ON CERTAIN TANNAKIAN CATEGORIES OF INTEGRABLE CONNECTIONS OVER KÄHLER MANIFOLDS

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ABSTRACT. Given a compact Kähler manifold X, it is shown that pairs of the form (E, D), where E is a trivial holomorphic vector bundle on X, and D is an integrable holomorphic connection on E, produce a neutral Tannakian category. The corresponding pro-algebraic affine group scheme is studied. In particular, it is shown that this pro-algebraic affine group scheme for a compact Riemann surface determines uniquely the isomorphism class of the Riemann surface.

CONTENTS

1. INTRODUCTION

A question of Ghys asks the following: Is there a pair of the form (M, D), where M is a compact Riemann surface of genus at least two, and D is an irreducible holomorphic

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$\mathrm{SL}(2,\mathbb{C})$–connection on the rank two trivial holomorphic vector bundle $\mathcal{O}_M^{\oplus 2}$, such that the image of the monodromy homomorphism for D is contained in a cocompact lattice of $\mathrm{SL}(2,\mathbb{C})$. The motivation for this question comes from the study of compact quotients of $\mathrm{SL}(2,\mathbb{C})$ by lattices. Such quotients are compact non-Kähler manifolds. While they can't contain a complex surface [HM, p. 239, Theorem 2], it is not known whether they can contain compact Riemann surfaces of genus $g > 1$. A positive answer to Ghys' question would provide a nontrivial holomorphic map from the Riemann surface M to the quotient of $\mathrm{SL}(2,\mathbb{C})$ by the cocompact lattice containing the image of the monodromy homomorphism for D. In fact the two problems are equivalent (see [CDHL] for explanations for the origin of Ghys' question).

Let us also mention that a related question of characterizing rank two holomorphic vector bundles $\mathcal{V}$ over a compact Riemann surface, such that for some holomorphic connection on $\mathcal{V}$ the image of the associated monodromy homomorphism is Fuchsian, was raised in [Ka, p. 556] (there this question is attributed to Bers).

The above, still open questions of Bers and Ghys, and some related questions in [CDHL, Ka], motivated us to investigate the holomorphic connections on a trivial holomorphic vector bundle. Answering a question asked in [CDHL], examples of irreducible holomorphic $\mathrm{SL}(2,\mathbb{C})$–connection with real monodromy, on the trivial holomorphic $\mathrm{SL}(2,\mathbb{C})$–bundle over a compact Riemann surface, were constructed in [BDH].

Here we consider integrable holomorphic connections on trivial holomorphic vector bundles over a compact Kähler manifold X. The purpose of the above discussions is to demonstrate the significance of holomorphic connections on the trivial holomorphic vector bundles. While highly motivated by the above mentioned questions of Bers and Ghys, it should be clarified that the present work does not shed particular light on those questions which remain open, but provides a study of the relevant category of integrable holomorphic connections on trivial holomorphic vector bundles.

Once we fix a base point $x_0 \in X$ in order to define a fiber functor, using the Tannakian category theory it is shown that the category of integrable holomorphic connections on trivial holomorphic vector bundles produces a quotient of the pro-algebraic completion $\varpi(X,\, x_0)$ of the fundamental group $\pi_1(X,\, x_0)$ (the details are in Section 3); this quotient of $\varpi(X,\, x_0)$ is denoted by $\Theta(X,\, x_0)$. Then we prove a Torelli type Theorem with respect to $\Theta(X,\, x_0)$, for compact Riemann surfaces and also for compact complex tori.

The main results of Section 3 and Section 4 are the following:

(1) *For compact Kähler manifolds X and Y, the natural homomorphism*

$$\Theta(X,\, x_0)\times\Theta(Y,\, y_0) \longrightarrow \Theta(X\times Y,\, (x_0,\, y_0))$$

is an isomorphism. (See Proposition 3.9.)

(2) *Let $\beta : X \longrightarrow Y$ be an orientation preserving diffeomorphism between compact Riemann surfaces such that the corresponding homomorphism*

$$\beta_\natural : \varpi(X,\, x_0) \longrightarrow \varpi(Y,\, \beta(x_0))$$

descends to a homomorphism from $\Theta(X,\, x_0)$ *to* $\Theta(Y,\, \beta(x_0))$. *Then the two Riemann surfaces* X *and* Y *are isomorphic.* (See Theorem 4.1.)

(3) *Let* $\varphi : \mathbb{T} \longrightarrow \mathbb{S}$ *be a diffeomorphism between two compact complex tori such that the corresponding homomorphism*

$$\varphi_* : \varpi(\mathbb{T},\, x_0) \longrightarrow \varpi(\mathbb{S},\, \varphi(x_0))$$

descends to a homomorphism from $\Theta(\mathbb{T},\, x_0)$ *to* $\Theta(\mathbb{S},\, \varphi(x_0))$. *Then there is a biholomorphism*

$$\mathbb{T} \longrightarrow \mathbb{S}$$

which is homotopic to the map φ. (See Proposition 4.2.)

In Section 5 we consider integrable holomorphic connections on holomorphic vector bundles over X which decompose into a direct sum of holomorphic line bundles. Using the Tannakian category theory in a similar way, we show that this category also produces a quotient $\Delta(X,\, x_0)$ of the pro-algebraic completion $\varpi(X,\, x_0)$ of the fundamental group $\pi_1(X,\, x_0)$. Then we adapt our methods in Section 4 in order to prove the same Torelli type theorems for $\Delta(X,\, x_0)$.

As is well-known, the category of vector bundles easily fails to be abelian, but as observed as far back as [Se, Proposition 3.1], semi-stability can be brought in to mend this failure and produce interesting *abelian* categories of vector bundles. Section 6 connects the group scheme Θ to such an idea by means of the category of pseudostable vector bundles (stemming from Simpsons's foundational work). We produce group schemes $\Sigma(X,\, x_0)$, accounting for connections on pseudostables, and $\pi^{\mathrm{S}}(X,\, x_0)$, accounting for pseudostables solely, and then establish a link between these and Θ; see Theorem 6.7.

2. Neutral Tannakian categories for a Kähler manifold

Let X be a compact connected Kähler manifold of complex dimension d. Fix a Kähler form ω on X. The *degree* of a torsionfree coherent analytic sheaf F on X is defined to be

$$\mathrm{degree}(F) := (c_1(\det F) \cup \omega^{d-1}) \cap [X] \in \mathbb{R}\,.$$

Here and elsewhere, we write c_i for the i–th Chern class in $H^{2i}_{dR}(X,\, \mathbb{R})$ and define the holomorphic line bundle $\det F$ following [Ko, Ch. V, § 6]. The real number

$$\mu(F) := \frac{\mathrm{degree}(F)}{\mathrm{rank}(F)}$$

is called the *slope* of F.

Fix a base point $x_0 \in X$. Let

$$\phi : \pi_1(X,\, x_0) \longrightarrow \varpi(X,\, x_0) \tag{2.1}$$

be the pro-algebraic completion of the fundamental group $\pi_1(X,\, x_0)$. We recall that $\varpi(X,\, x_0)$ is a pro-algebraic affine group scheme over $\mathbb{C}$ which is uniquely characterized by the following property: for any homomorphism

$$\gamma : \pi_1(X,\, x_0) \longrightarrow G$$

to a complex affine algebraic group G, there is a unique algebraic homomorphism

$$\widehat{\gamma} : \varpi(X, x_0) \longrightarrow G$$

such that $\widehat{\gamma} \circ \phi = \gamma$. There are mainly two equivalent constructions of $\varpi(X, x_0)$: one by means of the Tannakian category of finite dimensional representations of $\pi_1(X, x_0)$ and Tannakian duality [DMOS] and the other by Freyd's adjoint functor theorem applied to the $\mathbb{C}$–points functor from group schemes to groups [Fr, p. 84, 3.J].

We shall recall from [Si2, p. 70] three neutral Tannakian categories associated to the pointed Kähler manifold (X, x_0) which, in particular, furnish a Tannakian description of $\varpi(X, x_0)$.

Let $\mathcal{C}_{dR}(X)$ denote the category whose objects are pairs of the form (E, D), where E is a holomorphic vector bundle on X and D is an integrable holomorphic connection on E (see [At2] for holomorphic connections). Morphisms from (E, D) to (E', D') are all holomorphic homomorphisms of vector bundles $h : E \longrightarrow E'$ that intertwine the connections D and D', meaning $D' \circ h = (h \otimes \mathrm{Id}_{\Omega^1_X}) \circ D$ as differential operators from E to $E' \otimes \Omega^1_X$ with Ω^1_X being the holomorphic cotangent bundle of X. This category is equipped with the operators of direct sum, tensor product and dualization. More precisely, $\mathcal{C}_{dR}(X)$ is a rigid abelian tensor category (see [DMOS, p. 118, definition 1.14] for rigid abelian tensor categories).

Remark 2.1. It is known that any coherent analytic sheaf on X admitting a holomorphic connection is locally free (see [BGHE, p. 211, Proposition 1.7]); it should be clarified that although this proposition in [BGHE] is stated only for $\mathcal{O}_X$–modules with an integrable holomorphic connection (same as a $\mathcal{D}_X$–module), its proof uses only the Leibniz rule which is valid for holomorphic connections.

It is straightforward to check that the above category $\mathcal{C}_{dR}(X)$, equipped with the faithful fiber functor that sends any object (E, D) to the fiber $E|_{x_0}$ over $x_0 \in X$, defines a neutral Tannakian category (see [DMOS, p. 138, Definition 2.19], [Sa], [Si2, p. 67], [No, p. 76] for neutral Tannakian category). Given any neutral Tannakian category, a theorem of Saavedra Rivano associates to it a pro-algebraic affine group scheme over $\mathbb{C}$ [DMOS, p. 130, Theorem 2.11] (and the remark following [DMOS, p. 138, Definition 2.19]), [Sa], [No, p. 77, Theorem 1.1], [Si2, p. 69]. Therefore, the neutral Tannakian category $\mathcal{C}_{dR}(X)$ corresponds to a pro-algebraic affine group scheme over $\mathbb{C}$.

Let $\mathcal{C}_B(X)$ denote the category whose objects are all finite dimensional complex representations of $\pi_1(X, x_0)$. Using the tautological fiber functor, it defines a neutral Tannakian category. The pro-algebraic affine group scheme over $\mathbb{C}$ corresponding to $\mathcal{C}_B(X)$, by the above mentioned theorem of Saavedra Rivano ([Sa], [DMOS, p. 130, Theorem 2.11]), is the group scheme $\varpi(X, x_0)$ in (2.1) [Si2, p. 69, Lemma 6.1].

A Higgs bundle on X is a pair of the form (E, θ), where E is a holomorphic vector bundle of X and $\theta \in H^0(X, \mathrm{End}(E) \otimes \Omega^1_X)$ with $\theta \bigwedge \theta = 0$ [Si1], [Si2]; the holomorphic section θ is called a *Higgs field* on E. A Higgs bundle (E, θ) is called *stable* (respectively, *semistable*) if

$$\mu(F) < \mu(E) \ \ \text{(respectively, } \ \mu(F) \leq \mu(E))$$

for every coherent analytic subsheaf $F \subset E$ with $0 < \text{rank}(F) < \text{rank}(E)$ and

$$\theta(F) \subset F \otimes \Omega^1_X .$$

A Higgs bundle (E, θ) is called *polystable* if it is a direct sum of stable Higgs bundles of same slope.

Let $\mathcal{C}_{Dol}(X)$ denote the category whose objects are Higgs bundles (E, θ) such that $ch_2(E) \cup \omega^{d-2} = 0$ and there is a filtration of holomorphic subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_i \subset \cdots \subset E_{\ell-1} \subset E_\ell = E$$

satisfying the following three conditions:

(1) $\theta(E_j) \subset E_j \otimes \Omega^1_X$ for all $1 \leq j \leq \ell$,
(2) $\text{degree}(E_j/E_{j-1}) = 0$ for all $1 \leq j \leq \ell$, and
(3) the Higgs bundle $(E_j/E_{j-1}, \theta)$ is stable for all $1 \leq j \leq \ell$ (with a mild abuse abuse of notation, the Higgs field on E_j/E_{j-1} induced by θ is also denoted by θ).

