Université Pierre et Marie Curie

Mémoire d'Habilitation à Diriger des Recherches

Spécialité: Mathématiques

Manifestations algebro-géométriques du concept de groupe fondamental

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Ser poeta não é uma ambição minha
É a minha maneira de estar sozinho.

F. Pessoa, O guardador de rebanhos.

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Introduction

Il y a un temps pour planter, et un temps pour arracher ce qui a été planté; le présent texte est un résumé de mes recherches pendant les neuf dernières années dans un thème mathématique spécifique. Comme le quotient pages/temps met en évidence, les omissions sont nombreuses surtout dans ce qui concerne méthode, preuve et héritage. Et, plus sérieusement, la quantité d'espace dédiée à chaque thème ne fait pas justice ni à la difficulté, ni à la pertinence. De l'autre côté, les conclusions sont en bonne mesure.

Le thème de mes recherches ici tourne au tour de l'idée que, en géométrie, deux objets très communs — solutions de certaines équations différentielles et fonctions algébriques — peuvent être plus fructueusement comprises dans la présence d'un “groupe”. Cette ligne de pensée a une longue et éminente généalogie et mes efforts étaient de constater comment cette belle image peut survivre dans un monde algébrique abstrait. Dans un tel monde, plusieurs piliers de l'expérience précédente sont retirés: il n'y a pas un groupe fondamental, les solutions sont trop abondantes ou trop rares, les fonctions ne peuvent plus être determinées par leurs valeurs, etc. Ces piliers sont alors remplacés par des objets moins évidents, et c'est là où je commence. Plus spécifiquement, le rôle du groupe fondamental est joué par le schéma en groupes Tannakian, les solutions reçoivent une part mineure, et la notion de fonction algébrique est légèrement modifiée.

Je tache d'être plus technique. Nous nous donnons un schéma algébrique X sur un corps k et nous nous concentrons sur les $\mathcal{D}_X$-modules qui sont de type fini sur $\mathcal{O}_X$. Ici, $\mathcal{D}_X$ est le faisceau de tous les opérateurs différentiels sur X et relatifs à k, un objet qui est aisément mesuré par le biais de l'exemple suivant.

Exemple. Si $X = \mathbb{A}^1$ et x est la fonction coordonnée, alors

$$\mathcal{D}_X = \bigoplus_{q\geq 0} \mathcal{O}_X \cdot \partial^{[q]},$$

où

$$\partial^{[q]}(x^m) := \binom{m}{q} x^{m-q}$$

est un remplaçant formel de

$$\frac{1}{q!}\frac{\mathrm{d}^q}{\mathrm{d}x^q}.$$

Nous observons alors, avec N. Saavedra, que dans la présence de quelques restrictions sur X, la catégorie précédente $\mathcal{D}_X$-**mod** est équivalente à la catégorie des représentations d'un schéma en groupes affine, ou groupe pro-algébrique, Π_X, le schéma en groupes fondamental Tannakien de X. (Pour plus d'informations, voir Section 1.2.3 et Section 1.3.) Bien sûr, si $k = \mathbb{C}$, alors la théorie a des piliers plus économiques, mais quand k est un corps de caractéristique positive (ou même un AVD) le schéma en groupes fondamental s'avère très utile étant donné qu'il fournit des bonnes intuitions et des bonnes directions. Mais attention: Dans plusieurs cas il semble presque impossible de décrire effectivement cet objet. (Les variétés abéliennes sont, comme d'habitude, une heureuse exception.)

Je décrirai brièvement le contenu de chaque une de mes contributions apparues dans ce mémoire; elles sont organisées, en chapitres, par similarité technique.

Dans [dS07a], qui est revu à la Section 2.1, j'ai publié mes recherches au tour des propriétés basiques de Π_X (voir plus haut). Les résultats étaient fondamentaux parce que, à l'exception du merveilleux text [Gi75] de Gieseker, cet objet n'avait pas attiré de l'attention auparavant. Ma prochaine entreprise dans ce même thème a pris beaucoup de temps pour mûrir [dS12b]. Elle concerne la suite exacte d'homotopie pour le schéma en groupes fondamental et est révisée dans la Section 2.2. Il avait essentiellement un sommet à conquérir: obtenir, à la manière naturelle de [SGA1], l'égalité

$$\text{Image de } \Pi_{X_s} \quad = \quad \text{Noyau de } \Pi_X \to \Pi_S,$$

où $X \to S$ est un morphisme propre et lisse entre des k-schémas lisses sur un corps algébriquement clos k et X_s est la fibre sur un point fermé. Cette question a promu la découverte d'objets "algébro-différentio-géométriques" intéressants. J'ai eu la chance de trouver des idées proches à celles de E. Cartan et C. Ehresmann, voir également la section 1.1 pour une meilleure explication de cette dernière phrase.

Les recherches sur le comportement des objets lors de la "réduction modulo des premiers" sont très importantes en mathématiques et les travaux [dS09], [dS08] et [dS11] s'intéressent à une face de ce domaine. Ce thème commun occupe le Chapitre 3. Soit (R, π, k, K) un anneau de valuation discrète complet et de caractéristique mixte $(0, p)$. On choisit une algèbre R-adique "lisse" A. Parmi les $\mathcal{D}_{A\otimes K/K}$-modules il a ceux qui sont induits par les $\mathcal{D}_{A/R}$-modules (le lecteur est invité à interpréter $\mathcal{D}$ comme dans l'Exemple de la page 5) et chacun de ces derniers donne lieu à deux groupes algébriques différents: un sur K et un sur k. En symboles: Soit $\mathcal{M}$ un $\mathcal{D}_{A/R}$-module sans π-torsion et de type fini sur A. Il donne lieu à deux groupes algébriques: G_K sur K et G_k sur k. La quête d'un groupe algébrique $G(\mathcal{M}) = G$ sur R (un schéma en groupes) interpolant entre G_K et G_k était la principale motivation de [dS09]. En plus, la construction de $G(\mathcal{M})$ devrait être faite par le biais d'une certaine catégorie associée à $\mathcal{M}$ et devrait imiter celle du groupe de monodromie ou de Galois différentiel. Comme j'explique brièvement dans la Section 3.1, qui revoit [dS09], $G(\mathcal{M})$ est le produit d'un théorème de reconstruction tannakienne appliqué à une sous-catégorie des $\mathcal{D}_{A/R}$-modules associée à $\mathcal{M}$, cf. éq. (3.1).

D'un autre côté, la géométrie algébrique du XX siècle a montré que inverser le processus de réduction modulo des premiers est un sujet qui devrait être considéré. Il est normalement appelé "théorie des relèvements", et dans [dS11] j'ai suivi la théorie de déformations de Grothendieck-Schlessinger pour traiter ce thème. Le travail est revu à la section 3.2 et est centré sur deux problèmes spécifiques: trouver un relèvement "minimal", et puis estimer le "nombre" des relèvements. Voici quelques détails. Soient R et A comme dans le paragraphe précédent; on se donne également un $\mathcal{D}_{A\otimes k/k}$-module M. Par des résultats de Katz et Berthelot (cf. Théorème 19 et Théorème 31) le relèvement de M dans un $\mathcal{D}_{A/R}$-module en caractéristique 0 se fait assez aisément. Mais un tel relèvement, noté $\mathcal{M}$ dorénavant, peut hériter une complexité inutile, ce qui veut dire que le groupe de Galois différentiel de $\mathcal{M} \otimes K$ peut être démesurément grand. Cela veut dire que le groupe $G(\mathcal{M})$ du paragraphe précédent peut avoir une réduction modulo π qui dévie du groupe de Galois différentiel du $\mathcal{D}_{A\otimes k/k}$-module M. On se pose naturellement la question sur la possibilité de trouver un relèvement $\mathcal{M}$ tel que $G(\mathcal{M})$ soit le plus petit possible; ma réponse est transcrite dans le Théorème 32.

Le nombre des relèvements en caractéristique nulle que notre $\mathcal{D}_{A\otimes k/k}$-module peut avoir devient alors une question inévitable. La façon dont je la traite est basée sur la théorie formelle des déformations d'après Schlessinger, et l'idée qu'on devrait restreindre l'attention à des relèvements ayant des solutions dans un certain anneau préalablement choisit, cf. la Définition 34. La principale conclusion est présentée par le Théorème 35.

Le point de vue différentiel n'est pas le seul sur lequel j'ai choisi de méditer et maintenant j'écris à propos de mes efforts dans le domaine de la géométrie au tour des "fonctions algébriques". L'utilité de l'introduction d'un groupe dans cette branche a presque deux cents ans (Puiseux, Riemann), mais mon point de vue est tiré de celui degagé par Madhav Nori dans [N76]. Il est inspiré par les merveilleux travaux de Weil, Narasimhan et Seshadri [W38], [NS65]. Un des aspects de la théorie des fonctions algébriques qui n'a pas été touché par le succès spectaculaire de la théorie du groupe fondamental de Grothendieck était celle des revêtements inséparables. Nori explique comment étudier ce derniers en appliquant la (à l'époque) nouvelle théorie des catégories tannakiennes et a ainsi ouvert des nouvelles directions.

Soit X/k une variété projective sur un corps algébriquement clos de caractéristique $p > 0$. Nori introduit une catégorie de fibrés vectoriels sur X, appelée $\mathbf{EF}(X)$, qui est équivalente à la catégorie des représentations d'un schéma en groupes pro-fini $\Pi^{\text{EF}}(X)$. Il montre que $\Pi^{\text{EF}}(X)$ classifie essentiellement tous les fibrés principaux finis sur X. (Quelques détails se trouvent dans la Section 1.3.) Notre contribution (je parle d'un travail en collaboration avec I. Biswas) à la théorie de Nori était de proposer une définition plus simple de $\mathbf{EF}(X)$, voir Théorème 4.1. Cette dernière détecte que un phénomène topologique — devenir trivial après un revêtement — suffit pour définir $\mathbf{EF}(X)$. Conduit par cette condition assez simple, j'ai découvert les travaux de Mehta-Subramanian [MS02], [MS08] et Pauly [P07] qui affirment que, contrairement à l'intuition commune, le nombre de fibrés vectoriels rendus triviaux par un certain revêtement ramifié fini de X peut être "très grand" pourvu que ce revêtement soit *inséparable.* De façon explicite, il y a des exemples de courbes projectives lisses X possédant un *nombre infini* de classes d'isomorphisme de fibrés vectoriels stables qui deviennent triviaux sur le revêtement donné par le Frobenius $F : X \to X$. Mon travail à propos de ce phénomène, qui de nos jours porte le nom de "Conjecture de Nori" (grâce à une conjecture faite dans [N82]), est le sujet de la section 4.2. Mes découvertes dans ce domaine sont d'une nature computationnelle. Elles étaient catalysées par l'introduction d'une algèbre de type fini (voir Définition 44 et les lignes qui suivent) dont les représentations décrivent tous les fibrés vectoriels trivialisés par $F : X \to X$. Sa construction n'est rien d'autre qu'une application directe du théorème bien connu de Cartier sur la p-courbure, mais son introduction ouvre la possibilité de traiter le thème d'un point de vue calculatoire en suivant [Le08]. Cela, à son tour, permet la découverte de plusieurs exemples (cf. Théorème 48) où la conjecture de Nori s'avère fausse. (Il est bien possible que ça fausseté soit la règle, et non l'exception.)

J'ai séparé le texte en quatre chapitres; bien que les chapitres de 2 à 4 sont directs et courent droit dans le coeur de mes contributions, j'ai réservé le chapitre 1 pour un exposé moins technique. Il contient les piliers pour le travail mentionné dans cette introduction. Si ce mémoire serait trouvé par les yeux d'un mathématicien sans expérience antérieure dans ce domaine, c'est le chapitre 1 qui doit être lu en premier.

Introduction

There is a time to plant and a time to pluck what is planted; the present text is a summary of my researches in the past nine years working on a particular mathematical topic. As the ratio pages/time evidences, there are many omissions mostly in what concerns method, proof and inheritance. And, more seriously, the amount of space allocated to each topic does not in general make justice to relevance and difficulty. On the other hand, conclusions are in fit measure.

The theme of my researches presented here revolves around the idea that, in geometry, two very common objects — solutions to certain differential equations and algebraic functions — can be better understood in the presence of "a group." This line of thought has a long and distinguished pedigree and my efforts have been to see how this beautiful image can survive in an abstract algebraic world. In this world, many pillars from former experience are removed: there is no fundamental group, there are too few or too many solutions, algebraic functions are not determined by their values anymore, etc. These are then replaced by less obvious objects, and this is where I begin. More specifically, the role of the fundamental group is played by the *Tannakian fundamental group scheme*, solutions receive a smaller part, and the notion of an algebraic functions is slightly modified.

Let me be more technical. We are given an algebraic scheme X over some field k and we concentrate on $\mathcal{D}_X$-modules which are of finite type over $\mathcal{O}_X$. Here $\mathcal{D}_X$ is the sheaf of rings of all differential operators on X relative to k, an object which is quickly understood by means of the following example. (More material on this can be found in section 1.1.)

Example. If $X = \mathbb{A}^1$ with coordinate function x, then

$$\mathcal{D}_X = \bigoplus_{q\geq 0} \mathcal{O}_X \cdot \partial^{[q]},$$

where

$$\partial^{[q]}(x^m) := \binom{m}{q} x^{m-q}$$

is a formal equivalent of

$$\frac{1}{q!}\frac{\mathrm{d}^q}{\mathrm{d}x^q}.$$

We then observe with N. Saavedra that under some restrictions on X, the aforementioned category $\mathcal{D}_X$-**mod** is equivalent to the category of representations of an affine group scheme, or pro-algebraic group, Π_X, the *Tannakian fundamental group scheme of X*. (For more information, see Section 1.2.3 and Section 1.3.) Of course, if $k = \mathbb{C}$, then the theory has more economic pillars, but when k is a field of positive characteristic (or even a DVR), then the fundamental group scheme becomes quite useful since it gives us good directions and intuitions. But beware: in many concrete cases it seems almost impossible to effectively describe this object. (Abelian varieties are, as usual, a happy exception [dS07a].)

I will now briefly explain the content of each one of my contributions appearing in this report; they are organized, in chapters, by similarity in technique.

In [dS07a], which is review in Section 2.1, I published my researches concerning basic properties of Π_X (introduced above). The results were foundational since, with the exception of Gieseker's marvelous text [Gi75], this object had not attracted previous attention. My following endeavour on this same topic took much longer to mature [dS12b]. It dealt with the so-called homotopy exact sequence of the fundamental group scheme and is reviewed in Section 2.2. There was essentially one peak to be conquered: obtain, as is done in [SGA1], the equality

$$\text{Image of } \Pi_{X_s} \quad = \quad \text{Kernel of } \Pi_X \to \Pi_S,$$

where $X \to S$ is a proper and smooth morphism of smooth k-schemes over an algebraically closed field k, and X_s is the fibre over a closed point. This question promoted the discovery of interesting algebraic-differential-geometric objects. I was fortunate to encounter ideas closer to those of E. Cartan and C. Ehresmann, see also section 1.1 for a better explanation of this last sentence.

