THE ACTION OF THE ETALE FUNDAMENTAL GROUP SCHEME ON THE CONNECTED COMPONENT OF THE ESSENTIALLY FINITE ONE

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ABSTRACT. We follow the pattern in [Ota17, Section 4] to define an action of the etale fundamental group *scheme* $\pi^{\text{ét}}(X)$ on the local component of the essentially finite fundamental group scheme $\pi(X)$ of Nori. We show that the associated representation is faithful when X is a curve of genus $\geqslant 2$.

1. INTRODUCTION

Let X be a connected, proper and reduced algebraic scheme over a perfect field k, and x a k-rational point of X. In his seminal work [Nor76], M. V. Nori detected that a full subcategory of the category of vector bundles on X can be used to produce, via the Tannakian correspondence, an affine group scheme $\pi(X, x)$ over k which, colloquially speaking, classifies torsors with finite structural group. If the characteristic of k is positive, $\pi(X, x)$ possesses two relevant canonical quotients: $\pi^{\text{ét}}(X, x)$, which is the largest pro-etale one, and $\pi^{\text{loc}}(X, x)$, which is the largest local one. By considering the kernel of the morphism $\pi(X, x) \to \pi^{\text{ét}}(X, x)$, we then obtain another local affine group scheme, call it $\pi(X, x)^{\text{o}}$, and the question concerning the relation between $\pi(X, x)^{\text{o}}$ and $\pi^{\text{loc}}(X, x)$ naturally arises.

In [EHS08], the authors explained that $\pi^{\text{loc}}(X, x)$ in fact only accounts for a small portion of $\pi(X, x)^{\text{o}}$ by showing that the latter actually contains information about $\pi^{\text{loc}}(X')$ for *all* "geometric" etale coverings $X' \to X$ (see Theorem 3.5 of op. cit. for a precise statement). Further, in [EH10] it was noticed that $\pi(X, x)$ is a semi-direct product of $\pi(X, x)^{\text{o}}$ with $\pi^{\text{ét}}(X, x)$, and, when X is a smooth projective curve, the action of $\pi^{\text{ét}}(X, x)$ on $\pi(X, x)^{\text{o}}$ is trivial if and only if X has genus at most 1 (see Corollary 2.3 and Propostion 2.4 of op. cit.).

The work [EHS08] inspired Otabe [Ota17] to show that, in case k is of characteristic zero, his "semi-finite fundamental group scheme" $\pi^{\text{EN}}(X, x)$ [Ota17, Section 2.4] produces a *faithful* action of $\pi^{\text{ét}}(X, x)$ on its unipotent radical provided X is a smooth curve of genus at least two (see [Ota17, Theorem 4.12]).

We wish to demonstrate here that Otabe's point of view can give interesting information in the case of positive characteristic. Our main finding is that the action of $\pi^{\text{ét}}(X, x)$ on $\pi(X, x)^{\text{o}}$ is faithful if X is a geometrically connected, smooth and projective curve of genus at least two. See Section 4, specially Theorem 4.7.

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We now briefly describe the contents of this article. In Section 2 we review Nori's theory and some of its later developments. In Section 3 we slightly modify the presentation leading to Theorem 3.5 of [EHS08] so that we can easily state and prove our main result, Theorem 4.3 of Section 4. It is perhaps useful to note that Theorem 4.3 has a more portable consequence, which we present as Theorem 4.7. The proof of Theorem 4.3 requires an exercise which is carried out in Section 5.

Notations, conventions and generalities.

1.1. **Conventions.**

General. We let k denote a field. The category of finite dimensional vector spaces over it is denote by k**-mod**.

On vector bundles. A vector bundle is a locally free coherent sheaf of finite rank. If $x : \mathrm{Spec}\, K \to X$ is a point of a scheme X and V is a vector bundle on X, we write $V|_x$ for the K-vector space x^*V. A vector bundle V over X is said to be trivial if it is isomorphic to some $\mathcal{O}_X^{\oplus r}$.

On group schemes. Unless otherwise stated, group schemes are defined over k. For an affine group scheme G (over k), we write $k[G]$ instead of $\Gamma(G, \mathcal{O}_G)$. Given an affine group scheme G, the category of all its *finite dimensional* representations is denoted by $\mathrm{Rep}_k(G)$. An arrow $q : G \to H$ of affine group schemes is called a quotient morphism if it is faithfully flat. We use constantly the fact that $q : G \to H$ is a quotient morphism if and only if the associated arrow $k[H] \to k[G]$ is injective [Wa70, Chapter 14].

On Abelian varieties. For an abelian variety A, we let $[m] : A \to A$ stand for multiplication by m. The kernel of $[m]$ is denoted by $A[m]$.

1.2. **Generalities on adjunctions in the category of affine group schemes.** Let $\mathbf{G}$ be the category of affine group schemes. In this section, we explain how to treat in more robust fashion the process of "taking the largest quotient having a certain property".

We first note that

($\star$) $\mathbf{G}$ is stable under all small limits (use the standard criterion [Mac98, V.2, Corollary 2]),

($\star\star$) and that each arrow $f : G \to H$ can be decomposed uniquely as

$$G \xrightarrow{q} I \xrightarrow{i} H,$$

where i is a closed embedding and q is a quotient morphism.

Let $u : \mathbf{A} \to \mathbf{G}$ be a full subcategory of $\mathbf{G}$ enjoying the ensuing properties:

P1. The category $\mathbf{A}$ is small complete and u preserves all small limits.

P2. If A belongs to $\mathbf{A}$ and $i : H \to A$ is a closed embedding, then H also belongs to $\mathbf{A}$.