Notice that the above second condition implies that $\text{degree}(E) = 0$.

In [BG] such Higgs bundles are called pseudostable.

A homomorphism from (E, θ) to (E', θ') is a holomorphic homomorphism

$$h : E \longrightarrow E'$$

such that $\theta' \circ h = (h \otimes \text{Id}_{\Omega^1_X}) \circ \theta$ as homomorphisms from E to $E' \otimes \Omega^1_X$. It is known that $\mathcal{C}_{Dol}(X)$ is a rigid abelian tensor category [Si2, p. 70]. This category $\mathcal{C}_{Dol}(X)$ admits the faithful fiber functor that sends any object (E, θ) to the fiber $E|_{x_0}$. In other words, $\mathcal{C}_{Dol}(X)$ is a neutral Tannakian category.

The two categories $\mathcal{C}_{dR}(X)$ and $\mathcal{C}_B(X)$ are equivalent by the Riemann–Hilbert correspondence that assigns to a flat connection the corresponding monodromy representation. Using fundamental theorems of Corlette, [Co], and Simpson, [Si1], in [Si2] Simpson proved that the category $\mathcal{C}_{Dol}(X)$ is equivalent to $\mathcal{C}_{dR}(X)$ (see [Si2, p. 36, Lemma 3.5], [Si2, p. 70]). Therefore, all these three neutral Tannakian categories, namely $\mathcal{C}_{dR}(X)$, $\mathcal{C}_B(X)$ and $\mathcal{C}_{Dol}(X)$, produce the same pro-algebraic affine group scheme over $\mathbb{C}$ using the theorem of Saavedra Rivano mentioned earlier ([Sa], [DMOS, p. 130, Theorem 2.11]). In other words, each of these three neutral Tannakian categories produces the pro-algebraic affine group scheme $\varpi(X, x_0)$ in (2.1).

It should be mentioned that when X is a smooth complex projective variety, and the cohomology class of the closed form ω is rational, then the category $\mathcal{C}_{Dol}(X)$ coincides with the category of semistable Higgs bundles (E, θ) on X with $ch_2(E) \cup \omega^{d-2} = 0$ and $\text{degree}(E) = 0$ [Si2, p. 39, Theorem 2].

3. The Tannakian subcategory $\mathcal{T}_{dR}$ of $\mathcal{C}_{dR}$

3.1. **The Tannakian category and the associated group scheme.** As before, X is a compact connected Kähler manifold of complex dimension d.

Definition 3.1. Let $\mathcal{T}_{dR}(X)$ be the full subcategory of $\mathcal{C}_{dR}(X)$ (defined in Section 2) whose objects are all the couples $(E,\, D)$ satisfying the condition that the holomorphic vector bundle E is holomorphically trivial.

Clearly, $\mathcal{T}_{dR}(X)$ is stable under tensor products and duals. The identity object $(\mathcal{O}_X,\, \mathrm{d})$, where d is the de Rham differential, certainly is an object of $\mathcal{T}_{dR}(X)$. In addition $\mathcal{T}_{dR}(X)$ is stable under quotients, as shown by the next proposition.

Proposition 3.2. *Let* $(E,\, D) \,=\, (\mathcal{O}_X^{\oplus r},\, D)$ *be an object of* $\mathcal{T}_{dR}(X)$ *and*

$$q\,:\,(\mathcal{O}_X^{\oplus r},\, D)\,\longrightarrow\,(E',\, D')$$

an epimorphism in $\mathcal{C}_{dR}(X)$*. Then the holomorphic vector bundle* E' *is also trivial.*

Proof. We note that $c_1(E')\,=\,0$ because E' carries an integrable connection [At2, pp. 192–193, Theorem 4], [GH, p. 141, Proposition]. Let r' be the rank of E', and introduce $\mathrm{Gr}(r, r')$, the Grassmann manifold of r' dimensional quotients of $\mathbb{C}^r$. Note that the trivial bundle $\mathcal{O}^{\oplus r}_{\mathrm{Gr}(r,r')}$ comes with a tautological quotient of rank r', call it $\mathcal{O}^{\oplus r}_{\mathrm{Gr}(r,r')}\,\longrightarrow\, U$. Then, q induces a morphism $f\,:\,X\,\longrightarrow\,\mathrm{Gr}(r, r')$ such that $E'\,=\,f^*U$. In particular, we have $c_1(f^*(\det U))\,=\,c_1(E')\,=\,0$. The fact that $\det(U)$ is ample implies that f is a constant map [GH, p. 177]. Hence, E' is holomorphically trivial. □

Corollary 3.3. *The full subcategory* $\mathcal{T}_{dR}(X)$ *of* $\mathcal{C}_{dR}(X)$ *is an abelian subcategory.*

Proof. Let $\alpha\,:\,(E,\, D)\,\longrightarrow\,(E_1,\, D_1)$ be an arrow in $\mathcal{T}_{dR}(X)$. Now, $\mathrm{Image}(\alpha)$ is a subconnection, call it $(I,\, D_1)$, of $(E_1,\, D_1)$ and we have an epimorphism

$$\alpha\,:\,(E,\, D)\,\longrightarrow\,(I,\, D_1)$$

in $\mathcal{C}_{dR}(X)$. We note that I is locally free (see Remark 2.1). We conclude from Proposition 3.2 that $(I,\, D_1)$ is an object of $\mathcal{T}_{dR}(X)$. Again from Proposition 3.2 it follows that $\mathrm{coker}(\alpha)$ is an object of $\mathcal{T}_{dR}(X)$ because it is a quotient of an object of $\mathcal{T}_{dR}(X)$.

To prove that $\mathrm{kernel}(\alpha)$ is in $\mathcal{T}_{dR}(X)$, consider the dual connections $(E^*,\, D^*)$ and $(E_1^*,\, D_1^*)$ of $(E,\, D)$ and $(E_1,\, D_1)$ respectively. Let

$$\alpha^*\,:\,(E_1^*,\, D_1^*)\,\longrightarrow\,(E^*,\, D^*)$$

be the dual of the homomorphism α. The above argument gives that $\alpha^*(E_1^*)\,\subset\, E^*$ is a trivial subbundle, and the quotient $E^*/\alpha^*(E_1^*)$ is also a trivial holomorphic vector bundle. Hence

$$\mathrm{kernel}(\alpha)\,=\,(E^*/\alpha^*(E_1^*))^*$$

is a trivial holomorphic vector bundle. Therefore, $\mathrm{kernel}(\alpha)$, equipped with the integrable holomorphic connection induced by D, is also in $\mathcal{T}_{dR}(X)$.

Hence, the standard criterion for a subcategory to be an abelian subcategory can be applied [Fr, Theorem 3.41]. □

Let

$$\Theta(X,\, x_0) \tag{3.1}$$

be the affine group scheme over $\mathbb{C}$ corresponding to $\mathcal{T}_{dR}(X)$ via [DMOS, p. 130, Theorem 2.11] by means of the exact functor $(E,\, D) \longmapsto E|_{x_0}$. From Proposition 3.2 and the standard criterion [DMOS, Proposition 2.21, p.139] we conclude that the natural arrow of group schemes

$$\mathbf{q}_X \,:\, \varpi(X,\, x_0) \longrightarrow \Theta(X,\, x_0) \tag{3.2}$$

is a quotient homomorphism, where $\varpi(X,\, x_0)$ is the group scheme in (2.1).

Clearly, for any given holomorphic map $f : Y \longrightarrow X$, we obtain tensor functors

$$f^{\#} \,:\, \mathcal{C}_{dR}(X) \longrightarrow \mathcal{C}_{dR}(Y)$$

and

$$f^{\#} \,:\, \mathcal{T}_{dR}(X) \longrightarrow \mathcal{T}_{dR}(Y)$$

defined by $(E,\, D) \longmapsto (f^*E,\, f^*D)$. If, in addition, we let $y_0 \in Y$ be a point which is taken to x_0, we derive homomorphisms of group schemes

$$f_\natural \,:\, \varpi(Y,\, y_0) \longrightarrow \varpi(X,\, x_0)$$

and

$$f_\natural : \Theta(Y,\, y_0) \longrightarrow \Theta(X,\, x_0)\,.$$

Using the equivalence of categories between $\mathcal{C}_{Dol}(X)$ and $\mathcal{C}_{dR}(X)$ mentioned in Section 2, the above subcategory $\mathcal{T}_{dR}(X)$ of $\mathcal{C}_{dR}(X)$ gives a Tannakian full subcategory of $\mathcal{C}_{Dol}(X)$. It is natural to ask the following:

Question 3.4. *What is a direct description of the Tannakian full subcategory of $\mathcal{C}_{Dol}(X)$ corresponding to the subcategory $\mathcal{T}_{dR}(X)$ of $\mathcal{C}_{dR}(X)$?*

Answering Question 3.4 amounts to classifying all Higgs bundles $(E,\, \theta)$ such that the holomorphic vector bundle V underlying the flat connection $(V,\, D)$ corresponding to $(E,\, \theta)$ is trivial. The problem is that the holomorphic structure of V depends on $(E,\, \theta)$ in a rather intricate manner.

3.2. **Characters of Θ: multiplicative and additive.** Let $\mathbb{G}_m$ be the multiplicative group of nonzero scalars and $\mathbb{G}_a$ the additive group of scalars. Let G be an affine group scheme over $\mathbb{C}$. Two of the most basic abstract groups associated to it are its group of characters $\mathrm{Hom}(G,\, \mathbb{G}_m)$ and its group of additive characters $\mathrm{Hom}(G,\, \mathbb{G}_a)$ (notations are those of [Wa, p. 5, 1.2]). These are denoted respectively by $\mathbb{X}(G)$ and $\mathbb{X}_a(G)$ in what follows. We note that $\mathbb{X}_a(G)$ comes with the extra structure of a $\mathbb{C}$-vector space. Clearly, $\mathbb{X}(G)$ is the group of isomorphism classes of rank one representations of G. In turn, the standard immersion of $\mathbb{G}_a$ into $\mathrm{GL}(2, \mathbb{C})$ permits us to view $\mathbb{X}_a(G)$ as the vector space

$$\mathrm{Ext}_G(\mathbf{1},\, \mathbf{1})$$

parametrizing extensions of the rank one trivial representation by itself.

We shall record some observations on characters and additive characters of $\varpi(X, x_0)$ and $\Theta(X, x_0)$. In what follows,

$$\mathrm{Pic}^\tau(X)$$

is the subgroup of $\mathrm{Pic}(X)$ formed by the classes of holomorphic line bundles on X with vanishing first Chern class. Recall that according to our convention, Chern classes are real. The subgroup $\mathrm{Pic}^0(X)$ of classes having vanishing *integral* first Chern class is of finite index in $\mathrm{Pic}^\tau(X)$ as we learn from the theorem of the base.

