

THE HOMOTOPY EXACT SEQUENCE FOR THE FUNDAMENTAL GROUP SCHEME AND INFINITESIMAL EQUIVALENCE RELATIONS

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Dedicated to the memory of Vikram Bhagvandas Mehta

ABSTRACT. Let $f : X \rightarrow S$ be a morphism of smooth and connected schemes over an algebraically closed field k . What can be said about the relation between the three fundamental group schemes in sight: $\Pi(X)$, $\Pi(S)$ and $\Pi(\text{Fibre})$? This work addresses this question under the extra assumption that f is smooth, projective and geometrically connected. Moreover, it wishes to reach a natural answer by employing a method of proof analogous to the one present by its precursor: Exposé X of SGA1. Our approach is then based on a new Tannakian criterion for studying exact sequences, and infinitesimal equivalence relations. The latter objects are inspired by Ehresmann's image of connections on general fibre spaces. The proofs make no distinction between the cases $\text{char}(k) = 0$ and $\text{char}(k) > 0$.

1. INTRODUCTION

In this work we wish to convey a method and a means to prove the following.

Theorem 1. *Fix an algebraically closed field k . Let X and S be smooth and connected k -schemes. Let $f : X \rightarrow S$ be a projective and smooth morphism having geometrically connected fibres. Let x_0 be a k -point of X and denote by s_0 its image in S . Then*

- (1) *The homomorphism of fundamental group schemes $b : \Pi(X, x_0) \rightarrow \Pi(S, s_0)$ is faithfully flat.*
- (2) *If X_0 stands for the fibre of f above s_0 , then the image of $a : \Pi(X_0, x_0) \rightarrow \Pi(X, x_0)$ is precisely the kernel of b .*

The fundamental group scheme $\Pi(X, x_0)$ alluded to in the statement is the one obtained by Tannakian duality [DM82] from the category of $\mathcal{D}_{X/k}$ -modules which are \mathcal{O}_X -coherent [dS07, 2.1.1]. Here, $\mathcal{D}_{X/k}$ is the ring of differential operators of [EGA IV₄, 16.8,39ff] (which equals the expected one if $\text{char}(k) = 0$). See §6.1 for more information concerning Π .

In SGA 1, exposé X, the reader will find a result similar to that of Theorem 1, labelled the “homotopy exact sequence”, where $\Pi(-, -)$ is replaced by Grothendieck's “algebraic fundamental group.” Our strategy consists in following the natural transposition of the proof presented in SGA and face the inherent difficulties, including the immateriality of $\text{char}(k)$. There, since the quotient of a profinite group π by an open subgroup U still has a

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meaning in terms of finite π -sets, the most obvious set theoretical criterion, Lemma 6.11 on p.114 of [SGA1, V], makes perfect sense. The proof then makes use of this criterion and a method to “push-forward” finite etale coverings [SGA1, X, 1.2, p.202]. It is therefore reasonable to say that there are two main structures sustaining the “homotopy exact sequence.” The understanding of these in the context of the fundamental group scheme is what motivates the present research.

As is well-known, the theory of Tannakian categories meets an impediment in dealing with the fact that homogeneous spaces of algebraic groups are not necessarily affine. Although this difficulty is not found in the theory of SGA1 (see the previous paragraph), in the problem we are tackling this is precisely what forces the introduction of new methods. In showing that $\Pi(S, s_0)$ is the cokernel of $a : \Pi(X_0, x_0) \rightarrow \Pi(X, x_0)$ we are led to understand properties of something which might not be an affine scheme with a group scheme action, viz. $\text{Coker}(a)$. Hence, it becomes necessary to consider quasi-projective schemes and projective spaces endowed with actions of the group schemes in sight. This addresses the part of the problem concerning the extension of the Tannakian methods (see Section 4).

In the geometric theory employed to interpret the group theoretical directions, we turn to the study of projective stratified schemes and to the construction of a right adjoint to the obvious pull-back functor

$$(1) \quad f^* : \{\text{stratified schemes over } S\} \longrightarrow \{\text{stratified schemes over } X\}.$$

(Strictly speaking, we content ourselves with the construction of a certain “universal object” instead of an adjoint, see Definition 76.) As we are mainly interested in stratified *schemes*, and not just stratified modules, it seemed pointless to study morphism between topoi; we opted to interpret the imagined adjoint as a manifestation of the Weil restriction [BLR90, 7.6,191ff]. A blunt definition of the latter is simple enough to grasp: if $Z \rightarrow X$ is a scheme, its Weil restriction to S , f_*Z , is the scheme of S -*families* of sections $\sigma : X \rightarrow Z$. Following the necessary adjustments, we arrive at the scheme of Section 10.

In order to treat stratified schemes, we thought important to bring in the concept of connections as imagined by Ehresmann [E50] [CLN85, Chapter 5]. Ehresmann’s connections are connections on fibre bundles which do not necessarily have an affine typical fibre and are best treated via the theory of foliations. Foliations then steer us to certain equivalence relations which are trivial in the topological sense and account only for nilpotent elements in the sheaves of rings: *the infinitesimal equivalence relations*, see Section 7. To get a good hold of these, the reader is advised to read Example 24 before anything. One advantage offered by working with equivalence relations is the possibility of making algebraic geometric constructions as if they were banally set theoretical. Another, is the clear geometric way they give us the stratification on the scheme mentioned in the previous paragraph. Indeed, that scheme parametrizes certain families of sections, and we can imagine two among these to be equivalent if their images in the total space have equivalent points. See Definition 51 and Definition 53.

Here is the route to the text.

Section 3 presents a proof of part (1) of Theorem 1. It is a straightforward exercise.

In Section 4 we introduce two criteria based on actions on projective spaces for a short sequence of affine group schemes to be exact. As said before, this is parallel to the criterion presented in [SGA1, V, §6] and the appearance of projective spaces is suggested by the fact that homogeneous spaces are quasi-projective.

Section 5 deals with stratified schemes. It insists on the point of view that these stratified schemes are simply schemes endowed with an action of a “formal groupoid”, see §5.2. This idea seems to go back to Berthelot and Illusie, see [B74, Ch. II].