P3. If A belongs to $\mathbf{A}$ and $q : A \to H$ is a quotient morphism, then H also belongs to $\mathbf{A}$.

Under such conditions, it is a direct consequence of Freyd's theorem [Mac98, V.6, Theorem 2] that u has a left adjoint $G \mapsto G^{\mathbf{A}}$. Furthermore:

Lemma 1.1. *The unit morphism $\eta_G : G \to u(G^{\mathbf{A}})$ is always a quotient morphism while the co-unit $\varepsilon_A : (uA)^{\mathbf{A}} \to A$ is always an isomorphism.*

Proof. Let $G \stackrel{q}{\to} I \stackrel{i}{\to} u(G^{\mathbf{A}})$ be the decomposition of η_G predicted by $(\star\star)$. Then, $I \in \mathbf{A}$ and it follows that $q : G \to I$ is universal from G to u, so that i is an isomorphism. The second claim follows immediately from [Mac98, IV.3, Theorem 1]. □

This justifies the following standard terminology:

Definition 1.2. If G is an affine group scheme, then the arrow $G \to G^{\mathbf{A}}$ is called the largest quotient of G in $\mathbf{A}$.

Let now $v : \mathbf{B} \to \mathbf{G}$ be a second subcategory enjoying P1–P3. From Lemma 1.1 and stability under quotients, we conclude that $B^{\mathbf{A}} \in \mathbf{B}$ for all $B \in \mathbf{B}$; one easily sees that $(-)^{\mathbf{A}} : \mathbf{B} \to \mathbf{A} \cap \mathbf{B}$ is left adjoint to the inclusion $\mathbf{A} \cap \mathbf{B} \to \mathbf{B}$. This being so, the composition

$$\mathbf{G} \xrightarrow{(-)^{\mathbf{B}}} \mathbf{B} \xrightarrow{(-)^{\mathbf{A}}} \mathbf{A} \cap \mathbf{B}$$

is adjoint to the inclusion $\mathbf{A} \cap \mathbf{B} \to \mathbf{G}$ since "left adjoint of a composition is the composition of the left adjoints" [Mac98, IV.8, Theorem 1]. Consequently, employing [Mac98, IV.1, Corollary 1] we have

Lemma 1.3. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be categories as above. Then, the compositions*

$$\mathbf{G} \xrightarrow{(-)^{\mathbf{A}}} \mathbf{A} \xrightarrow{(-)^{\mathbf{B}}} \mathbf{A} \cap \mathbf{B} \quad \text{and} \quad \mathbf{G} \xrightarrow{(-)^{\mathbf{B}}} \mathbf{B} \xrightarrow{(-)^{\mathbf{A}}} \mathbf{A} \cap \mathbf{B}$$

are naturally isomorphic. Moreover, they are also naturally isomorphic to

$$\mathbf{G} \xrightarrow{\quad (-)^{\mathbf{A} \cap \mathbf{B}} \quad} \mathbf{A} \cap \mathbf{B}$$

□

As is customary, if $\mathbf{A}$ is the category of abelian affine group schemes, respectively local affine group schemes, then $G^{\mathbf{A}}$ is denoted by G^{ab}, respectively G^{loc}. They are then, in the spirit of Definition 1.2 above, called the largest abelian quotient, respectively the largest local quotient, of G.

2. The essentially finite fundamental group scheme

In this section, we make a leisurely introduction to the essentially finite group scheme; it serves mainly to help us establish notation and to introduce the reader to our mode of thought. Besides the seminal text [Nor76], the reader should consult [EHS08] for detailled information.

In what follows, k stands for a perfect field of characteristic $p > 0$. Let X be a proper, reduced and connected algebraic scheme over k. In [Nor76], Nori introduced two important classes of vector bundles: the *finite* and the *essentially finite*. He also *came close* to another one which has an important place in organizing the theory: the Nori-semistables (this terminology seems to come from [Meh09]). A vector bundle V on X is said to be *Nori-semistable* if it becomes semistable and of degree zero when pulled back along *any* non-constant morphism $\gamma : C \to X$ from a smooth and projective curve (see the Definition after Proposition 3.4 in [Nor76]). The finite vector bundles are those V for which the set

$$\left\{ \begin{array}{c} \text{isomorphism classes of indecomposable} \\ \text{direct summands of } V^{\otimes 1}, V^{\otimes 2}, \ldots \end{array} \right\}$$

is *finite* (see the Definition after Lemma 3.1 in [Nor76]). It turns out that all finite vector bundles are Nori-semistable and that the category of Nori-semistables – any morphism of vector bundle being an arrow – is abelian. This fact allows one to consider all the Nori-semistables of the form W/W', where $W' \subset W$ are both subobjects of a common finite V, and show that the resulting category, with the evident tensor product, is a tensor category over k in the sense of [Del90, 1.2]. This is the category of *essentially finite* vector bundles, which is denoted in what follows by $\mathcal{C}^{\mathrm{EF}}(X)$.