Investigations on how algebraic objects "reduce modulo primes" are quite important in pure mathematics and the works [dS09], [dS08] and [dS11] are concerned with one facet of this domain. This common theme occupies chapter 3. Let (R, π, k, K) be a complete DVR of mixed characteristic $(0, p)$ and consider an R-adic "smooth" algebra A. Among all possible $\mathcal{D}_{A\otimes K/K}$-modules there are those which are induced by $\mathcal{D}_{A/R}$-modules (the reader is invited to take $\mathcal{D}$ similar to the one in Example on p.8) and each one of these gives rise to two distinct algebraic groups: one over K and one over k. In symbols: Let $\mathcal{M}$ be a $\mathcal{D}_{A/R}$-module which is free of π-torsion and of finite type over A. It gives rise to two algebraic groups: G_K over K and G_k over k. The quest for some algebraic group $G(\mathcal{M}) = G$ over R (a group scheme) interpolating between G_K and G_k was the main motivation of [dS09]. Moreover, the construction of $G(\mathcal{M})$ should be given by means of a certain category associated to $\mathcal{M}$ and should mimic that of the so-called monodromy group, or differential Galois group. As I swiftly explain in section 3.1, which reviews [dS09], $G(\mathcal{M})$ is a product of a Tannakian reconstruction theorem applied to a carefully chosen subcategory of $\mathcal{D}_{A/R}$-modules associated to $\mathcal{M}$, cf. eq. (3.1).

On the other hand, algebraic geometry of the XX century showed that reversing the process of "reduction modulo primes" is a topic which should be considered. It is usually named "lifting", and in [dS11] I followed Grothendieck-Schlessinger's deformation theory to treat this theme. The work is reviewed in section 3.2 and centers around two specific problems: find one "minimal" lifting, and then estimate the "amount" of liftings. Here are the particulars. Let R and A be as in the previous paragraph and assume that we are given a $\mathcal{D}_{A\otimes k/k}$-module M. It turns out that a result of Katz generalized by Berthelot (cf. Theorem 19 and Theorem 31) allows the lifting of M to a $\mathcal{D}_{A/R}$-module in characteristic zero quite easily. But such a lifting, call it $\mathcal{M}$, can inherit needless complexity, which precisely means that the differential Galois group of $\mathcal{M} \otimes K$ may be bigger than desired. This means that the algebraic group $G(\mathcal{M})$ over R mentioned in the previous paragraph can have a reduction modulo π which deviates from the differential Galois group of the $\mathcal{D}_{A\otimes k/k}$-module M. One then naturally inquires if it is possible to find a lifting $\mathcal{M}$ such that $G(\mathcal{M})$ is as small as possible; my answer is transcribed in Theorem 32.

The amount of liftings to characteristic zero that our $\mathcal{D}_{A\otimes k/k}$-module may have becomes then an obvious question. The way I proceeded to treat it is grounded in Schlessinger's formal deformation theory and the idea that one should restrict attention to liftings which have solutions in some preassigned ring, cf. Definition 34. The main conclusion is presented in Theorem 35.

The differential point of view is not the only one I chose to meditate on and now I write on my endeavors in the branch of geometry revolving around "algebraic functions." The utility of the introduction of a group in this domain has almost two hundred years, but my point of view is taken from the one put forward by Madhav Nori in [N76] which is inspired by the marvelous work of Weil, Narasimhan and Seshadri [W38], [NS65]. One aspect of the theory of algebraic functions which remained untouched by the spectacular success of Grothendieck's theory of the etale fundamental group was that of inseparable coverings. Nori explained how to study these by applying the newly created theory of Tannakian categories and founded new directions.

Let X/k be a projective variety over an algebraically closed field of characteristic $p > 0$. Nori introduced a category of vector bundles on X, call it $\mathbf{EF}(X)$, which is equivalent to the category of representations of a pro-finite group scheme $\Pi^{\mathrm{EF}}(X)$, and showed that $\Pi^{\mathrm{EF}}(X)$ essentially classifies all *finite* principal bundles over X. (Some details are in section 1.3 below.) Our contribution (this was work done in collaboration with I. Biswas) to Nori's theory is to propose a much simpler definition of $\mathbf{EF}(X)$, see Theorem 37 in Section 4.1. It simply detects that a topological phenomenon, that of becoming trivial after some covering, is enough to define $\mathbf{EF}(X)$. Led by this simple condition, I discovered the works of Mehta-Subramanian [MS02], [MS08] and Pauly [P07] which contend that, very much contrary to common intuition, the number of vector bundles which become trivial on some finite cover of X can be "very large" provided that the covering is *inseparable*. Explicitly, there are many examples of smooth and projective curves X possessing *infinitely* many isomorphism classes of stable vector bundles which become trivial after being pulled-back by the Frobenius $F : X \to X$. My work on this phenomenon, which nowadays runs under the name of "Nori's conjecture" (due to a conjecture in [N82]), is the subject of Section 4.2. My findings in this branch are of a computational nature. They are catalyzed by the introduction of a finitely generated algebra (see Definition 44 and following lines) whose representations describe all vectaor bundles trivialized by $F : X \to X$. Its construction is nothing more than a direct application of Cartier's well-known theorem on the p-curvature, but its introduction opens the possibility of treating the topic from a computational point of view following [Le08]. This in turn allowed the discovery of many other examples (cf. Theorem 48) where Nori's conjecture is false. (It is possible that the falsity of the conjecture is the rule and not the exception.)

I have separated the text into four chapters; while chapters 2 through 4 are quite direct and run straight into the heart of my contribution, I have reserved chapter 1 for a less technical report. It contains the basic pillars for the work mentioned in this introduction. Should this report be met by the eyes of a non specialist mathematician, it is chapter 1 that should be first read.

Notations and conventions

In all that follows, we shall work with the following notations and conventions.

1. If x is a point of a scheme, $\boldsymbol{k}(x)$ stands for the residue field of its local ring.

2. For any ring R, the ring of matrices of size $n \times n$ is denoted by $\mathrm{M}_n(R)$.

3. A *vector bundle* is a locally free sheaf of finite fixed rank over the structure sheaf. The category of vector bundles on some scheme X is denoted by $\mathbf{VB}(X)$.

4. If V is a vector bundle and W is a sub-$\mathcal{O}_X$-module of V such that V/W is also a vector bundle, then W is called a subbundle.

5. A trivial vector bundle is a vector bundle isomorphic to some $\mathcal{O}^r$. A vector bundle V on some X is trivialized by some $f : X' \to X$ if f^*V is trivial.

6. The general linear group scheme over some ring R will be denoted by $\mathbf{GL}_{n,R}$, or $\mathbf{GL}(V)$.

7. If G is a group scheme over some ring A, $\mathrm{Rep}_A(G)$ stands for the category of representations [J87] of G on A-modules of *finite presentation.* The analogous convention is in force whenever G is, instead of a group scheme, simply an abstract group.

8. A "DVR" is a discrete valuation ring.

9. If $X \to S$ is a relative scheme, we follow Grothendieck [EGA IV$_4$, 16] and denote by $\mathcal{D}_{X/S}$ the sheaf of relative differential operators of arbitrary order. If $\mathcal{M}$ is an $\mathcal{O}_X$-module, a stratification with respect to S is a structure of $\mathcal{D}_{X/S}$-module on $\mathcal{M}$ extending the given $\mathcal{O}_X$-module structure.

10. A variety over an algebraically closed field k is an integral scheme of finite type over $\mathrm{Spec}\, k$.

11. When speaking about linearly topologized rings, we employ the term "complete" to signify what EGA calls complete and separated.

12. If A is a ring which is complete with respect to the $\mathfrak{I}$-adic topology, the completion of $A[x_1, \ldots, x_m]$ with respect to the linear topology induced by $\mathfrak{I}$ will be denoted by $A\langle x_1, \ldots, x_m\rangle$.

Chapter 1

Standard facts and techniques around the objects of study

We shall present here the technical and conceptual basis for most constructions in this memoir. These concern the theory of $\mathcal{D}$-modules, stratifications, and the theory of neutral Tannakian categories. Put together, these theories furnish a fruitful environment to treat the ubiquitous concept of monodromy and the fundamental group in an algebraic manner.

In this chapter we made an effort to reach a skilled mathematician who is not an expert in this particular field of study.

1.1 $\mathcal{D}$-modules, systems of differential equations and stratifications

Let X be an open subset of the complex plane $\mathbb{C}$. For each holomorphic function f on X, we can fabricate another one by taking its derivative. In this way we obtain an operation on the set of all holomorphic functions and the theory of $\mathcal{D}$-modules is simply a way to furnish an algebraic garment to this very simple and profound procedure. We give the set of holomorphic functions on X, $\mathcal{O}(X)$, the structure of a $\mathbb{C}$-algebra, and we consider inside the huge ring $\mathrm{End}_{\mathbb{C}}(\mathcal{O}(X))$ the subring D_X generated by $\mathcal{O}(X)$ and the derivation operator $f \mapsto f'$. Bringing on all the mass of what is nowadays common mathematical education, we have:

Definition 1. Let X be a complex manifold and $U \subset X$ an open. Write

$$\mathcal{D}_X(U) = \begin{array}{c}\text{subring of } \mathrm{End}_{\mathbb{C}}(\mathcal{O}_U) \text{ generated by multiplication by holomorphic} \\ \text{functions on } U \text{ and Lie derivations along holomorphic vector fields on } U.\end{array}$$

The association $U \mapsto \mathcal{D}_X(U)$ defines a quasi-coherent sheaf of associative $\mathcal{O}_X$-algebras.

If we return to the naive description offered above, it is then manifest that systems of linear differential equations

$$\begin{aligned} f_1' &= a_{11}f_1 + a_{12}f_2 \\ f_2' &= a_{21}f_1 + a_{22}f_2 \end{aligned}$$

give rise to $\mathcal{D}_X$-modules so that the interest behind these algebraic objects is immediately justified for the mathematician. Moreover, in the general setting of Definition 1, one also obtains an interpretation of $\mathcal{D}_X$-modules as systems of linear partial differential equations which, furthermore, satisfy "integrability" conditions, see Proposition 4.

Definition 2 (Integrability). Let X be an open subset of $\mathbb{C}^m$ and write ∂_{z_k} for $\frac{\partial}{\partial z_k}$. A system of m linear partial differential equations

$$\begin{aligned} \partial_{z_k} u_1 &= a_{k,11} \cdot u_1 + \ldots + a_{k,1n} \cdot u_n \\ &\cdots\cdots \\ \partial_{z_k} u_n &= a_{k,n1} \cdot u_1 + \ldots + a_{k,nn} \cdot u_n, \end{aligned} \tag{1.1}$$

where $A_1 = (a_{1,ij}), \ldots, A_m = (a_{m,ij})$ are holomorphic, is called integrable if

$$\partial_{z_k} A_\ell + A_k A_\ell = \partial_{z_\ell} A_k + A_\ell A_k \tag{1.2}$$

for all $k, \ell = 1, \ldots, m$.

Remark 3. The term "integrability" is a contracted version of "complete integrability" which plainly means that we have *existence* and *uniqueness* once initial conditions are fixed. See [D69, X.9].

It is easily verified that, if $X \subset \mathbb{C}^m$ is an open subset, then $\mathcal{D}(X)$ is the quotient of the free algebra $\mathcal{O}(X)\{p_1, \ldots, p_m\}$ by the relations

$$\begin{array}{lll} p_i p_j - p_j p_i & = 0, & \forall\, i, j \\ p_i f - f p_i & = \partial_{z_i}(f), & \forall\, i, \forall\, f \in \mathcal{O}(X). \end{array}$$

With this description, the following proposition is clearly true.

Proposition 4. *Let X be an open subset of $\mathbb{C}^n$. A structure of $\mathcal{D}_X$-module on $\mathcal{O}_X^n$ is equivalent to an integrable system of partial linear differential equations as in Definition 2. More precisely, if $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathcal{O}_X^n$ and*

$$\frac{\partial}{\partial z_k} \cdot \mathbf{e}_j = \sum_{i=1}^n a_{k,ij} \cdot \mathbf{e}_i,$$

the system associated to the matrices $A_k = (a_{k,ij})$ is integrable and conversely. □

These simple results show that a small piece inside the theory of partial differential equations possesses an algebraic formulation. It is the exploration of the interactions between the analysis, the algebra, and the geometry fostered by this setting that consumes most of our energies. In fact, we chose to lower the role of analysis in the above picture so that the algebraic and geometric features take the lead. It is by this choice that we turn to *the theory of connections* and to Grothendieck's theory of *algebraic differential operators.* (We could recklessly attach the eponym 'Cartan–Ehresmann–Levi-Civita' to "the theory of connections".) Let us begin with a swift and tendencious presentation of the theory of connections.

Above we made explicit that, for an open subset X of $\mathbb{C}^m$, the structure of a $\mathcal{D}_X$-module on $\mathcal{O}_X^n$ amounts to a certain system of partial differential equations. But there is another powerful image which allows us to describe these objects. This is the notion of a *connection* and can be defined simply as a system of linear partial differential equations as in (1.1) by forgetting the integrability equations (1.2). As this misses the inherent geometric picture we prefer the ensuing definition. In it, we have adopted the term "1-connection" to prevent the reflex of some readers to cause confusion.

Definition 5. Let X be a complex manifold and denote by $\mathfrak{D}$ the ideal of X sitting diagonally inside $X \times X$. Let $P_X^1 \subset X \times X$ be the complex space cut out by $\mathfrak{D}^2$; the projections to X are denoted by p_0, p_1. A 1-connection on a $\mathcal{O}_X$-module $\mathcal{M}$ is simply an isomorphism $p_1^*\mathcal{M} \simeq p_0^*\mathcal{M}$ which, when restricted to X, induces the identity isomorphism of $\mathcal{M}$.

The geometric content of this definition can be vivified by noting that for each Zariski tangent vector $v : \operatorname{Spec}\mathbb{C}[\varepsilon] \to X$ based at a point x, a 1-connection induces an isomorphism $v^*\mathcal{M} \simeq x^*\mathcal{M}$, and this gives back the image "there is a way of comparing fibres above neighboring points." We note that this is roughly what E. Cartan had in mind [Ma07, §2]. Moreover, once all ornaments are removed, i.e. take X to be an open in $\mathbb{C}^m$ and $\mathcal{M} = \mathcal{O}_X^n$, the foregoing definition asks simply for a $m \times m$ matrix of 1-forms since $\Omega_X^1 = \operatorname{Ker}\mathcal{O}_{P_X^1} \to \mathcal{O}_X$. This is the data necessary to define the system (1.1).