Lemma 3.5. *The map*

$$H^0(X, \Omega^1_X) \longrightarrow \mathbb{X}(\Theta(X, x_0))$$

sending a form α to the isomorphism class $(\mathcal{O}_X, \mathrm{d} + \alpha)$ is an isomorphism. The group $\mathbb{X}(\varpi(X, x_0))$ sits in a short exact sequence

$$1 \longrightarrow \mathbb{X}(\Theta(X, x_0)) \longrightarrow \mathbb{X}(\varpi(X, x_0)) \longrightarrow \mathrm{Pic}^\tau(X) \longrightarrow 1\,.$$

Proof. The first claim is very simple and its verification is omitted. Then, if (L, ∇) is an integrable connection with L a holomorphic line bundle, the interpretation of $c_1(L)$ as curvature form [GH, p. 141, Proposition] assures that $L \in \mathrm{Pic}^\tau(X)$, and we obtain the arrow

$$\mathbb{X}(\varpi(X, x_0)) \longrightarrow \mathrm{Pic}^\tau(X)\,.$$

The kernel is precisely the group of (isomorphism classes of) objects in $\mathcal{T}_{dR}(X)$ of rank one, which is $\mathbb{X}(\Theta(X, x_0))$. That any class in $\mathrm{Pic}^\tau(X)$ carries an integrable connection is explained by [LT, p. 40, Corollary 1.3.12]. □

We shall now be concerned with the additive characters or, with the vector spaces

$$\mathbb{X}_a(\varpi(X, x_0)) \simeq \mathrm{Ext}_{\mathcal{C}_{dR}(X)}(\mathbf{1}, \mathbf{1})$$

and

$$\mathbb{X}_a(\Theta(X, x_0)) \simeq \mathrm{Ext}_{\mathcal{T}_{dR}(X)}(\mathbf{1}, \mathbf{1})\,,$$

where $\mathbf{1}$ is the identity object of the pertinent categories.

Let $H^1_{dR}(X, \mathbb{C})$ be $\mathbb{H}^1(X, \Omega^\bullet_X)$ [GH, p. 446], which is of course canonically isomorphic to $H^1(X, \mathbb{C})$ [GH, p. 448], and let

$$\lambda_X \,:\, H^0(X, \Omega^1_X) \longrightarrow H^1_{dR}(X, \mathbb{C}) \tag{3.3}$$

be the natural arrow obtained from the fact that global holomorphic one forms on X are closed. Here we use that X is Kähler.

Lemma 3.6. *There are functorial isomorphisms of vector spaces*

$$\mathrm{Ext}_{\mathcal{C}_{dR}(X)}(\mathbf{1}, \mathbf{1}) \xrightarrow{\sim} H^1_{dR}(X, \mathbb{C})$$

and

$$\mathrm{Ext}_{\mathcal{T}_{dR}(X)}(\mathbf{1}, \mathbf{1}) \xrightarrow{\sim} H^0(X, \Omega^1_X)\,.$$

Under these isomorphisms, the canonic arrow

$$\mathrm{Ext}_{\mathcal{T}_{dR}(X)}(\mathbf{1},\, \mathbf{1}) \longrightarrow \mathrm{Ext}_{\mathcal{C}_{dR}(X)}(\mathbf{1},\, \mathbf{1})$$

corresponds to the homomorphism λ_X *in* (3.3).

Proof. Let $(E,\, \nabla) \in \mathcal{C}_{dR}(X)$ be an extension of $(\mathcal{O}_X,\, \mathrm{d})$ by itself. Let $\mathfrak{U} = \{U_i\}$ be an open covering of X on which we have $e_i \in H^0(U_i,\, E)$ mapping to $1 \in \mathcal{O}_X(U_i)$. Consider the holomorphic 1–form θ_i on U_i such that $\nabla(e_i) = 1 \otimes \theta_i$. Then, on $U_i \cap U_j$, we have $e_i = a_{ij} \cdot 1 + e_j$ with a_{ij} a holomorphic function. The element

$$(\theta_i,\, a_{ij}) \in C^0(\mathfrak{U},\, \Omega^1_X) \oplus C^1(\mathfrak{U},\, \mathcal{O}_X)$$

defines a 1-cocycle of the total Čech complex associated to the bicomplex

$$(C^p(\mathfrak{U},\, \Omega^q_X))_{0 \le p,q} :$$

the form θ_i is closed and $\mathrm{d}a_{ij} = \theta_i - \theta_j$. Hence, $\{(\theta_i,\, a_{ij})\}$ gives an element of $H^1_{dR}(X,\, \mathbb{C})$, call it $[E,\, \nabla]$. Note that a 1-cocycle with coefficients in GL_2 representing the vector bundle E is given by the matrices $\begin{pmatrix} 1 & -a_{ij} \\ 0 & 1 \end{pmatrix}$ so that E is isomorphic to $\mathcal{O}_X^2$ if and only if there exists, after eventually passing to a finer covering, a 0–cochain (h_i) with

$$h_j - h_i = -a_{ij} \,.$$

In this case, $[E,\, \nabla]$ belongs to the subspace $H^0(X,\, \Omega^1_X) \subset H^1_{dR}(X,\, \mathbb{C})$.

It is a lengthy but straightforward verification to show that

$$\mathrm{Ext}_{\mathcal{C}_{dR}(X)}(\mathbf{1},\, \mathbf{1}) \longrightarrow H^1_{dR}(X,\, \mathbb{C})\,, \quad (E,\, \nabla) \longmapsto [E,\, \nabla]$$

is actually bijective. □

3.3. Properties of the pro-algebraic group scheme Θ.

Lemma 3.7. *Let* X *and* Y *be compact connected Kähler manifolds and* V *a holomorphic vector bundle on* $X \times Y$ *of rank* r *such that*

- *the restriction of* V *to* $\{x\} \times Y$ *is holomorphically isomorphic to* $\mathcal{O}_Y^{\oplus r}$ *for every* $x \in X$*, and*
- *the restriction of* V *to* $X \times \{y\}$ *is holomorphically isomorphic to* $\mathcal{O}_X^{\oplus r}$ *for every* $y \in Y$*.*

Then V *is holomorphically isomorphic to* $\mathcal{O}_{X\times Y}^{\oplus r}$*.*

Proof. Fix a point $y_0 \in Y$. For any point $x \in X$, the restriction V_x of V to $\{x\} \times Y$ trivializable. Hence the evaluation to y_0 of sections of V_x

$$H^0(Y,\, V_x) \longrightarrow V_{x,y_0}\,, \quad \sigma \longmapsto \sigma(y_0)$$

is an isomorphism. This implies that for the natural projection

$$p^X \,:\, X \times Y \longrightarrow X\,,$$

the direct image of V to X is

$$p_*^X V \,=\, V^{y_0} \,:=\, V|_{X\times\{y_0\}}\,. \tag{3.4}$$

The vector bundle V^{y_0} is supposed to be holomorphically trivial. Fixing r linearly independent holomorphic sections $\{s_1,\, \cdots,\, s_r\}$ of V^{y_0}, we get a holomorphic trivialization of V^{y_0}. Using the isomorphism in (3.4), each s_i produces a section

$$\widetilde{s}_i \,\in\, H^0(X,\, p_*^X V) \,=\, H^0(X\times Y,\, V)\,.$$

Now these holomorphic sections $\{\widetilde{s}_1,\, \cdots,\, \widetilde{s}_r\}$ of V trivialize V holomorphically. $\square$

Proposition 3.8. *The group scheme $\Theta(X,\, x_0)$ is the projective limit of connected algebraic groups.*

Proof. Consider an algebraic quotient $\Theta(X,\, x_0) \,\longrightarrow\, \Theta'$. We need to show that Θ' is connected and this amounts to showing that if $\Theta' \,\longrightarrow\, G$ is an algebraic quotient morphism to a finite group G, then G is the trivial group. Now, write R for the left regular representation of G and note that the multiplication operation $R\otimes R \,\longrightarrow\, R$ and the identity $\mathbb{C} \,\longrightarrow\, R$ are G–equivariant.

Since G is a quotient of $\Theta(X,\, x_0)$, any G–module gives a $\Theta(X,\, x_0)$–module. In particular, R gives a $\Theta(X,\, x_0)$–module. Let $(\mathcal{R},\, \nabla)$ be the object in the category $\mathcal{T}_{dR}(X)$ corresponding to this $\Theta(X,\, x_0)$–module R. We note that $\mathcal{R}$ is the trivial holomorphic vector $\mathcal{O}_X\otimes_{\mathbb{C}} R$ on X with fiber R. In particular, the rank of $\mathcal{R}$ is $|G|$. The above homomorphism $\mathbb{C} \,\longrightarrow\, R$ produces an arrow $\mathcal{O}_X \,\longrightarrow\, \mathcal{R}$. The above mentioned multiplication operation $R\otimes R \,\longrightarrow\, R$ produces another arrow $\mathcal{R}\otimes\mathcal{R} \,\longrightarrow\, \mathcal{R}$. Endowed with these, $\mathcal{R}$ becomes a locally free $\mathcal{O}_X$–algebra. Note, in addition, that giving $\mathcal{R}\otimes\mathcal{R}$ the tensor product connection, the arrow $\mathcal{R}\otimes\mathcal{R} \,\longrightarrow\, \mathcal{R}$ is also horizontal.

Let $f\,:\, Y \,\longrightarrow\, X$ be the analytic spectrum of $\mathcal{R}$ (see [SC, Exposé 19, Definition 2] and [Fi, 1.14-15]); the morphism f is finite and $f_*\mathcal{O}_Y \,=\, \mathcal{R}$. In addition, it is not hard to see that f is a *local biholomorphism* so that Y is a compact complex *manifold*; here are the details. Let $x \,\in\, X$ be arbitrary. We have a G–equivariant isomorphism of R-algebras

$$R\otimes R \,\longrightarrow\, R^{\oplus|G|}$$

which allows us to see that the $\mathcal{R}_x$–algebra $\mathcal{R}_x\otimes_{\mathcal{O}_{X,x}}\mathcal{R}_x$ (multiplication on the left) is isomorphic to $\mathcal{R}_x^{\oplus|G|}$. Consequently, $\mathcal{O}_{X,x} \,\longrightarrow\, \mathcal{R}_x$ is an étale morphism by flat descent [Ra, II, Proposition 4, p. 14]. Let now $y\in Y$ be above x. We know that $\mathcal{O}_{Y,y}$ is, as an $\mathcal{O}_{X,x}$–algebra, the localization of $\mathcal{R}_x$ at some maximal ideal [SC, Exposé 19, Proposition 6]. So, the natural arrow $\mathcal{O}_{X,x} \,\longrightarrow\, \mathcal{O}_{Y,y}$ is an isomorphism [Ra, VII, Proposition 3, p. 76].

Finally, Y is connected since any idempotent

$$e \,\in\, H^0(X,\, \mathcal{R})$$

is horizontal so that it produces a horizontal arrow $\mathcal{O}_X \,\longrightarrow\, \mathcal{R}$ and then a G–invariant element of R.

Because

$$\begin{aligned} 1 &= \dim H^0(Y, \mathcal{O}_Y) \\ &= \dim H^0(X, \mathcal{R}) \\ &= |G| \cdot \dim H^0(X, \mathcal{O}_X), \end{aligned}$$

we conclude that $|G| = 1$. □

Proposition 3.9. *Let (X, x_0) and (Y, y_0) be pointed compact Kähler manifolds. Set $P = X \times Y$, and let $i : X \longrightarrow P$ (respectively, $j : Y \longrightarrow P$) stand for the immersion defined by $x \longmapsto (x, y_0)$ (respectively, $y \longmapsto (x_0, y)$). Then, the arrow of affine group schemes*

$$(i_\natural, j_\natural) : \Theta(X, x_0) \times \Theta(Y, y_0) \longrightarrow \Theta(P, (x_0, y_0))$$

is an isomorphism.

Proof. Let p (respectively, q) be the natural projection of P to X (respectively, Y). In what follows, we abandon reference to the base points.

We know that the natural morphism

$$(i_\natural, j_\natural) : \varpi(X) \times \varpi(Y) \longrightarrow \varpi(P)$$

is an isomorphism.

We claim that $i_\natural : \Theta(X) \longrightarrow \Theta(P)$ is a closed and normal immersion.

First note that $i_\natural$ is closed since

$$p_\natural \circ i_\natural : \Theta(X) \longrightarrow \Theta(X)$$

is the identity map. Next, $i_\natural$ is normal since $i_\natural : \varpi(X) \longrightarrow \varpi(P)$ is a normal immersion and we posses the following commutative diagram

$$\begin{array}{ccc} \varpi(X) & \xrightarrow{i_\natural} & \varpi(P) \\ \mathbf{q}_X \Big\downarrow & & \mathbf{q}_P \Big\downarrow \\ \Theta(X) & \xrightarrow{i_\natural} & \Theta(P) \end{array}$$

in which the vertical arrows are the quotient morphisms. This proves the claim.