Section 6 takes up the idea of interpreting stratifications as actions of a formal groupoid of Section 5 and applies it to obtain instances where a stratified *proper* scheme “comes from the fundamental group scheme”, see Proposition 14. There the reader will also find a result which lets us conclude that stratified proper schemes are locally trivial for the fppf topology, see Lemma 12. This resembles the fact that stratified vector bundles are always locally free [BO78, Proposition 2.16].

The text then takes a different stand starting from Section 7, where we introduce infinitesimal equivalence relations (i.e.r.’s). This is one possible analogue of foliations [CLN85] in the algebraic geometric setting. Such a path follows what seems to be Ehresmann’s departure point for studying connections [E50], although this view seems to have lost appeal in our particular branch due to the simplicity of the standard purely “abelian” definition “ $\nabla : E \rightarrow E \otimes \Omega^1$ ”. Usually, to link the Theory of Foliations to Algebraic Geometry [E87, 3.1], [Mi87], one reinterprets the foliation as a sheaf of differential operators but our choice insists in promoting the idea of orbits of vector fields.

Infinitesimal equivalence relations (i.e.r.’s) are simply equivalence relations which are trivial whenever points take values in *reduced* schemes. In Section 7 we only make definitions and prove basic properties of these. The introduction of the *linearization* (Definition 25) of an i.e.r. is particularly useful in the rest of the work, while the notion of total complementarity (Definition 29) shows how i.e.r.’s relate to stratifications. It should become clear to the reader that the notion of i.e.r. can be more useful than that of a stratification.

In the ensuing section, Section 8, we introduce a more specific and adapted kind of infinitesimal equivalence relation: the differentially smooth ones (Definition 31). These are more sophisticated analogues of foliations and mimic Grothendieck’s definition of differential smoothness [EGA IV₄, 16.10]. In this important section, we observe that differentially smooth i.e.r.’s are locally given by formal actions of formal affine spaces and describe these precisely on the level of completions, see §§8.2, 8.3. This is parallel to the local interpretation of foliations as actions of affine spaces, see the appendix on the Frobenius Theorem in [CLN85] and the proof of Theorem 2.7 on p. 117 of [ESB05].

Section 9 is devoted to the study of two fundamental properties of the theory: invariance and tangency. Continuing with the formal analysis made in §§8.2 and 8.3, we study the shape of the complete local rings of an invariant subscheme of maximal codimension (a leaf), see Corollary 46. This study is a preparation to the analysis of the scheme of tangent sections.

In Section 10 we introduce the main character in the proof of Theorem 1: *the relative scheme of tangent sections*, $H_f(Z)$, or simply *the scheme of tangent sections*. See Definition 49 and the following lines. In the above notation, this scheme is the natural candidate for a “push-forward” of a stratified scheme $Z \rightarrow X$ along $f : X \rightarrow S$.

$$\begin{array}{ccc} Z & \dashrightarrow & H_f(Z) \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & S \end{array}$$

Its construction follows Grothendieck’s analysis of the Weil restriction by means of Hilbert schemes [TDTE, IV, §4 c, p. 19ff]. We note nonetheless that the construction of the scheme of tangent sections works in a slightly more general setting than that of a stratification.

Building on the formal description of invariant subschemes of maximal codimension (§§8.2 and 8.3), we show that under natural assumptions the scheme of tangent sections $H_f(Z)$ is *proper* over S (Theorem 50). This characteristic usually escapes the Weil restriction.

In Section 11 we are concerned with the stratified structure of the scheme of tangent sections H of the previous paragraph. In accordance with our principle that i.e.r.’s are more suitable objects than stratifications, we look for an i.e.r. on H . In doing so, we are led to present an equivalence relation among closed subschemes of a scheme endowed with an i.e.r. (Definition 53). This equivalence relation is the evident one: two closed subschemes are equivalent if their “orbits” intersect.

Section 12 concentrates in showing that the i.e.r. previously proposed turns the scheme of tangent sections $H_f(Z)$ into a universal object from f^* to Z , see Definition 76. This property, though less popular, is the heart of “adjointness.” Briefly, it implies that morphisms from an i.e.r. (T, Ψ) over S , viz. a stratified S -scheme, to $H_f(Z)$ are in bijection with morphisms of i.e.r.’s $f^*(T, \Psi) \rightarrow Z$, see Corollary 58. From that, we deduce directly that the formation of H commutes with base change, see Corollary 59.

Section 13 concentrates in showing that, under certain conditions, the infinitesimal equivalence relation defined in $H_f(Z)$ (see Section 11) comes from an S -stratification. This is condensed in Corollary 69 and stated in terms of “total complementarity” of infinitesimal equivalence relations (Definition 29). The base of the argument is Proposition 63, which uses Hensel’s Lemma.

In Section 14 we study the scheme $H_f(Z)$ in the case where S is a point. The obvious goal, in light of “the base change property” written in Corollary 59, is to gain knowledge concerning the fibres of the scheme of tangent sections over a positive dimensional base. The main result is Proposition 74. To arrive at its statement, one important assumption on the stratified scheme Z is made: it “comes from the fundamental group scheme”, see the preamble of §14.2 and the lines following Proposition 74. It should be noted that by imagining everything in terms of Ehresmann’s Theorem [CLN85, Theorem 3, Ch. 5,p.99]

$$\left\{ \begin{array}{l} \text{left actions of} \\ \text{the fundamental group} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{fibre bundles with} \\ \text{Ehresmann connections} \end{array} \right\},$$

the statement of Proposition 74 is no surprise.

In Section 15 we put the developed material together to furnish a proof of Theorem 1(2). As mentioned earlier, the idea is to employ that the (relative) scheme of tangent sections $H_f(Z)$ of a certain stratified scheme $Z \rightarrow X$ (Section 10) is a “right adjoint” to—or better: a universal object from— f^* of eq. (1). (Such a property was concreted in Section 12.) Through general nonsense (Lemma 77) we are then able to compare the fibre of $H_f(Z)$ above $s_0 \in S(k)$ with the scheme of fixed points $\mathbb{P}(V)^{\text{Ker}(b)}$; here V is some representation of $\Pi(X)$. Such a maneuver is possible because $\mathbb{P}(V)^{\text{Ker}(b)}$ is universal from

$$\text{Res}(b) : \{ \text{schemes with } \Pi(S, s_0) \text{ action} \} \longrightarrow \{ \text{schemes with } \Pi(X, x_0) \text{ action} \}$$

to $\mathbb{P}(V)$. This comparison, jointly with the depiction of $H_f(Z)|_{s_0}$ from Section 14 allows us to verify the conditions for $\text{Ker}(b) = \text{Im}(a)$ of Section 4.

