R(\Gamma)^\wedge$ the structure of a $R$-linear $\otimes$-category.

The categories $\mathrm{Rep}_R(\Gamma)$ and $\mathrm{Rep}_R(\Gamma)^\wedge$ are related by two functors

$$\mathfrak{F} : \mathrm{Rep}_R(\Gamma) \longrightarrow \mathrm{Rep}_R(\Gamma)^\wedge$$

and

$$\mathfrak{L} : \mathrm{Rep}_R(\Gamma)^\wedge \longrightarrow \mathrm{Rep}_R(\Gamma).$$

On the level of modules these are defined by

$$\mathfrak{F}(E) = (R_n \otimes_R E, \text{canonic}) \quad \text{and} \quad \mathfrak{L}(E_n, \sigma_n) = \varprojlim_n E_n.$$

(That the projective limit above is of finite type over $R$ is proved in [EGA $0_{\mathrm{I}}$, 7.2.9].) Needless to say, $\mathfrak{L}$ and $\mathfrak{F}$ are inverse $\otimes$-equivalences of $R$-linear categories, see 7.2.9-11 and 7.7.1 in [EGA, $0_{\mathrm{I}}$].

**Remark 5.9.** The additive category $\mathrm{Rep}_R(\Gamma)^\wedge$ is abelian and $\mathfrak{F}$ respects that structure, but kernels *are not* defined effortlessly because for an arrow $(\varphi_n) : (E_n) \to (F_n)$, the projective system of kernels $(\mathrm{Ker}\,\varphi_n)$ is not necessarily an object of $\mathrm{Rep}_R(\Gamma)^\wedge$. For example, one can consider the obvious object $(R_n)$ of $\mathrm{Rep}_R(\Gamma)^\wedge$ and the arrow $(\varphi_n) : (R_n) \to (R_n)$ of multiplication by $t$; then the kernel is certainly trivial, while $\mathrm{Ker}(\varphi_0) = R_0$.

5.7. **The main result.** We suppose here that $R = \mathbb{C}[\![t]\!]$ and return to the setting of Section 5.2, where, among others, we considered a *smooth* morphism

$$f : X \longrightarrow S.$$

We assume in addition that $f$ is *proper* and that $X_0$ is *connected*. Note that, the latter condition implies that each $|X_n^{\mathrm{an}}|$ is also connected [SGA1, XII Proposition 2.4].

Following the model of Definition 5.2, we use the $\mathbf{N}$-indexed family of categories $\mathcal{D}(X_n^{\mathrm{an}}/S_n^{\mathrm{an}})$-**mod** to produce

$$\mathcal{D}(X^{\mathrm{an}}/S^{\mathrm{an}})\text{-}\mathbf{mod}^\wedge.$$

(Here $X^{\mathrm{an}}$ and $S^{\mathrm{an}}$ are just graphical signs carrying no mathematical meaning.) Since $X_n^{\mathrm{an}}$ is proper over $S_n^{\mathrm{an}}$ [SGA1, XII, Proposition 3.2, p. 245], Proposition 5.6 produces $\otimes$-equivalences of $R_n$-linear categories

$$(-)^{\mathrm{an}} : \mathcal{D}(X_n/S_n)\text{-}\mathbf{mod} \xrightarrow{\sim} \mathcal{D}(X_n^{\mathrm{an}}/S_n^{\mathrm{an}})\text{-}\mathbf{mod},$$

which in turn give rise to a $\otimes$-equivalence of $R$-linear categories

$$\mathfrak{A} : \mathcal{D}(X/S)\text{-}\mathbf{mod}^\wedge \xrightarrow{\sim} \mathcal{D}(X^{\mathrm{an}}/S^{\mathrm{an}})\text{-}\mathbf{mod}^\wedge.$$

Let $x$ be an $R$-point of $X$ and write $x_n$ for the corresponding $R_n$-point of $X_n$. Clearly, each $x_n$ induces the same point, call it $x_0$, on the topological space $|X_0^{\mathrm{an}}| = |X_n^{\mathrm{an}}|$. Let $\Gamma$ be the fundamental group of $|X_0^{\mathrm{an}}|$ based at $x_0$. From Corollary 5.8 we have $\otimes$-equivalences of $R_n$-linear categories:

$$\mathbb{M}_{x_0} \circ \mathbf{Sol} : \mathcal{D}(X_n^{\mathrm{an}}/S_n^{\mathrm{an}})\text{-}\mathbf{mod} \xrightarrow{\sim} \mathrm{Rep}_{R_n}(\Gamma),$$

which in turn induce an equivalence

$$\mathfrak{M}_{x_0} : \mathcal{D}(X^{\mathrm{an}}/S^{\mathrm{an}})\text{-}\mathbf{mod}^\wedge \xrightarrow{\sim} \mathrm{Rep}_R(\Gamma)^\wedge.$$

We have therefore a diagram

$$\text{(6)}\qquad \mathcal{D}(X/S)\text{-}\mathbf{mod} \xrightarrow{\mathrm{D}\Phi} \mathcal{D}(X/S)\text{-}\mathbf{mod}^{\wedge} \xrightarrow{\mathfrak{A}} \mathcal{D}(X^{\mathrm{an}}/S^{\mathrm{an}})\text{-}\mathbf{mod}^{\wedge} \xrightarrow{\mathfrak{M}_{x_0}}$$

$$\xrightarrow{\mathfrak{M}_{x_0}} \mathrm{Rep}_R(\Gamma)^{\wedge} \xrightarrow{\mathfrak{L}} \mathrm{Rep}_R(\Gamma),$$

where each arrow is a $\otimes$-equivalence of $R$-linear categories. We can therefore say:

**Theorem 5.10.** *The $R$-linear $\otimes$-categories $\mathcal{D}(X/S)$-**mod** and $\mathrm{Rep}_R(\Gamma)$ are equivalent by means of the composition of* (6) *above.* □

Since we are also interested in the differential Galois groups associated to objects in $\mathcal{D}(X/S)$-**mod** (understood in the sense of [DHdS17, section 7]), we complement Theorem 5.10 by explaining how the aforementioned equivalence relates fibre functors. As we observed after Proposition 5.6 and Corollary 5.8, $x_n^* \simeq x_n^{\mathrm{an}*} \circ (-)^{\mathrm{an}}$ and $x_n^{\mathrm{an}*} \simeq \omega\mathbb{M}_{x_0}\mathbf{Sol}$. This implies that $\omega\mathfrak{M}_{x_0}\mathfrak{A}$ is $\otimes$-isomorphic to $(\mathcal{E}_n) \mapsto (x_n^*\mathcal{E}_n)$ so under the above equivalence the forgetful functor $\mathrm{Rep}_R(\Gamma) \to R$-**mod** and $x^* : \mathcal{D}(X/S)$-**mod** $\to R$-**mod** are $\otimes$-isomorphic.

**Corollary 5.11.** *If $\Gamma$ has an infinite cyclic quotient, then the group scheme $\mathcal{N}$ of Section 4.3 appears as the full differential Galois group of some $(\mathcal{E}, \nabla) \in \mathcal{D}(X/S)$-**mod**$^{\circ}$.*

*Proof.* Let $\Pi$ be the Tannakian envelope of $\Gamma$ (Definition 4.1). Using [DH17, 3.2.1(ii)] and then Proposition 4.15 we produce a faithfully flat morphism $u : \Pi \to \mathcal{N}$. Let now $E$ be a free $R$-module of finite rank and $\rho : \mathbf{G}_a \times \mathbf{G}_m \to \mathbf{GL}(E)$ a faithful representation. We have a commutative diagram in $(\mathbf{FGSch}/R)$

$$\begin{array}{ccc} \Pi & \longrightarrow & \mathbf{GL}(E) \\ {\scriptstyle u}\downarrow & & \uparrow{\scriptstyle \rho} \\ \mathcal{N} & \longrightarrow & \mathbf{G}_a \times \mathbf{G}_m; \end{array}$$

from [DHdS17, Proposition 4.10] we know that the full subcategory $\langle E \rangle_{\otimes}$ of $\mathrm{Rep}_R(\Pi)$ is equivalent to $\mathrm{Rep}_R(\mathcal{N})$ under $u$. Now we translate these observations to the category $\mathcal{D}(X/S)$-**mod**. □

## 6. Prudence of flat group schemes and projectivity of the underlying Hopf algebra

In this section, $R$ is assumed to be a *complete* DVR. In what follows, $G$ is a flat group scheme over $R$, which is *not* assumed to be of finite type. We shall investigate under which conditions $R[G]$ is a projective $R$-module. Our method is based on the following fundamental result.

**Theorem 6.1** (cf. [Ka52, page 338])**.** *Let $M$ be a flat $R$-module whose associated $K$-module $M \otimes K$ has countable rank. Then $M$ is free if and only if $\cap\pi^n M = (0)$.* □

Therefore, all efforts should be concentrated on understanding the *divisible ideal*, which we now introduce.

**Definition 6.2.** Let $M$ be an $R$-module. We denote by $\mathfrak{d}(M)$ the submodule $\cap\pi^n M$, and call it the divisible submodule of $M$. If $A$ is an $R$-algebra, $\mathfrak{d}(A)$ is an ideal, which we call the divisible ideal of $A$. If $X = \operatorname{Spec} A$, we write alternatively $\mathfrak{d}(X)$ to mean $\mathfrak{d}(A)$ and let $\mathrm{Haus}(X)$ stand for the closed subscheme of $X$ defined by $\mathfrak{d}(X)$.

It is quite easy to see that $\mathfrak{d}(G)$ cuts out a closed subgroup scheme (see Proposition 6.12). This being so, in order to study how far $\mathfrak{d}(G)$ is from being (0), the natural idea is, following Chevalley, to search semi-invariants. This is the crucial idea behind our notion of *prudence*, see Definition 6.7.

Section 6.1 explains what we understand by semi-invariants in the context of group schemes over $R$. In it we also derive basic results on this concept which prove useful further ahead. In Section 6.2 we introduce the notion of a prudent group scheme; this property is compared to the vanishing of the divisible ideal in Section 6.3. The function of Section 6.4 is to relate prudence to other existing properties (appearing in [SGA3$_{\rm new}$], [Ra72] and [DH17]). Finally, Section 6.5 links prudence of differential Galois groups to Grothendieck's existence theorem of projective geometry.

6.1. **Generalities on semi-invariants.** Let $V$ be an object of $\mathrm{Rep}_R(G)$; comultiplication $V \to V \otimes_R R[G]$ is denoted by $\rho$.