Of course, Definition 5 is neither the original nor the standard one, but it is a translation into Grothendickean terms of the initial concept of the pioneers. Furthermore, the integrability condition imposed on the matrices $(a_{k,ij})$ of (1.1) can be recovered by a "*cocycle*" or "*transitivity*" rule on the isomorphism $p_0^*\mathcal{M} \simeq p_1^*\mathcal{M}$ and involves the complex space $P_X^2 \subset X \times X$ cut out by $\mathfrak{D}^3$.

Lemma 6. *Let X be an open of $\mathbb{C}^m$ and $\varepsilon : p_1^*\mathcal{M} \xrightarrow{\sim} p_0^*\mathcal{M}$ a 1-connection on $\mathcal{M} = \mathcal{O}_X^n$. Let (θ_{ij}) be an $n \times n$ matrices of global 1-forms on X defined by $\varepsilon(1 \otimes \mathbf{e}_j) = \sum \mathbf{e}_i \otimes \theta_{ij}$. Then the matrices A_k appearing in $(\theta_{ij}) = \sum A_k\,\mathrm{d}z_k$ satisfy the integrability condition* (1.1) *if and only if there exists an isomorphism*

$$\varepsilon_2 : P_X^2 \underset{p_1,X}{\times} \mathcal{M} \xrightarrow{\sim} \mathcal{M} \underset{X,p_0}{\times} P_X^2$$

lifting ε and satisfying the following transitivity condition. Let

$$q_0 : P_X^1 \times_X P_X^1 \longrightarrow P_X^2,$$
$$q_1 : P_X^1 \times_X P_X^1 \longrightarrow P_X^2,$$
$$\delta : P_X^1 \times_X P_X^1 \longrightarrow P_X^2,$$

be induced respectively by

$$(x, y; y, z) \longmapsto (x, y)$$
$$(x, y; y, z) \longmapsto (y, z)$$
$$(x, y; y, z) \longmapsto (x, z).$$

The transitivity condition is

$$\delta^*(\varepsilon_2) = q_0^*(\varepsilon_2) \circ q_1^*(\varepsilon_2).$$

The proof of this result can probably be extracted from [Be74, II], but a simple and long local computation also works. (One direction is explicitly stated in [Be74, p.130].) It is based on the fact that the complex space P_X^2 is none other than $(|X|, \mathcal{P}_X^2)$, where the sheaf of $\mathbb{C}$-algebras $\mathcal{P}_X^2$ is, when regarded as an $\mathcal{O}_X$-algebra via the projection p_0, simply $\mathcal{O}_X[\mathrm{d}x_1, \ldots, \mathrm{d}x_m]/(\mathrm{d}x_1, \ldots, \mathrm{d}x_m)^3$.

In view of Lemma 6, the following definition makes perfect sense.

Definition 7. Let $X \to S$ be a relative scheme and let $\mathcal{M}$ be a $\mathcal{O}_X$-module. Let $\mathfrak{D}$ be the ideal of X sitting diagonally in $X \times_S X$ and let P_X^μ be the subscheme of $X \times_S X$ cut out by $\mathfrak{D}^{\mu+1}$. The projection $P_X^\mu \to X$ onto the first factor, respectively second factor, is denoted by p_0, respectively p_1.

1. A μ-connection on $\mathcal{M}$ is an isomorphism

$$\varepsilon_\mu : p_1^*\mathcal{M} \xrightarrow{\sim} p_0^*\mathcal{M}$$

which induces the identity when pulled-back along the diagonal $X \to P_X^\mu$. See [Be74, II.1.2.1].

2. The data of a μ-connection ε_μ for each $\mu \geq 1$ is called a pseudo-stratification if ε_μ is induced by $\varepsilon_{\mu+1}$ through the canonical closed embedding $P_X^\mu \subset P_X^{\mu+1}$. See [Be74, II.1.2.1].

3. A pseudo-stratification $\{\varepsilon_\mu\}$ is called a stratification when the following "cocyle" or "transitivity" rule is verified. Let $\mathfrak{D}(2)$ stand for the ideal of $X \to X \times_S X \times_S X$ and let $P_X^\mu(2)$ be the subscheme cut out by $\mathfrak{D}(2)^{\mu+1}$. Define

$$\begin{array}{ll} q_0^\mu : P_X^\mu(2) \longrightarrow P_X^\mu, & (x,y,z) \mapsto (x,y), \\ q_1^\mu : P_X^\mu(2) \longrightarrow P_X^\mu, & (x,y,z) \mapsto (y,z), \\ \delta^\mu : P_X^\mu(2) \longrightarrow P_X^\mu, & (x,y,z) \mapsto (x,z). \end{array}$$

The cocycle (or transitivity) rule is

$$\delta^*(\varepsilon_\mu) = q_0^{\mu*}(\varepsilon_\mu) \circ q_1^{\mu*}(\varepsilon_\mu), \qquad \forall \mu.$$

See [Be74, II.1.3.1] and [Be74, II.1.3.3].

In addition to providing a clear algebraic picture to emulate the original idea of a connection, the study of the "infinitesimal neighbourhoods" P_X^μ allows us to review differential operators from an algebraic viewpoint.

Let $j : X \to X \times_S X$ be the diagonal immersion of X and let $\mathfrak{D}$ be the ideal of $j^{-1}(\mathcal{O})$ defining it. Let

$$\mathcal{P}_X^\mu := j^{-1}(\mathcal{O})/\mathfrak{D}^{\mu+1}$$

be the sheaf of rings of P_X^μ. This sheaf is the target of two evident morphisms of rings, $d_0, d_1 : \mathcal{O}_X \to \mathcal{P}_X^\mu$, and the dual

$$\mathcal{D}_{X,\mu} := \mathcal{H}om_X\left(\mathcal{P}_X^\mu, \mathcal{O}_X\right)$$

with respect to its $\mathcal{O}_X$-module structure defined by d_0 furnishes the "correct analogue" of the ring of differential operators of Definition 1 as suggests the following.

Proposition 8 (EGA IV_4, 16.8 and 16.11.2)**.** *1. Given $\mu, \nu \in \mathbb{N}$, there exists a morphism of $\mathcal{O}_S$-modules $\mathcal{D}_{X,\mu} \times \mathcal{D}_{X,\nu} \to \mathcal{D}_{X,\mu+\nu}$ which endows the ring*

$$\varinjlim_\mu \mathcal{D}_{X,\mu}$$

with the structure of a quasi-coherent $\mathcal{O}_X$-algebra.

2. For each $\partial \in \mathcal{H}om_X(\mathcal{P}_X^\mu, \mathcal{O}_X)$ and each $f \in \mathcal{O}_X$, define $\nabla_\partial(f) := \partial \circ d_1(f)$. This association induces a morphism of $\mathcal{O}_X$-rings $\mathcal{D}_X \to \mathcal{E}nd_S(\mathcal{O}_X)$.

3. *If $x_1, \ldots, x_m$ are functions on X such that $\mathrm{d}x_1, \ldots, \mathrm{d}x_m$ form a basis of $\Omega^1_{X/S}$, then $\mathcal{D}_{X,\mu}$ is free on*

$$\left\{\partial^{[q]} \,:\, q \in \mathbb{N}^m,\ q_1 + \ldots + q_m \leq \mu\right\}$$

and $\partial^{[q]}$ is "$\frac{1}{q!}\frac{\partial^q}{\partial x^q}$", i.e.

$$\partial^{[q]} x^\alpha = \binom{\alpha}{q} x^{\alpha - q}.$$

It is then a matter of unravelling definitions to arrive at.

Theorem 9. 1. *Given a stratification $\{\varepsilon_\mu\}$ on the $\mathcal{O}_X$-module $\mathcal{M}$, the rule which to every differential operator*

$$\partial \in \mathcal{D}_{X,\mu} := \mathcal{H}om_X\left(\mathcal{P}^\mu_X, \mathcal{O}_X\right)$$

associates the endomorphism

$$\mathcal{M} \xrightarrow{\varepsilon_\mu} \mathcal{M} \underset{\mathcal{O}_X}{\otimes} \mathcal{P}^\mu_X \xrightarrow{\mathrm{id}\otimes\partial} \mathcal{M}$$

gives $\mathcal{M}$ the structure of a $\mathcal{D}_X$-module. (Here we have abused notation in the definition of the first arrow.)

2. *If X is smooth over S, then this construction is an equivalence between the category of stratifications and $\mathcal{D}_X$-modules.*

3. *Assume that X is a smooth $\mathbb{C}$-scheme. Then a 1-connection ε on a coherent $\mathcal{O}_X$-module $\mathcal{M}$ possessing a lifting to a transitive 2-connection (as in Lemma 6) can be extended to a stratification.*

The proof of (1) and (2) can be found in [BO78, 2.11]. The proof of (3) is a consequence of [BO78, 2.15] and the fact that, at least in characteristic zero, the existence of a transitive 2-connection (as in Lemma 6) is equivalent to the standard notion of a "connection with vanishing curvature". The verification of this last claim follows from a long and straightforward computation, identical to the one alluded to after Lemma 6.

We have now showed that an analytic concept – that of the differential system (1.1) – has an algebraic manifestation which is fully explorable in the realm of abstract algebra.

1.2 Monodromy, fundamental groups, coverings and Tannakian categories

1.2.1 The idea of monodromy and the fundamental group

Let $X \subset \mathbb{C}^m$ be an open domain. When studying solutions to the integrable system

$$\begin{aligned} \partial_{z_k} u_1 &= a_{k,11} \cdot u_1 + \ldots + a_{k,1n} \cdot u_n \\ \cdots\cdots & \\ \partial_{z_k} u_n &= a_{k,n1} \cdot u_1 + \ldots + a_{k,nn} \cdot u_n, \end{aligned} \tag{1.1}$$

of linear PDEs, the phenomenon of monodromy is one of the most prominent. Let us fix $x_0 \in X$ and, by means of the integrability assumption, let $\boldsymbol{y}_0 : U_0 \to \mathbb{C}^n$ be the unique solution to (1.1) in an open ball U_0 about x_0 satisfying the initial condition $\boldsymbol{y}_0(x_0) = \boldsymbol{c}_0$. Given a loop $\gamma : [0,1] \to X$ based at x_0, we successively cover γ by open balls $U_0, U_1, \ldots, U_r$ and find solutions $\boldsymbol{y}_i : U_i \to \mathbb{C}^n$ which, furthermore, coincide on the intersections. (That such a construction is possible follows from the existence and uniqueness of solutions.) Since $\boldsymbol{y}_r$ will be a solution in a neighbourhood of x_0, we can pose the evident question: Is it true that $\boldsymbol{y}_0 = \boldsymbol{y}_r$ in $U_0 \cap U_r$? As there is place for only one solution with a given initial value, the answer of the previous questions boils down to checking the value $\boldsymbol{y}_r(x_0)$. The *principle of monodromy* says that if γ can be continuously deformed(=is homotopic) in X to the constant loop, then $\boldsymbol{y}_r(x_0) = \boldsymbol{y}_0(x_0)$. In fact, if δ is a loop based at x_0 which can be continuously deformed into γ, then, in the process of continuation described above, the final solutions coincide near x_0.

This is one of the reasons which led Poincaré to introduce the fundamental group* of X based at x_0 and note that, since solutions form a vector space, the above theory produces a *linear* representation on the vector space $\mathbb{C}^n$ of all possible initial values. More precisely, we have a homomorphim ϱ from the fundamental group $\pi_1(X, x_0)$ to $\mathrm{GL}_n(\mathbb{C})$ which is characterized by $\boldsymbol{y}_r = \varrho_\gamma \cdot \boldsymbol{y}_0$. This homomorphism is nowadays called the monodromy representation of the system of differential equations and the image of ϱ is called monodromy group.

1.2.2 Unramified coverings

The notion of unramified covering is essentially a geometric way, due to Riemann, to deal with “multivalued functions.” This topic is therefore already algebraic in nature as it deals in particular with algebraic functions. It was well served for may years and continues to be so. But Riemann’s real goal was to treat *ramified coverings*, and the unramfied ones are simply versions of the latter obtained by discarding a small amount of information. In abstract algebraic geometry, in contrast, passing from “ramified” to “unramified” can represent a total lost of control due to the existence of Frobenius’ morphism. Let us be more specific.

If $P(z, w) \in \mathbb{C}[z, w]$, it is clear that except for a finite number of points $z_1, \ldots, z_s \in \mathbb{C}$, called the ramification points, the equation $P(z, w) = 0$ will have exactly $\deg_w P$ solutions in w: the study of how these solutions vary on the *unramified part* $\mathbb{C} \setminus \{z_1, \ldots, z_s\}$ brings a lot of information about P. If P is now a polynomial in two variables over a field of positive characteristic, then it may happen that the solutions to $P(z, w) = 0$ carry no interesting information whatsoever. Another point of view is then desirable.

Following Riemann’s work, the essentially function theoretic approach set loose more geometric ideas and a new garment was found. An unramfied covering $Y \to X$ of topological spaces is, near any $x \in X$, of the form $U \times S$ for some discrete set S. Moreover, attention should be restricted to the case in which $\mathrm{Aut}(Y/X)$ acts *transitively* on the fibres; these are called, for the sake of discussion, *principal coverings* of X.

At this point, the algebraic vein produces examples of “principal coverings” which bear little resemblance to the geometric case. This is so because in positive characteristic, group schemes whose underlying topological space is a point abound [MAV, §11]. In [N76] Nori set out to study these “principal coverings” in algebraic geometry.

Definition 10. A Nori covering† of X is a finite morphism $Y \to X$ which is obtained in the following fashion. There exists a finite group scheme G, a principal homogeneous

*See §12 of [Po85] and also the now famous third paragraph on p.101 of [Po21].

†This terminology is not standard and one is bound to run into the term “essentially finite covering.”

G-bundle $P \to X$ (for the fpqc topology) and a left action of G on a finite k-scheme L such that Y is isomorphic to the quotient of $P \times L$ by the right action $(p, \ell) \cdot g = (pg, g^{-1}\ell)$.

Remark 11. In the definition we used the notion of quotient of a scheme by some group action, which is a slippery concept masterfully resolved by Grothendieck's descent theory. As this is too large a subject to discuss here, we simply refer the reader to [DG70, III, §2-§4] or [J87, Part 1, Chapter 5].

In essence, Nori coverings are simply principal homogeneous spaces under finite group schemes. It is then clear that, by construction, the understanding of these is intimately related to group objects; the fruitful idea of associating groups to algebraic functions is really a tautology in this setting.