Clearly, the same arguments apply to $j_\natural : \Theta(Y) \longrightarrow \Theta(P)$.

Let

$$\chi : \Theta(P) \longrightarrow Q$$

be the cokernel of $i_\natural : \Theta(X) \longrightarrow \Theta(P)$ [Wa, 16.3, Theorem]. It is easy to see that the kernel of the composition

$$\Theta(Y) \xrightarrow{j_\natural} \Theta(P) \xrightarrow{\chi} Q$$

is trivial; indeed, $q_\natural : \Theta(P) \longrightarrow \Theta(Y)$ factors as

$$\Theta(P) \xrightarrow{\chi} Q \xrightarrow{r} \Theta(Y)$$

since $q_\natural \circ i_\natural$ is trivial, and $\mathrm{id}_{\Theta(Y)} = q_\natural \circ j_\natural$. We conclude that $\mathrm{Im}(i_\natural) \cap \mathrm{Im}(j_\natural) = \{e\}$. Now, a well-known lemma from group theory says that

$$(i_\natural,\, j_\natural) \,:\, \Theta(X) \times \Theta(Y) \longrightarrow \Theta(P)$$

is a normal monomorphism of group schemes [Bou, I.4.9, p. 48, Proposition 15]. As such, it is a closed immersion [Wa, 15.3, Theorem]. Finally, since

$$(i_\natural,\, j_\natural) \,:\, \varpi(X) \times \varpi(Y) \longrightarrow \varpi(P)$$

is a quotient morphism, we conclude that

$$(i_\natural,\, j_\natural) \,:\, \Theta(X) \times \Theta(Y) \longrightarrow \Theta(P)$$

is also one. Hence, $\Theta(X) \times \Theta(Y) \simeq \Theta(P)$. □

Take a pointed compact Kähler manifold $(X,\, x_0)$. Let

$$\Gamma \subset H^0(X,\, \Omega^1_X)^* \tag{3.5}$$

be the image of the homomorphism $H_1(X,\, \mathbb{Z}) \longrightarrow H^0(X,\, \Omega^1_X)^*$ that sends any

$$\gamma \in H_1(X,\, \mathbb{Z})$$

to the element of the dual vector space $H^0(X,\, \Omega^1_X)^*$ defined by the integral

$$\nu \longmapsto \int_\gamma \nu\,, \quad \nu \in H^0(X,\, \Omega^1_X)\,.$$

Let $\mathrm{A}(X) = H^0(X,\, \Omega^1_X)^*/\Gamma$ be the Albanese variety of X. Let

$$\mathcal{A} \,:\, X \longrightarrow \mathrm{A}(X) \tag{3.6}$$

be the Albanese map defined by

$$\mathcal{A}(x)(\nu) = \int_{x_0}^{x} \nu\,;$$

this does not depend on the choice of path from x_0 to x. We have $H^0(X,\, \Omega^1_X) = H^0(\mathrm{A}(X),\, \Omega^1_{\mathrm{A}(X)})$, and the homomorphism

$$\alpha \,:\, H^0(\mathrm{A}(X),\, \Omega^1_{\mathrm{A}(X)}) \longrightarrow H^0(X,\, \Omega^1_X) \tag{3.7}$$

defined by $\xi \longmapsto \mathcal{A}^*\xi$, for all $\xi \in H^0(\mathrm{A}(X),\, \Omega^1_{\mathrm{A}(X)})$, is an isomorphism. Hence the homomorphism

$$\overline{\alpha} \,:\, H^1(\mathrm{A}(X),\, \mathcal{O}_{\mathrm{A}(X)}) = \overline{H^0(\mathrm{A}(X),\, \Omega^1_{\mathrm{A}(X)})} \longrightarrow \overline{H^0(X,\, \Omega^1_X)} = H^1(X,\, \mathcal{O}_X) \tag{3.8}$$

is an isomorphism.

In view of the isomorphism α in (3.7) we conclude that there is a natural bijection between the holomorphic connections on $\mathcal{O}_X^{\oplus r}$ and the holomorphic connections on $\mathcal{O}_{\mathrm{A}(X)}^{\oplus r}$; this isomorphism sends a holomorphic connection D on $\mathcal{O}_{\mathrm{A}(X)}^{\oplus r}$ to the holomorphic connection $\mathcal{A}^*D$ on $\mathcal{O}_X^{\oplus r} = \mathcal{A}^*\mathcal{O}_{\mathrm{A}(X)}^{\oplus r}$.

If D is integrable, then $\mathcal{A}^*D$ is integrable. Notice that the converse need not be true in general. To construct examples of non-integrable D such that $\mathcal{A}^*D$ is integrable, we recall that there are examples of $(M,\, \mathbb{D})$, where M is a compact Riemann surface and $\mathbb{D}$

is an irreducible holomorphic connection on $\mathcal{O}_M^{\oplus 2}$ [CDHL], [BD], [BDH] (any holomorphic connection on a Riemann surface is automatically integrable). Since the fundamental group of a compact complex torus $\mathbb{T}$ is abelian, there is no irreducible integrable connection on a rank two bundle over $\mathbb{T}$. Also, the pullback of a reducible connection is reducible. Therefore, for any $(M,\, \mathbb{D})$ as above, the corresponding holomorphic connection on the Albanese variety $\mathrm{A}(M)$ is not integrable.

Note that Γ in (3.5) is a quotient of $\pi_1(X,\, x_0)$, because $H_1(X,\, \mathbb{Z})$ is a quotient of $\pi_1(X,\, x_0)$, and Γ is a quotient of $H_1(X,\, \mathbb{Z})$.

Proposition 3.10. *Let D be an integrable holomorphic connection on $\mathcal{O}_X^{\oplus r}$. Then there is an integrable holomorphic connection $\widetilde{D}$ on $\mathcal{O}_{\mathrm{A}(X)}^{\oplus r}$ such that*

$$(\mathcal{O}_X^{\oplus r},\, D) \,=\, (\mathcal{A}^*\mathcal{O}_{\mathrm{A}(X)}^{\oplus r},\, \mathcal{A}^*\widetilde{D})\,,$$

where $\mathcal{A}$ is the map in (3.6)*, if and only if the monodromy representation*

$$\rho_D \,:\, \pi_1(X,\, x_0) \,\longrightarrow\, \mathrm{GL}(r, \mathbb{C})$$

for D factors through the quotient group Γ of $\pi_1(X,\, x_0)$.

Proof. We have $\pi_1(\mathrm{A}(X)) \,=\, \Gamma$, and the homomorphism

$$\mathcal{A}_* \,:\, \pi_1(X,\, x_0) \,\longrightarrow\, \pi_1(\mathrm{A}(X))$$

induced by the Albanese map $\mathcal{A}$ in (3.6) actually coincides with the quotient homomorphism $\pi_1(X,\, x_0) \,\longrightarrow\, \Gamma$. Therefore, if

$$(\mathcal{O}_X^{\oplus r},\, D) \,=\, (\mathcal{A}^*\mathcal{O}_{\mathrm{A}(X)}^{\oplus r},\, \mathcal{A}^*\widetilde{D})$$

for some integrable holomorphic connection $\widetilde{D}$ on $\mathcal{O}_{\mathrm{A}(X)}^{\oplus r}$, then the monodromy representation $\rho_D \,:\, \pi_1(X,\, x_0) \,\longrightarrow\, \mathrm{GL}(r, \mathbb{C})$ for D factors through the quotient group Γ of $\pi_1(X,\, x_0)$.

To prove the converse, assume that the monodromy representation $\rho_D \,:\, \pi_1(X,\, x_0) \,\longrightarrow\, \mathrm{GL}(r, \mathbb{C})$ for D does factor through the quotient group Γ of $\pi_1(X,\, x_0)$. Therefore, the representation

$$\rho'_D \,:\, \Gamma \,=\, \pi_1(\mathrm{A}(X),\, \mathcal{A}(x_0)) \,\longrightarrow\, \mathrm{GL}(r, \mathbb{C}) \tag{3.9}$$

given by ρ_D produces a pair $(V,\, \widetilde{D})$, where

- V is a holomorphic vector bundle of rank r on $\mathrm{A}(X)$,
- $\widetilde{D}$ is an integrable holomorphic connection on V, and
- $(\mathcal{A}^*V,\, \mathcal{A}^*\widetilde{D}) \,=\, (\mathcal{O}_X^{\oplus r},\, D)$.

Consequently, to prove the proposition it suffices to show that

$$V \,\simeq\, \mathcal{O}_{\mathrm{A}(X)}^{\oplus r}\,. \tag{3.10}$$

Consider $\mathbb{C}^r$ as a Γ–module using the homomorphism ρ'_D (in (3.9)) together with the standard action of $\mathrm{GL}(r, \mathbb{C})$ on $\mathbb{C}^r$. Since Γ is an abelian group, we contend that the

Γ–module $\mathbb{C}^r$ decomposes as

$$\mathbb{C}^r = \bigoplus_{i=1}^{m} L_i \otimes U_i\,, \tag{3.11}$$

where L_i (respectively, U_i) is a one-dimensional (respectively, unipotent) representation of Γ. Indeed, let S be an indecomposable summand of the Γ–module $\mathbb{C}^r$ and, for any given $\gamma \in \Gamma$, consider the decomposition of S into generalized eigenspaces: $\bigoplus_i S_i$. Since every $\delta \in \Gamma$ commutes with γ, we can say that every S_i is invariant under Γ; it follows that γ has a single eigenvalue in S (recall that S is indecomposable). Associating to each γ the previous eigenvalue, we get a homomorphism $\Gamma \longrightarrow \mathbb{C}^*$ and hence a one dimensional representation L. Now, $L^\vee \otimes S$ is indecomposable and has only one eigenvalue, namely 1; here $L^\vee$ denotes the dual of L. Simultaneous triangularization — which follows easily from the fact that Γ is abelian — now shows that $L^\vee \otimes S$ is unipotent thus establishing the decomposition in (3.11).

From (3.11) we obtain a decomposition of the connection $(V,\, \widetilde{D})$ as

$$(V,\, \widetilde{D}) = \bigoplus_{i=1}^{m} \mathcal{L}_i \otimes \mathcal{U}_i\,, \tag{3.12}$$

where $\mathcal{L}_i$ stands for a rank one integrable connection and $\mathcal{U}_i$ stands for a connection with unipotent monodromy. Note that, since $\mathcal{U}_i$ has unipotent monodromy, each $\mathcal{L}_i$ is a sub-connection of V and hence $\mathcal{A}^*\mathcal{L}_i$ is a sub-connection of $\mathcal{A}^*V$. Using a dualization and Proposition 3.2, we can say that the vector bundle underlying $\mathcal{A}^*\mathcal{L}_i$ is trivial. Since the homomorphism

$$\mathrm{Pic}^0(\mathrm{A}(X)) \longrightarrow \mathrm{Pic}^0(X)\,, \quad L \longmapsto \mathcal{A}^*L\,,$$

where $\mathcal{A}$ is the map in (3.6), is an isomorphism, we conclude that

$$\text{the underlying holomorphic line bundle of } \mathcal{L}_i \text{ in (3.12) is trivial} \tag{3.13}$$

for all $1 \leq i \leq m$.

Next, we note that the connection $\mathcal{A}^*\mathcal{U}_i$ is a sub-connection of $\mathcal{A}^*(V) \otimes (\mathcal{A}^*\mathcal{L}_i^\vee)$ and hence, applying a dualization and Proposition 3.2, from (3.13) we conclude that

$$\mathcal{A}^*\mathcal{U}_i \in \mathcal{T}_{dR}(X)\,.$$

The holomorphic vector bundle underlying $\mathcal{U}_i$ will be denoted by U_i. The integrable holomorphic connection on U_i defining $\mathcal{U}_i$ will be denoted by D_i. Let s be the rank of U_i.