The article closes with Section 16, which tackles the question of checking affinity of certain automorphism group schemes of projective schemes. It is written to permit the application of Proposition 14, which elucidates when a stratified scheme “comes from the fundamental group scheme.”

We end the introduction with a brief review of recent literature on the “exact sequence.” In [H11] the reader will find a proof of Theorem 1 for group schemes *associated* to $\Pi(-, -)$ and under the assumption that $\text{char}(k) > 0$. In [Z13], employing the complex analytic case, Lei Zhang gives a proof of Theorem 1 under the assumption $\text{char}(k) = 0$. The case where Π is replaced by Nori’s fundamental group scheme was the subject of the incomplete work [EHV].

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2. TERMINOLOGY

- (1) Given a morphism of ringed spaces $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$, we denote the underlying morphism of topological spaces by $|f|$ and the accompanying morphism of rings $|f|^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ by $f^\#$.
- (2) For any scheme X , we let X_{red} stand for the reduced closed subscheme associated to X . See [EGA I, 5.1.4,p.128]. If A is a ring, we let A_{red} be the quotient of A by its nilradical, so that $\text{Spec}(A_{\text{red}}) = \text{Spec}(A)_{\text{red}}$. Concerning this topic, we follow the conventions of EGA I.
- (3) Let B be a base scheme, S be a B -scheme and G a B -group scheme. Endow S with the trivial G -action. A G -torsor over S consists of an S -scheme $\psi : P \rightarrow S$ together with an action $\mu : P \times_B G \rightarrow P$ on the right of G such that (i) ψ is faithfully flat and quasi-compact, and (ii) the obvious morphism $P \times G \rightarrow P \times_S P$ is an isomorphism.

- (4) Let G and H be affine group schemes over a field. A homomorphism $u : G \rightarrow H$ is called a quotient morphism if the induced arrow on rings $u^\# : \mathcal{O}(H) \rightarrow \mathcal{O}(G)$ is faithfully flat. Due to [Wa, Theorem 14.1, p.109], it suffices that $u^\#$ be injective.
- (5) Given a group scheme G over a field and a closed subgroup H , we let G/H stand for the fppf sheaf associated to the obvious functor [DG, III,§3, 1.4, p. 324]. If G is affine and algebraic, this sheaf is represented by an algebraic scheme [DG, III,§3, 5.4, p.341]. If G is simply affine and H is normal, then the obvious fpqc sheaf G/H is represented by an affine group scheme, as we learn from [SGA3, VI_B, 11.17].

3. SURJECTIVITY OF $b : \Pi(X, x_0) \rightarrow \Pi(S, s_0)$

We maintain the notations of Theorem 1. Let $\mathcal{D}_\bullet\text{-mod}$ stand for the category of \mathcal{O}_\bullet -coherent $\mathcal{D}_{\bullet/k}$ -modules. We remind the reader that, due to smoothness of X and S , if $\mathcal{E} \in \mathcal{D}\text{-mod}$, then \mathcal{E} is locally free over \mathcal{O} [BO78, Proposition 2.16].

We wish to show that the functor

$$f^* : \mathcal{D}_S\text{-mod} \longrightarrow \mathcal{D}_X\text{-mod}$$

satisfies

- (i) f^* is fully faithful, and
- (ii) the essential image of $\mathcal{D}_S\text{-mod}$ is stable under subobjects.

Together with [DM82, 2.21,p.139], this shows part (1) of Theorem 1. We begin by noting that since $f_*\mathcal{O}_X = \mathcal{O}_S$, the functor f^* from the category of locally free \mathcal{O}_S -modules to the category of locally free \mathcal{O}_X -modules is full and faithful. Thus, to show (i), we need to prove: Given \mathcal{E}, \mathcal{F} in $\mathcal{D}_S\text{-mod}$ and an \mathcal{O}_S -linear morphism $\varphi : \mathcal{E} \rightarrow \mathcal{F}$ such that $f^*\varphi$ is \mathcal{D}_X -linear, then φ is \mathcal{D}_S -linear. This follows from direct local computations. Indeed, let $(s_1, \dots, s_m) : U \rightarrow \mathbb{A}_k^m$ be etale coordinates on an open subset of S . Let $V \subseteq X$ be an open subset mapping into U and possessing functions x_1, \dots, x_n such that $(\mathbf{x}, \mathbf{s}) : V \rightarrow \mathbb{A}_k^{m+n}$ is etale. Fix an arbitrary $\mathbf{p} \in \mathbb{N}^m$ and let $\partial^{[\mathbf{p}]} \in \mathcal{D}_S(U)$ be the differential operator satisfying $\partial^{[\mathbf{p}]}(\mathbf{s}^{\mathbf{q}}) = \binom{\mathbf{q}}{\mathbf{p}} \cdot \mathbf{s}^{\mathbf{q}-\mathbf{p}}$, see [EGA IV₄, 16.11.2,p.54]. Let $\tilde{\partial}^{[\mathbf{p}]} \in \mathcal{D}_X(V)$ be the differential operator satisfying $\tilde{\partial}^{[\mathbf{p}]}(\mathbf{s}^{\mathbf{q}}) = \binom{\mathbf{q}}{\mathbf{p}} \cdot \mathbf{s}^{\mathbf{q}-\mathbf{p}}$ and $\tilde{\partial}^{[\mathbf{p}]}(\mathbf{x}^{\mathbf{q}}) = 0$ [EGA IV₄, 16.11.2,p.64]. It is not hard to show that for each $e \in \mathcal{E}(U)$, we have

$$\tilde{\partial}^{[\mathbf{p}]} \cdot (1 \otimes e) = 1 \otimes \partial^{[\mathbf{p}]}e.$$

Using that f is faithfully flat, we conclude that φ is \mathcal{D}_S -linear.