1.2.3 Tannakian categories

Maintaining our intentions to look at the theory of $\mathcal{D}$-modules and coverings by algebraic means, we comment on the topic of Tannakian categories. These furnish a purely algebraic replacement for monodromy and the fundamental group. Our discussion makes few incursions on technical properties – the reader is refereed to [DM82] for those – but renders noticeable the main ideas.

Tannakian categories were invented by Saavedra and Grothendieck to understand and algebraize a phenomenon discovered by Tadao Tannaka [T38]: Given a compact Lie group K, (a) the subring of continuous functions $K \to \mathbb{C}$ generated by the matrix coefficients of continuous representations on finite dimensional vector-spaces is the ring of regular functions of an algebraic group $G/\mathbb{C}$, and (b) the Lie group K can be reconstructed from G, see [Ch46, Chapter VI]. Put together with the theory of categories, these findings give rise to two mathematical questions:

i) To what extent is a group determined by its linear representations? This is known as the reconstruction problem.

ii) When is an abelian category the category of representations of a group? This is known as the inversion problem.

These summarize the purpose of Tannakian categories. We shall only discuss (ii) in what follows.

Let us fix a field k and a group Γ. To understand the question of inversion, we have to make evident particular properties of $\mathrm{Rep}_k(\Gamma)$. Some structures immediately spring to mind: direct sums, tensor products, existence of kernels, cokernels etc. It turns out that a "tensor product functor" $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ on a k-linear abelian category enjoying a certain number of pleasant properties (see [DM82, §1]) permits the problem of inversion to come close to a satisfactory answer. However, there is still need for a capital property, which is the existence of a a functor $\omega : \mathcal{T} \to k\text{-}\mathbf{mod}$ – named a (neutral) *fibre functor* – enjoying the ensuing properties: (i) ω is faithful and k-linear, (ii) under ω the tensor product functor on $\mathcal{T}$ "corresponds" to the usual tensor product. (Details are in Definition 1.8, p. 113 of [DM82].)

The data of the functors $\otimes$ and ω, together with all their conveniences, hence permits a fulfilling answer to the problem of inversion: the triple $(\mathcal{T}, \otimes, \omega)$ is then christened a *neutral Tannakian category over the field* k, cf. [DM82, Definition 2.19]. We then have:

Theorem 12 (Saavedra,[DM82], 2.11)**.** *Let* $(\mathcal{T}, \otimes, \omega)$ *be a neutral Tannakian category over the field* k*. Then there exists an affine group scheme* G *over* k *such that* ω *induces an equivalence between* $\mathcal{T}$ *and the subcategory* $\mathrm{Rep}_k(G)$ *of* k-**mod***. The group scheme* G *is called the Tannakian fundamental group scheme associated to* $\mathcal{T}$ *via* ω*.* □

Remark 13. The existence of a fibre functor $\omega : \mathcal{T} \to k$-**mod** is sometimes too much to ask; one therefore weakens this condition and requires ω to take values on k'-**mod** for some extension k'/k instead. This gives rise to the general notion of Tannakian category, cf. [DM82, Definition 3.7, Remark 3.9] and [D90, 2.8]. We will have no use for them in this work.

Example 14. Let Γ be an abstract group. The category of complex linear representations $\mathrm{Rep}_{\mathbb{C}}(\Gamma)$ is certainly neutral Tannakian and thus is equivalent to the category of representations of an affine group scheme Γ^{alg} called the algebraic hull of Γ. A more constructive description of Γ^{alg} is based on the following. Given a homomorphism $\varrho : \Gamma \to \mathrm{GL}(V) = \mathbf{GL}(V)(\mathbb{C})$, we obtain a closed reduced subgroup scheme $G \subset \mathbf{GL}(V)$ by taking the Zariski closure of $\mathrm{Im}(\varrho)$ in $\mathbf{GL}(V)$. These groups form a projective system of group schemes and Γ^{alg} is the limit.

Our preferred example of this situation is the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules on a complex manifold X, call it $\mathcal{D}_X$-**mod**. As suggested by the link between the systems (1.1) and $\mathcal{D}$-modules (cf. Section 1.1), and the definition of mondromy given in Section 1.2.1, the category $\mathcal{D}_X$-**mod** is equivalent to the category of linear representations of the fundamental group $\pi_1(X, x_0)$. (The best exposition on this theme is probably the opening pages of [Del70].) Therefore, the Tannakian fundamental group of $\mathcal{D}_X$-**mod** associated to the fibre functor x_0^* is $\pi_1(X, x_0)^{\mathrm{alg}}$. Moreover, if $\mathcal{M}$ is an object of $\mathcal{D}_X$-**mod**, then the Zariski closure of the monodromy group of §1.2.1 is simply the image of $\pi_1(X, x_0)^{\mathrm{alg}}$ in $\mathbf{GL}(x_0^*\mathcal{M})$.

This example shows how to study the algebraic manifestation of monodromy and the fundamental group. The applications to the theory of Nori coverings as definition in Section 1.2.2 have a different flavour and are considered only in Section 1.3.2.

1.3 Fundamental group schemes: encounter of the previous theories

Let now k be an algebraically closed field and let and X be a connected algebraic k-scheme. We comment on the two main ways of applying Tannakian categories to find algebraic structures paralleling the phenomena described in the previous sections.

1.3.1 Analogy with systems of differential equations

We introduce algebraic analogies to the theory in Section 1.2.1.

Denote by $\mathbf{str}(X/k)$ the category whose objects are couples $(\mathcal{M}, \{\varepsilon_\mu\})$ consisting of a coherent $\mathcal{O}_X$-module and a stratification (see Definition 7), and whose arrows are homomorphisms of $\mathcal{O}_X$-modules respecting the stratification. If X happens to be smooth over k, then $\mathbf{str}(X/k)$ is equivalent to the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules (Theorem 9).

Already in [S72, VI,1.2.2] it is observed that the category $\mathbf{str}(X/k)$ endowed with its obvious tensor functor is neutral Tannakian by means of the pull-back functor x_0^* : $\mathbf{str}(X/k) \to k$-**mod**, where $x_0 \in X(k)$. (The proof of this fact is omitted in [S72], but the reader can find a very clear one in [BO78, Proof of 2.16].) Hence we can make the following.

Definition 15. The fundamental group scheme of X based at the point x_0 is the Tannakian fundamental group scheme associated to $\mathbf{str}(X/k)$ via x_0^*. It is denoted by $\Pi(X, x_0)$.

Since the category of representations of $\Pi(X, x_0)$ is, by construction, the category $\mathbf{str}(X/k)$, we arrive at a picture similar to that of Section 1.2.1. (To repeat: if X/k is smooth stratifications and $\mathcal{D}_X$-modules are one and the same.)

1.3.2 Analogy with unramified coverings

We now discuss the point of view suggested by the theory mentioned in Section 1.2.2. It should be repeated that Tannakian categories are not really necessary to produce a group to help the theory here, see [N82, Chapter II].

If the theory of unramified coverings can be though along the lines of Galois' theory of fields, then the theory of Nori coverings parallels that of inseparable extensions. But this is not the most appealing aspect of [N76]: it is the pedigree it shows in treating the subject in the light of the theorems of Weil [W38] and Narasimhan–Seshadri [NS65], and the use of Tannakian categories. The following definition is the starting point.

Definition 16. Assume that X is proper over k. A vector bundle E on X is called Nori-semistable if for each arrow from a smooth and projective curve $\varphi : C \to X$, the vector bundle $\varphi^* E$ is semistable and of degree zero. A Nori-semistable vector bundle E is finite if the number of isomorphism classes of indecomposables appearing in the Krull-Schmidt decomposition of $E^{\otimes 1}, E^{\otimes 2}, \ldots$ is *finite.* The vector bundle E is essentially finite if it can be written as E'/E'' with E' and E'' Nori-semistable subbundles of some finite vector bundle V.

From now on *we take X to be proper over k* and write $\mathbf{EF}(X)$ to signify the category of essentially finite vector bundles.

Theorem 17 ([N76]). *Let $x_0 \in X(k)$ be a point. Then the the triple $(\mathbf{EF}(X), \otimes_{\mathcal{O}_X}, x_0^*)$ is a neutral Tannakian category. The Tannakian fundamental group scheme associated to it is profinite and "classifies" principal bundles over X with finite structure group.*

For the precise meaning of the term "classifies" employed in the statement, the reader is referred to [N76, 3.11,p.40].

It should be mentioned that after the extensive generalizations of the theorems of Weil [W38] and Narasimhan-Seshadri [NS65], the fundamental group predicted by Theorem 17 turned out to be the largest profinite quotient of a much bigger fundamental group scheme defined by A. Langer [L11] and V.B. Mehta [M09].

Chapter 2

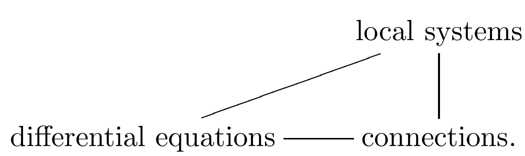
$\mathcal{D}$-modules on smooth algebraic varieties

In this chapter we explain our part in understanding $\mathcal{D}$-modules on a smooth scheme over a field k. There are essentially two texts to be commented: [dS07a] and [dS12b]. The first is entirely concerned with the case of characteristic $p > 0$, while the second has the merit of handling both $p = 0$ and $p > 0$ indistinctly.

Throughout this Chapter, we let k be an algebraically closed field of characteristic $p \geq 0$, and X a smooth and connected scheme over k.

2.1 Foundational material on $\mathcal{D}$-modules over schemes of positive characteristic

Let $\mathcal{D}_X$-**mod** stand for the category of $\mathcal{D}_X$-modules which are $\mathcal{O}_X$-coherent on X. By Theorem 9 we know that $\mathcal{D}_X$-**mod** is equivalent to the category of stratifications, $\mathbf{str}(X/k)$, discussed in Sections 1.1 and 1.3. We then have the lower edge of the archetypal triangle:



Although the omission of "local systems" in Chapter 1 was possible, it can only be justified by economical expedient. The reason is due to the reach of Katz's theorem, which begins this section. This result provides a means to deal with $\mathcal{D}_X$-modules by employing $\mathcal{O}_X$-modules and the Frobenius morphism: the F-divided sheaves. *Until the rest of the section, we assume that k is of characteristic $p > 0$.*

Definition 18 (F-divided sheaf)**.** The category of F-divided sheaves $\mathbf{Fdiv}(X)$ is the category whose:

Objects are families $\{E_i, \sigma_i\}_{i\in\mathbb{N}}$, where each E_i is a coherent $\mathcal{O}_X$-module and σ_i is an isomorphism from F^*E_{i+1} to E_i.

Arrows are projective systems $\alpha_i : E_i \to E_i'$ of $\mathcal{O}_X$-linear which respect the associated isomorphisms between pull-backs by Frobenius.

Any F-divided sheaf $\{E_i, \sigma_i\}$ gives rise to a $\mathcal{D}_X$-module structure on E_0 in the following way. Let $x_1, \ldots, x_n$ be etale local coordinates on some open set of X and let $\partial^{[q]}$ stand for the differential operators described in Proposition 8. For each $\partial^{[q]}$ with $q_1 + \ldots + q_n < p^m$ and each local section e of E_0, we define

$$\partial^{[q]}(e) = \sum_j \partial^{[q]}(a_j) \cdot e_j,$$

where $e = \sum_j a_j e_j$ is an expression of e in terms of sections e_j from E_m. Or, less analytically, we decree that the sections of E_0 induced by those of E_m are to be annihilated by all $\partial^{[q]}$ with $q_1 + \ldots + q_n < p^m$. (For details, cf. [Gi75, pp.3-4].)

Theorem 19 ([Gi75], Theorem 1.3)**.** *Let $\mathcal{D}_X$-***mod** *stand for the category of $\mathcal{D}_X$-modules which are $\mathcal{O}_X$-coherent. Then the above association defines an equivalence of categories between* **Fdiv**(X) *and $\mathcal{D}_X$-***mod***.*

It should be noted that this theorem can assume a much more general form and that the above construction can be reinterpreted — with the help of the equivalence between $\mathcal{D}$-modules and stratifications — as a formal consequence of usual faithfully flat descent [Be12]. This generality will be employed decisively in the study of $\mathcal{D}$-modules in mixed characteristic, see Chapter 3.

The category **Fdiv**(X) possesses a natural tensor product obtained "term-by-term" and the presence of an F-division forces all the $\mathcal{O}_X$-modules in question to be locally free [dS07a, Lemma 6]. These together with the choice of a point $x_0 \in X(k)$ endow **Fdiv**(X) with the structure of a neutral Tannakian category which, furthermore, corresponds to the homonymic structure on $\mathcal{D}_X$-**mod** under the equivalence of Theorem 19.

Fortunately, the category **Fdiv**(X) is much more amenable than $\mathcal{D}_X$-**mod** and one can obtain clear proofs of the following properties of the affine group scheme $\Pi(X, x_0)$ of Definition 15.

Theorem 20 ([dS07a])**.**

1. *The Frobenius morphism $\mathcal{O}(\Pi(X, x_0)) \to \mathcal{O}(\Pi(X, x_0))$ is an isomorphism.*
2. *Any quotient of $\Pi(X, x_0)$ is smooth.*
3. *If X is proper and U is a unipotent quotient of $\Pi(X, x_0)$, then U is pro-etale.*
4. *If X is an abelian variety, then $\Pi(X, x_0)$ is abelian.*
5. *Even if X is an elliptic curve over $\overline{\mathbb{F}}_p$, the affine group scheme $\Pi(X, x_0)$ does not "base-change well".*
6. *Assume that k is endowed with a non-trivial non-archimedean absolute value, that X is proper and that the analytic space X^{an} can be "uniformized." Then the algebraic hull (see Example 14) of the topological fundamental group $\pi_1^{\mathrm{top}}(X^{\mathrm{an}})$ appears as a quotient of $\Pi(X, x_0)$.*

Property (1) together with the fact that a reduced algebraic group is smooth [DG70, II.5.2.1] immediately implies (2); the latter seems to be the most basic (it already appears in a different setting in [MvdP].) It establishes a strong contrast to Nori's theory of the fundamental group scheme (see Section 1.3 and Chapter 4). Item (5) is a direct consequence of the fact that $\Pi(X, x_0)$ has too many characters and (6) follows from the fact that $\pi_1^{\mathrm{top}}(X^{\mathrm{an}})$ "parametrizes" $\mathcal{D}_X$-modules which, when regarded as $\mathcal{D}_{X^{\mathrm{an}}}$-modules in X^{an}, become trivial on some *admissible open cover*. (On this topic, in addition to [dS07a], the reader will profit from [vdP12].)