**Definition 6.3.** We say that $\mathbf{v} \in V$ is a semi-invariant if $R\mathbf{v}$ is a subrepresentation [Ja03, Part I, 2.9]. The set of all semi-invariants in $V$ is denoted by $\mathrm{SI}_G(V)$.

The following results are all simple algebraic exercises.

**Lemma 6.4.** *The element $\mathbf{v} \in V$ is a semi-invariant if and only if $\rho(\mathbf{v}) = \mathbf{v} \otimes \theta$ for some $\theta \in R[G]$.*

*Proof.* If $R\mathbf{v}$ is a subrepresentation, then $\rho(\mathbf{v})$ belongs to the $R$-submodule $R\mathbf{v} \otimes_R R[G]$ of $V \otimes R[G]$, and hence $\rho(\mathbf{v}) = \mathbf{v} \otimes \theta$ for some $\theta \in R[G]$. On the other hand, if $\rho(\mathbf{v}) = \mathbf{v} \otimes \theta$ for some $\theta \in R[G]$, then $\rho$ factors through $R\mathbf{v} \otimes R[G]$, which amounts to being a subrepresentation [Ja03, Part I, 2.9]. □

**Remarks 6.5.** We refrain from calling $\theta$ a group-like element because we *do not* assume that $R\mathbf{v}$ is isomorphic to $R$. Note that [SGA 3, Exposé VI$_B$] only uses the notion of semi-invariant in the case of a base field, see Definition 11.15 on p. 420 in there.

The set of semi-invariants is not usually stable under addition, so that it cannot be an $R$-submodule. On the other hand, it is stable under scalar multiplication (apply Lemma 6.4).

**Lemma 6.6.** *The following claims are true.*

1. *Let $\varphi : V \to W$ be a morphism of representations of $G$. The image of $\mathrm{SI}_G(V)$ lies in $\mathrm{SI}_G(W)$.*
2. *Let $H$ be another flat affine group scheme over $R$ and $f : G \to H$ a morphism. Then, for any $V \in \mathrm{Rep}_R(H)$, we have $\mathrm{SI}_H(V) \subset \mathrm{SI}_G(V)$.*
3. *Under the notations of the previous item, assume that $f^* : R[H] \to R[G]$ is injective and that $V$ belongs to $\mathrm{Rep}_R^\circ(H)$. Then $\mathrm{SI}_H(V) = \mathrm{SI}_G(V)$.*

*Proof.* The verification of (1) and (2) follows the obvious method and we omit it.

(3) Let $\mathbf{v} \in V$ be $G$-semi-invariant. We assume to begin that $\mathbf{v}$ is the first element of an ordered basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ of $V$. Write $\sigma : V \to V \otimes R[H]$ for the coaction and let $\sigma(\mathbf{v}_j) = \sum_i \mathbf{v}_i \otimes \sigma_{ij}$ with $\sigma_{ij} \in R[H]$. Our hypothesis jointly with Lemma 6.4 then implies that

$$\sum_i \mathbf{v}_i \otimes f^*(\sigma_{i1}) = \mathbf{v}_1 \otimes \theta$$

for some $\theta \in R[G]$. This shows that $f^*(\sigma_{i1}) = 0$ if $i > 1$ and, consequently, that $\sigma_{i1} = 0$ if $i > 1$. This implies, that $\mathbf{v}$ is $H$-semi-invariant (again by Lemma 6.4).

Now, for a general $\mathbf{v} \in V \setminus \{0\}$, there exists an ordered basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ and some non-negative integer $m$ such that $\pi^m \mathbf{v}_1 = \mathbf{v}$. We then see that $\mathbf{v}_1$ is also $G$-semi-invariant:

There exists $\theta \in R[G]$ such that $(\mathrm{id}_V \otimes f^*) \circ \sigma(\pi^m \mathbf{v}_1) = \pi^m \mathbf{v}_1 \otimes \theta$, and hence $(\mathrm{id}_V \otimes f^*) \circ \sigma(\mathbf{v}_1) = \mathbf{v}_1 \otimes \theta$ as $\pi^m$ is not a zero-divisor in $V \otimes R[G]$; Lemma 6.4 proves that $\mathbf{v}_1$ is semi-invariant for $G$. From the previous considerations, $\mathbf{v}_1$ is $H$-semi-invariant, so that $\mathbf{v} = \pi^m \mathbf{v}_1$ is also $H$-semi-invariant. □

6.2. **The concept of prudence.** We are now able to isolate the property of $\mathrm{Rep}_R(G)$ which forces the ideal $\mathfrak{d}(G)$ to vanish.

**Definition 6.7.** Let $V \in \mathrm{Rep}_R^\circ(G)$. We say that $V$ is prudent if an element $\mathbf{v}$ whose reductions modulo $\pi^{n+1}V$ are semi-invariant for all $n$ must necessarily be semi-invariant. We say that $G$ is prudent if each $V \in \mathrm{Rep}_R^\circ(G)$ is prudent.

The definition of prudence admits the following reformulation (the proof is quite simple, given the material developed so far, and we omit it).

**Lemma 6.8.** *Let $V \in \mathrm{Rep}_R^\circ(G)$. Assume that each $\mathbf{v} \in V$ which*

- *belongs to a basis, and*
- *has a reduction modulo $\pi^{n+1}V$ which is semi-invariant for all $n$,*

*is a semi-invariant. Then $V$ is prudent.* □

The following result shall be employed in deriving deeper consequences in Section 6.3.