Given a k-point x of X, the functor $V \mapsto V|_x$ (see section 1.1) from $\mathcal{C}^{\mathrm{EF}}(X)$ to k-**vect** is exact and faithful, so that the main result of Tannakian theory [DM82, 2.11, p.130] constructs an affine group scheme over k, usually called the Nori or *essentially finite fundamental group scheme* $\pi(X, x)$, and an equivalence of tensor categories

$$\mathcal{C}^{\mathrm{EF}}(X) \xrightarrow{\sim} \mathrm{Rep}_k(\pi(X, x)), \quad V \longmapsto V|_x.$$

Let us now elaborate on a useful notion. Given $V \in \mathcal{C}^{\mathrm{EF}}(X)$, let $\langle V \rangle_\otimes$ stand for the full subcategory of $\mathcal{C}^{\mathrm{EF}}(X)$ whose objects are subquotients of finite direct sums of vector bundles of the form $V^{\otimes a} \otimes V^{*\otimes b}$. Then,

$$\bullet|_x : \langle V \rangle_\otimes \longrightarrow \mathrm{Rep}_k(\pi(X, x))$$

defines an equivalence between $\langle V \rangle_\otimes$ and the category $\mathrm{Rep}_k(\pi(X, V, x))$ of a certain quotient $\pi(X, V, x)$ of $\pi(X, x)$ [DM82, 2.21,p.139]. This quotient turns out to be a *finite* group scheme, a fact which can be grasped by looking at the definition of a finite vector bundle and [DM82, 2.20(a), p.138]. In addition, there exists a $\pi(X, V, x)$-torsor which "gives back" V, see the beginning of Section 3 for further information.

The full subcategory of $\mathcal{C}^{\mathrm{EF}}(X)$ consisting of those V for which $\pi(X, V, x)$ is etale, respectively local, will be denoted by $\mathcal{C}^{\text{ét}}(X)$, respectively $\mathcal{C}^{\mathrm{loc}}(X)$. Accordingly, objects of $\mathcal{C}^{\text{ét}}(X)$, respectively of $\mathcal{C}^{\mathrm{loc}}(X)$, are called etale, respectively local, vector bundles. By means of the criterion [DM82, Proposition 2.21, p.139] and the fact that etale and local finite group schemes are stable under quotient morphism, the functor $\bullet|_x$ induces an equivalence between $\mathcal{C}^{\text{ét}}(X)$, respectively $\mathcal{C}^{\mathrm{loc}}(X)$, and a *quotient* $\pi^{\text{ét}}(X, x)$, respectively $\pi^{\mathrm{loc}}(X, x)$, of $\pi(X, x)$. Needless to say, the affine group scheme $\pi^{\mathrm{loc}}(X, x)$, respectively $\pi^{\text{ét}}(X, x)$, is a projective limit of finite and local group schemes, respectively finite and etale group schemes.

The relation between $\pi^{\text{ét}}(X, x)$ and its celebrated predecessor, the etale fundamental group of [SGA1] is quite simple: Let $\overline{k}$ be an algebraic closure of k, and write $\overline{X} = X \otimes_k \overline{k}$. Then, using the obvious geometric point $\overline{x} : \mathrm{Spec}\, \overline{k} \to \overline{X}$, we construct the geometric fundamental group $\pi_1(\overline{X}, \overline{x})$ of $\overline{X}$. Since $\overline{x}$ actually comes from a k-rational point, $\pi_1(\overline{X}, \overline{x})$ has a continuous action of $\mathrm{Gal}(\overline{k}/k)$, and by the construction of [DG70, II, §5, no. 1.7] we can associate to $\pi_1(\overline{X}, \overline{x})$ a profinite group scheme. This is $\pi^{\text{ét}}(X, x)$. As we shall have no use for this characterization here, we omit the verifications. (Note that this relation is incorrectly stated in [DM82, 2.34] and partially explained in [EHS08, Remarks 2.10].)

We end this section with a result which is left implicit in most works on the subject.

Lemma 2.1. *Let E be a vector bundle over X, and K be a finite and separable extension of k. Then E is essentially finite if and only if $E \otimes K$ is essentially finite over $X \otimes K$. Moreover, the same statement is true if we replace "essentially finite" by "local" or "etale".*

Proof. Only the "if" statement needs attention, so assume that $E \otimes K$ is essentially finite. We can therefore find a finite group scheme G (over K), a G-torsor $P \to X \otimes K$, and a monomorphism $E \otimes K \to \mathcal{O}_P^{\oplus r}$. Now, according to [No82, Chapter II, Propsoition 5, p.89], P can be chosen to come from X, that is, $P = P_0 \otimes K$, where $P_0 \to X$ is a torsor under a certain finite group scheme. Consequently, we obtain a monomorphism of $\mathcal{O}_X$-modules $E \to \mathcal{O}_{P_0}^{\oplus r} \otimes K$; as E is certainly Nori-semistable, we conclude that E is essentially finite. The proof of the last claim follows the same method, since we can replace G with a local, or etale finite group scheme. □

3. The kernel of $\pi(X) \to \pi^{\text{ét}}(X)$

We maintain the notations and terminology of section 2, but *omit reference to the base point* x in speaking about fundamental group schemes. In what follows we briefly review some results of [EHS08], including one of its main outputs, Theorem 3.5 on p. 389. In fact, we shall, with an eye to future applications, use a slightly different path to arrive at [EHS08, Theorem 3.5]; while in op. cit. the authors rely on a category of all vector bundles "coming from the local ones on the universal covering" [EHS08, Definition 3.3], we prefer to use "truncated" versions of this category (called $\langle \psi_* \mathcal{C}^{\text{loc}}(X') \rangle_\otimes$). See the discussion after Definition 3.3 below as well as the remark after equations (3.7).