Many more remarkable properties of $\Pi(X, x_0)$ can be derived by using the category $\mathbf{Fidv}(X)$ instead of $\mathcal{D}_X$-$\mathbf{mod}$, but we mention only [EM10]. (In fact, it would not be far from the truth to say that almost all relevant facts concerning $\Pi(X, x_0)$ in positive characteristic stem from $\mathbf{Fidv}(X)$! But see the next section for a counter-argument.)

2.2 The homotopy exact sequence for the fundamental group scheme

Here we explain the basic ideas and methods behind [dS12b]. Let $f : X \to S$ be a smooth, proper and geometrically connected morphism of smooth schemes over k, an algebraically closed field of *arbitrary* characteristic. Let $x_0 \in X(k)$ be above $s_0 \in S(k)$. We are interested in determining the cokernel and the kernel of the natural morphism

$$f_\# : \Pi(X, x_0) \longrightarrow \Pi(S, s_0).$$

More precisely, as experience from the topological case suggest [St51, 17.3], we wish to show that

(a) $f_\#$ is surjective and

(b) $\mathrm{Ker}\, f_\#$ is the image of $\Pi(X_{s_0}, x_0)$.

The proof should be the "natural" one, i.e. paralleling that of the analogous fact for the etale fundamental group appearing algebraically in [SGA1, Exp. V, Proposition 6.11] (cf. Lemma 22 below) and geometrically in [SGA1, Exp. X, Corollary 1.4]. To accomplish this task is the objective of [dS12b].

Theorem 21. *Under the above assumptions, conditions (a) and (b) hold true.*

The verification of Claim (a) above is quite easy and all the difficulty lies in interpreting $\mathrm{Ker}\, f$. A careful reading of [SGA1, X] shows that the key property allowing the equality

$$\mathrm{Im}\, \pi_1^{\mathrm{et}}(X_{s_0}, x_0) \to \pi^{\mathrm{et}}(X, x_0) = \mathrm{Ker}\, \pi_1^{\mathrm{et}}(X, x_0) \to \pi_1^{\mathrm{et}}(S, s_0)$$

is the existence of a means to obtain from each etale covering $Z \to X$ a "universal" etale covering $f_*Z \to S$, cf. [SGA1,X, 1.2]. In fact, this property is just a geometric manifestation of the following very simple fact.

Lemma 22. *Let* $K \stackrel{a}{\to} \Pi \stackrel{b}{\to} H$ *be homomorphisms of groups satisfying* $\mathrm{Ker}\, b \supset \mathrm{Im}\, a$. *Then,* $\mathrm{Ker}\, b = \mathrm{Im}\, a \Leftrightarrow \mathrm{Ker}\, b$ *acts trivially on* $\Pi/\mathrm{Im}\, a \Leftrightarrow$ *the action of* Π *on the left of* $\Pi/\mathrm{Im}\, a$ *is induced by a set with an action of* $\Pi/\mathrm{Ker}\, b$.

Once this fact surfaces, we are assigned the task of understanding the geometric analogue of $\Pi/\mathrm{Im}\, a$. This leads us to consider *stratified schemes* instead of stratified vector bundles: a difficulty which is not present in SGA1 due to the fact that quotients are pro-finite in that theory. Since we deal with algebraic groups, we should restrict attention to *projective stratified schemes* because, as Chevalley showed more than 60 years ago, quotients of algebraic groups are quasi-projective varieties.

Theorem 23. *Let $g : Z \to X$ be a projective and smooth stratified scheme over X. Then, there exists a stratified scheme $H_f(Z) \to S$ enjoying the following properties:*

i) There is a morphism $e : f^*H_f(Z) \to Z$ of stratified schemes over X.

ii) If $W \to S$ is any stratified scheme over S whose pull-back f^*W is the source of an arrow of stratified schemes $f^*W \to Z$, then there exists a unique arrow $W \to H_f(Z)$ of stratified schemes over S which, when lifted back to X, factors $f^*W \to Z$. More concisely, $e : f^*H_f(Z) \to Z$ is universal from f^* to Z. (See [dS12a, Corollary 58].)

iii) The morphism $H_f(Z) \to S$ is projective. (See [dS12a, Theorem 50].)

The construction of $H_f(Z)$ mimics Grothendieck's construction of the Weil restriction functor considered in [TDTE, IV,19ff]. Here is how he proceeded: given the morphism $Z \to X$ of schemes, define the S-scheme functor f_*Z by

$$T \to S \quad \longmapsto \quad \text{Sections of } Z_T \to X_T.$$

By imposing conditions on $X \to S$, Grothendieck shows that the above functor is representable by a scheme over S; our contribution is to introduce differential structures throughout [dS12a, Section 10] and show that the associated representability problem has a solution.

The learned geometer will profit by thinking of $H_f(Z)$ *as a local system over S whose fibre is the fixed part of $Z|_{X_s}$ under the action of* $\Pi(X_s)$. Of course, the imaginative simplicity of this last statement requires some effort which was made in Sections 11 to 14 of [dS12b]. The rigorous statement ensues.

Proposition 24 ([dS12b], Corollary 59 and Proposition 74)**.** *Assume that the stratified scheme Z comes from a left action of $\Pi(X, x_0)$ on a projective scheme P. Then the fibre of the stratified scheme $H_f(Z) \to S$ above s_0 is isomorphic to the closed subscheme of P which is fixed by $\Pi(X_{s_0})$, i.e. $P^{\Pi(X_{s_0})}$.*

Let us, for the sake of discussion, call a stratified scheme $g : Z \to X$ over X *affinely controled* if it comes from an action of $\Pi(X)$ on the fiber Z_{x_0}. If we were to work on the complex-analytic category, this would mean that the action homomorphism stemming from Ehresmann's theory [E50, p.37] (or [CLN85, Chapter 5])

$$\pi_1^{\text{top}}(X) \longrightarrow \text{Aut}(Z_{x_0})$$

factors through some linear algebraic subgroup of $\text{Aut}(Z_{x_0})$. Among affinely controlled stratified S-schemes, there is a much simpler candidate for the function of $H_f(Z)$, namely, the stratified scheme obtained from the $\Pi(S, s_0)$-scheme

$$\text{closed subscheme of } Z_{x_0} \text{ fixed by the kernel of } f_\# : \Pi(X, x_0) \to \Pi(S, s_0).$$

Therefore, given the construction of $H_f(Z)$ and the interpretations of its fibre, and given the simple criterion hinted by Lemma 22, we are led to show that $H_f(Z)$ is an affinely controlled stratified scheme over S. This results in a theory which detects affinely controlled projective stratified schemes $H \to S$.

Proposition 25 ([dS12b], Proposition 14)**.** *Let $\pi : H \to S$ be a projective morphism. Assume that H comes with a stratification relative to S. Then, if the group scheme $\text{Aut}_k(H_{s_0})$ is affine, $H \to S$ is affinely controlled.*

Given this last result, we are then able to show that the simple criterion alluded to in Lemma 22 can be applied to show that $\text{Ker}\, f_\#$ is the image of $\Pi(X_{s_0}, x_0)$, hence proving Theorem 21.

Chapter 3

$\mathcal{D}$-modules in Mixed Characteristic

The most basic question which affronts those interested in differential Galois theory when arriving in the setting of mixed characteristic is the relation between the two possible differential Galois groups in sight: one in characteristic zero and the other in characteristic p. Since in positive characteristic one is bound to run into algebraic-geometric objects with very few physical points – non-reduced objects – the Tannakian point of view imposes itself. In [dS08], [dS09] and [dS11] we wrote down our investigations concerning the above problematic. They are strongly influenced by the work of Matzat and van der Put [MvdP] and represent our effort in building bridges between algebraic geometer and those working in differential Galois theory.

It should be said that this is probably the hardest part of this monograph to be reviewed and explained due to the infancy of differential Galois theory of mixed characteristic. The author of the present lines believes that a more serious review of literature needs to be made, and foundational material on Tannakian categories over DVRs needs to be carried out. (A glance at [DH13] should make clear that this sort of work needs to be done.) As this is far from our intention in writing the present lines, we leave this work to a future opportunity [DHdS].

3.1 The reduction of the differential Galois group

Let $\mathfrak{o}$ be a complete DVR with residue field k of characteristic $p > 0$, quotient K field of characteristic 0, and uniformizer ϖ. Let us assume furthermore that k is algebraically closed. We fix a flat, topologically of finite type and adic $\mathfrak{o}$-algebra A; by ξ we understand an $\mathfrak{o}$-point of $\operatorname{Spf} A$. Furthermore, we assume that A has etale coordinates $x_1, \ldots, x_d$ over $\mathfrak{o}$. Write $A_k := A \otimes k$ and $A_K := A \otimes K$.

Let $\mathcal{D}$ be the ring of all continuous $\mathfrak{o}$-linear differential operators on A: it is obtained just as in [EGA $\textsc{iv}_4$, §16] once we replace the tensor product $A \otimes_{\mathfrak{o}} A$ by $A \widehat{\otimes}_{\mathfrak{o}} A$ when constructing the ring of principal parts. Associated to it are the rings $\mathcal{D}_k$ and $\mathcal{D}_K$.

By an abuse of notation justified in Section 1.1, we let $\mathbf{str}(A/\mathfrak{o})$ stand for the category of left $\mathcal{D}$-modules which are of finite type over A. We write $\mathbf{str}^{\#}(A/\mathfrak{o})$ to denote those which are flat A-modules and employ analogous notations for $\mathcal{D}_k$- and $\mathcal{D}_K$-modules. Note that there are obvious functors from $\mathbf{str}(A/\mathfrak{o})$ to $\mathbf{str}(A_k/k)$ and to $\mathbf{str}(A_K/K)$ which we denote respectively by $(-)_k$ and $(-)_K$.

We note that $\mathbf{str}(A_k/k)$ has already been given the structure of neutral Tannakian category in Section 1.3.1. Similarly, $\mathbf{str}(A_K/K)$ can be given such a structure. It then follows that for each $M \in \mathbf{str}(A_k/k)$, respectively $\mathcal{M} \in \mathbf{str}(A_K/K)$, the category

$$\langle M\rangle_\otimes = \left\{ N \in \mathbf{str}(A_k/k) \ : \ \begin{array}{c} \text{there exists some generalized tensor} \\ T = \bigoplus (M^\vee)^{\otimes \nu_j} \otimes M^{\otimes \mu_j}, \text{ a subobject } M' \subset T \\ \text{and an epimorphism } M' \to N \text{ of } \mathcal{D}_k\text{-modules} \end{array} \right\},$$

respectively

$$\langle \mathcal{M}\rangle_\otimes = \left\{ \mathcal{N} \in \mathbf{str}(A_K/K) \ : \ \begin{array}{c} \text{there exists some generalized tensor} \\ \mathcal{T} = \bigoplus (\mathcal{M}^\vee)^{\otimes \nu_j} \otimes \mathcal{M}^{\otimes \mu_j}, \text{ a subobject } \mathcal{M}' \subset \mathcal{T} \\ \text{and an epimorphism } \mathcal{M}' \to \mathcal{N} \text{ of } \mathcal{D}_K\text{-modules} \end{array} \right\},$$

is equivalent to the category of representations of an *affine algebraic* group scheme over k, respectively over K. These groups are the *differential Galois groups*; they will be called to mind by means of the "function" $\mathrm{Gal}(-)$.

In [MvdP, §8], the question of studying differential Galois groups over $\mathfrak{o}$ is put forward. The authors make a conjecture, Conjecture 8.5, which, experience tells, contains more than meets the eye.

Let $M \in \mathbf{str}(A_K/K)$; Matzat and van der Put inquire wether it is possible to find a $\mathcal{D}$-module $\mathcal{M}$ "equivalent" to M having the ensuing properties: (**C1**) The differential Galois group $\mathrm{Gal}(M)$ is the generic fibre of $\mathrm{Gal}(\mathcal{M})$ and $\mathrm{Gal}(\mathcal{M}_k)$ can be identified to a closed subgroup of $\mathrm{Gal}(\mathcal{M}) \otimes k$; (**C2**) If $\mathrm{Gal}(M)$ is finite, then the previous identification $\mathrm{Gal}(\mathcal{M}_k) \subset \mathrm{Gal}(\mathcal{M}) \otimes k$ is an isomorphism. It turns out that the adjective "equivalent" employed before cannot be taken to equal "isomorphic", as the examples in [MvdP, 8.6] suggest and the second paragraph on p.97 of [dS09] explains.

In [dS09] we chose to start from $\mathcal{M} \in \mathbf{str}^\#(A/\mathfrak{o})$, not some $M \in \mathbf{str}(A_K/K)$, and investigate if it is possible to find a "Tannakianly" defined differential Galois group $\mathrm{Gal}(\mathcal{M})$ enjoying (**C1**) and (**C2**). We were then led to the following.

Theorem 26. *There exists a flat and affine group scheme over* $\mathfrak{o}$, $\Pi = \Pi(\mathcal{M}, \xi)$, *enjoying the following properties.*

i. The category $\mathrm{Rep}^\#_{\mathfrak{o}}(\Pi)$ *of representations of* Π *on free* $\mathfrak{o}$*-modules of finite rank is monoidally equivalent to a full subcategory* $\langle \mathcal{M}\rangle^s_\otimes$ *(see below) of* $\mathbf{str}^\#(A/\mathfrak{o})$.

ii. The group Π *is of finite type over* $\mathfrak{o}$,

iii. The generic fibre of Π *is* $\mathrm{Gal}(\mathcal{M}_K)$ *and the affine group scheme* $\mathrm{Gal}(\mathcal{M}_k)$ *is a closed subgroup of* $\Pi \otimes k$.

iv. If Π *is finite, then* $(\Pi \otimes k)_{\mathrm{red}} \simeq \mathrm{Gal}(\mathcal{M}_k)$.

In connection with this theorem, we recall that

$$\mathrm{Obj}\,\langle \mathcal{M}\rangle^s_\otimes = \left\{ \mathcal{N} \in \mathbf{str}^\#(A/\mathfrak{o}) \ : \ \begin{array}{l} \text{there exists some generalized tensor} \\ \mathcal{T} = \bigoplus (\mathcal{M}^\vee)^{\otimes \nu_j} \otimes \mathcal{M}^{\otimes \mu_j}, \\ \text{a subobject } \mathcal{M}' \subset \mathcal{T} \text{ with } \mathcal{T}/\mathcal{M}' \text{ torsion free} \\ \text{and an epimorphism } \mathcal{M}' \to \mathcal{N} \end{array} \right\}. \quad (3.1)$$

In other words, we do not consider all sub-objects of tensor powers of $\mathcal{M}$ to construct the group Π, but only those which are "subbundles."