**Lemma 6.9.** *Let $f : G \to H$ be a morphism in $(\mathbf{FGSch}/R)$ such that the associated morphism of Hopf algebras $f^* : R[H] \to R[G]$ is injective. Then, if $G$ is prudent, so is $H$.*

*Proof.* Let $V \in \mathrm{Rep}_R^\circ(H)$ and consider an element $\mathbf{v} \in V$ whose reduction modulo $\pi^{n+1}V$ is $H$-semi-invariant for any given $n$. From Lemma 6.6-(2), we have

$$\mathrm{SI}_H(V/\pi^{n+1}) \subset \mathrm{SI}_G(V/\pi^{n+1}).$$

This forces, as $G$ is prudent, $\mathbf{v}$ to be $G$-semi-invariant. But by Lemma 6.6-(3), we know that $\mathbf{v}$ is then $H$-semi-invariant. □

**Example 6.10.** Let $\Gamma$ be an abstract group with Tannakian envelope $\Pi$ (see Section 4.1 for definitions). We contend that $\Pi$ is prudent. Let $V$ be a free $R$-module of rank $r$ affording a linear representation of $\Gamma$. Let $\mathbf{v} \in V$ be such that, for any given $n \in \mathbf{N}$, $\mathbf{v} + \pi^{n+1}V$ is semi-invariant. We pick a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ of $V$ over $R$ such that $\mathbf{v} = \pi^m \mathbf{v}_1$ for some $m \in \mathbf{N}$. We know that $R(\mathbf{v} + \pi^n V)$ is a subrepresentation, which means that this very same $R$-module is stable under the action of $\Gamma$. Hence, if $[\gamma_{ij}] \in \mathrm{GL}_r(R)$ is the matrix associated to $\gamma$ (with respect to the basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$), we can find $c_n \in R$ such that $\sum_i \pi^m \gamma_{i1} \mathbf{v}_i \equiv c_n \mathbf{v} \mod \pi^n V$. Then,

$$\sum_{i>1} \pi^m \gamma_{i1} \mathbf{v}_i \equiv \pi^m \mathbf{v}_1 [c_n - \gamma_{11}] \mod \pi^n V.$$

This shows that $\pi^m \gamma_{i1} \equiv 0 \mod \pi^n$ for each $i > 1$. Since $n$ is arbitrary, we conclude that $\gamma_{i1} = 0$ for $i > 1$. This says that $R\mathbf{v}_1$ is invariant under $\Gamma$. Hence, $R\mathbf{v}$ is also invariant under $\Gamma$.

6.3. **Prudence and the divisible ideal.** Our objective here is to establish:

**Theorem 6.11.** *If $G$ is prudent, then $\mathfrak{d}(G) = 0$.*

Before embarking on a proof, we note a fundamental fact.

**Proposition 6.12.** *The closed subscheme of $G$ defined by $\mathfrak{d}(G)$,* $\mathrm{Haus}(G)$, *is in fact a closed and flat subgroup scheme.*

The proof of this Proposition follows easily from item (3) of

**Lemma 6.13.** *Let $M$ and $N$ be flat $R$-modules.*

1. *Assume that $\mathfrak{d}(M) = (0)$, and let $E \subset M$ be finitely generated. Then $E^{\rm sat}$ is free.*
2. *If $\mathfrak{d}(M) = (0)$ and $\mathfrak{d}(N) = (0)$, then $\mathfrak{d}(M \otimes_R N) = (0)$.*
3. *The divisible submodule of $M \otimes N$ is simply $\mathfrak{d}(M) \otimes N + M \otimes \mathfrak{d}(N)$.*

*Proof.* (1) Clearly $E \otimes K \xrightarrow{\sim} E^{\rm sat} \otimes K$, and we can apply Theorem 6.1 to conclude.

(2) Let $t \in \mathfrak{d}(M \otimes N)$. Then, there exist a free and saturated submodule $U$ of $M$ and a free and saturated submodule $V$ of $N$ such that $t \in U \otimes V$. Now $U \otimes V$ is saturated in $M \otimes N$, and hence $\mathfrak{d}(M \otimes N) \cap (U \otimes V) = \mathfrak{d}(U \otimes V)$. Therefore, $t = 0$ since $\mathfrak{d}(U \otimes V) = (0)$.

(3) This is a simple consequence of (2). □

We now head towards the proof of Theorem 6.11. In fact, this theorem is a consequence of the ensuing result, which therefore becomes the object of our efforts. (To state it, we employ the notion of the diptych [DHdS17, Section 4].)