The proof of (ii) also follows from a local calculation. Let $\mathcal{V}' \in \mathcal{D}_S\text{-mod}$ and let \mathcal{E} be a subobject in $\mathcal{D}_X\text{-mod}$ of $f^*\mathcal{V}'$. Let $s \in S$ be a closed point. Since $\mathcal{V}'|_{X_s}$ is a trivial \mathcal{D}_{X_s} -module, it follows that $\mathcal{E}|_{X_s}$ is also trivial as a \mathcal{D}_{X_s} -module. In particular, $\mathcal{E}|_{X_s}$ is trivial as an \mathcal{O}_{X_s} -module; this shows that $\dim H^0(X_s, \mathcal{E}|_{X_s}) = \text{rank } \mathcal{E}$. By [MAV, Corollary 2,p.50], $\mathcal{V} := f_*\mathcal{E}$ is locally free and the restriction to X_s of the canonical arrow of \mathcal{O}_X -modules

$$\theta : f^*\mathcal{V} \longrightarrow \mathcal{E}$$

is an isomorphism. As $f^*\mathcal{V}$ and \mathcal{E} are locally free, it follows that θ is an isomorphism. Since f^* is a fully faithful functor from the category of locally free \mathcal{O}_S -modules to the category of locally free \mathcal{O}_X -modules, there exists a unique \mathcal{O}_S -linear arrow $\sigma : \mathcal{V} \rightarrow \mathcal{V}'$ such that $f^*\sigma$ is the composition $f^*\mathcal{V} \xrightarrow{\sim} \mathcal{E} \hookrightarrow f^*\mathcal{V}'$. By faithful flatness of f , σ is a monomorphism.

We now need to show that \mathcal{V} possesses a \mathcal{D}_S -module structure which turns σ into a morphism of \mathcal{D}_S -modules. Put differently, we need to show that $\text{Im}(\sigma) \subseteq \mathcal{V}$ is invariant under \mathcal{D}_S . Once verified this statement, the proof of (ii) is finished.

We will reduce the work to a local calculation using etale coordinates. Let $(s_1, \dots, s_m) : U \rightarrow \mathbb{A}_k^m$ be etale coordinates on an open affine of S . Let $V \subseteq X$ be an open affine subset mapping into U and possessing functions x_1, \dots, x_n such that $(\mathbf{x}, \mathbf{s}) : V \rightarrow \mathbb{A}_k^{m+n}$ is etale. For each multi-index $\mathbf{p} \in \mathbb{N}^m$, respectively $(\mathbf{p}, \mathbf{q}) \in \mathbb{N}^m \times \mathbb{N}^n$, we write $\partial^{[\mathbf{p}]}$, resp. $\tilde{\partial}^{[\mathbf{p}, \mathbf{q}]}$, for the differential operators on U , resp. V , obtained from the functions \mathbf{s} , resp. \mathbf{s} and \mathbf{x} , as in [EGA IV₄, 16.11.2, p.54]. Moreover, we assume that \mathcal{V} is free on the basis v_1, \dots, v_h , while \mathcal{V}' is free on the basis $\sigma(v_1), \dots, \sigma(v_h), v'_1, \dots, v'_c$; this assumption poses no serious impediment since f is faithfully flat and $\text{Coker}(f^*\sigma)$ is locally free due to its \mathcal{D}_X -module structure. Then

$$\begin{aligned} 1 \otimes \partial^{[\mathbf{p}]} \sigma(v_i) &= \tilde{\partial}^{[\mathbf{p}, \mathbf{0}]} \cdot (1 \otimes \sigma(v_i)) \\ &= \sum_{j=1}^h \xi_{ij} \cdot (1 \otimes \sigma(v_j)), \end{aligned}$$

where $\xi_{ij} \in \mathcal{O}_X(V)$. On the other hand, there exist functions $\alpha_{ij}, \beta_{il} \in \mathcal{O}_S(U)$ such that $\partial^{[\mathbf{p}]} \sigma(v_i) = \sum \alpha_{ij} \cdot \sigma(v_j) + \sum \beta_{il} \cdot v'_l$, so that $\xi_{ij} = f^\# \alpha_{ij}$ and $f^\# \beta_{il} = 0$. By flatness of $f^\#$, we conclude that $\beta_{il, P} = 0$ for each $P \in f(V)$. This shows that $\text{Im}(\sigma)$ is invariant under \mathcal{D}_S .

4. CRITERION FOR EXACTNESS USING PROJECTIVE REPRESENTATIONS

We fix an algebraically closed field k . All morphisms and objects are over k . Let

$$K \xrightarrow{a} \Pi \xrightarrow{b} H$$

be homomorphisms of affine group schemes such that $b \circ a$ is the trivial homomorphism and b is a quotient morphism.

4.1. We define a property concerning the morphisms a and b .

Property 4.1: For all affine algebraic quotients $q : \Pi \rightarrow G$, and all closed subgroup schemes $I \leq G$,

$$(G/I)^K \neq \emptyset \Rightarrow \text{Ker}(b) \text{ acts trivially on } G/I.$$

Remark 2. Let $N \trianglelefteq \Pi$ be a closed subgroup. Let $q : \Pi \rightarrow G$ be a quotient morphism of affine group schemes. Then N acts trivially on $G/I \Leftrightarrow q : N \rightarrow G$ factors through I . The proof of this fact is an exercise using [DG, III, §3,1.5, p.324] and the fact that $\mathcal{O}(\Pi)$ is a direct limit of faithfully flat $\mathcal{O}(G)$ -algebras [Wa, 3.3 and 14.1].

Lemma 3. *Property 4.1 holds* $\Leftrightarrow \text{Ker}(b) \subseteq \text{Im}(a)$.

Proof. (\Rightarrow). Let $q : \Pi \rightarrow G$ be an affine and algebraic quotient. Let $I \leq G$ be the image of $q \circ a : K \rightarrow G$. There exists a K -invariant element in G/I determined by the image of the identity

$$\mathrm{Spec}(k) \xrightarrow{e_G} G \longrightarrow G/I.$$

So $\mathrm{Ker}(b)$ acts trivially on G/I . Hence, $q : \mathrm{Ker}(b) \rightarrow G$ factors through I . The implication then follows, not without a small effort, from the fact that the Hopf algebra $\mathcal{O}(\Pi)$ is a direct limit of Hopf algebras which are finitely generated over k [Wa, 3.3].