Given $V \in \mathcal{C}^{\text{EF}}(X)$, *an inverse* of the equivalence

$$\bullet|_x : \langle V \rangle_\otimes \longrightarrow \mathrm{Rep}_k(\pi(X,V))$$

constructed on Section 2 produces a principal $\pi(X,V)$-bundle

$$\psi_V : X_V \longrightarrow X$$

together with a k-point x_V on the fibre of ψ_V above x. Moreover, our inverse equivalence is just the contracted product functor

$$\mathscr{L}_{X_V} : \mathrm{Rep}_k(\pi(X,V)) \longrightarrow \langle V \rangle_\otimes. \tag{3.1}$$

(See [Sa74, I.4.4.2] for the existence an inverse to $\bullet|_x$ which is a tensor functor and [Nor76, p.11] for the construction of X_V.)

Let us fix $V \in \mathcal{C}^{\text{ét}}(X)$ and simplify notations by writing

$$\boxed{X' := X_V, \quad \psi = \psi_V, \quad G = \pi(X,V), \quad x' = x_V.}$$

Two simple features of X' are immediately remarked: X' is reduced and proper (ψ is finite and etale), and X' is "Nori"-reduced, that is, $\Gamma(X', \mathcal{O}_{X'}) = k$, see [No82, Proposition 3, p. 87]. We are then allowed to consider $\mathcal{C}^{\text{loc}}(X')$, and set out to investigate its relation to $\mathcal{C}^{\text{EF}}(X)$. Using the proof of Theorem 2.9 in [EHS08] (see also the paragraph preceding Lemma 2.8 on p. 384), we can say the following.

Theorem 3.2. *For each $E' \in \mathcal{C}^{\text{EF}}(X')$, the vector bundle $\psi_*(E')$ is also essentially finite on X.* □

Hence, we obtain a functor

$$\psi_* : \mathcal{C}^{\text{loc}}(X') \longrightarrow \mathcal{C}^{\text{EF}}(X)$$

which, it turns out, allow us to understand the category of representations of the kernel

$$\mathrm{Ker}\, \pi(X,x) \longrightarrow \pi^{\text{ét}}(X,x).$$

Until now, this is exactly the point of view in [EHS08]; let us start making minor changes.

Definition 3.3. Given any finite set of objects in $\mathcal{C}^{\mathrm{EF}}(X)$, call it s, we let $\langle s\rangle_\otimes$ stand for the full subcategory

$$\langle \oplus_{W\in s} W\rangle_\otimes$$

of $\mathcal{C}^{\mathrm{EF}}(X)$. If S is an arbitrary set of objects in $\mathcal{C}^{\mathrm{EF}}(X)$, we let $\langle S\rangle_\otimes$ stand for the full subcategory having

$$\bigcup_{s\subset S\text{ finite}} \langle s\rangle_\otimes$$

as objects.

We now apply the above definition to the set of objects of $\psi_*\mathcal{C}^{\mathrm{loc}}(X')$ to obtain a full subcategory

$$\langle \psi_*\mathcal{C}^{\mathrm{loc}}(X')\rangle_\otimes$$

of $\mathcal{C}^{\mathrm{EF}}(X)$. It only takes a moment's thought to see that, with the evident structures, $\langle \psi_*\mathcal{C}^{\mathrm{loc}}(X')\rangle_\otimes$ is a tensor category [Del90, 1.2], and that the functor $\bullet|_x : \langle \psi_*\mathcal{C}^{\mathrm{loc}}(X')\rangle_\otimes \to k$**-vect** is faithful and exact. Through the main existence theorem [DM82, 2.11, p.130], we see that $\bullet|_x$ defines an affine group scheme

$$\pi(X, \mathcal{C}^{\mathrm{loc}}(X'))$$

and an equivalence

$$\langle \psi_*\mathcal{C}^{\mathrm{loc}}(X')\rangle_\otimes \xrightarrow{\sim} \mathrm{Rep}_k(\pi(X, \mathcal{C}^{\mathrm{loc}}(X'))).$$

In addition, [DM82, 2.21, p.139] allows us to see $\pi(X, \mathcal{C}^{\mathrm{loc}}(X'))$ as a *quotient* of $\pi(X)$.

Proposition 3.4. *Let V and $\psi : X' \to X$ be as above. Then the following claims are true.*

(1) *A vector bundle $E \in \mathcal{C}^{\mathrm{EF}}(X)$ belongs to $\langle \psi_*\mathcal{C}^{\mathrm{loc}}(X')\rangle_\otimes$ if and only if ψ^*E belongs to $\mathcal{C}^{\mathrm{loc}}(X')$.*

(2) *The vector bundle V belongs to $\langle \psi_*\mathcal{C}^{\mathrm{loc}}(X')\rangle_\otimes$ and the resulting morphism*

$$\pi(X\,;\,\mathcal{C}^{\mathrm{loc}}(X')) \longrightarrow \pi(X, V) = G$$

is a quotient morphism.

(3) *Each $E \in \mathcal{C}^{\mathrm{loc}}(X)$ belongs to $\langle \psi_*\mathcal{C}^{\mathrm{loc}}(X')\rangle_\otimes$ and the resulting morphism*

$$\pi(X\,;\,\mathcal{C}^{\mathrm{loc}}(X')) \longrightarrow \pi^{\mathrm{loc}}(X)$$

is a quotient morphism. In particular, $\pi^{\mathrm{loc}}(X)$ is the largest local quotient of $\pi(X\,;\,\mathcal{C}^{\mathrm{loc}}(X'))$.