**Theorem 6.14.** *Assume that $G$ is prudent and let $V \in \mathrm{Rep}_R^\circ(G)$ be arbitrary. Consider the diptych of $\rho$*

$$\begin{array}{ccc} G & \xrightarrow{\ \rho\ } & \mathbf{GL}(V) \\ {\scriptstyle faithfully\ flat}\downarrow & & \uparrow{\scriptstyle closed\ immersion} \\ \Psi' & \longrightarrow & \Psi, \end{array}$$

*so that $\Psi' \otimes K = \Psi \otimes K$ is of finite type over $K$. Then $\mathfrak{d}(\Psi') = 0$.*

*Proof that Theorem 6.14 ⇒ Theorem 6.11.* Let $a \in \mathfrak{d}(G)$. Endowing $R[G]$ with its right-regular action and using "local finiteness" [Ja03, Part I, 2.13], we can find a subcomodule $V \subset R[G]$ which contains $a$ and is finitely generated as an $R$-module. Since $R[\Psi]$ is constructed as the image of the obvious morphism $R[\mathbf{GL}(V)] \to R[G]$, and $R[\Psi']$ is the saturation of $R[\Psi]$ in $R[G]$ (see [DHdS17, Section 4]), we conclude that $a \in R[\Psi']$. Now, because $R[\Psi']$ is saturated in $R[G]$, we have $\mathfrak{d}(\Psi') = \mathfrak{d}(G) \cap R[\Psi']$. If Theorem 6.14 is true, then $a = 0$. □

We need some lemmas to establish Theorem 6.14. In these results we employ the notion of closure of a closed subscheme of a the generic fibre [EGA IV$_2$, 2.8, 33ff] as well as the constructions of Neron blowups appearing in Section 2.2.

**Lemma 6.15.** *Let $f : \mathcal{G}' \to \mathcal{G}$ be a morphism of $(\mathbf{FGSch}/R)$. We suppose that $\mathcal{G}$ is of finite type and that $f \otimes K$ is an isomorphism. Let $H \subset \mathcal{G}$ stand for the closure of*

$$\mathrm{Haus}(\mathcal{G}') \otimes K \subset \mathcal{G}' \otimes K = \mathcal{G} \otimes K$$

*in $\mathcal{G}$. Then $f$ factors through*

$$\mathcal{N}_H^\infty(\mathcal{G}) \longrightarrow \mathcal{G}.$$

*Proof.* Algebraically, $H$ is cut out by the ideal

$$I = \{a \in R[\mathcal{G}] \,:\, 1 \otimes a \in K \otimes \mathfrak{d}(\mathcal{G}')\}.$$

By definition, $R[\mathcal{N}_H^\infty(\mathcal{G})]$ is the $R$-subalgebra of $K[\mathcal{G}]$ obtained by adjoining to $R[\mathcal{G}]$ all elements of the form $\pi^{-n} \otimes a$, where $a \in I$. We then need to prove that for each $a \in I$ and each $n \in \mathbf{N}$, there exists $\delta_n \in R[\mathcal{G}']$ such that $a = \pi^n \delta_n$. So let $a \in I$. By construction, $1 \otimes a$ belongs to the ideal $K \otimes \mathfrak{d}(\mathcal{G}')$. Hence, there exists some $m \in \mathbf{N}$ and some $\delta \in \mathfrak{d}(\mathcal{G}')$ such that $1 \otimes a = \pi^{-m} \otimes \delta$. This shows that $\pi^m a \in \mathfrak{d}(\mathcal{G}')$. Now, for every $n \in \mathbf{N}$, we pick $\delta_n$ such that $\pi^{n+m}\delta_n = \pi^m a$, so that $\pi^n \delta_n = a$.

□

**Lemma 6.16.** *Let $\mathcal{H} \to \mathcal{G}$ be a closed embedding of flat group schemes of finite type over $R$. If $\mathcal{N}_{\mathcal{H}}^{\infty}(\mathcal{G})$ is prudent, then $\mathcal{H} = \mathcal{G}$.*

*Proof.* From Lemma 6.17 below, we can find $V \in \mathrm{Rep}_R^{\circ}(\mathcal{G})$ and a line $R\mathbf{v} \subset V$ such that

$$\mathbf{Stab}_{\mathcal{G}}(R\mathbf{v}) = \mathcal{H}.$$

The $\mathcal{H}$-semi-invariance of $\mathbf{v}$ forces $\mathbf{v} + \pi^{n+1}V \in V/\pi^{n+1}$ to be $\mathcal{H}$-semi-invariant for all $n$. (Use Lemma 6.6-(1), if anything.) But since for each $n$ the base change

$$\mathcal{N}_{\mathcal{H}}^{\infty}(\mathcal{G}) \otimes R_n \longrightarrow \mathcal{G} \otimes R_n$$

factors through a morphism

$$\mathcal{N}_{\mathcal{H}}^{\infty}(\mathcal{G}) \otimes R_n \longrightarrow \mathcal{H} \otimes R_n,$$

and this is the central point, we conclude that $\mathbf{v} + \pi^{n+1}V$ is $\mathcal{N}_{\mathcal{H}}^{\infty}(\mathcal{G})$-semi-invariant. (The arrow $\mathcal{N}_{\mathcal{H}}^{\infty}(\mathcal{G}) \otimes R_n \to \mathcal{H} \otimes R_n$ is even an isomorphism [DHdS17, Corollary 5.11].) As $\mathcal{N}_{\mathcal{H}}^{\infty}(\mathcal{G})$ is assumed prudent, $\mathbf{v}$ is $\mathcal{N}_{\mathcal{H}}^{\infty}(\mathcal{G})$-semi-invariant as well. But then, Lemma 6.6-(3) guarantees that $\mathbf{v} \in \mathrm{SI}_{\mathcal{G}}(V)$. This implies that $R\mathbf{v}$ is stable under $\mathcal{G}$, so that $\mathcal{G} = \mathcal{H}$. □

**Lemma 6.17.** *Let $\mathcal{H} \to \mathcal{G}$ be a closed embedding in* $(\mathbf{FGSch}/R)$. *If $\mathcal{G}$ is of finite type, then, there exists a certain $V \in \mathrm{Rep}_R^{\circ}(\mathcal{G})$ and an element $\mathbf{v} \in V$ such that*

$$\mathbf{Stab}_{\mathcal{G}}(R\mathbf{v}) = \mathcal{H}.$$

*Proof.* This can be proved as in the case of a ground field (see the Theorem on p. 121 of [Wa79, 16.1]). For the convenience of the reader, we check the details.