A few words concerning the proper place of the above result are in order. After discovering Theorem 26, Y. André informed us that an economic candidate for Π is the *schematic closure* of $\mathrm{Gal}(\mathcal{M}_K)$ inside $\mathbf{GL}(\xi^*\mathcal{M})$: this would give property (iii) of the theorem directly [K90, §2.4]. Also, we learned that Saavedra [S72] had proposed an effective theory of Tannakian reconstruction for monoidal categories over general noetherian rings. It is this theory which is employed in [A01, 3.2] to construct differential Galois groups over Dedekind domains. (To arrive at Theorem 26, we leaned on a theory due to M. Nori which was written up by A. Bruguières [Bru]. In [Bru] there is no mention to Saavedra's work on this specific topic and it seems that the task of harmonizing things will be done by P. H. Hai.) However, the differential Galois groups constructed by [A01] are not necessarily of finite type over $\mathfrak{o}$ [A01, 3.2.1.5], and what we propose in Theorem 26 is to take the image of André's differential Galois group inside the corresponding linear algebraic group and give it a Tannakian interpretation. With hindsight, in [dS09] we examine under which conditions on Tannakian categories we have closed embeddings of associated group schemes, cf. Definition 10 of [dS09] and [DH13, §3] for a complete treatment. In this way we obtain *algebraic* differential Galois groups together with Tannakian information. Moreover, one should understand to what extent the knowledge of $\Pi(\mathcal{M})$ helps in determining the "full" differential Galois group of [A01, 3.2] and its reduction (we have reserves about the veracity of [A01, 3.3.1.1]). This is work in progress [DHdS].

We now wish to review our contribution to the conjecture in [MvdP] which was written down in [dS08]. For a given $M \in \mathbf{str}(A_K/K)$ with *finite differential Galois group*, we wish to find some $\mathcal{N} \in \mathbf{str}^{\#}(A/\mathfrak{o})$ such that $\mathcal{N}_K$ is *equivalent* to M and such that $\mathrm{Gal}(\mathcal{N}_k) \simeq \Pi(\mathcal{N}) \otimes k$. As said above, the subtlety here is to define the term "equivalent" properly. In the work [dS08] we *started* from a certain $\mathcal{M} \in \mathbf{str}^{\#}(A/\mathfrak{o})$ such that $\mathrm{Gal}(\mathcal{M}_K)$ is finite: that is, we avoided the question of existence of a certain integral model. We then obtained, by desingularizing the canonical universal principal $\Pi(\mathcal{M})$-torsor above $\mathrm{Spec}\, A$,

$$\mathrm{Spec}\, B_{\mathcal{M}} \longrightarrow \mathrm{Spec}\, A,$$

a certain $\mathcal{D}$-module $\mathcal{N}$ with pleasant properties.

The reader is informed that in what follows we have omitted some technical conditions which the ring A has to enjoy for the proofs to work. (See [dS08, Section 2.1].)

Theorem 27 ([dS08], Theorem 5). *Let* $\mathcal{N}$ *be the algebra of the normalisation of the* $\Pi(\mathcal{M})$*-torsor* $B_{\mathcal{M}}$. *Then* $\mathcal{N}$ *has a canonical structure of a* $\mathcal{D}$*-module. Its differential Galois group* $\Pi(\mathcal{N})$ *is etale over* $\mathfrak{o}$, *and satisfies*

$$\Pi(\mathcal{N}) \otimes k \simeq \mathrm{Gal}(\mathcal{N}_k)$$

Moreover, the categories $\langle \mathcal{N}_K \rangle_{\otimes}$ *and* $\langle \mathcal{M}_K \rangle_{\otimes}$ *coincide.*

In other words, even though $B_{\mathcal{M}}$ does not need to be etale over A, its normalisation is. This remarkable property is a simple consequence of the fact that $B_{\mathcal{M}}$ is a subring of a ring of power series as is explained by the following.

Lemma 28 ([dS08], Lemma 6). *Let* $\mathfrak{p}_\xi$ *be the prime ideal of the point* $\xi \in \mathrm{Spec}\, A$ *and let* $\widehat{A}$ *stand for its* $\mathfrak{p}_\xi$*-adic completion. Let* B *be a finite flat extension of* A *contained in* $\widehat{A}$ *and assume that*

i) B_K/A_K *is etale and*

ii) after a finite extension K'/K, $B_{K'}$ is Galois over $A_{K'}$.

Then the normalisation of B is an etale extension of A.

Instead of explaining the proof, we prefer to reproduce an example from [dS08] whose abstraction is the whole purpose of the latter text.

Example 29. Let $p = 3$, $\varpi = \sqrt{-3}$, $\mathfrak{o} = W(\overline{\mathbb{F}}_p)[\varpi]$, $A = \mathfrak{o}\langle x\rangle$ and $\mathfrak{p}_\xi = (x)$. It is an elementary exercise to note that the power series $e = \exp(3\varpi x)$ belongs to A. Let $B = A[Y]/(Y^3 - e)$; we will see in the following lines that B is not normal and that its normalization C is etale over A.

The equation $Y^3 - e$ has an evident solution in $\widehat{A} = \mathfrak{o}[\![x]\!]$, call it y, so that we get an injective homomorphism $\sigma : B \to \widehat{A}$. Note that $y = 1 + \varpi z$, for some $z \in \widehat{A}$. Using the equation $y^3 - e = 0$, we obtain

$$F(z) = z^3 - \varpi z^2 - z + \frac{e-1}{3\varpi} = 0,$$

which implies that z is integral over A. Note that the A-algebra $C = A[Z]/(F)$ is etale because the special fibre is the coordinate ring of an Artin-Schreier covering of $\mathbb{A}^1_k$. In particular C is the normalization of B.

It is natural to enquire if it is possible to achieve such a simple desingularization result without the finiteness hypothesis.

3.2 Is it possible to lift $\mathcal{D}$-modules without loosing too much information on Galois groups?

In section 3.1 we reviewed our contribution to the theme of "reduction of the differential Galois group." This circle of ideas promptly induces the modern algebraic geometer to investigate liftings: how can one pass from positive to zero characteristic? This is the object of [dS11], which draws inspiration from the work of Matzat, van der Put, Berthelot, and Mazur (see [MvdP], [Mt06], [Be00], [Be12], [Maz89]).

We maintain the notations and conventions of section 3.1. Let M be a $\mathcal{D}_k$-module which is of finite type as an A_k-module, i.e. $M \in \mathbf{str}(A_k/k)$. The first and most obvious question one can ask once attention has been driven in the above direction is : *Are there any $\mathcal{D}$-modules which are flat over A and which reduce modulo ϖ to M?* In [MvdP], Theorem 8.4, Matzat and van der Put gave a direct and simple answer to this question. The idea is to forget about the the ring $\mathcal{D}_k$ and to think of elements in $\mathbf{str}(A_k/k)$ by means of their Frobenius division and then look for an analogue of this interpretation in characteristic zero to arrive at an object of $\mathbf{str}^\#(A/\mathfrak{o})$. (The method is not conspicuous in [MvdP], but it is in [Mt06].) The aforementioned analogue can be built up on the foundations laid by the Frobenius descent theory of [Be00] and is the subject of [Be12]. Let us delve into the details.

Let us fix an isomorphism of rings $\sigma : \mathfrak{o} \to \mathfrak{o}$ satisfying $\sigma a \equiv a^p \mod \varpi$. (That such a σ exists in all common cases follows from the theory of Witt-vectors.) Since A is formally smooth over $\mathfrak{o}$, there exists a σ-linear homomorphism of rings $F : A \to A$ lifting the absolute Frobenius morphism $A_k \to A_k$, i.e.

$$Fa \equiv a^p \mod \varpi.$$

To render the situation more standardized, we introduce the $\mathfrak{o}$-algebra $A^{(m)}$; as a ring it is simply A, but the structure morphism from $\mathfrak{o}$ is defined by $\mathfrak{o} \stackrel{\sigma^{-m}}{\to} \mathfrak{o} \to A$. In this way, $F^m : A^{(m+s)} \to A^{(s)}$ is $\mathfrak{o}$-linear.

Definition 30. The category of F-divided modules over A, denoted $\mathbf{Fdiv}(A/\mathfrak{o})$, has

as objects: families $\{\mathcal{M}_i, \alpha_i\}_{i\in\mathbb{N}}$, where $\mathcal{M}_i$ a finitely generated $A^{(i)}$-module and $\alpha_i : F^*\mathcal{M}_{i+1} \to \mathcal{M}_i$ is an isomorphism of $A^{(i)}$-modules;

as arrows: families of morphisms of modules $\{\psi_i\}_{i\in\mathbb{N}}$ which are compatible with the given isomorphisms.

Then, in simplified form, Berthelot's main result in [Be12], Theorem 2.6, reads:

Theorem 31. *There exists an equivalence of categories between* $\mathbf{str}(A/\mathfrak{o})$ *and* $\mathbf{Fdiv}(A/\mathfrak{o})$. *This equivalence preserves tensor products and is "compatible" with Katz's equivalence outlined in Section 2.1.*

Therefore, if $\{M_i, \alpha_i\}$ is the F-division associated to the $\mathcal{D}_k$-module M over A_k, all we need to do is to lift each M_i to a flat $A^{(i)}$-module, and then lift the isomorphisms α_i. This gives us a $\mathcal{D}$-module, which is flat over A, and which reduces modulo ϖ to M.

But if the answer to the this first question reveals a simple structure, it pays no attention to the *complexity of the lifted* $\mathcal{D}$*-module*. If one starts with an uncomplicated $\mathcal{D}_k$-module M, there is no a priori way to see this facility in the lifted $\mathcal{M} \in \mathbf{str}^{\#}(A/\mathfrak{o})$. This is where [dS11] finds its utility by noting that it is not M that needs to be lifted, but the universal torsor.

Let $G_{\mathfrak{o}}$ be an affine smooth group scheme over $\mathfrak{o}$ whose reduction modulo ϖ, call it G_k, is the differential Galois group the $\mathcal{D}_k$-module M. Let

$$\operatorname{Spec} B_M \longrightarrow \operatorname{Spec} A_k$$

be the universal G_k-torsor associated to M and let

$$\operatorname{Spec} \mathcal{B} \longrightarrow \operatorname{Spec} A$$

be a lifting to a $G_{\mathfrak{o}}$-torsor. Moreover, we assume that the $\mathcal{D}_k$-module structure of B_M has also been lifted to a $\mathcal{D}$-module structure on $\mathcal{B}$. Such a maneuver is possible because of Berthelot's above mentioned theorem and the smoothness of $G_{\mathfrak{o}}$, see [dS11, §4.2] for details. The problem of lifting M to a $\mathcal{D}$-module in characteristic zero is then transformed into the problem of lifting the representation of G_k associated to M to a representation of $G_{\mathfrak{o}}$ which is flat as a $\mathfrak{o}$-module. Although such a lifting is not always possible, see [J87, Part II, 2.16] to concoct a counterexample, its existence is by no means an exception. It happens, for example, if $G_{\mathfrak{o}}$ is split reductive and the second cohomology group of G_k with values in the adjoint representation of $G_k \to \mathbf{GL}(\xi_k^* M)$ vanishes [dS11, §5.3]. Other interesting cases can be extracted from [J87, Part II, 5.5, 5.6].

Theorem 32. *We maintain the notations and conventions introduced above. Furthermore, we assume that each connected component of* $G_{\mathfrak{o}}$ *dominates* $\operatorname{Spec} \mathfrak{o}$.

If the representation $G_k \to \mathbf{GL}(\xi_k^* M)$ *associated to* M *possesses a lifting to a representation* $G_{\mathfrak{o}} \to \mathbf{GL}_{r,\mathfrak{o}}$, *then* M *can be lifted to* $\mathcal{M} \in \mathbf{str}^{\#}(A/\mathfrak{o})$ *which satisfies* $\Pi(\mathcal{M}, \xi) \simeq G_{\mathfrak{o}}$. ☐

Remark 33. It can be worth studying examples of liftings obtained from Theorem 31 in the cases where the lifting of $G_k \to \mathbf{GL}(\xi_k^* M)$ does not exist.

Having obtained information on the existence of liftings of a $\mathcal{D}_k$-module with reasonably controlled differential Galois groups, an ensuing question which arises naturally, and corresponds to the second part of [dS11], revolves around the *number* of such possible liftings. At this point, the problem is treated from the point of view of Schlessinger's formal deformation theory.

Let $\mathcal{M}$ be a flat A-module of finite type and assume that its reduction modulo ϖ is endowed with a $\mathcal{D}_k$-module structure $\nabla_k : \mathcal{D}_k \to \mathrm{End}_k(\mathcal{M}_k)$. What are the possible $\mathcal{D}$-module structures on $\mathcal{M}$? Or, less ambitiously, what are the possible $\mathcal{D}_\Lambda$-module structures on $\mathcal{M}_\Lambda$, where Λ is some local Artin $\mathfrak{o}$-algebra with residue field k. We are now in the format of Schlessinger's theory, and the study of the functor

$$\mathrm{Def}^+_{\nabla_k} : \mathcal{C} \longrightarrow \mathbf{Set}, \qquad \Lambda \longmapsto \left\{ \begin{array}{l} \mathcal{D}_\Lambda\text{-module structures on } \mathcal{M}_\Lambda \\ \text{which induce } \nabla_k \text{ upon reduction} \end{array} \right\}$$

imposes. (Here $\mathcal{C}$ is the category of Artin algebras possessing the aforementioned properties.) In truth, quite differently from what happens to Mazur's theory of deformation of Galois representations [Maz89], the functor $\mathrm{Def}^+_{\nabla_k}$ carries too much information, and no direct study seems possible. (This is due to the fact that $\mathrm{Ext}^1_{\mathcal{D}_k}(\mathcal{M}_k, \mathcal{M}_k)$ can be enormous.) The following definition then made investigations accessible.

Definition 34 ([dS11], Definition 30)**.** Let $A \to \mathcal{B}$ be a faithfully flat morphism of rings and assume that $\mathcal{B}$ is an algebra in the monoidal category $\mathrm{Ind}\,\mathbf{str}^{\#}(A/\mathfrak{o})$. Call a deformation $(\mathcal{M}_\Lambda, \nabla_\Lambda)$ $\mathcal{B}$-periodic if there exists an isomorphism of $\mathcal{D}_\Lambda$- and $\mathcal{B}_\Lambda$-modules

$$\mathcal{B}_\Lambda \otimes \mathcal{M}_\Lambda \simeq \mathcal{B}_\Lambda^{\oplus r}.$$

We then have another "deformation problem":

$$\mathrm{Def}^+_{\mathcal{B}} : \mathcal{C} \longrightarrow \mathbf{Set}, \qquad \Lambda \longmapsto \left\{ \begin{array}{l} \mathcal{B}\text{-periodic } \mathcal{D}_\Lambda\text{-module structures on } \mathcal{M}_\Lambda \\ \text{which induce } \nabla_k \text{ upon reduction} \end{array} \right\}.$$

This "deformation problem" has an interesting solution now.