Proof. (1) The proof goes as that of [EHS08, Lemma 2.8, p.384]. Let $E = \psi_*(E')$, where E' is a local vector bundle. Using the cartesian square

$$\begin{array}{ccc} X'\times G & \xrightarrow{\alpha} & X' \\ {\scriptstyle \mathrm{pr}}\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\ X' & \xrightarrow[\psi]{} & X, \end{array}$$

where α is the action morphisms, we conclude that $\psi^*E \simeq \mathrm{pr}_*\alpha^*E'$. But, after a possible extension of the base field, $X'\times G$ becomes a disjoint sum of copies of X' while $\mathrm{pr}_*\alpha^*E'$ becomes a sum of vector bundles of the shape g^*E', where $g \in \mathrm{Aut}(X')$. Hence, $\psi^*E \in \mathcal{C}^{\mathrm{loc}}(X')$. (Here we have implicitly used Lemma 2.1.) For a general $E \in \langle \psi_*\mathcal{C}^{\mathrm{loc}}(X')\rangle_\otimes$, the definition says that $E \in \langle \psi_*(E')\rangle_\otimes$ for some $E' \in \mathcal{C}^{\mathrm{loc}}(X')$. But then, as $\psi^* : \mathcal{C}^{\mathrm{EF}}(X) \to \mathcal{C}^{\mathrm{EF}}(X')$ is an exact tensor functor, ψ^*E belongs to $\langle\psi^*(\psi_*E')\rangle_\otimes$, which is a subcategory of $\mathcal{C}^{\mathrm{loc}}(X')$, as $\pi(X) \to \pi^{\mathrm{loc}}(X)$ is a quotient morphism.

Now let $E \in \mathcal{C}^{\mathrm{EF}}(X)$ be such that ψ^*E belongs to $\mathcal{C}^{\mathrm{loc}}(X')$. Since ψ is faithfully flat, the "unit" $E \to \psi_*\psi^*(E)$ is a monomorphism, and consequently E belongs to $\langle\psi_*(\psi^*E)\rangle_\otimes$. By definition, this says that E lies in $\langle\psi_*\mathcal{C}^{\mathrm{loc}}(X')\rangle_\otimes$.

(2) The first claim is a consequence of (1) and the fact that ψ^*V is trivial. Since $\langle\psi_*\mathcal{C}^{\mathrm{loc}}(X')\rangle_\otimes$ is a full subcategory of $\mathcal{C}^{\mathrm{EF}}(X')$ which is stable under subquotients, the standard criterion [DM82, 2.21, p.139] guarantees the veracity of the second statement once applied to the inclusion $\langle V\rangle_\otimes \subset \langle\psi_*\mathcal{C}^{\mathrm{loc}}(X')\rangle_\otimes$.

(3) This is again a simple application of (1) and the criterion [DM82, 2.21, p.139]. □

At this point, we wish to describe the kernel of

$$\pi(X\,;\,\mathcal{C}^{\mathrm{loc}}(X')) \longrightarrow G = \pi(X, V),$$

which is the statement paralleling [EHS08, Theorem 3.5]. From Proposition 3.4-(1), we obtain from ψ^* a morphism

$$\psi_\# : \pi^{\mathrm{loc}}(X') \longrightarrow \pi(X\,;\,\mathcal{C}^{\mathrm{loc}}(X')). \tag{3.5}$$

(Recall that X' has a k-point x' above x.) The translation of [EHS08, Theorem 3.5] in our setting is:

Theorem 3.6. *The morphism $\psi_\#$ of* (3.5) *is in fact the kernel of* $\pi(X\,;\,\mathcal{C}^{\mathrm{loc}}(X')) \to G$. *Put differently, we have an exact sequence*

$$1 \longrightarrow \pi^{\mathrm{loc}}(X') \longrightarrow \pi(X\,;\,\mathcal{C}^{\mathrm{loc}}(X')) \longrightarrow G \longrightarrow 1.$$

Proof. Firstly, we note that $\psi_\#$ is a closed embedding. So let $E' \in \mathcal{C}^{\mathrm{loc}}(X')$; by definition $\psi_*(E')$ belongs to $\langle\psi_*\mathcal{C}^{\mathrm{loc}}(X')\rangle_\otimes$ and since the "co-unit" $\psi^*(\psi_*E') \to E'$ is an epimorphism, the criterion [DM82, 2.21(b), p.139] immediately proves the statement.

We then verify that conditions (iii-a) to (iii-c) of Theorem A.1 on p. 396 of [EHS08] are true. In fact, only (iii-a) and (iii-b) need attention, since the argument above already shows that (iii-c) holds.

Let $E \in \mathcal{C}^{\mathrm{EF}}(X)$ become trivial when pulled back to X'. Then, faithfully flat descent shows that E lies in the image of the contracted product $\mathscr{L}_{X'}$ of (3.1). Hence, E belongs to $\langle V\rangle_\otimes$. This is condition (iii-a) of [EHS08, Theorem A1].

Let A be the $\mathcal{O}_X$-coherent algebra $\psi_*(\mathcal{O}_{X'})$ and let E be an object of $\langle\psi_*\mathcal{C}^{\mathrm{loc}}(X')\rangle_\otimes$. Let H be the space $H^0(X, A\otimes_{\mathcal{O}_X} E)$, $\delta \in H^0(X, A\otimes A^\vee)$ be the global section associated to id_A, and

$$\mathrm{ev} : H\otimes_k A^\vee \longrightarrow E$$

the evaluation. Since each $h \in H$ is the image of $h\otimes\delta$ under

$$\mathrm{ev}\otimes\mathrm{id}_A : (H\otimes_k A^\vee)\otimes_{\mathcal{O}_X} A \longrightarrow E\otimes A,$$