(\Leftarrow). Assume that $\mathrm{Ker}(b) = \mathrm{Im}(a)$. Consequently, $\mathrm{Im}(a) \trianglelefteq \Pi$. Let $q : \Pi \rightarrow G$ be a faithfully flat homomorphism and let $I \leq G$ be a closed subgroup such that $(G/I)^K \neq \emptyset$. We claim that $q : \mathrm{Ker}(b) \rightarrow G$ factors through $I \rightarrow G$. Let $\mathfrak{p} : G \rightarrow G/I$ be the canonical morphism. Since G/I is algebraic over k , and $\mathfrak{p}(k) : G(k) \rightarrow (G/I)(k)$ is surjective, there exists $g \in G(k)$ such that $\mathfrak{p}(g) \in (G/I)(k)$ is K -invariant. It is also possible to assume that $g = q(\gamma)$ for some $\gamma \in \Pi(k)$. Now for any k -algebra R and any $y \in K(R)$, we have

$$\begin{aligned} \mathfrak{p}(g^{-1} \cdot qa(y) \cdot g) &= g^{-1} \cdot \mathfrak{p}(qa(y) \cdot g) \\ &= g^{-1} \cdot \mathfrak{p}(g) \\ &= \mathfrak{p}(e). \end{aligned}$$

As the inverse image of $\mathfrak{p}(e)$ under $\mathfrak{p}(R) : G(R) \rightarrow (G/I)(R)$ is $I(R)$ [DG, III§3,1.5, p.324], it follows that $g^{-1} \cdot qa \cdot g : K \rightarrow G$ factors through I . Using that $\mathrm{Im}(a) \trianglelefteq \Pi$ and that $g = q(\gamma)$, it is easy to conclude that $qa : K \rightarrow G$ factors through I . \square

4.2. Let us define another property. The arrows $a : K \rightarrow \Pi$ and $b : \Pi \rightarrow H$ are as before.

Property 4.2: For any finite dimensional representation V of Π ,

$$\mathbb{P}(V)^{\mathrm{Im}(a)}(k) \subseteq \mathbb{P}(V)^{\mathrm{Ker}(b)}(k).$$

Here we employ the notations and definitions of [DG, II, §1,3.4, p.163] (or [J87, Part 1, 2.6, 27ff]) concerning schemes of fixed points. It is opportune to observe that Property 4.2 can be translated as forcing the equality of topological spaces

$$\left| \mathbb{P}(V)^{\mathrm{Ker}(b)} \right| = \left| \mathbb{P}(V)^{\mathrm{Im}(a)} \right|,$$

as these fixed point functors are all represented by closed subschemes of $\mathbb{P}(V)$ [DG, II, §1, 3.6(d), p. 165].

Lemma 4. *Property 4.2 \Leftrightarrow property 4.1.*

Proof. (4.2 \Rightarrow 4.1). Let $q : \Pi \rightarrow G$ be a faithfully flat homomorphism to an algebraic group scheme, and let $I \leq G$ be a closed subgroup. We assume that $(G/I)^K \neq \emptyset$. By Chevalley's theorem, there exists a representation V of G and an equivariant immersion $G/I \rightarrow \mathbb{P}(V)$, as indicates [DG] in the proof of Theorem 5.4 in §3 of Chapter III. Denote by \mathfrak{p} the canonical morphism of schemes $G \rightarrow G/I$ and let $g \in G(k)$ be such that $\mathfrak{p}(g)$ is K -invariant. By hypothesis, $\mathfrak{p}(g)$ is also $\mathrm{Ker}(b)$ -invariant. This means that for each k -algebra R and each $y \in \mathrm{Ker}(b)(R)$, we have

$$\begin{aligned} q(y) \cdot \mathfrak{p}(g) &= \mathfrak{p}(g) \Rightarrow (g^{-1} \cdot q(y) \cdot g) \cdot \mathfrak{p}(e) = \mathfrak{p}(e) \\ &\Rightarrow g^{-1} \cdot q \cdot g : \mathrm{Ker}(b) \rightarrow G \text{ factors through } I, \end{aligned}$$

due to [DG, III, §3,1.5, p.324]. As $q : \Pi(k) \rightarrow G(k)$ is surjective and $\text{Ker}(b)$ is normal, it follows that $q : \text{Ker}(b) \rightarrow G$ factors through I . By Remark 2, $\text{Ker}(b)$ acts trivially on G/I .

(4.2 \Leftarrow 4.1). By virtue of Lemma 3 this is clear. \square

5. STRATIFIED SCHEMES

5.1. Groupoids and actions of groupoids. Let \mathcal{C} be a category; in the sequel, we assume that all displayed fibered products exist in \mathcal{C} . Let

$$G \begin{array}{c} \xrightarrow{p_0} \\ \rightrightarrows \\ \xleftarrow{p_1} \end{array} S, \quad \delta : G \times_{p_0, S, p_1} G \longrightarrow G, \quad \sigma : G \longrightarrow G, \quad \varepsilon : S \longrightarrow G$$

be a groupoid in \mathcal{C} [MM03, 5.1]. The source of δ , the object of “composable” arrows, will be denoted by $G(2)$. This object is also the source of other two arrows:

$$\begin{aligned} \chi_0 : G(2) &\longrightarrow G &= &\text{first projection} \\ \chi_1 : G(2) &\longrightarrow G &= &\text{second projection.} \end{aligned}$$

These arrows satisfy:

$$(2) \quad p_0 \circ \delta = p_0 \circ \chi_1, \quad p_1 \circ \delta = p_1 \circ \chi_0, \quad p_1 \circ \chi_1 = p_0 \circ \chi_0,$$

$$(3) \quad p_0 \varepsilon = p_1 \varepsilon = \text{Id}_S, \quad p_0 \sigma = p_1, \quad p_1 \sigma = p_0.$$

Remark 5. There is some discrepancy in the literature concerning the definition of the object of “composable” arrows. We have followed the one which preserves graphical simplicity, saying that $\delta(g, g')$ should be thought of as $g \circ g'$. This is what [MM03] adopts as definition. On the other hand, following Grothendieck [TDTE, 212–07ff], it is common to work with the switched object of “composable” arrows, i.e. $G \times_{p_1, S, p_0} G$. Berthelot [B74] also follows this convention. Further ahead we will find convenient to follow it.

We now want to study left actions of G . Given an object $\pi : E \rightarrow S$ over S , let

$$\alpha : G \times_{p_0, S, \pi} E \longrightarrow E$$

be an arrow satisfying

A1. $\pi \alpha(g, y) = p_1(g)$ for all $g \in G$ and $y \in E$.