Theorem 35 ([dS11], Corollaries 34 and 35)**.** *Assume that* $\mathrm{End}_{\mathcal{D}_k}(\mathcal{M}_k) = k$ *and that* $\mathcal{B}_k$ *is the (algebra of the) universal torsor for the differential Galois group* G_k *of* $\mathcal{M}_k$*. Moreover, we suppose that*

$$\dim_k \mathrm{Ext}^1_{G_k}(\rho_k, \rho_k) < \infty,$$

where ρ_k *is the representation of* G_k *attached to* $\mathcal{M}_k$*.*

Then, the functor obtained from $\mathrm{Def}^+_{\mathcal{B}}$ *by identifying isomorphic deformations is pro-representable by a noetherian complete local* $\mathfrak{o}$*-algebra whose relative Zariski tangent space has the dimension of the* k*-vector-space* $\mathrm{Ext}^1_{G_k}(\rho_k, \rho_k)$*.*

In less precise terms, once some extension space is finite dimensional, we obtain a *finite dimensional* "moduli space" for the problem of periodic deformations. Although not quite explicit in the literature, spaces of extensions of the kind considered in Theorem 35 are often finite dimensional, even in positive characteristic, as soon as unipotent groups are set aside. In fact, $\mathrm{Ext}^1_G(V, V)$ is always finite dimensional if G is reductive so that the hypothesis in Theorem 35 are general enough.

In addition, it turns out that the moduli space in question is a manifestation of something purely group-theoretic.

Theorem 36 ([dS11], Theorem 38)**.** *Assume that* $\mathrm{End}(\mathcal{M}_k) = k$*, that the differential Galois group* G_k *of* $\mathcal{M}_k$ *lifts to a flat group scheme* $G_{\mathfrak{o}}$ *over* $\mathfrak{o}$*, and that* $\mathrm{Spec}\,\mathcal{B}$ *is a torsor under* $G_{\mathfrak{o}}$ *lifting the* G_k*-torsor* $\mathrm{Spec}\,\mathcal{B}_k$*. Then the moduli space alluded to in Theorem 35 is isomorphic to the moduli space of isomorphism classes of deformations of the representation* ρ_k *of* G_k *attached to* $\mathcal{M}_k$*.*

Chapter 4

Nori's theory of essentially finite vector bundles

Let k be an algebraically closed field and X a proper and reduced scheme over k. Nori detects that a full subcategory of $\mathbf{VB}(X)$ can be used to produce, via the Tannakian correspondence, an affine group scheme over k which classifies all principal homogeneous spaces with finite structural group, see Section 1.3.2.

This theory, after a period of inactivity of almost 30 years, became the centre of much attention starting in the beginning of the new millennium. It did not only serve as a concrete theme of mathematical research, but also as a vivid source of methods and behaviors to follow.

Below we review our contributions to this theory which have appeared in print in [BdS11], [BdS12], [dS12a].

4.1 Characterizing essential finiteness differently

Let X be a projective variety over the algebraically closed field k. Recall from Section 1.3.2 that Nori defined a full subcategory $\mathbf{EF}(X)$ of $\mathbf{VB}(X)$, called the essentially finite bundles (cf. Definition 16), such that, for a given $x_0 \in X(k)$, the triple $(\mathbf{EF}(X), \otimes_{\mathcal{O}_X}, x_0^*)$ is a neutral Tannakian category. Moreover, the associated Tannakian fundamental group scheme, henceforth denoted by $\Pi^{\mathrm{EF}}(X, x_0)$, is pro-finite, cf. Theorem 17.

Our contribution to the *general* study of $\mathbf{EF}(X)$, written down in [BdS11] and [BdS12], was to characterize the essentially finite vector bundles in terms of trivializations. This is:

Theorem 37 ([BdS11])**.** *Assume that X is smooth. Then a vector bundle E is essentially finite if and only if there exists a proper surjective morphism $f : X' \to X$ such that f^*E is trivial.*

We then obtain a less technical definition of the essentially finite vector bundles. This characterization is also appealing in the sense of descent theory (to be understood as a generalization of topology), since in topology one can always imagine representations of the fundamental group as being vector bundles trivialized by the universal cover. This can be made more precise via:

Corollary 38. *A finite morphism $Y \to X$ is a Nori covering (see Definition 10) if and only if there exists a proper and surjective morphism $f : X' \to X$ trivializing Y.*

Proof. We show only how to prove the 'if' clause. In this case, we know from Theorem 37 that the coherent $\mathcal{O}_X$-algebra $\mathcal{O}_Y$ is an essentially finite vector bundle, which means that it is an algebra in the category $\mathbf{EF}(X)$. Such an algebra corresponds to a finite scheme L with a left action of $\Pi^{\mathrm{EF}}(X, x_0)$; obviously there exists a finite quotient G of $\Pi^{\mathrm{EF}}(X, x_0)$ and an action of G on the left of L "giving back" the action of $\Pi^{\mathrm{EF}}(X, x_0)$. From G we obtain a principal G-bundle $P \to X$ such that $P \times^G L \simeq Y$. □

The proof of Theorem 37 relies on A. Langer's results on the S-fundamental group scheme and the version of the Lefschetz theorem on hyperplane sections in this context [L11]. The latter theorem allows to restrict attention to the case $\dim X = 1$, where the verification yields easily. It should be noted that with a simpler proof and under the assumption that X is only normal and projective, [AM11] have obtained Theorem 37.

It is perhaps useful to observe that the idea to put forward such a simple characterization as in Theorem 37 comes from our previous work on $\mathcal{D}_X$-modules over rationally connected varieties [BdS13]. Let us assume, in this paragraph, that X is furthermore separably rationally connected. In [BdS13], whose main result is surpassed by [EM10], we observed that an $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module must be trivial, as its restriction to any rational curve is trivial. By means of a universal scheme parametrizing rational curves, triviality on each $\mathbb{P}^1$ translates into triviality after some proper and surjective pull-back, hence the idea to the statement of Theorem 37. Moreover, as one sees from the this application of Theorem 37, it is important to allow the morphism $X' \to X$ in its statement to be *as general as possible*. (It is because of this that the above result is not contained in [BP11], as the introduction of [AM11] says.)

In [BdS12] we gathered some remarks issuing from [BdS11] and placed it in perspective. The theme is "the number" of vector bundles trivialized by *a particular finite morphism* $X' \to X$ or, in other words, the category

$$\mathbf{EF}(X'/X) = \{\text{vector bundles in } \mathbf{VB}(X) \text{ which become trivial on } X'\}.$$

Proposition 39 ([BdS12], Theorem 8)**.** *If the extension of function fields induced by $X' \to X$ is separable, then $\mathbf{EF}(X'/X)$ is equivalent to the category of representations of a finite etale group scheme.*

As we will explain in Section 4.2 below, the work of Mehta and Subramanian shows that, if $X' \to X$ is *inseparable*, the number of stable vector bundles in $\mathbf{EF}(X'/X)$ is in close connection with the problem of "base-change" for Π^{EF}. For the time being, we simply state the following result whose proof is to be found on Section 3.4.1 of [BdS12]. We chose to reproduce it here because it imprints yet another strong contrast between the theory of the etale fundamental group scheme and that of Π^{EF}.

Proposition 40. *Let $X' \to X$ be a finite and surjective morphism from a variety X'. Consider the following claims.*

i. The affine group scheme $\Pi^{\mathrm{EF}}(X'/X, x_0)$ associated to the category $\mathbf{EF}(X'/X)$ is not finite.

ii. There are infinitely many isomorphism classes of stable vector bundles in $\mathbf{EF}(X'/X)$.

iii. The universal torsor for $\Pi^{\mathrm{EF}}(X, x_0)$ is not reduced as a scheme.

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii), and the implication (iii) $\Rightarrow$ (i) does not need to hold.

This result explains that the theory of fields should be of little use in studying the non-etale part of Π^{EF}. The mechanism behind this proposition is quite simple: Since the universal principal $\Pi^{\text{EF}}(X'/X,x_0)$-bundle $P\to X$ factors the arrow $X'\to X$, we conclude that if P is reduced, then the rank of $\Pi^{\text{EF}}(X'/X,x_0)$ is bounded by the degree of $X'\to X$ (since $P\to X$ is pro-finite).

4.2 Estimating computationally the finiteness of F-trivial vector bundles

On page 89 of [N82], Nori conjectures that the group scheme $\Pi^{\text{EF}}(X)$ "base changes correctly." The precise meaning of this condition is that for *any* extension of algebraically closed fields $k\subset k'$, and any normal proper variety X/k, the natural morphism

$$\Pi^{\text{EF}}(X\otimes_k k', x_0\otimes_k k')\longrightarrow \Pi^{\text{EF}}(X,x_0)\otimes_k k' \tag{BC}$$

is an isomorphism. Due to [SGA1, X], this question needs only to be addressed if k is of positive characteristic $p>0$. From here on we assume that $k\subset k'$ is an extension of algebraically closed fields of characteristic $p>0$.

It was in [MS02] that Mehta and Subramanian reinterpreted what it means for (BC) to be an isomorphism and obtained the first counterexample to the above conjecture. They observed that if (BC) was to be an isomorphism, then the *stable* F-trivial vector bundles (see below) on $X\otimes_k k'$ would all come from X [MS02, Proposition 3.1, p.148]. The counterexample then explored the well-known fact that if X is a *singular cuspidal curve*, then the k-group scheme $\text{Pic}^0(X)$ contains a copy of $\mathbb{G}_{a,k}$. The simple nature of the construction suggests that more important than impeding the veracity of Nori's conjecture, the work [MS02] allowed the understanding of the mechanism which would force (BC) to be an isomorphism. This was made even clearer on [MS08], where the authors slightly change the theme to the study of the *local fundamental group scheme* $\Pi^{\text{loc}}(X,x_0)$. This group scheme is obtained, via Tannakian duality, by means of the full subcategory

$$\mathbf{FT}(X)=\left\{E\in\mathbf{VB}(X)\ :\ \begin{array}{l}F^{t*}E\text{ is isomorphic to a trivial}\\ \text{vector bundle for some } t\in\mathbb{N}\end{array}\right\}$$

of $\mathbf{EF}(X)$. Although some discrepancy can be found in the literature, the above displayed vector bundles are called *Frobenius-trivial* or *F-trivial*. The main achievement of [MS08] reads.

Theorem 41 (Mehta-Subramanian). *Write $S(X,r,t)$ for the set of isomorphism classes of stable vector bundles on X of rank r which are trivialized by F^t. Then the natural base-change morphism*

$$\Pi^{\text{loc}}(X\otimes_k k')\longrightarrow \Pi^{\text{loc}}(X)\otimes_k k'$$

is an isomorphism for every algebraically closed extension k' of k if and only if $S(X,r,t)$ is finite for all $r,t\geq 0$.

Note that

$$\Pi^{\text{loc}}(X,x_0)=\varprojlim_t \Pi^{\text{loc},t}(X,x_0),$$

where each $\Pi^{\text{loc},t}$ is an affine group scheme obtained from the neutral Tannakian category of vector bundles trivialized by F^t and $S(X,r,t)$ is none other than the set of isomorphism classes of *simple* $\Pi^{\mathrm{loc},t}(X,x_0)$*-modules.* To substantiate the relevance of the finiteness condition on the set $S(X,r,t)$, we include the ensuing digression, which simply mimics the ideas of these two authors.

Let $K \subset K'$ be an extension of algebraically closed fields *of arbitrary characteristic* and Γ be a finitely generated abstract group. Denote by $(\Gamma/K)^{\mathrm{alg}}$ the algebraic hull of Γ over K (see Example 14), and by $S(\Gamma,K,r)$ the set of isomorphism classes of *simple* linear representations of Γ on $K^{\oplus r}$.

Proposition 42 (compare [H93]). *The natural morphism of K'-group schemes*

$$(\Gamma/K')^{\mathrm{alg}} \longrightarrow (\Gamma/K)^{\mathrm{alg}} \otimes_K K'$$

is an isomorphism for every algebraically closed extension K' of K if and only if $S(\Gamma,K,r)$ is finite for all r.

Proof. In the sequel we will use that $(\Gamma/K')^{\mathrm{alg}} \to (\Gamma/K)^{\mathrm{alg}} \otimes_K K'$ is an isomorphism if and only if every $\varrho' \in \mathrm{Rep}_{K'}(\Gamma)$ is a sub-quotient of some $\varrho \otimes K'$ with $\varrho \in \mathrm{Rep}_K(\Gamma)$. We will also employ the following results from geometric representation theory. Let $\mathfrak{M} = \operatorname{Spec} R$ be the representation scheme of the group algebra $K\Gamma$ as defined in [Mo80, §1]. It is naturally endowed with an action of $\mathbf{GL}_r$ and, according to Artin, see [A69, 12.6, p.559] or [Mo80, Theorem 2.5,p.203], an orbit is closed if and only if it corresponds to a semi-simple representation. Now geometric invariant theory tells us that the ring of invariants $R^{\mathbf{GL}_r}$ is finitely generated and that the fibres of the quotient

$$\pi : \mathfrak{M} \longrightarrow \mathfrak{N} := \operatorname{Spec} R^{\mathbf{GL}_r}$$

are in bijection with the closed orbits of $\mathbf{GL}_r$. (The statement concerning closed orbits, though well-known, is never explicitly mentioned in the common literature. Its proof can be extracted from [Ne78, Theorem 3.5,p.61]. Beware that we are not saying that each fibre is a closed orbit!)

Assume that for *each* r, the set $S(\Gamma,K,r)$ is finite. This implies that there are only finitely many isomorphisms classes of semi-simple representations $\Gamma \to \mathrm{GL}_r(K)$. Therefore, we conclude that the scheme $\mathfrak{N}$ has only finitely many K-points. Hence $\mathfrak{N}$ is a finite affine K-scheme, and consequently the obvious map $\mathfrak{N}(K) \to \mathfrak{N}(K')$ is a bijection. Let now $\varrho' \in \mathrm{Rep}_{K'}(\Gamma)$ be semi-simple of rank r regarded as a K'-point of the scheme $\mathfrak{M}\otimes K'$. As the obvious homomorphism

$$R^{\mathbf{GL}_r} \otimes K' \longrightarrow \big(R \otimes K'\big)^{\mathbf{GL}_{r,K'}},$$

is bijective, we conclude that the image of ϱ' in $\operatorname{Spec}(R \otimes K')^{\mathbf{GL}_{r,K'}}$ is induced by a K'-point of $\mathfrak{N}$. It then must come from a K-point on $R^{\mathbf{GL}_r}$. This entails that ϱ' is isomorphic to $\varrho \otimes K'$ for some $\varrho \in \mathrm{Rep}_K(\Gamma)$.