Since $\mathcal{U}_i$ is an integrable connection with unipotent monodromy, there is a filtration of holomorphic subbundles

$$0 = V_0 \subset V_1 \subset \cdots \subset V_i \subset \cdots \subset V_{r-1} \subset V_s = U_i \tag{3.14}$$

such that

(1) $\mathrm{rank}(V_i) = i$ for all $1 \leq i \leq s$,
(2) every quotient V_i/V_{i-1} is $\mathcal{O}_{\mathrm{A}(X)}$,
(3) each V_i is preserved by D_i, and

(4) for every $1 \leq i \leq s$, the connection on V_i/V_{i-1} induced by D_i is the trivial connection on $\mathcal{O}_{\mathrm{A}(X)}$ given by the de Rham differential.

Let

$$0 = \mathcal{A}^*V_0 \subset \mathcal{A}^*V_1 \subset \cdots \subset \mathcal{A}^*V_i \subset \cdots \subset \mathcal{A}^*V_{s-1} \subset \mathcal{A}^*V_s = \mathcal{A}^*U_i = \mathcal{O}_X^{\oplus s} \quad (3.15)$$

be the filtration of $\mathcal{A}^*U_i = \mathcal{O}_X^{\oplus s}$ obtained by pulling back the filtration in (3.14) using the map $\mathcal{A}$. We know that

$$(\mathcal{A}^*V_i)/(\mathcal{A}^*V_{i-1}) = \mathcal{A}^*(V_i/V_{i-1}) = \mathcal{O}_X$$

for all $1 \leq i \leq s$.

We will prove that the filtration in (3.15) splits holomorphically.

For any $1 \leq j \leq s$, let $\mathcal{S}_j = \mathcal{O}_X \subset \mathcal{O}_X^{\oplus r}$ be the j–th factor in the direct sum $\mathcal{O}_X^{\oplus s}$. Let ψ_j denote the following composition of homomorphisms:

$$\mathcal{S}_j \hookrightarrow \mathcal{O}_X^{\oplus s} = \mathcal{A}^*V_s \longrightarrow (\mathcal{A}^*V_s)/(\mathcal{A}^*V_{s-1}) = \mathcal{A}^*(V_s/V_{s-1}) = \mathcal{O}_X$$

(see (3.15)), where $\mathcal{A}^*V_s \longrightarrow (\mathcal{A}^*V_s)/(\mathcal{A}^*V_{s-1})$ is the natural quotient map. For some $1 \leq j_0 \leq s$, the above homomorphism ψ_{j_0} is nonzero. This j_0 is evidently an isomorphism. Hence we conclude that

$$\mathcal{O}_X^{\oplus s} = \mathcal{S}_{j_0} \oplus \left(\bigoplus_{j=1, j\neq j_0}^{s} \mathcal{S}_j \right) = \mathcal{A}^*V_s = \mathcal{A}^*V_{s-1} \oplus \mathcal{O}_X \,; \quad (3.16)$$

the above direct summand $\mathcal{O}_X \subset \mathcal{A}^*V_s$ is the image of $\mathcal{S}_{j_0}$ by the identification in (3.15) between $\mathcal{A}^*V_s$ and $\mathcal{O}_X^{\oplus s}$. Applying [At1, p. 315, Theorem 2] to the decomposition in (3.16) we conclude that

$$\mathcal{O}_X^{\oplus (s-1)} = \bigoplus_{j=1, j\neq j_0}^{s} \mathcal{S}_j = \mathcal{A}^*V_{s-1}$$

Repeating the above argument, after replacing $\mathcal{O}_X^{\oplus s}$ (respectively, $\mathcal{A}^*V_s$) by $\mathcal{O}_X^{\oplus (s-1)}$ (respectively, $\mathcal{A}^*V_{s-1}$), we conclude that

$$\mathcal{A}^*V_{s-1} = \mathcal{A}^*V_{s-2} \oplus \mathcal{O}_X \,.$$

Now proceeding inductively we conclude that the filtration in (3.15) splits holomorphically.

The homomorphism

$$H^1(\mathrm{A}(X),\, \mathcal{O}_{\mathrm{A}(X)}) \longrightarrow H^1(X,\, \mathcal{O}_X)\,, \quad \theta \longmapsto \mathcal{A}^*\theta$$

is an isomorphism. Since the extensions of $\mathcal{O}_{\mathrm{A}(X)}$ by $\mathcal{O}_{\mathrm{A}(X)}$ are parametrized by the cohomology $H^1(\mathrm{A}(X),\, \mathcal{O}_{\mathrm{A}(X)})$, from this and the fact that the filtration in (3.15) splits holomorphically it follows that the filtration in (3.14) splits holomorphically. This proves that $U_i = \mathcal{O}_{\mathrm{A}(X)}^{\oplus s}$. $\square$

Let us now study the case of "abelian" Kähler manifolds by first recalling certain fundamental facts from the theory of affine group schemes.

Let U be an algebraic affine and unipotent group scheme over $\mathbb{C}$ [Wa, 8.3]: there is a closed immersion of U into some group of strict upper triangular matrices. Endowing the nilpotent Lie algebra Lie(U) with its Baker–Campbell–Hausdorff multiplication, it is known that

$$\exp \,:\, \mathrm{Lie}(U) \longrightarrow U$$

is an isomorphism of group schemes [Ho, Theorem XVII.4.2, p. 232.]; in particular, if U is in addition a commutative group scheme, then $U \simeq \mathbb{G}_a^r$ for some r. Moreover, $r = \dim_{\mathbb{C}} \mathrm{Hom}(U, \mathbb{G}_a)$ [Wa, Theorem 8.4]. Said differently,

$$U \simeq \mathrm{Hom}(U, \mathbb{G}_a)\,.$$

Still in the topic of affine group schemes, for any given abstract abelian group Λ, we shall denote by Diag(Λ) the diagonalizable group scheme corresponding to the abstract group Λ as explained in [Wa, 2.2]; on the level of $\mathbb{C}$-points, Diag(Λ) is just Hom(Λ, $\mathbb{C}^\times$).

Proposition 3.11. *Suppose that* $\pi_1(X, x_0)$ *is abelian. Then*

$$\Theta(X, x_0) \simeq H^0(X, \Omega^1_X) \times \mathrm{Diag}(H^0(X, \Omega^1_X))\,.$$

Proof. Since $\pi_1(X, x_0)$ is abelian, the group scheme $\varpi(X, x_0)$ is abelian. This implies that $\Theta(X, x_0)$ is abelian and hence is a product of a unipotent U and a diagonal group scheme $\mathbb{D}$ [Wa, p. 70, Theorem, 9.5]. The arguments made in Lemma 3.5 and Lemma 3.6 jointly with the preliminary material on group schemes recalled above now allow us to explicitly determine U and $\mathbb{D}$ as wanted. Indeed, the group of characters of $U \times \mathbb{D}$ (respectively, additive characters) is simply the group of characters of $\mathbb{D}$ (respectively, additive characters of U) as explained in [Wa, Chapter 8], Corollary in 8.3 and Exercise 6. □

4. Riemann surfaces and compact complex tori

4.1. **Neutral Tannakian category for a Riemann surface.** Let X and Y be two compact connected Riemann surfaces of common genus g, with $g \geq 1$. Fix a point $x_0 \in X$. Let

$$\beta \,:\, X \longrightarrow Y \tag{4.1}$$

be a C^∞ orientation preserving diffeomorphism. Let

$$\widehat{\beta} \,:\, \pi_1(X, x_0) \longrightarrow \pi_1(Y, \beta(x_0))$$

be the homomorphism of fundamental groups induced by β. It produces a unique algebraic homomorphism

$$\beta_\natural \,:\, \varpi(X, x_0) \longrightarrow \varpi(Y, \beta(x_0)) \tag{4.2}$$

such that $\beta_\natural \circ \phi = \phi_Y \circ \widehat{\beta}$, where ϕ is the homomorphism in (2.1) and

$$\phi_Y \,:\, \pi_1(Y, \beta(x_0)) \longrightarrow \varpi(Y, \beta(x_0))$$

is the similar homomorphism for Y.

We will say that the homomorphism $\beta_\natural$ in (4.2) descends to a homomorphism from $\Theta(X, x_0)$, constructed in (3.1), to $\Theta(Y, \beta(x_0))$ if there is a homomorphism

$$\beta'_\natural \,:\, \Theta(X, x_0) \longrightarrow \Theta(Y, \beta(x_0))$$

such that

$$\beta'_\natural \circ \mathbf{q}_X \,=\, \mathbf{q}_Y \circ \beta_\natural\,,$$

where $\mathbf{q}_X$ is the homomorphism in (3.2) and

$$\mathbf{q}_Y \,:\, \varpi(Y,\, \beta(x_0)) \,\longrightarrow\, \Theta(Y,\, \beta(x_0))$$

is the similar homomorphism for Y, while $\beta_\natural$ is constructed in (4.2).

Theorem 4.1. *Assume that the homomorphism $\beta_\natural$ in (4.2) descends to a homomorphism from $\Theta(X,\, x_0)$ to $\Theta(Y,\, \beta(x_0))$. Then the two Riemann surfaces X and Y are isomorphic.*

Proof. Let $\beta'_\natural \,:\, \Theta(X,\, x_0) \,\longrightarrow\, \Theta(Y,\, \beta(x_0))$ be a homomorphism such that the diagram

$$\begin{array}{ccc} \varpi(X,\, x_0) & \xrightarrow{\beta_\natural} & \varpi(Y,\, \beta(x_0)) \\ \mathbf{q}_X \Big\downarrow & & \mathbf{q}_Y \Big\downarrow \\ \Theta(X,\, x_0) & \xrightarrow{\beta'_\natural} & \Theta(Y,\, \beta(x_0)) \end{array} \tag{4.3}$$

is commutative. Let

$$\begin{array}{ccc} \mathbb{X}_a(\Theta(Y,\, \beta(x_0))) & \longrightarrow & \mathbb{X}_a(\Theta(X,\, x_0)) \\ \lambda^2 \Big\downarrow & & \lambda^1 \Big\downarrow \\ \mathbb{X}_a(\varpi(Y,\, \beta(x_0))) & \xrightarrow{\rho} & \mathbb{X}_a(\varpi(X,\, x_0)) \end{array} \tag{4.4}$$

be the corresponding commutative diagram of vector spaces (see Section 3.2).

Now, according to Lemma 3.6,

$$\mathbb{X}_a(\Theta(X,\, x_0)) \,=\, H^0(X,\, \Omega^1_X) \quad \text{and} \quad \mathbb{X}_a(\varpi(X,\, x_0)) \,=\, H^1_{dR}(X,\, \mathbb{C})\,.$$

Similarly, we have

$$\mathbb{X}_a(\Theta(Y,\, \beta(x_0))) \,=\, H^0(Y,\, \Omega^1_Y) \quad \text{and} \quad \mathbb{X}_a(\varpi(Y,\, \beta(x_0))) \,=\, H^1_{dR}(Y,\, \mathbb{C})\,.$$

The homomorphism λ^1 in (4.4) is the natural inclusion of $H^0(X,\, \Omega^1_X)$ in $H^1_{dR}(X,\, \mathbb{C})$ given by the fact that any holomorphic 1–form on X is closed. The homomorphism

$$\rho \,:\, H^1_{dR}(Y,\, \mathbb{C}) \,=\, \mathbb{X}_a(\varpi(Y,\, \beta(x_0))) \,\longrightarrow\, \mathbb{X}_a(\varpi(X,\, x_0)) \,=\, H^1_{dR}(X,\, \mathbb{C}) \tag{4.5}$$

in (4.4) coincides with the pullback homomorphism

$$\beta^* \,:\, H^1_{dR}(Y,\, \mathbb{C}) \,\longrightarrow\, H^1_{dR}(X,\, \mathbb{C})\,, \quad c \longmapsto \beta^* c\,, \tag{4.6}$$

where β is diffeomorphism in (4.1).