A2. $\alpha(\varepsilon(\pi(y)), y) = y$ for all $y \in E$.

Condition A1 amounts to an arrow

$$\theta := (\text{pr}_G, \alpha) : G \times_{p_0, S, \pi} E \longrightarrow G \times_{p_1, S, \pi} E$$

of G -objects. Due to (3), pulling-back θ by $\varepsilon : S \rightarrow G$ defines an arrow $E \rightarrow E$ and condition A2 just means that this is id_E . Pulling-back θ by χ_1 , we derive an arrow of $G(2)$ objects

$$\chi_1^*(\theta) : G(2) \times_{p_0 \chi_1, S, \pi} E \longrightarrow G(2) \times_{p_1 \chi_1, S, \pi} E, \quad (g, h, y) \longmapsto (g, h, \alpha(h, y)).$$

Pulling-back θ by χ_0 , we obtain

$$\chi_0^*(\theta) : G(2) \times_{p_0 \chi_0, S, \pi} E \longrightarrow G(2) \times_{p_1 \chi_0, S, \pi} E, \quad (g, h, y) \longmapsto (g, h, \alpha(g, y)).$$

Pulling-back θ by δ , we obtain

$$\delta^*(\theta) : G(2) \times_{p_0\delta, S, \pi} E \longrightarrow G(2) \times_{p_1\delta, S, \pi} E, \quad (g, h, y) \longmapsto (g, h, \alpha(\delta(g, h), y))$$

By means of eq. (2), we see that $\delta^*(\theta)$ and $\chi_0^*(\theta) \circ \chi_1^*(\theta)$ have the same source and target in \mathcal{C} . The definition presented on [MM03, p.125] can now be given the following translation.

Definition 6. We say that α is a left action of G on $\pi : E \rightarrow S$ if, together with A1 and A2, we have

$$\mathbf{A3.} \quad \delta^*(\theta) = \chi_0^*(\theta) \circ \chi_1^*(\theta).$$

Lemma 7. *Let α be a left action of G on $\pi : E \rightarrow S$. Then the arrow θ is an isomorphism.*

Proof. We assume that the category \mathcal{C} is a category of sets. Let $\theta(g, y) = \theta(g', y')$; by definition $g = g'$ and $\alpha(g, y) = \alpha(g, y')$. As $\alpha(\sigma(g), \alpha(g, z)) = \alpha(\varepsilon(\pi(z)), z) = z$, we obtain $y = y'$. Thus θ is injective. We now prove surjectivity. Let $(g, y) \in G \times E$ be such that $p_1(g) = \pi(y)$. Since $p_0(\sigma(g)) = p_1(g)$, we can consider the element $y' := \alpha(\sigma(g), y)$. Now, $\pi(y') = p_1(\sigma(g)) = p_0(g)$. Then $\alpha(g, y') = \alpha(g, \alpha(\sigma(g), y)) = y$. \square

In view of this easy lemma, Definition 8 below concords with Definition 6.

Definition 8. An action of G on $\pi : E \rightarrow S$ is an isomorphism of G -objects

$$\theta : G \times_{p_0, S, \pi} E \xrightarrow{\sim} G \times_{p_1, S, \pi} E$$

which satisfies $\varepsilon^*(\theta) = \text{id}_E$ and axiom A3.

In more functional fashion, an action θ of the groupoid G on $\pi : E \rightarrow S$ determines for each $g \in G(U) = \text{Hom}(U, G)$ an U -isomorphism

$$(4) \quad \theta_g : p_0(g)^{-1}E \xrightarrow{\sim} p_1(g)^{-1}E$$

and these satisfy

$$\begin{aligned} \theta_{g'g} &= \theta_{g'} \circ \theta_g, \quad \text{if } p_0(g') = p_1(g), \text{ and} \\ \theta_{\varepsilon(s)} &= \text{id}_{s^{-1}E} \quad \text{for each } s \in S(U). \end{aligned}$$

5.2. The formal diagonal groupoid and stratifications [B74, 1.1, p. 82ff]. Let us fix a noetherian separated scheme B , and S a B -scheme locally of finite type and separated. Let **Sch** stand for the category of locally noetherian B -schemes and **FSch** for the category of locally noetherian formal B -schemes. We regard **Sch** as a full subcategory of **FSch** [EGA I, 10.4.8, p.186], and the latter as a full subcategory of $\mathbf{Set}^{\text{Sch}^{\text{op}}}$ following [Mac98, Ch X, §6, Prp.2, p.246] and [EGA I, 10.6, 188ff]. Even though general fibre products do not exist in **FSch**, it will certainly be the case for all such products considered in the sequel. (It is profitable to recall that if \mathfrak{X} is an *adic* formal scheme with ideal of definition \mathcal{J} such that $(|\mathfrak{X}|, \mathcal{O}/\mathcal{J})$ is a locally noetherian scheme, and the ideal $\mathcal{J}/\mathcal{J}^2$ is of finite type as a \mathcal{O}/\mathcal{J} -module, then \mathfrak{X} is locally noetherian [EGA I₀, 7.2.6, p.64].) References to B will be omitted as much as possible, e.g. \times will stand for \times_B .

We wish to harmonize Berthelot's terminology in [B74, Ch.II] with ours, so *our groupoids will have the switched object of "composable" arrows for the moment*. See Remark 5.

Let $S' := S \times S$ and $S'' := S \times S \times S$. Denote the obvious projections $S' \rightarrow S$ by \hat{p}_0 and \hat{p}_1 . Define the following morphisms

$$\begin{aligned} \hat{q}_0 : S'' &\longrightarrow S', (s_1, s_2, s_3) \mapsto (s_1, s_2) \\ \hat{q}_1 : S'' &\longrightarrow S', (s_1, s_2, s_3) \mapsto (s_2, s_3) \\ \hat{s} : S'' &\longrightarrow S', (s_1, s_2, s_3) \mapsto (s_1, s_3). \end{aligned}$$

Note that by means of the identification

$$(5) \quad S'' = S' \times_{\hat{p}_1, S, \hat{p}_0} S', \quad (s_1, s_2, s_3) \longmapsto [(s_1, s_2); (s_2, s_3)],$$