we conclude that $\mathrm{ev}\otimes\mathrm{id}_A$ induces a surjection on global sections. This means that $\psi^*(\mathrm{ev})$ induces a surjection on global sections. A fortiori, $\psi^*(\mathrm{Im}(\mathrm{ev})) \to \psi^*E$ induces a surjection on global sections, which implies that any morphism from $\mathcal{O}_{X'}$ to ψ^*E factors through $\psi^*(\mathrm{Im}(\mathrm{ev}))$. Now, $\langle\psi_*\mathcal{C}^{\mathrm{loc}}(X')\rangle_\otimes$ is stable under quotients and A is an object of it; this shows that $\mathrm{Im}(\mathrm{ev})$ lies in $\langle\psi_*\mathcal{C}^{\mathrm{loc}}(X')\rangle_\otimes$. Then, since $\psi^*(A^\vee)$ is a trivial vector bundle, we can say that $\psi^*(\mathrm{Im}(\mathrm{ev}))$

is equally trivial. In conclusion, $\psi^*(\mathrm{Im}(\mathrm{ev}))$ is the largest trivial subobject of ψ^*E, which is condition (iii-b) of [EHS08, Theorem A1].

□

Now let us order $\mathcal{C}^{\text{ét}}(X)$ in the following way: $W < W'$ if $W \in \langle W' \rangle_\otimes$. Using the direct sum of vector bundles, we see that the resulting partially ordered set is directed, and we obtain a directed system of exact sequences

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \pi^{\mathrm{loc}}(X_{W'}) & \longrightarrow & \pi(X\,;\,\mathcal{C}^{\mathrm{loc}}(X_{W'})) & \longrightarrow & \pi(X, W') & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \pi^{\mathrm{loc}}(X_{W}) & \longrightarrow & \pi(X\,;\,\mathcal{C}^{\mathrm{loc}}(X_{W})) & \longrightarrow & \pi(X, W) & \longrightarrow & 1. \end{array}$$

Taking the limit and using that

$$\pi^{\mathrm{loc}}(X_{W'}) \longrightarrow \pi^{\mathrm{loc}}(X_W)$$

is always a quotient morphism [EHS08, Proposition 3.6, p.390], we arrive at an exact sequence

$$1 \longrightarrow \varprojlim_W \pi^{\mathrm{loc}}(X_W) \longrightarrow \varprojlim_W \pi(X\,;\,\mathcal{C}^{\mathrm{loc}}(X_W)) \longrightarrow \varprojlim_W \pi(X, W) \longrightarrow 1.$$

Note that the rightmost term is a *proetale* affine group scheme, while the leftmost is a *local* affine group scheme. In addition, by looking at the categories of representations, we see that the natural morphisms

$$\pi(X) \longrightarrow \varprojlim_W \pi(X\,;\,\mathcal{C}^{\mathrm{loc}}(X_W)) \quad \text{and} \quad \pi^{\text{ét}}(X) \longrightarrow \varprojlim_W \pi(X, W)$$

are isomorphisms. Hence, borrowing the notation of [Wa70, Ch. 6, Exercise 7], we conclude that

$$\begin{aligned} \pi(X)^{\mathrm{o}} &:= \text{identity component of } \pi(X) \\ &= \varprojlim \pi^{\mathrm{loc}}(X_W). \end{aligned} \tag{3.7}$$

This is precisely [EHS08, Theorem 3.5], as the category $\mathcal{D}$ appearing on [EHS08, Definition 3.3] is just the representation category of $\varprojlim_W \pi^{\mathrm{loc}}(X_W)$.

4. The action of $\pi^{\text{ét}}(X)$ on $\pi(X)^{\mathrm{o}}$

We work in the setting described in the beginning of section 3; in particular, k is a perfect field of characteristic $p > 0$, X is a proper, reduced and connected algebraic k-scheme, and $\psi : X' \to X$ is a torsor under the finite and etale group scheme G.

Since the kernel of the morphism $\pi(X\,;\,\mathcal{C}^{\mathrm{loc}}(X')) \to G$ appearing in Theorem 3.6 is the local affine group scheme $\pi^{\mathrm{loc}}(X')$, it is not hard to see, using [Wa70, 6.8, Lemma], that

$$\pi(X\,;\,\mathcal{C}^{\mathrm{loc}}(X'))_{\mathrm{red}} \xrightarrow{\sim} G.$$

We then obtain an action of G on $\pi^{\mathrm{loc}}(X')$ by group automorphisms. Our next goal is to understand under which circumstances this action is "faithful."

Proposition 4.1. *Let* $H \subset G$ *be a subgroup scheme acting trivially on* $\pi^{\text{loc}}(X')$. *Then the natural morphism*

$$\pi^{\text{loc}}(X') \longrightarrow \pi^{\text{loc}}(X'/H)$$

is an isomorphism. (We use the image of x' *on* X'/H *as base-point for constructing* $\pi^{\text{loc}}(X'/H)$*.)*

Proof. We adopt the notations implied by the following diagram:

$$\begin{array}{ccc} & & X' \\ & \psi \swarrow & \downarrow \rho \\ & & X'/H \\ & & \downarrow \sigma \\ & & X. \end{array}$$

Note that $\rho : X' \to X'/H$ is an H-torsor so that we can apply Proposition 3.4-(1) to conclude that σ takes objects of $\langle \psi_* \mathcal{C}^{\text{loc}}(X') \rangle_\otimes$ to $\langle \rho_* \mathcal{C}^{\text{loc}}(X') \rangle_\otimes$.