For the isomorphism β^* in (4.6), we have

$$\beta^*(H^1(Y,\, \mathbb{Z})) \,=\, H^1(X,\, \mathbb{Z})\,.$$

Furthermore, since the diffeomorphism β is orientation preserving, it takes the natural symplectic pairing on $H^1_{dR}(Y,\, \mathbb{C})$ defined by

$$c_1 \otimes c_2 \,\longmapsto\, \int_Y c_1 \wedge c_2 \,\in\, \mathbb{C} \tag{4.7}$$

to the corresponding symplectic pairing on $H^1_{dR}(X,\, \mathbb{C})$.

The Jacobian $J(Y)$ of Y coincides with the following quotient:

$$(H^1_{dR}(Y,\, \mathbb{C})/H^0(Y,\, \Omega^1_Y))/H^1(Y,\, \mathbb{Z}) \;=\; J(Y)\,,$$

and the natural principal polarization on $J(Y)$ is constructed using the pairing in (4.7) and the complex structure of $H^1_{dR}(Y,\, \mathbb{C})$. More precisely, the holomorphic tangent bundle $TJ(Y)$ of $J(Y)$ is the trivial holomorphic vector bundle

$$J(Y) \times \overline{H^0(Y,\, \Omega^1_Y)} \;\longrightarrow\; J(Y)$$

with fiber $\overline{H^0(Y,\, \Omega^1_Y)} \;=\; H^1(Y,\, \mathcal{O}_Y)$. The Hermitian form on $\overline{H^0(Y,\, \Omega^1_Y)}$ defined by

$$\overline{c}_1 \otimes \overline{c}_2 \;\longmapsto\; -\sqrt{-1}\int_Y \overline{c}_1 \wedge c_2 \;\in\; \mathbb{C}\,, \quad c_1,\, c_2 \;\in\; H^0(Y,\, \Omega^1_Y)$$

produces the canonical principal polarization on $J(Y)$.

We noted above that the $\mathbb{C}$–linear isomorphism β^* in (4.6) takes $H^1(Y,\, \mathbb{Z})$ to isomorphically $H^1(X,\, \mathbb{Z})$ and takes the symplectic pairing in (4.7) to the corresponding pairing for X. Since

$$\rho \;=\; \beta^*\,,$$

where ρ is the homomorphism in (4.5) (and (4.4)), from the commutativity of the diagram in (4.4) we now conclude that $J(Y)$ is isomorphic to the Jacobian $J(X)$ of X as a principally polarized abelian variety. Now from the standard Torelli theorem (see [ACGH, Ch. VI, § 3, pp. 245–246]) we conclude that X is isomorphic to Y. $\square$

4.2. **Neutral Tannakian category for a compact complex torus.** Let $\mathbb{T}$ be a compact complex torus of complex dimension d. The group scheme $\varpi(\mathbb{T},\, x_0)$ is abelian because $\pi_1(\mathbb{T},\, x_0)$ is so. Hence the quotient $\Theta(X,\, x_0)$ of $\varpi(\mathbb{T},\, x_0)$ is also abelian. As in the proof of Theorem 4.1 we consider the corresponding additive character spaces $\mathbb{X}_a(\varpi(\mathbb{T},\, x_0))$ and $\mathbb{X}_a(\Theta(\mathbb{T},\, x_0))$ and their Lie algebras. From Lemma 3.6 we have

$$\mathbb{X}_a(\varpi(\mathbb{T},\, x_0)) \;=\; H^1_{dR}(\mathbb{T},\, \mathbb{C})$$

and

$$\mathbb{X}_a(\Theta(\mathbb{T},\, x_0)) \;=\; H^0(\mathbb{T},\, \Omega^1_{\mathbb{T}})\,,$$

where $\Omega^1_{\mathbb{T}}$ is the holomorphic cotangent bundle of $\mathbb{T}$. Consider the linear map

$$\Psi \;:\; H^0(\mathbb{T},\, \Omega^1_{\mathbb{T}}) \;=\; \mathbb{X}_a(\Theta(\mathbb{T},\, x_0)) \;\longrightarrow\; \mathbb{X}_a(\varpi(\mathbb{T},\, x_0)) \;=\; H^1_{dR}(\mathbb{T},\, \mathbb{C}) \tag{4.8}$$

induced by the homomorphism $\mathbf{q}_{\mathbb{T}}$ in (3.2). We note that Ψ coincides with the natural inclusion of $H^0(\mathbb{T},\, \Omega^1_{\mathbb{T}})$ in $H^1_{dR}(\mathbb{T},\, \mathbb{C})$ given by the fact that any holomorphic 1–form on $\mathbb{T}$ is actually closed.

Let $\mathbb{S}$ be a compact complex torus of complex dimension d, and let

$$\varphi \;:\; \mathbb{T} \;\longrightarrow\; \mathbb{S}$$

be a diffeomorphism. Let

$$\varphi_\natural \;:\; \varpi(\mathbb{T},\, x_0) \;\longrightarrow\; \varpi(\mathbb{S},\, \varphi(x_0)) \tag{4.9}$$

be the homomorphism corresponding to φ. We say that $\varphi_\natural$ descends to a homomorphism from $\Theta(\mathbb{T}, x_0)$ to $\Theta(\mathbb{S}, \varphi(x_0))$ if there is a homomorphism

$$\varphi'_\natural \,:\, \Theta(\mathbb{T}, x_0) \longrightarrow \Theta(\mathbb{S}, \varphi(x_0))$$

such that the diagram

$$\begin{array}{ccc} \varpi(\mathbb{T}, x_0) & \xrightarrow{\varphi_\natural} & \varpi(\mathbb{S}, \varphi(x_0)) \\ \mathbf{q}_{\mathbb{T}}\Big\downarrow & & \mathbf{q}_{\mathbb{S}}\Big\downarrow \\ \Theta(\mathbb{T}, x_0) & \xrightarrow{\varphi'_\natural} & \Theta(\mathbb{S}, \varphi(x_0)) \end{array} \tag{4.10}$$

is commutative, where $\mathbf{q}_{\mathbb{T}}$ and $\mathbf{q}_{\mathbb{S}}$ are the projections in (3.2).

Proposition 4.2. *Let $\varphi \,:\, \mathbb{T} \longrightarrow \mathbb{S}$ be a diffeomorphism such that the homomorphism $\varphi_\natural$ in* (4.9) *descends to a homomorphism from $\Theta(\mathbb{T}, x_0)$ to $\Theta(\mathbb{S}, \varphi(x_0))$. Then there is a biholomorphism*

$$\widetilde{\varphi} \,:\, \mathbb{T} \longrightarrow \mathbb{S}$$

which is homotopic to the map φ.

Proof. Let $\varphi'_\natural \,:\, \Theta(\mathbb{T}, x_0) \longrightarrow \Theta(\mathbb{S}, \varphi(x_0))$ be the homomorphism such the diagram in (4.10) is commutative. Let

$$\begin{array}{ccccccc} H^0(\mathbb{S}, \Omega^1_{\mathbb{S}}) & = & \mathbb{X}_a(\Theta(\mathbb{S}, \varphi(x_0))) & \longrightarrow & \mathbb{X}_a(\Theta(\mathbb{T}, x_0)) & = & H^0(\mathbb{T}, \Omega^1_{\mathbb{T}}) \\ & & \xi^2\Big\downarrow & & \xi^1\Big\downarrow & & \\ H^1_{dR}(\mathbb{S}, \mathbb{C}) & = & \mathbb{X}_a(\varpi(\mathbb{S}, \varphi(x_0))) & \xrightarrow{\delta} & \mathbb{X}_a(\varpi(\mathbb{T}, x_0)) & = & H^1_{dR}(\mathbb{T}, \mathbb{C}) \end{array} \tag{4.11}$$

be the commutative diagram of $\mathbb{C}$–linear maps corresponding to (4.10).

Let

$$\varphi^* \,:\, H^1_{dR}(\mathbb{S}, \mathbb{C}) \longrightarrow H^1_{dR}(\mathbb{T}, \mathbb{C}), \quad c \longmapsto \varphi^* c \tag{4.12}$$

be the pullback map. The homomorphism δ in (4.11) coincides with the homomorphism φ^* in (4.12). Therefore, from (4.11) we conclude that

$$\varphi^*(H^0(\mathbb{S}, \Omega^1_{\mathbb{S}})) \,=\, H^0(\mathbb{T}, \Omega^1_{\mathbb{T}})\,. \tag{4.13}$$

We also have

$$\varphi^*(H^1(\mathbb{S}, \mathbb{Z})) \,=\, H^1(\mathbb{T}, \mathbb{Z}) \tag{4.14}$$

because φ is a diffeomorphism.

We consider $\mathbb{T}$ (respectively, $\mathbb{S}$) as a complex abelian Lie group by taking x_0 (respectively, $\varphi(x_0)$) to be the identity element of $\mathbb{T}$ (respectively, $\mathbb{S}$). From (4.13) and (4.14) it follows immediately that the homomorphism φ^* in (4.12) induces a holomorphic isomorphism

$$\begin{aligned} \widetilde{\varphi}^\vee \,:\, \mathbb{S}^\vee &:= (H^1_{dR}(\mathbb{S}, \mathbb{C})/H^0(\mathbb{S}, \Omega^1_{\mathbb{S}}))/H^1(\mathbb{S}, \mathbb{Z}) \\ &\longrightarrow (H^1_{dR}(\mathbb{T}, \mathbb{C})/H^0(\mathbb{T}, \Omega^1_{\mathbb{T}}))/H^1(\mathbb{T}, \mathbb{Z}) \,=\, \mathbb{T}^\vee\,; \end{aligned}$$

here $\mathbb{S}^\vee \,=\, \mathrm{Pic}^0(\mathbb{S})$ is the dual torus of $\mathbb{S}$, and $\mathbb{T}^\vee \,=\, \mathrm{Pic}^0(\mathbb{T})$ is the dual torus of $\mathbb{T}$. Let

$$\widetilde{\varphi} \,:\, \mathbb{T} \,=\, (\mathbb{T}^\vee)^\vee \longrightarrow (\mathbb{S}^\vee)^\vee \,=\, \mathbb{S}$$

be the dual of the above homomorphism $\widetilde{\varphi}^\vee$.

From the construction of the above homomorphism $\widetilde{\varphi}$ it is evident that the pullback homomorphism

$$\widetilde{\varphi}^* : H^1_{dR}(\mathbb{S}, \mathbb{C}) \longrightarrow H^1_{dR}(\mathbb{T}, \mathbb{C}), \quad c \longmapsto \widetilde{\varphi}^* c$$

coincides with the homomorphism φ^* in (4.12). This implies that the two maps $\widetilde{\varphi}$ and φ are homotopic. □

5. Completely decomposable vector bundles

As before, X is a compact connected Kähler manifold. We define and study in this section a subcategory of $\mathcal{C}_{dR}(X)$ given by the following:

Definition 5.1. Let $\mathcal{D}_{dR}(X)$ be the full subcategory of $\mathcal{C}_{dR}(X)$ whose objects are pairs $(E,\, D)$ satisfying the condition that the holomorphic vector bundle E is a direct sum of holomorphic line bundles.

It is straightforward to check that $\mathcal{D}_{dR}(X)$ is stable under tensor products and duals. In order to prove that $\mathcal{D}_{dR}(X)$ is abelian we will need the next two lemmas.

Lemma 5.2. *Let L be a holomorphic line bundle on X. Then L admits a holomorphic connection if and only if $c_1(L) = 0$. If L admits a holomorphic connection, then any holomorphic connection on L is integrable.*

Let L_i, $1 \leq i \leq r$, be holomorphic line bundles on X such that the holomorphic vector bundle $\bigoplus_{i=1}^r L_i$ admits a holomorphic connection. Then each L_i admits a holomorphic connection; in other words, $c_1(L_i) = 0$.