Let $V \subset R[\mathcal{G}]$ be a subcomodule of the right regular representation which, as an $R$-module, is of finite type and contains generators of the ideal $I$ cutting out $\mathcal{H}$. Let $W := V \cap I$ and note that $W$ is saturated in $V$ since $I \subset R[\mathcal{G}]$ is. Let then $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ be a basis of $V$ over $R$ such that $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a basis of $W$; in this case $R[\mathcal{G}]\mathbf{v}_1 + \cdots + R[\mathcal{G}]\mathbf{v}_r = I$. Write $\Delta\mathbf{v}_j = \sum \mathbf{v}_i \otimes a_{ij}$, so that $\mathbf{Stab}_{\mathcal{G}}(W)$ is cut out by the ideal $J = (a_{ij} \,:\, i > r,\ j \le r)$, see the proof of the last Lemma in [Wa79, 12.4]. We now claim that $J = I$, hence proving that $\mathcal{H} = \mathbf{Stab}_{\mathcal{G}}(W)$.

First, since $\mathbf{v}_j = (\varepsilon \otimes \mathrm{id}) \circ \Delta\mathbf{v}_j$, we see, using that $\varepsilon(\mathbf{v}_1) = \ldots = \varepsilon(\mathbf{v}_r) = 0$, that $\mathbf{v}_j = \sum_{i=r+1}^{s} \varepsilon(\mathbf{v}_i) a_{ij}$, which shows that $I \subset J$. Now, let $\psi : R[\mathcal{G}] \to R[\mathcal{H}]$ be the projection and consider the $R[\mathcal{H}]$-comodule defined by $(\mathrm{id} \otimes \psi)\Delta : R[\mathcal{G}] \to R[\mathcal{G}] \otimes R[\mathcal{H}]$. (This corresponds to the action of $\mathcal{H}$ on $\mathcal{G}$ by multiplication on the right.) Then, we know that $V$ and $I$ are $R[\mathcal{H}]-$subcomodules, so that $(\mathrm{id} \otimes \psi)\Delta(I) \subset I \otimes R[\mathcal{H}]$ and $(\mathrm{id} \otimes \psi)\Delta(V) \subset V \otimes R[\mathcal{H}]$. This proves that $(\mathrm{id} \otimes \psi)\Delta(W) \subset W \otimes R[\mathcal{H}]$, because $R[\mathcal{H}]$ is $R$-flat [Mat89, Theorem 7.4(i)]. Therefore, since $\mathbf{v}_1, \ldots, \mathbf{v}_r \in W$, we see that $(\mathrm{id} \otimes \psi)\Delta(\mathbf{v}_j) \in W \otimes R[\mathcal{H}]$ provided that $1 \le j \le r$. But $(\mathrm{id} \otimes \psi)\Delta(\mathbf{v}_j) = \sum_{i \le r} \mathbf{v}_i \otimes \psi(a_{ij}) + \sum_{i > r} \mathbf{v}_i \otimes \psi(a_{ij})$, and hence $\psi(a_{ij}) = 0$, if $i > r$ and $j \le r$. This proves the inclusion $J \subset I$, and our claim.

The fact that $W$ can be chosen of rank one follows from $\mathbf{Stab}_{\mathcal{G}}(W) = \mathbf{Stab}_{\mathcal{G}}(\wedge^s W)$ [Wa79, Appendix 2]. □

*Proof of Theorem 6.14.* Since $G$ is prudent, Lemma 6.9 guarantees that $\Psi'$ is prudent. Let $H \subset \Psi$ be the closure of $\mathrm{Haus}(\Psi') \otimes K \subset \Psi \otimes K$ in $\Psi$. Lemma 6.15 shows that $\Psi' \to \Psi$ factors as

$$\Psi' \longrightarrow \mathcal{N}_H^{\infty}(\Psi) \longrightarrow \Psi.$$

Another application of Lemma 6.9 ensures that $\mathcal{N}_H^{\infty}(\Psi)$ is prudent because $\Psi'$ is. As $\Psi$ is of finite type, Lemma 6.16 tells us that $H = \Psi$. Hence, the closed embedding $\mathrm{Haus}(\Psi') \subset \Psi'$ becomes an isomorphism when base-changed to $K$. This is only possible if $\mathrm{Haus}(\Psi') = \Psi'$, and we are done. □

6.4. **Prudence and some other characterizations.** Fundamental as it is, Kaplansky's theorem has the drawback of needing the hypothesis on the rank. Using Raynaud-Gruson's notion of Mittag-Leffler modules [Ra72, Ch. 2, §2, Definition 3] (see also [RG71, Part 2]), we can put our knowledge in a more efficient structure and examine two other distinctive features which were singled out in [DH17] (repeated in Definition 6.18 below) and in [SGA3$_{\rm new}$, VI$_B$, 11]. We recall that $G$ is a flat group scheme over the complete discrete valuation ring $R$.