There are now two exact sequence in sight (see Theorem 3.6),

$$(*) \qquad 1 \longrightarrow \pi^{\text{loc}}(X') \longrightarrow \pi(X'/H\,;\, \mathcal{C}^{\text{loc}}(X')) \longrightarrow H \longrightarrow 1$$

and

$$(**) \qquad 1 \longrightarrow \pi^{\text{loc}}(X') \longrightarrow \pi(X\,;\, \mathcal{C}^{\text{loc}}(X')) \longrightarrow G \longrightarrow 1.$$

The above observation assures that they are related by the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi^{\text{loc}}(X') & \longrightarrow & \pi(X'/H\,;\, \mathcal{C}^{\text{loc}}(X')) & \longrightarrow & H & \longrightarrow & 1 \\ & & \sim \downarrow & & \sigma_\# \downarrow & & \text{inclusion} \downarrow & & \\ 1 & \longrightarrow & \pi^{\text{loc}}(X') & \longrightarrow & \pi(X\,;\, \mathcal{C}^{\text{loc}}(X')) & \longrightarrow & G & \longrightarrow & 1, \end{array}$$

where the arrow $\sigma_\#$ is constructed from the functor $\sigma^* : \langle \psi_* \mathcal{C}^{\text{loc}}(X') \rangle_\otimes \to \langle \rho_* \mathcal{C}^{\text{loc}}(X') \rangle_\otimes$. The relevance of this relation is that it shows that the action of H on $\pi^{\text{loc}}(X')$ stemming from the sequence $(*)$ coincides, once all identifications are unraveled, with the action of H on $\pi^{\text{loc}}(X')$ derived from $(**)$. (The reader wishing to run a careful verification should profit from the fact that the action of G, respectively of H, is really an action of $\pi(X\,;\, \mathcal{C}^{\text{loc}}(X'))_{\text{red}}$, respectively $\pi(X'/H\,;\, \mathcal{C}^{\text{loc}}(X'))_{\text{red}}$.) The assumption of the proposition then implies that the action of H on $\pi^{\text{loc}}(X')$ arising from $(*)$ is trivial. From this, we derive a retraction

$$r : \pi(X'/H\,;\, \mathcal{C}^{\text{loc}}(X')) \longrightarrow \pi^{\text{loc}}(X')$$

which exhibits $\pi^{\text{loc}}(X')$ as the largest local quotient of $\pi(X'/H\,;\, \mathcal{C}^{\text{loc}}(X'))$. But by Proposition 3.4(b), the largest local quotient of $\pi(X'/H\,;\, \mathcal{C}^{\text{loc}}(X'))$ is $\pi^{\text{loc}}(X'/H)$, and therefore

$$\pi^{\text{loc}}(X') \simeq \pi^{\text{loc}}(X'/H).$$

(It is not hard to see that this morphism is in fact the canonical one.) □

We now want to show that the conclusion in the statement of Proposition 4.1 *cannot* take place if X is a "hyperbolic curve". For that, we only need to study the *largest commutative quotient* of the local fundamental group scheme and apply the following result.

Proposition 4.2. *Let* C *be a smooth, geometrically connected and projective one-dimensional* k*-scheme (a "curve"),* c *a* k*-rational point on* C*,* m *a positive integer, and* $\mathrm{Jac}(C)$ *the Jacobian of* C*. Then, the largest quotient of* $\pi^{\mathrm{loc}}(C,c)$ *which is commutative, finite and annihilated by* p^m *is isomorphic to* $\mathrm{Jac}(C)[p^m]^{\mathrm{loc}}$.

Proof. To ease notation, we write J in place of $\mathrm{Jac}(C)$. Let

$$\varphi : C \longrightarrow J$$

be the Abel-Jacobi (or Albanese) morphism sending c to the origin e. Then, we arrive at a commutative diagram

$$\begin{array}{ccc} \pi(C,c) & \xrightarrow{\varphi_\#} & \pi(J,e) \\ \downarrow{\scriptstyle q} & \nearrow{\scriptstyle \alpha} & \\ \pi(C,c)^{\mathrm{ab}}, & & \end{array}$$

in which the arrow α is an isomorphism [An11, Corollary 3.8]. Hence, as explained in Section 1.2,

$$\begin{aligned}\left[\pi(C,c)^{\mathrm{loc}}\right]^{\mathrm{ab}} &\simeq \left[\pi(C,c)^{\mathrm{ab}}\right]^{\mathrm{loc}} \\ &\simeq \pi(J,e)^{\mathrm{loc}}.\end{aligned}$$

Now let $\mathbf{K}$ be the full subcategory of the category of affine group schemes defined by those which are commutative, finite and annihilated by p^m. Then, using Lemma 5.1 below and the notations of Section 1.2, we see that

$$\left[\pi(J,e)^{\mathrm{loc}}\right]^{\mathbf{K}} = \left[\pi(J,e)^{\mathbf{K}}\right]^{\mathrm{loc}} \simeq J[p^m]^{\mathrm{loc}}.$$

□

Theorem 4.3. *If our* X *is a smooth, geometrically connected and projective curve of genus at least two, then no non-trivial subgroup scheme of* G *acts trivially on* $\pi^{\mathrm{loc}}(X')$.

Proof. Let H be as in the statement of Proposition 4.1. Then, the fact that $\pi^{\mathrm{loc}}(X')$ and $\pi^{\mathrm{loc}}(X'/H)$ are isomorphic implies, via Proposition 4.2, that

$$\begin{array}{c}\text{Tangent space}\\ \text{at the origin}\end{array}\ \mathrm{Jac}(X') \simeq \begin{array}{c}\text{Tangent space}\\ \text{at the origin}\end{array}\ \mathrm{Jac}(X'/H).$$

Therefore, X' and X'/H have the same genus (which is the dimension of the tangent space to the Jacobian [Mi86b, Proposition 2.1]). The Riemann-Hurwitz formula then shows that H is trivial. □

We now wish to obtain from Theorem 4.3 a statement which is easier to carry.