Proof. Since X is Kähler, if L admits a holomorphic connection, then $c_1(L) = 0$ [At2, pp. 192–193, Theorem 4]. If $c_1(L) = 0$, then L admits an integrable holomorphic connection ∇^L whose monodromy lies in U(1) see [LT, Corollary 1.3.12, p.40]; note also that any holomorphic line bundle admits a Hermitian–Einstein metric [UY], [LT, p. 61, Theorem 3.0.1].

If L admits a (integrable) holomorphic connection ∇^L, then any holomorphic connection on L is of the form $\nabla^L + \beta$, where $\beta \in H^0(X,\, \Omega^1_X)$. The curvature of $\nabla^L + \beta$ is $\mathrm{d}\beta$, because ∇^L is integrable. Since any holomorphic 1–form on X is closed, we conclude that $\nabla^L + \beta$ is integrable.

Let $p_j : \bigoplus_{i=1}^r L_i \longrightarrow L_j$ be the projection to the j–th factor. Then for any holomorphic connection D on $\bigoplus_{i=1}^r L_i$, the composition of homomorphisms

$$L_j \hookrightarrow \bigoplus_{i=1}^r L_i \stackrel{D}{\longrightarrow} (\bigoplus_{i=1}^r L_i) \otimes \Omega^1_X \stackrel{p_j \otimes \mathrm{Id}_{\Omega^1_X}}{\longrightarrow} L_j \otimes \Omega^1_X$$

is a holomorphic connection on L_j. □

Lemma 5.3. *Let L_i, $1 \leq i \leq r$, and $\mathcal{L}_j$, $1 \leq j \leq \ell$, be holomorphic line bundles on X such that $c_1(L_i) = 0 = c_1(\mathcal{L}_j)$ for all i, j. Let*

$$f : \bigoplus_{i=1}^{r} L_i \longrightarrow \bigoplus_{j=1}^{\ell} \mathcal{L}_j$$

be a holomorphic homomorphism. Then the following two hold:

(1) *If* kernel$(f) \neq 0$, *then* kernel(f) *is a direct sum of holomorphic line bundles of vanishing first Chern class.*
(2) *If* cokernel$(f) \neq 0$, *then* cokernel(f) *is a direct sum of holomorphic line bundles of vanishing first Chern class.*

Proof. Since X is Kähler every holomorphic line bundle L' on X with $c_1(L') = 0$ admits a flat unitary connection. In fact, there is a unique flat unitary connection on L'; the flat Hermitian structure is unique up to a constant scalar multiplication. Equip each L_i (respectively, $\mathcal{L}_j$) with an integrable holomorphic connection ∇^i (respectively, $\widetilde{\nabla}^j$) such that the monodromy lies in U(1). Therefore, $\bigoplus_{i=1}^r \nabla^i$ is a flat unitary connection on $\bigoplus_{i=1}^r L_i$, and $\bigoplus_{j=1}^\ell \widetilde{\nabla}^j$ is a flat unitary connection on $\bigoplus_{j=1}^\ell \mathcal{L}_j$. Since any flat unitary connection is a Hermitian–Einstein connection with respect to any Kähler metric, $\bigoplus_{i=1}^r \nabla^i$ and $\bigoplus_{j=1}^\ell \widetilde{\nabla}^j$ are Hermitian–Einstein connections on $\bigoplus_{i=1}^r L_i$ and $\bigoplus_{j=1}^\ell \mathcal{L}_j$ respectively.

If A and B are holomorphic vector bundles on X with $c_1(A) = 0 = c_1(B)$, and are equipped with Hermitian–Einstein connections ∇^A and ∇^B respectively, then every holomorphic homomorphism $f' : A \longrightarrow B$ is flat with respect to the connection on the holomorphic vector bundle Hom(A, B) induced by ∇^A and ∇^B (see [LT, p. 50, Theorem 2.2.1]). Therefore, we conclude that the holomorphic homomorphism f in the lemma intertwines the connections $\bigoplus_{i=1}^r \nabla^i$ on $\bigoplus_{i=1}^r L_i$ and $\bigoplus_{j=1}^\ell \widetilde{\nabla}^j$ on $\bigoplus_{j=1}^\ell \mathcal{L}_j$ (equivalently, it is flat with respect to the connection on Hom$(\bigoplus_{i=1}^r L_i, \bigoplus_{j=1}^\ell \mathcal{L}_j)$ induced by $\bigoplus_{i=1}^r \nabla^i$ and $\bigoplus_{j=1}^\ell \widetilde{\nabla}^j$). Consequently, kernel$(f)$ is preserved by the connection $\bigoplus_{i=1}^r \nabla^i$ on $\bigoplus_{i=1}^r L_i$ and image(f) is preserved by the connection $\bigoplus_{j=1}^\ell \widetilde{\nabla}^j$ on $\bigoplus_{j=1}^\ell \mathcal{L}_j$. From Remark 2.1 we know that both kernel(f) and image(f) are locally free. Therefore, kernel(f) is a holomorphic subbundle of $\bigoplus_{i=1}^r L_i$, and cokernel(f) is a quotient bundle of $\bigoplus_{j=1}^\ell \mathcal{L}_j$. Note that cokernel$(f)$ has an integrable holomorphic connection induced by $\bigoplus_{j=1}^\ell \widetilde{\nabla}^j$, because image$(f)$ is preserved by $\bigoplus_{j=1}^\ell \widetilde{\nabla}^j$.

Since kernel(f) and cokernel(f) both admit integrable holomorphic connections, we have

$$c_1(\text{kernel}(f)) = 0 = c_1(\text{cokernel}(f))$$

(see [At2, pp. 192–193, Theorem 4]), and hence we conclude that

$$\text{degree}(\text{kernel}(f)) = 0 = \text{degree}(\text{cokernel}(f)) \tag{5.1}$$

with respect to any Kähler form on X.

Any holomorphic line bundle on X is stable. Since $\bigoplus_{i=1}^r L_i$ is a direct sum of stable bundles of slope zero, we conclude that $\bigoplus_{i=1}^r L_i$ is a polystable vector bundle. Any subbundle of degree zero of a polystable vector bundle of degree zero is again polystable. Therefore, from (5.1) we conclude that kernel(f) is a polystable vector bundle of degree zero, if kernel(f) is nonzero. Assume that kernel(f) is nonzero, so kernel(f) is a polystable vector bundle of degree zero. Hence kernel(f) is a direct sum of stable vector bundle of degree zero. If S is a stable subbundle of kernel(f) of degree zero, then S is also a stable subbundle of $\bigoplus_{i=1}^r L_i$ of degree zero, because kernel(f) is a subbundle of $\bigoplus_{i=1}^r L_i$. But any stable subbundle of $\bigoplus_{i=1}^r L_i$ of degree zero is holomorphically isomorphic to L_i for some $1 \leq i \leq r$. Therefore, we conclude that kernel(f) is a direct sum of holomorphic line bundles of vanishing first Chern class, if kernel$(f) \neq 0$.

Next we note that $\bigoplus_{j=1}^{\ell} \mathcal{L}_j$ is a polystable vector bundle of degree zero, and cokernel(f) is a quotient bundle of $\bigoplus_{j=1}^{\ell} \mathcal{L}_j$ of degree zero. Hence cokernel(f) is a polystable vector bundle of degree zero, if cokernel(f) is nonzero. Assume that cokernel(f) is nonzero. So cokernel(f) is a polystable vector bundle of degree zero. Therefore, cokernel(f) is a direct sum of stable vector bundles of degree zero. Any stable quotient of cokernel(f) of degree zero is also a stable quotient of $\bigoplus_{j=1}^{\ell} \mathcal{L}_j$ of degree zero, because cokernel(f) is a quotient of $\bigoplus_{j=1}^{\ell} \mathcal{L}_j$. But any stable quotient of $\bigoplus_{j=1}^{\ell} \mathcal{L}_j$ of degree zero is holomorphically isomorphic to $\mathcal{L}_j$ for some $1 \leq j \leq \ell$. Therefore, if cokernel$(f) \neq 0$, then cokernel(f) is a direct sum of holomorphic line bundles of vanishing first Chern class. □

Using Lemma 5.3 it can be deduced that the category $\mathcal{D}_{dR}(X)$ is abelian.

Fix a point $x_0 \in X$. Equip the category $\mathcal{D}_{dR}(X)$ with the exact functor to the category of finite dimensional complex vector spaces defined by $(E,\, D) \longmapsto E|_{x_0}$. Let

$$\Delta(X,\, x_0) \tag{5.2}$$

be the affine group scheme over $\mathbb{C}$ corresponding to $\mathcal{D}_{dR}(X)$ [DMOS, p. 130, Theorem 2.11]. Note that $\Delta(X,\, x_0)$ is the target of a morphism of affine group schemes

$$\varpi(X,\, x_0) \longrightarrow \Delta(X,\, x_0)\,, \tag{5.3}$$

where $\varpi(X,\, x_0)$ is the group scheme in (2.1). According to the standard criterion [DMOS, Proposition 2.21, p. 139], amplified by [BHdS, Lemma 2.1], using Lemma 5.3 it follows that the homomorphism in (5.3) is a quotient morphism.

The homomorphism $\mathbf{q}_X$ in (3.2) factors through the quotient $\Delta(X,\, x_0)$ and gives a quotient homomorphism

$$\mathbf{p}_X \,:\, \Delta(X,\, x_0) \longrightarrow \Theta(X,\, x_0)\,. \tag{5.4}$$

For a pointed compact Kähler manifold $(Y,\, y_0)$ it can be shown that the natural homomorphism

$$\Delta(X,\, x_0) \times \Delta(Y, y_0) \longrightarrow \Delta(X \times Y,\, (x_0,\, y_0))$$

is an isomorphism; it's proof is similar to that of Proposition 3.9.

The homomorphism of additive character $\mathbb{C}$–vector spaces

$$\mathbb{X}_a(\Theta(X,\, x_0)) \longrightarrow \mathbb{X}_a(\Delta(X,\, x_0))$$

induced by $\mathbf{p}_X$ in (5.4) is an isomorphism as follows from Lemma 5.3 and the interpretation of these vector spaces as extension groups (see Section 3.2). In view of this, examining the proofs of Theorem 4.1 and Proposition 4.2 we conclude that these two results remain valid if $\Theta(X, x_0)$ is replaced by the group scheme $\Delta(X, x_0)$ in (5.2).

6. Connections on pseudostable vector bundles

We shall now relate the group scheme $\Theta(X, x_0)$ of Section 3.1 to another group scheme which deals only with a full subcategory of vector bundles.

Definition 6.1. A holomorphic vector bundle E over X is said to be pseudostable (see [BG]) if the Higgs vector bundle $(E, 0)$ is an object of the category $\mathcal{C}_{Dol}(X)$ introduced in Section 2. In other words, E is pseudostable if and only if

- $\text{degree}(E) = 0$,
- $ch_2(E) \wedge \omega^{d-2} = 0$, and
- there exists a filtration of E by holomorphic *subbundles*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{\ell-1} \subset E_\ell = E$$

  in which every E_j/E_{j-1}, $1 \leq j \leq \ell$, is stable and of degree zero.

(Actually the third condition implies the first condition.) The full subcategory of pseudostable vector bundles on X will be denoted by $\text{Vect}_0^{\text{s}}(X)$.

The following theorem, which is based on the works of Simpson and Corlette, is proved in [Si2] (see [Si2, pp. 35–37, (3.4.1)–(3.4.5)] and [Si2, p. 70]).