\hat{q}_0 and \hat{q}_1 correspond to the first and second projections respectively. Finally, let $\hat{\varepsilon} : S \rightarrow S'$ stand for the diagonal morphism and $\hat{s} : S' \rightarrow S'$ for the “switch” morphism. We then obtain an obvious groupoid, called sometimes the kernel groupoid [MM03, 5.1(2), p.113]:

$$(6) \quad S' \begin{array}{c} \xrightarrow{\hat{p}_0} \\ \xrightarrow{\hat{p}_1} \end{array} S, \quad \hat{\delta} : S'' \longrightarrow S', \quad \hat{s} : S' \longrightarrow S', \quad \hat{\varepsilon} : S \longrightarrow S'.$$

We now pass to formal schemes. Let $|\Delta'|$, respectively $|\Delta''|$, stand for the closed subspace of S' , respectively S'' , corresponding to the diagonal. Let

$$\widehat{P}_S = \text{completion of } S' \text{ along } |\Delta'| \quad \text{and} \quad \widehat{P}_S(2) = \text{completion of } S'' \text{ along } |\Delta''|.$$

(See [EGA I, 10.8, 194ff].)

Since $\hat{\varepsilon}(S) \subseteq |\Delta'|$, $\hat{s}(|\Delta'|) \subseteq |\Delta'|$, $\hat{q}_i(|\Delta''|) \subseteq |\Delta'|$, and $\hat{\delta}(|\Delta''|) \subseteq |\Delta''|$, we deduce morphisms

$$p_i : \widehat{P}_S \longrightarrow S, \quad \varepsilon : S \longrightarrow \widehat{P}_S, \quad \delta : \widehat{P}_S(2) \longrightarrow \widehat{P}_S, \quad \text{etc.}$$

Note that eq. (5) and [EGA I, 10.9.7, p.200] allow us to identify

$$\widehat{P}_S(2) = \widehat{P}_S \times_{p_1, S, p_0} \widehat{P}_S,$$

q_0 with the first projection, and q_1 with the second. (In particular, the fibre product in question exists in **FSch**.) It is interesting to note that the restriction $\delta : P_S^\nu \times_S P_S^\nu \rightarrow \widehat{P}_S$ does not factor through P_S^ν ; a fact which hides the property that the composition of differential operators of order ν will not usually be of order ν , see [EGA , 16.8.9.1, p. 43]. By “functoriality of completions” [EGA I, 10.9.3, p.199], the arrows in (6) together with the identification $\widehat{P}_S(2) = \widehat{P}_S \times_{p_1, p_0} \widehat{P}_S$ give us a groupoid

$$\widehat{P}_S \begin{array}{c} \xrightarrow{p_0} \\ \xrightarrow{p_1} \end{array} S, \quad \delta : \widehat{P}_S \times_{p_1, p_0} \widehat{P}_S \longrightarrow \widehat{P}_S, \quad \varepsilon : S \longrightarrow \widehat{P}_S, \quad s : \widehat{P}_S \longrightarrow \widehat{P}_S$$

in **FSch**, which we call the *formal diagonal groupoid*. We now want to make explicit the fact that actions of the formal diagonal groupoid are simply the stratifications of [B74].

Let P_S^ν be the ν -th infinitesimal neighbourhood of the diagonal in $S \times S$, see [EGA IV₄, 16.1.2, p.5], [EGA IV₄, 16.3.1, p.14]. Note that

$$\widehat{P}_S = \varinjlim_{\nu} P_S^\nu$$

as explains [EGA I, 10.6]. The schemes P_S^ν come with two evident morphisms to S , p_0' and p_1' , and one from S , $\varepsilon^\nu : S \rightarrow P_S^\nu$. Analogously, let $P_S^\nu(2)$ denote the ν -th infinitesimal

neighbourhood of the diagonal in $S'' = S \times S \times S$. Write q_0', q_1' and δ' for the morphisms from $P_S'(2)$ to P_S' induced by $\dot{q}_0, \dot{q}_1, \dot{\delta}$ above. Berthelot [B74, p.96] defines a stratification of π to be a family of compatible isomorphisms

$$\theta_\nu : p_0^{\nu*} E \xrightarrow{\sim} p_1^{\nu*} E$$

of P_S' -schemes which satisfies

$$\varepsilon^{\nu*}(\theta_\nu) = \text{id}_E, \quad \delta^{\nu*}(\theta_\nu) = q_1^{\nu*}(\theta_\nu) \circ q_0^{\nu*}(\theta_\nu), \quad \forall \nu \in \mathbb{N}.$$

(Note that in [B74, p.84], Berthelot prefers to work with the algebras instead of schemes, so, in the preceding discussion, our q_i' is his $\text{Spec } q_i'$.) Due to [EGA I, 10.6.7, p.191] and [EGA I, 10.7.4], such families are in bijection with isomorphism

$$\theta : E \times_{S, p_0} \widehat{P}_S \xrightarrow{\sim} \widehat{P}_S \times_{p_1, S} E$$

in $\mathbf{FSch}/\widehat{P}_S$ satisfying

$$\varepsilon^*(\theta) = \text{id}_E$$

and

$$(7) \quad \delta^*(\theta) = q_1^*(\theta) \circ q_0^*(\theta).$$

If we now *reinstate our conventions of §5.1* and switch the factors in the objects of “composable” arrows, i.e. define $\widehat{P}_S(2)$ as $\widehat{P}_S \times_{p_0, p_1} \widehat{P}_S$, then q_0 corresponds to what was χ_1 in §5.1 and q_1 to χ_0 ; consequently eq. (7) is equivalent to condition A3 of §5.1. All these trivial observations only find their *raison d'être* in sustaining the following.

Lemma 9. *Let $\pi : E \rightarrow S$ be a morphism of schemes. The data of a B -linear stratification of π in the sense of [B74, p.96] is a left action (see Definition 8) of the formal diagonal groupoid on E . \square*

The category of B -linearly stratified schemes over S will be denoted by $\mathbf{StrS}(S/B)$, and when no risk of confusion presents, we shall drop the reference to the base B . In the same spirit of economy and simplification, we shall adopt the following convention of Berthelot.

Convention 10. Whenever considering a fibre product in which one of the factors is P_S' (or \widehat{P}_S) we will put the symbol P_S' on the *left* to mean that the *second projection* is being considered, and on the *right*, if the first projection is being used to form the product. E.g. $P_S' \times_S X$ means that we consider P_S' as a scheme over S via p_1' in order to form the fibre product.