Let $\mathcal{G}$ be an affine group scheme over k and M a vector space affording a representation of $\mathcal{G}$. If M is finite dimensional and $\mathcal{G}$ is algebraic, we say that M is faithful if the obvious morphism $\mathcal{G} \to \mathbf{GL}(M)$ is a closed embedding or, equivalently, its kernel is trivial [Wa70, 15.3, Theorem]. We now translate this last condition in terms of the coaction $\rho : M \to M \otimes k[\mathcal{G}]$ for future usage. Define a *modified coefficient* of the representation M as any element of the form

$$(u \otimes \mathrm{id}) \circ \rho(m) - u(m) \cdot 1 \in k[\mathcal{G}],$$

where $m \in M$ and $u \in \mathrm{Hom}(M, k)$. (We leave to the reader the simple task of justifying the term "modified coefficient".) Then, the kernel of $\mathcal{G} \to \mathbf{GL}(M)$ is trivial if and only if the modified coefficients generate the augmentation ideal of $k[\mathcal{G}]$.

Note that the definition of modified coefficient makes perfect sense for a general representation, finite or infinite dimensional, of a general affine group scheme $\mathcal{G}$. Hence, the following encompasses the above definition.

Definition 4.4. Let $\mathcal{G}$ be an affine group scheme and M a vector space affording a representation of $\mathcal{G}$. We say that M is faithful if the modified coefficient of M generate the augmentation ideal of $k[\mathcal{G}]$.

Remark 4.5. The concept "faithful representation" is not really well established in the literature on group schemes. On the other hand, a representation M of $\mathcal{G}$ is faithful if and only if no closed non-trivial subgroup scheme of $\mathcal{G}$ acts trivially. (This is because, quite generally, the ideal generated by the modified coefficients is a Hopf-ideal.)

Let A be a directed set and $\{\mathcal{G}_\alpha; q_{\beta\alpha}\}$ a projective system indexed by A. We assume that the transition morphisms, $q_{\beta\alpha}$ are all faithfully flat and let $\mathcal{G}$ stand for the limit $\varprojlim \mathcal{G}_\alpha$.

Lemma 4.6. *Let M be a vector space affording a representation of the affine group scheme $\mathcal{G}$. Assume that for each $\alpha \in A$, there exists some $\beta \geqslant \alpha$, and a faithful representation M_β of $\mathcal{G}_\beta$ which can be $\mathcal{G}$-equivariantly embedded in M. Then M is a faithful representation of $\mathcal{G}$.* □

Proof. Let f be an element of the augmentation ideal of $k[\mathcal{G}]$. Clearly f belongs to the augmentation ideal of some $k[\mathcal{G}_\alpha]$. Let M_β be as in the statement. It then follows that f, which also belongs to the augmentation ideal of $k[\mathcal{G}_\beta]$, can be expressed as a sum $\sum x_i f_i$, where f_i is a modified coefficient of M_β. Since M_β embeds $\mathcal{G}$-equivariantly in M, it is easy to see that each f_i is also a modified coefficient of M. □

Employing this language and the identification (3.7), we can translate Theorem 4.3 as follows.

Theorem 4.7. *Let our X be a smooth, geometrically connected and projective curve of genus at least two. Then, the representation of $\pi^{\text{ét}}(X)$ on $k[\pi(X)^{\text{o}}]$ is faithful.* □

5. An exercise on the fundamental group scheme of an abelian variety

Let A be an abelian variety over k. If m and q are positive integers, then multiplication by q on $A[n]$ induces a morphism $A[qm] \to A[m]$ which is in fact faithfully flat. This allows us to define the affine group scheme $TA := \varprojlim A[n]$. Paralleling the Lang-Serre theorem [SGA1, Exposé XI, 2.1], Nori showed in [No83] that the essentially finite fundamental group scheme $\pi(A)$ of A based at the identity is just TA. The following is a very simple consequence of this fact.

Lemma 5.1. *Let m be a positive integer. The obvious arrow $\pi(A) = TA \to A[p^m]$ is universal from $\pi(A)$ to the category of finite, commutative group schemes which are annihilated by p^m.*

Proof. To ease notation, let π stand for $\pi(A) = TA$. Let $\alpha : \pi \to H$ be an arrow to some H which is finite, commutative and annihilated by p^m. We then have a commutative diagram

$$\begin{array}{ccc} \pi & \longrightarrow & A[p^\nu] \\ & {\scriptstyle\alpha}\searrow & \downarrow{\scriptstyle\beta} \\ & & H, \end{array}$$

where $\nu > m$ and the horizontal arrow is the obvious one. As p^m annihilates H and $[p^m] : A[p^\nu] \to A[p^{\nu-m}]$ is a quotient morphism (see [Mi86a, §8] for details), we conclude that $A[p^{\nu-m}] \subset \operatorname{Ker}\beta$. Using the exact sequence

$$0 \longrightarrow A[p^{\nu-m}] \longrightarrow A[p^\nu] \longrightarrow A[p^m] \longrightarrow 0,$$

we obtain an arrow

$$\begin{array}{ccc} A[p^\nu] & \longrightarrow & A[p^m] \\ \downarrow & \swarrow & \\ H. & & \end{array}$$

This arrow is unique since $\pi \to A[p^m]$ is a quotient morphism. □

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