Theorem 6.2 ([Si2, (3.4.1)–(3.4.5)]). *The following three statements hold.*

1. *The category* $\text{Vect}_0^{\text{s}}(X)$ *is abelian and stable under the tensor product of vector bundles.*
2. *Any* $E \in \text{Vect}_0^{\text{s}}(X)$ *carries a canonical holomorphic integrable connection.*
3. *If* X *is projective, and the cohomology class of* ω *is rational, then* $\text{Vect}_0^{\text{s}}(X)$ *coincides with the category of all semistable vector bundles* E *such that*

$$ch_1(E) \wedge \omega^{d-1} = ch_2(E) \wedge \omega^{d-2} = 0$$

   (see [Si2, p. 39, Theorem 2] *for it).*

Remark 6.3. As mentioned earlier in Section 2, the category $\mathcal{C}_{Dol}(X)$ is equivalent to $\mathcal{C}_{dR}(X)$ [Si2, p. 36, Lemma 3.5]. Take $(E, \theta) \in \mathcal{C}_{Dol}(X)$, and let $(V, \nabla) \in \mathcal{C}_{dR}(X)$ be the object corresponding to (E, θ) by the equivalence of categories between $\mathcal{C}_{Dol}(X)$ and $\mathcal{C}_{dR}(X)$. Although the C^∞ vector bundles underlying E and V coincide, their holomorphic structure do not coincide in general. However, if $\theta = 0$, then the holomorphic structures of E and V coincide. Therefore, ∇ is a holomorphic integrable connection on $V = E$. We note that if E is polystable, then this holomorphic connection ∇ on E coincides with the unique holomorphic integrable connection on E whose monodromy is unitary. See [BS,

p. 4004, Theorem 3.1] for a generalization of this canonical connection on $E \in \mathrm{Vect}_0^{\mathrm{s}}(X)$ to the principal bundles over X.

Remark 6.4. Take $E_1,\, E_2 \in \mathrm{Vect}_0^{\mathrm{s}}(X)$. The canonical holomorphic integrable connections on E_1 and E_2 (see Remark 6.3) will be denoted by ∇^1 and ∇^2 respectively. Let

$$\Phi\, :\, E_1 \longrightarrow E_2$$

be a homomorphism of coherent analytic sheaves. Then Φ is a homomorphism from $(E_1,\, 0) \in \mathcal{C}_{Dol}(X)$ to $(E_2,\, 0) \in \mathcal{C}_{Dol}(X)$. Therefore, from the equivalence of categories between $\mathcal{C}_{Dol}(X)$ and $\mathcal{C}_{dR}(X)$ ([Si2, p. 36, Lemma 3.5]) we conclude that Φ is a homomorphism from $(E_1,\, \nabla^1) \in \mathcal{C}_{dR}(X)$ to $(E_2,\, \nabla^2) \in \mathcal{C}_{dR}(X)$. In particular, $\Phi(E_1) \subset E_2$ is preserved by the integrable holomorphic connection ∇^2 on E_2. Now setting E_1 to be the trivial line bundle $\mathcal{O}_X$ we conclude that every holomorphic section $s \in H^0(X,\, E_2)$ is flat (same as integrable) with respect to the integrable holomorphic connection ∇^2 on E_2. It also follows that the canonical connection on $E_1 \oplus E_2$ coincides with $\nabla^1 \oplus \nabla^2$. Moreover, the canonical connection on $E_1 \otimes E_2$ (respectively, $\mathrm{Hom}(E_1,\, E_2)$) coincides with the connection on $E_1 \otimes E_2$ (respectively, $\mathrm{Hom}(E_1,\, E_2)$) induced by ∇^1 and ∇^2. Also, the canonical connection on the dual vector bundle E_1^* coincides with the one induced by the connection ∇^1 on E_1. These properties of the canonical connection were crucial in the proofs of [BS, p. 4004, Theorem 3.1] and [BG, p. 20, Theorem 1.1].

It then follows that there exists an affine group scheme $\pi^{\mathrm{S}}(X,\, x_0)$ over $\mathbb{C}$ such that the functor

$$\bullet|_{x_0}\, :\, \mathrm{Vect}_0^{\mathrm{s}}(X) \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

induces an equivalence between $\mathrm{Vect}_0^{\mathrm{s}}(X)$ and the category of finite dimensional algebraic representations of $\pi^{\mathrm{S}}(X,\, x_0)$.

Following the path taken in the previous sections, we define $\mathcal{S}_{dR}(X)$ as the full subcategory of $\mathcal{C}_{dR}(X)$ consisting of those $(E,\, \nabla)$ such that E belongs to $\mathrm{Vect}_0^{\mathrm{s}}(X)$. This produces another group scheme

$$\Sigma(X,\, x_0) \tag{6.1}$$

whose category of representations is, via the functor $\bullet|_{x_0}$, simply $\mathcal{S}_{dR}(X)$.

Using the forgetful functor

$$U\, :\, \mathcal{S}_{dR}(X) \longrightarrow \mathrm{Vect}_0^{\mathrm{s}}(X)$$

and the natural inclusion

$$J\, :\, \mathcal{T}_{dR}(X) \longrightarrow \mathcal{S}_{dR}(X),$$

we arrive at morphisms

$$\pi^{\mathrm{S}}(X,\, x_0) \xrightarrow{\mathbf{u}_X} \Sigma(X,\, x_0) \xrightarrow{\mathbf{j}_X} \Theta(X,\, x_0)\,. \tag{6.2}$$

Clearly, $\mathbf{j}_X \circ \mathbf{u}_X$ is the trivial homomorphism.

We shall now explore some properties of $\mathbf{u}_X$. For that, we make a group theoretical interlude. No particular property concerning the ground field $\mathbb{C}$ is required so, in what follows, an "affine group scheme" should be interpreted as an "affine group scheme over a unspecified field."

From the existence of limits in the category of affine group schemes, we can make:

Definition 6.5. Let G be an affine group scheme and $H \subset G$ a closed subgroup scheme. The normal closure of H inside G is the intersection of all normal and closed subgroup schemes of G containing H.

Here are two simple features of the normal closure. The reader unfamiliar with the construction of arbitrary quotients in the theory of group schemes should consult chapters 15 and 16 of [Wa]. In particular, the verification of the following result is immediate once 15.4 and 16.3 in [Wa] are understood.

Lemma 6.6. *Let G be an affine group scheme, $H \subset G$ a closed subgroup scheme and N the normal closure of H in G.*

(1) *For each quotient morphism $q : G \longrightarrow Q$ of group schemes such that $q|_H$ is the trivial morphism, there exists a factorization*

$$\begin{array}{ccc} G & \xrightarrow{q} & Q \\ \downarrow & \nearrow_{\bar{q}} & \\ G/N. & & \end{array}$$

(2) *Let $D \subset G$ be a closed and normal subgroup scheme containing H. Suppose that D enjoys the property attributed to N in the previous item. Then $D = N$.*

Theorem 6.7. *The following three statements hold.*

(1) *In* (6.2)*, the morphism $\mathbf{u}_X$ is a closed immersion, and $\mathbf{j}_X$ is a quotient map.*
(2) *The normal closure of* $\mathrm{Im}(\mathbf{u}_X)$ *inside* $\Sigma(X, x_0)$ *is* $\mathrm{Ker}(\mathbf{j}_X)$.
(3) *The morphism $\mathbf{u}_X$ possesses a left inverse*

$$\mathbf{v}_X : \Sigma(X, x_0) \longrightarrow \pi^{\mathrm{S}}(X, x_0).$$

Proof. The morphism

$$\mathbf{q}_X : \varpi(X, x_0) \longrightarrow \Theta(X, x_0)$$

in (3.2) factors through $\mathbf{j}_X$ so that the fact that $\mathbf{q}_X$ is a quotient morphism shows that $\mathbf{j}_X$ is likewise. That $\mathbf{u}_X$ is a closed immersion follows from the standard criterion [DMOS, p. 139, Proposition 2.21] and Theorem 6.2.

To show that $\mathrm{Ker}(\mathbf{q}_X)$ is the normal closure of $\mathrm{Im}(\mathbf{u}_X)$ inside $\Sigma(X, x_0)$, we verify condition (2) in Lemma 6.6. Let

$$\rho : \Sigma(X, x_0) \longrightarrow G$$

be a quotient morphism such that $\rho \circ \mathbf{u}_X$ is trivial. Let

$$\xi : \mathrm{Rep}_{\mathbb{C}}(\Sigma(X, x_0)) \longrightarrow \mathcal{S}_{dR}(X)$$

be an inverse to the tensor equivalence $\bullet|_{x_0}$ [Sa, I.4.4] and let Res_ρ be the functor restricting representations from of G to $\Sigma(X, x_0)$. It follows that, for each $V \in \mathrm{Rep}_{\mathbb{C}}(G)$, the vector bundle underlying $\xi \circ \mathrm{Res}_\rho(V)$ is trivial, which means that $\xi \circ \mathrm{Res}_\rho$ is in fact a functor to $\mathcal{T}_{dR}(X)$. Passing to group schemes we obtain a morphism $\Theta(X, x_0) \longrightarrow G$ factoring ρ [Sa, II.3.3.1, 148ff].

Assigning to any pseudostable vector bundle $E \in \mathrm{Vect}_0^{\mathrm{s}}(X)$ the canonical integrable holomorphic connection on E (see Theorem 6.2(2)), a functor is obtained; call it

$$\mathbb{V} : \mathrm{Vect}_0^{\mathrm{s}}(X) \longrightarrow \mathcal{S}_{dR}(X).$$

This $\mathbb{V}$ is a tensor functor (see Remark 6.4). Hence $\mathbb{V}$ produces the section $\mathbf{v}_X$ to $\mathbf{u}_X$. □

Proposition 6.8. *Suppose that X is a compact Riemann surface of genus at least two. Then* $\mathrm{Im}(\mathbf{u}_X)$ *is not normal in* $\Sigma(X, x_0)$.

Proof. If $\mathrm{Im}(\mathbf{u}_X)$ is normal, then, for each representation

$$\rho : \Sigma(X, x_0) \longrightarrow \mathrm{GL}(V),$$

the subspace $V^{\pi^S(X,x_0)}$ is invariant under $\Sigma(X, x_0)$. This translates into the following property on the side of connections. Given $(E, \nabla) \in \mathcal{S}_{dR}(X)$, the largest trivial subbundle of E is preserved by ∇.

Let L be a non-trivial invertible sheaf affording an integrable holomorphic connection

$$D : L \longrightarrow L \otimes \Omega_X^1.$$

In addition, let us choose L satisfying $H^0(X, L \otimes \Omega_X^1) \neq 0$ (this is possible given the hypothesis on the genus). Let

$$\varphi \in H^0(X, L \otimes \Omega_X^1) \setminus \{0\}$$

and consider the connection ∇ on $\mathcal{O}_X \oplus L$ defined, on an unspecified open subset, by

$$\nabla(a, \ell) = (da, a\varphi + D\ell).$$

Then, $\mathcal{O}_X \oplus \{0\} \subset \mathcal{O}_X \oplus L$ is the largest trivial subbundle, and $\nabla(1, 0) = (0, \varphi)$, meaning that $\mathcal{O}_X \oplus \{0\}$ is not preserved by ∇. □

Remark 6.9. Let $\widetilde{\mathrm{Vect}^{\mathrm{s}}}_0(X)$ be the full subcategory of $\mathrm{Vect}_0^{\mathrm{s}}(X)$ defined by all holomorphic vector bundles in $\mathrm{Vect}_0^{\mathrm{s}}(X)$ that are direct sum of holomorphic line bundles. It is a Tannakian subcategory, and defines a quotient group scheme

$$\pi^{\mathrm{S}}(X, x_0) \longrightarrow \widetilde{\pi}^{\mathrm{S}}(X, x_0)$$

of $\pi^{\mathrm{S}}(X, x_0)$. A result similar to Theorem 6.7 can be deduced by replacing Σ with Δ (constructed in (5.2)) and π^S with $\widetilde{\pi}^{\mathrm{S}}$.

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