Assume now that $(\Gamma/K')^{\mathrm{alg}} \to (\Gamma/K)^{\mathrm{alg}} \otimes_K K'$ is an isomorphism, and let $\varrho' \in \mathrm{Rep}_{K'}(\Gamma)$ be simple. By what was said above, we can find $\varphi \in \mathrm{Rep}_K(\Gamma)$ such that ϱ' is a quotient of $\varphi \otimes K'$. This means that ϱ' comes as a factor of the Jordan-Hölder filtration of $\varphi \otimes K'$. Now the factors of the Jordan-Hölder filtration of a representation $\varphi \otimes K'$ are all of the form $\varrho \otimes K'$, since $- \otimes_K K'$ sends simple representations to simple representations. (This follows from Burnside's theorem [LA, XVII, Cor. 3.4, p.648]). In particular, all semi-simple objects of $\mathrm{Rep}_{K'}(\Gamma)$ come from $\mathrm{Rep}_K(\Gamma)$. Using the above introduced notations, we conclude that all K'-points of $\mathfrak{N} \otimes K'$ come from K-points of it, which means that $\mathfrak{N}$ is finite over K. From this, we see that there are only finitely closed orbits in $\mathfrak{M}$. Consequently, there are only finitely many semi-simple representations of Γ up to isomorphism. □

We return to the theory of vector bundles. Once the content of Theorem 41 was understood, C. Pauly used his knowledge of vector bundles in characteristic two to produce:

Theorem 43 (Pauly [P07])**.** *Assume that $p = 2$. There exist a family of projective and smooth curves over k such that the set of isomorphism classes of* stable *vector bundles of rank two which are trivialized by F^4 is infinite. These curves all have genus two.*

It is in this setting that our contribution to the nature of (BC) comes in. In [dS12a], we proceeded to the following construction, which we present in a slightly greater generality than in loc. cit. with a view to eventual applications for *families of curves*.

Let $f : X \to S$ be smooth, projective, and geometrically connected curve over an affine k-scheme $S = \operatorname{Spec} A$. Let **CT** be the category whose objects are integrable A-connections

$$\nabla : \mathcal{O}_X^n \longrightarrow \mathcal{O}_X^n \otimes \Omega^1_{X/S}$$

with vanishing p-curvature and whose arrows are just horizontal morphisms. When $A = k$, Cartier's theorem on the p-curvature [K70, Theorem 5.1] allows us identify **CT** with the category of vector bundles trivialized by F. Therefore, counting F-trivial vector bundles amounts to counting matrices of 1-forms which satisfy the "p-integrability condition."

To fix ideas, we assume that the direct image of $\Omega = \Omega^1_{X/S}$ is free on the basis $\theta_0, \ldots, \theta_{g-1}$ over S. Using the forms θ_i, we have the identification

$$\mathrm{M}_n(A)^{\times g} \xrightarrow{\sim} H^0(X, \mathrm{M}_n(\mathcal{O}_X) \otimes \Omega). \tag{4.1}$$

Let us also fix an open and affine subscheme $U = \operatorname{Spec} B \subset X$ which has an "etale coordinate" $x : U \to \mathbb{A}^1_S$. Write $\theta_{i,x} \cdot \mathrm{d}x = \theta_i|_U$, and denote by ∂_x the vector field associated to $\mathrm{d}x$.

On the set of S-connections on $\mathcal{O}_X^n$, we have the "direct sum connection" $\mathrm{d}_{\mathrm{triv}}$. Hence, for any other $\nabla : \mathcal{O}_X^n \to \mathcal{O}_X^n \otimes \Omega$ we obtain

$$R(\nabla) = \nabla - \mathrm{d}_{\mathrm{triv}} \in H^0(X, \mathrm{M}_n(\mathcal{O}_X) \otimes \Omega).$$

By means of the identification (4.1), we write $R(\nabla) = (R_0(\nabla), \ldots, R_{g-1}(\nabla))$. This establishes a bijection between the set of S-connections on $\mathcal{O}_X^n$ and elements of $\mathrm{M}_n(A)$. Given $\boldsymbol{R} = (R_0, \ldots, R_{g-1}) \in \mathrm{M}_n(A)^{\times g}$, the accompanying connection is denoted by $\mathrm{d}_{\boldsymbol{R}}$.

Let $\boldsymbol{T} = (T_0, \ldots, T_{g-1})$ be variables and write

$$L := \sum \theta_{i,x} T_i \in B\{\boldsymbol{T}\}.$$

(The curly brackets refer to non-commutative polynomials.) We let $\delta \in \operatorname{End}_A(B\{\boldsymbol{T}\})$ be defined by applying ∂_x to each coefficient of a non-commutative polynomial and embed $B\{\boldsymbol{T}\}$ into $\operatorname{End}_A(B\{\boldsymbol{T}\})$ via $\lambda_P : Q \mapsto PQ$. Applying Jacobson's formula [JLA, p.187] to

$$\Psi = (\delta + \lambda_L)^p$$

and using that $[\delta, \lambda_P] = \lambda_{\delta P}$, we see that

$$\begin{aligned}\Psi &= \lambda_{L^p} + \sum_{i=1}^{p-1} s_i(\delta, \lambda_L) \\ &= \lambda_{L^p} + \sum_{i=1}^{p-1} \tilde{s}_i(\lambda_L, \lambda_{\delta L}, \ldots, \lambda_{\delta^i L}).\end{aligned}$$

Here, $\tilde{s}_i$ is a non-commutative polynomial with coefficients in $\mathbb{F}_p$. In fact, if t is an indeterminate, then $i \cdot s_i(\delta, \lambda_L)$ is the coefficient of t^{i-1} in

$$\begin{aligned}\mathrm{ad}(t\delta + \lambda_L)^{p-1}(\delta) &= -\mathrm{ad}(t\delta + \lambda_L)^{p-2}(\lambda_{\delta L}) \\ &= -\lambda\left\{(t\delta + \mathrm{ad}_L)^{p-2}(\delta L)\right\},\end{aligned}$$

since $\mathrm{ad}(t\delta + \lambda_P)(\lambda_Q) = \lambda_{t\delta Q + [P,Q]}$. In particular, Ψ belongs to the image of λ. We write

$$\Psi = \lambda\left(\sum_{\boldsymbol{\nu}} \Psi_{\boldsymbol{\nu}} \cdot \boldsymbol{T}^{\boldsymbol{\nu}}\right)$$

and note that each $\Psi_{\boldsymbol{\nu}} \in B$ is a linear combinations over $\mathbb{F}_p$ of monomials

$$\partial_x^{\alpha_1}\theta_{i_1,x} \cdots \partial_x^{\alpha_m}\theta_{i_m,x},$$

with $\alpha_1 + \ldots + \alpha_m \leq p$. Let B' be an A-module such that $B \oplus B'$ is free [GR71, p.19] and pick a dual basis $\{\chi_j : j \in J\}$.

Definition 44. The p-curvature algebra associated to the basis $\theta_0, \ldots, \theta_{g-1}$, the affine open U and the coordinate x,

$$\Lambda = \Lambda(X/A; \boldsymbol{\theta}, U, x),$$

is the quotient of $A\{\boldsymbol{T}\}$ by the ideal generated by the non-commutative polynomials $\sum_{\boldsymbol{\nu}} \chi_j(\Psi_{\boldsymbol{\nu}})\boldsymbol{T}^{\boldsymbol{\nu}}$.

Remark 45. Considering then the finitely many non-zero coefficients $\chi_j(\Psi_{\boldsymbol{\nu}})$ appearing in each $\Psi_{\boldsymbol{\nu}}$, we can assure that the ideal defining Λ is finitely generated.

Remark 46. It is evident that Λ does not depend seriously on the choices made. Replacing $\boldsymbol{\theta}$ by another basis modifies $\boldsymbol{T}$ linearly. If we pick another dual basis of $B \oplus B'$, then the non-commutative polynomials defining Λ are replaced by linear combinations.

Let $\varrho : A\{\boldsymbol{T}\} \to \mathrm{M}_n(A)$ be a representation and write $\varrho T_i = R_i$. It is clear that the *B-linear* p-curvature endomorphism [K70, p.190]

$$\psi_p(\mathrm{d}_{\boldsymbol{R}})(\partial_x) : B^n \longrightarrow B^n$$

is simply

$$\sum_{\boldsymbol{\nu}} \Psi_{\boldsymbol{\nu}} \cdot \boldsymbol{R}^{\boldsymbol{\nu}} \in \mathrm{M}_n(B).$$

Therefore, $\psi_p(\mathrm{d}_{\boldsymbol{R}}) = 0 \Leftrightarrow \sum_{\boldsymbol{\nu}} \Psi_{\boldsymbol{\nu}} \cdot \boldsymbol{R}^{\boldsymbol{\nu}} = 0$. A simple exercise shows that $\sum_{\boldsymbol{\nu}} \Psi_{\boldsymbol{\nu}} \boldsymbol{R}^{\boldsymbol{\nu}} = 0$ $\Leftrightarrow \sum_{\boldsymbol{\nu}} \chi_j(\Psi_{\boldsymbol{\nu}})\boldsymbol{R}^{\boldsymbol{\nu}} = 0$ for all $j \in J$. This establishes:

Proposition 47. *Let $\varrho : \Lambda \to \mathrm{M}_n(A)$ be a representation. Then the connection $\mathrm{d}_{\varrho \boldsymbol{T}} : \mathcal{O}_X^n \to \mathcal{O}_X^n \otimes \Omega$ has vanishing p-curvature. Moreover, if $(\mathcal{O}_X^n, \mathrm{d}_{\boldsymbol{R}})$ has vanishing p-curvature, then the matrices $(R_0, \ldots, R_{g-1}) \in \mathrm{M}_n(A)$ define a representation of Λ on A^n.*

From the explicit construction of Λ, it is clear that for each k-point s of S, the algebra $\Lambda \otimes_A \boldsymbol{k}(s)$ is just the p-curvature algebra

$$\Lambda(X_s/k, \boldsymbol{\theta} \otimes \boldsymbol{k}(s), U \otimes \boldsymbol{k}(s), x \otimes \boldsymbol{k}(s))$$

which is considered in [dS12a]. There, it is argued that there is an equivalence of abelian k-linear abelian categories

$$\left\{ \begin{array}{l} \text{representations of } \Lambda(X_s/k, \ldots) \text{ on} \\ \text{finite dimensional } k\text{-vector-spaces} \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{vector bundles on } X_s \text{ which} \\ \text{are trivialized by } F \end{array} \right\}.$$

See [dS12a, Proposition 3, p.230]. Of course, a similar result is derived in our case of a relative curve, but the effect is somewhat diminished due to technical conditions.

But the theoretical point of view was not the priority in [dS12a]: we concentrated on computational aspects and, using [Le08], we concluded that for some specific examples in characteristic two, there are infinitely many *isomorphism classes of stable* vector bundles of rank two which are trivialized by F.

Theorem 48. *Let $k = \overline{\mathbb{F}}_2$ and assume that X is birational to an affine plane curve of equation*

$$\begin{array}{ll} y^2 + y = x^3 + x^{-3} & (\text{genus } X = 3) \\ y^2 + y = x^3 + x^{-5} & (\text{genus } X = 4) \\ y^2 + y = x^7 & (\text{genus } X = 3) \\ y^2 + y = x^9 & (\text{genus } X = 4) \\ y^2 + y = x^{11} & (\text{genus } X = 5) \end{array}$$

Then there exist infinitely many isomorphism classes of stable vector bundles of rank two trivialized by F. Consequently, the arrow (BC) ***is not*** *an isomorphism for the considered curves.* □

The first and second examples above are considered in [dS12a], the others run along the same lines. It is possible to obtain many more examples, all of them in characteristic two, where (BC) fails to be an isomorphism. We have not detected any serious theoretical reason behind these "pathologies", but we did go through a diagnostic process. We found that the p-rank plays no role, that the *pathologies only happen for hyperelliptic curves with very few Weierstrass points.* Although this direction is worth investigating, the only examples we know of curves in characteristic three with few Weierstrass points are of very high genus [H78], so that computations are unbearable. So far, we have not found any theoretical method to advance this line of thought.

Assume that S is now an affine variety over k, and write K for its field of functions. We then obtain an algebra $\Lambda \otimes K$ over K and by the method of [Le08] we can still ask whether or not there are infinitely many isomorphism classes of semi-simple representations of a given rank over $\overline{K}$.

Proposition 49. *a) Assume that there are* ***infinitely*** *many isomorphism classes of semi-simple n-dimensional representations of $\Lambda \otimes K$ over $\overline{K}$. Then, for almost all closed points $s \in S(k)$, the curve X_s has* ***infinitely many*** *isomorphism classes of stable F-trivial vector bundle of some rank $\le n$.*

b) *Assume that there are* ***finitely many*** *isomorphism classes of semi-simple n-dimensional representations of $\Lambda \otimes K$ over $\overline{K}$. Then, for almost all closed points $s \in S(k)$, the curve X_s has* ***finitely many*** *isomorphism classes of stable F-trivial vector bundle of some rank $\leq n$.*

Proof. (a) Let us assume that for some n there are infinitely many isomorphim classes of semi-simple representations $\varrho : \Lambda \otimes K \to \mathrm{M}_n(\overline{K})$. This means that the "trace algebra" of rank n associated to $\Lambda \otimes K$, call it $C_n(\Lambda \otimes K/K)$, has positive Krull dimension. (For information on the "trace" algebra, see [Le08, 2.8,p.3930ff] and [Pr74, 173ff].) But this algebra is simply $C_n(\Lambda/A) \otimes_A K$. A straightforward exercise shows the Krull dimension of $C_n(\Lambda/A) \otimes \boldsymbol{k}(s)$ is ≥ 1 for almost all k-points s of S. (It is not generally possible to remove "almost"!) Since $C_n(\Lambda/A) \otimes \boldsymbol{k}(s)$ is just $C_n(\Lambda \otimes \boldsymbol{k}(s)/\boldsymbol{k}(s))$, we arrive at our conclusion. The proof of (b) is similar. □

Instances where this observation proves fruitful should appear elsewhere; for example, we can look for explicit families of curves inside the moduli space and try to see if generically they are pathological. (So far we have only been able to find families where the A-algebra Λ is "constant".) Furthermore, contrary to our hope, the trace algebra $C_n(\Lambda)$ seems to be quite complicated so that not much can be inferred. (For example, it seems fruitless to assume $C_n(\Lambda)$ to be flat over A.)

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