**Definition 6.18** (cf. [DH17, Section 3.1])**.** A $G$-module $M$ is called *specially locally finite* if for each $G$-submodule $V \subset M$ whose underlying $R$-module is of finite type, the saturation $V^{\rm sat} \subset M$ is also of finite type (over $R$). The group scheme $G$ is specially locally finite if $R[G]_{\rm right}$ is specially locally finite.

As remarked already in [DH17, Proposition 3.1.5], the above property is closely related to the Mittag-Leffler condition. The ensuing result straightens this relation.

**Proposition 6.19.** *The following conditions are equivalent.*

(1) *The group scheme $G$ is prudent.*
(2) *The divisible ideal $\mathfrak{d}(G)$ is null.*
(3) *The group scheme $G$ is specially locally finite.*
(4) *The $R$-module $R[G]$ satisfies the Mittag-Leffler condition of Gruson-Raynaud.*
(5) *For each $V \in \mathrm{Rep}_R^\circ(G)$ and each finite subset $F \subset V$, the set*
$$\{W \text{ subrepresentation of } V \text{ containing } F\}$$
*has a least element.*

*Proof.* (1) ⇒ (2). This is the content of Theorem 6.11.

(2) ⇒ (3). We know from [DH17, Theorem 4.1.1] that $G$ is the limit in $(\mathbf{FGSch}/R)$ of a system
$$\{G_\lambda, \varphi_{\lambda,\mu} : G_\mu \to G_\lambda\}$$
where: (a) each $G_\lambda \otimes K$ is of finite type, and (b) each $\varphi_{\lambda\mu}$ is faithfully flat. Since $\mathfrak{d}(G) = 0$, we can certainly say that $\mathfrak{d}(G_\lambda) = 0$ for all $\lambda$, so that, by Kaplansky's Theorem (Theorem 6.1), $R[G_\lambda]$ is free as an $R$-module. Now, $R[G]_{\rm right}$ is the direct limit of a system of $G$-modules, $R[G]_{\rm right} = \varinjlim_\lambda R[G_\lambda]$, and, by Proposition 3.1.5-(ii) of [DH17], $R[G_\lambda]$ is specially locally finite. As each $R[G_\lambda]$ is saturated in $R[G]$, it becomes a simple matter to show that $R[G]$ is also specially locally finite.

(3) ⇒ (4). This is already explained in [DH17, Proposition 3.1.5-(i)].

(4) ⇒ (5). We write $G = \varprojlim_\lambda G_\lambda$ as in the proof of "(2) ⇒ (3)." Since the canonical morphism $G \to G_\lambda$ is faithfully flat, condition (4) and Corollary 2.1.6 of [RG71, Part 2, §1] guarantee that $R[G_\lambda]$ satisfies the Mittag-Leffler condition. The fact that $K[G_\lambda]$ is of finite type jointly with Proposition 1 of [Ra72, Chapter 2, §2] assure in turn that, for each $\lambda$, $R[G_\lambda]$ is a projective $R$-module.

Let now $V$ and $F$ be as in condition (5). It is possible to find some $\lambda$ such that $V$ actually comes from a representation of $G_\lambda$. Now [SGA3$_{\rm new}$, VI$_B$, 11.8.1, p.418] assures the existence of a $G_\lambda$-subrepresentation $W$ containing $F$ and not properly containing any other such $G_\lambda$-subrepresentation. Since any $G$-subrepresentation of $V$ must actually come from a representation of $G_\lambda$ [DH17, Theorem 4.1.2(i)], we are done.

(5) ⇒ (1). Let $V \in \mathrm{Rep}_R^\circ(G)$. If $\mathbf{v} \in V$ is such that its image in $V/\pi^{n+1}$ is a semi-invariant, it follows that $W_n := R\mathbf{v} + \pi^{n+1}V$ is a subrepresentation. Now, let $W \subset V$ be a subrepresentation containing $\mathbf{v}$ and not properly containing any other such. It then follows that $R\mathbf{v} \subset W \subset \cap W_n = R\mathbf{v}$, so that $R\mathbf{v}$ is a subrepresentation. By definition, $\mathbf{v}$ is semi-invariant, and $G$ must be prudent. □

6.5. **Applications to the full differential Galois group.** Let $f : X \to S$ be a smooth and proper morphism having integral fibres and admitting a section $\xi \in X(R)$. Given $\mathcal{E} \in \mathcal{D}(X/R)\text{-}\mathbf{mod}^\circ$, let $\mathrm{Gal}'(\mathcal{E})$ be the full differential Galois group as in Section 7 of [DHdS17]. (Notations are those of loc.cit.)

**Theorem 6.20.** *The ring of functions of the group* $\mathrm{Gal}'(\mathcal{E})$ *is a free* $R$*-module.*

*Proof.* For brevity, we write $G' = \mathrm{Gal}'(\mathcal{E})$ and $E = \xi^*\mathcal{E}$. We need to show that $G'$ is prudent to apply Theorem 6.11 and Theorem 6.1. So let $\mathcal{V} \in \mathcal{D}(X/S)\text{-}\mathbf{mod}^\circ$ belong to $\langle\mathcal{E}\rangle_\otimes$ and write $V = \xi^*\mathcal{V}$; this free $R$-module affords a representation of $G'$. Let now $\mathbf{v}$ be a non-zero element of $V$ whose image in $V_n = V/\pi^{n+1}$, call it $\mathbf{v}_n$, is semi-invariant. It is to prove that $R\mathbf{v}$ is also a subrepresentation of $V$. As Lemma 6.8 guarantees, there is no loss of generality in supposing that $\mathbf{v}$ belongs to a basis of $V$.

We denote by $\psi_n : L_n \to V_n$ the inclusion of $R\mathbf{v}_n$ (an $R$-module isomorphic to $R_n$) into $V_n$. The equivalence $\xi^* : \langle\mathcal{E}\rangle_\otimes \xrightarrow{\sim} \langle E\rangle_\otimes$ produces, from the commutative diagram with exact rows

$$\begin{array}{ccccccc} L_{n+1} & \xrightarrow{\pi^{n+1}} & L_{n+1} & \longrightarrow & L_n & \longrightarrow & 0 \\ \downarrow{\scriptstyle \psi_{n+1}} & & \downarrow{\scriptstyle \psi_{n+1}} & & \downarrow{\scriptstyle \psi_n} & & \\ V_{n+1} & \xrightarrow{\pi^{n+1}} & V_{n+1} & \longrightarrow & V_n & \longrightarrow & 0, \end{array}$$

a commutative diagram with exact rows in $\mathcal{D}(X/S)\text{-}\mathbf{mod}$:

$$\begin{array}{ccccccc} \mathcal{L}_{n+1} & \xrightarrow{\pi^{n+1}} & \mathcal{L}_{n+1} & \longrightarrow & \mathcal{L}_n & \longrightarrow & 0 \\ \downarrow{\scriptstyle \phi_{n+1}} & & \downarrow{\scriptstyle \phi_{n+1}} & & \downarrow{\scriptstyle \phi_n} & & \\ \mathcal{V}_{n+1} & \xrightarrow{\pi^{n+1}} & \mathcal{V}_{n+1} & \longrightarrow & \mathcal{V}_n & \longrightarrow & 0. \end{array}$$

By GFGA [Il05, Theorem 8.4.2], there exists an arrow of coherent sheaves

$$\phi : \mathcal{L} \longrightarrow \mathcal{V}$$

inducing $\phi_n$ for each $n$. Using Lemma 5.4 and the fact that each $\phi_n$ is a monomorphism we conclude that $\phi$ is also a monomorphism. Applying the claim made on the proof of essential surjectivity in Proposition 5.3, we conclude that $\mathcal{L}$ carries a structure of $\mathcal{D}(X/S)$-module. In addition, $\phi$ is then an arrow of $\mathcal{D}(X/S)$-modules, according to the claim made on the proof of fully faithfulness in Proposition 5.3. As we already remarked that $\phi$ is a monomorphism, we conclude that $\phi$ is really an arrow of $\langle\mathcal{E}\rangle_\otimes$.

Let $\psi : L \to V$ be an arrow in $\mathrm{Rep}_R(G')$ whose image under $\xi^*$ is $\phi$ so that $\psi$ induces $\psi_n$ upon reduction modulo $\pi^{n+1}$. It only takes a moment's thought to see that $R\mathbf{v} = \psi(L)$, so that $R\mathbf{v}$ is a subrepresentation of $V$. □

The various differential Galois groups can be put together to form the relative fundamental group scheme of $X$ based at $\xi$, see [DH17, Section 5.1]. If we denote this group scheme by $\Pi$, then $\Pi = \varprojlim_\lambda G_\lambda$, where each $R[G_\lambda]$ is a free $R$-module (by Theorem 6.20) and the transition arrows $G_\mu \to G_\lambda$ are faithfully flat (we use [DH17, Theorem 4.1.2(i)]). Then [SP, Tag 0AS7] tells us that

**Corollary 6.21.** *The ring of functions of the relative fundamental group scheme (cf.* [DH17, Section 5.1]*) of* $X/R$ *based at* $\xi$ *satisfies the Mittag-Leffler condition.* □

Let us end this section by putting together Theorem 6.20 and the results on unipotent group schemes explained in Section 3.

**Corollary 6.22.** *Assume that* $\mathcal{E}_K$ *is unipotent and that* $R$ *is of characteristic* $(0,0)$*. Then* $\mathrm{Gal}'(\mathcal{E})$ *is of finite type.*

*Proof.* We consider the morphism $\mathrm{Gal}'(\mathcal{E}) \to \mathrm{Gal}(\mathcal{E})$ from the full to the restricted differential Galois group (see Section 7 of [DHdS17]). Since $\mathcal{E}_K$ is unipotent, $\mathrm{Gal}(\mathcal{E}) \otimes K$ is unipotent and, a fortiori, connected [DG70, IV.2.2.5, p.487]. Hence, [To15, Theorem 2.11, p.187] shows that $\mathrm{Gal}(\mathcal{E})$ is unipotent over $R$. We now apply Corollary 3.5 and Theorem 6.20 to conclude that $\mathrm{Gal}'(\mathcal{E})$ is of finite type. □

**Remark 6.23.** Theorem 6.20 is certainly false in case $X$ is simply affine. See Example 7.11 in [DHdS17].

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