6. A NON-AFFINE TANNAKIAN CORRESPONDENCE

We fix an algebraically closed field k . In this section, \mathbf{Sch} , respectively \mathbf{FSch} , stands for the category of locally noetherian k -schemes, respectively locally noetherian formal k -schemes.

Let S be a smooth and connected k -scheme. We fix $s_0 \in S(k)$. Let \mathcal{D}_S stand for the ring of k -linear differential operators on S , see [EGA IV₄, 16.3.1, p.14] or [BO78, §2].

Our goal in this section is to study the relation between stratified S -schemes of Section 5 and \mathcal{O}_S -coherent \mathcal{D}_S -modules.

6.1. \mathcal{O}_S -coherent \mathcal{D}_S -modules. The category of \mathcal{D}_S -modules, $\mathcal{D}_S\text{-Mod}$, is an abelian category endowed with a tensor product. Its full subcategory consisting of those modules which are coherent over \mathcal{O}_S will be denoted $\mathcal{D}_S\text{-mod}$ in the sequel. The functor

$$s_0^* : \mathcal{D}_S\text{-mod} \longrightarrow k\text{-mod}, \quad \mathcal{M} \longmapsto \mathcal{M} \otimes \mathbf{k}(s_0)$$

endows $\mathcal{D}_S\text{-mod}$ with a structure of neutral Tannakian category [DM82, §2], and the affine group scheme associated to this data is denoted by $\Pi(S, s_0)$, see [DM82, Theorem 2.11, p. 130] for the construction of $\Pi(S, s_0)$, [Sav72, VI 1.2.2] for one of its earliest appearances, and [dS07] for more basic properties. It is common to call $\Pi(S, s_0)$ the stratified fundamental group scheme, and we will refer to it simply as *the fundamental group scheme of S* . To ease notation, we will write $\Pi(S)$ instead of $\Pi(S, s_0)$.

Let $\psi : \mathcal{U}_S \rightarrow S$ stand for the universal $\Pi(S)$ -torsor as constructed in [dS07, 2.3.2], or in [Nor76, §2]. This is the object of the ind-category of \mathcal{D}_S -modules corresponding to the left-regular representation of $\Pi(S)$. Note that the morphism ψ is *affine*, so that ψ amounts to the data of the quasi-coherent \mathcal{O}_S -algebra $\psi_*\mathcal{O}_{\mathcal{U}_S}$. This \mathcal{O}_S -algebra is in fact a particular kind of \mathcal{D}_S -module which we now set to describe.

Firstly, just as for affine group schemes,

$$(8) \quad \mathcal{U}_S = \varprojlim_{\lambda} \mathcal{U}_{\lambda},$$

where $\psi_{\lambda} : \mathcal{U}_{\lambda} \rightarrow S$ is a torsor under an algebraic affine groups $\Pi(S)_{\lambda}$ and the arrows in the projective systems are all morphisms of torsors. Each \mathcal{U}_{λ} is a stratified torsor over S , the meaning of which we recall. For each λ , there exists a k -linear stratification of ψ_{λ} ,

$$\{\alpha_{\lambda, \nu} : \mathcal{U}_{\lambda} \times_S P_S^{\nu} \longrightarrow \mathcal{U}_{\lambda} : \nu \in \mathbb{N}\},$$

such that the canonical isomorphism

$$\mathcal{U}_{\lambda} \times \Pi(S) \xrightarrow{\sim} \mathcal{U}_{\lambda} \times_S \mathcal{U}_{\lambda}, \quad (u, \gamma) \longmapsto (u, u \cdot \gamma)$$

sends the standard stratification on $\mathcal{U}_{\lambda} \times \Pi(S)$, i.e. the stratification induced by the stratification on \mathcal{U}_{λ} , to the product stratification on $\mathcal{U}_{\lambda} \times_S \mathcal{U}_{\lambda}$. In addition, the transition morphisms in the projective family \mathcal{U}_{λ} are morphisms of stratified schemes.

Of course, all these conditions can be translated in “abelian terms” upon passage to the category of quasi-coherent \mathcal{O}_S -algebras, this is explained in [dS07, 2.3.1, 2.3.2]. Due to the fact that each $\Pi(S)_{\lambda}$ -module is a direct limit of *finite dimensional* $\Pi(S)_{\lambda}$ -modules [Wa, 3.3, p.24], it follows that the \mathcal{D}_S -module $\mathcal{O}_{\mathcal{U}_{\lambda}}$ is a direct limit of \mathcal{O}_S -coherent \mathcal{D}_S -modules. A fortiori $\mathcal{O}_{\mathcal{U}_S} = \varinjlim_{\lambda} \mathcal{O}_{\mathcal{U}_{\lambda}}$ is a direct limit of objects in $\mathcal{D}_S\text{-mod}$. The role of the fundamental group scheme $\Pi(S)$ and \mathcal{U}_S is to control all \mathcal{D}_S -modules of this particular type.

6.2. Proper stratified schemes. Let $\mathbf{StrP}(S)$ stand for the full subcategory of $\mathbf{StrS}(S)$ (see Section 5) consisting of those objects whose underlying S -scheme is *proper*. One simple way to construct objects in this category is by *contraction* with \mathcal{U}_S . Choose a proper k -scheme F together with a representation $\rho : \Pi(S) \rightarrow \mathbf{Aut}_k(F)$; assume that the quotient of $\mathcal{U}_S \times F$ by the diagonal action of $\Pi(S)$ on the right is a scheme

$$\mathcal{U}_S \times^\rho F \longrightarrow S.$$

(A good source for learning the gymnastics around contracted products $\bullet \times^\rho \bullet$ is [J87, Part 1, Ch.5].) The stratification of $\mathcal{U}_S \times F$ “on the first factor” then gives a stratified S -scheme $\mathcal{U}_S \times^\rho F \rightarrow S$: this can be seen using the two following arguments. (i) There is an obvious stratification on the *pre-sheaf* $(\mathcal{U}_S \times F)/\Pi(S) : \mathbf{Sch}^o \rightarrow \mathbf{Set}$. (ii) Any stratification on a pre-sheaf induces one on the associated sheaf, due to the fact that “taking the associated sheaf commutes with limits.” (This last statement is explained in SGA4, see exposé II, Prp. 4.3, p.237 and exposé I, Def. 2.5, p.13.) Moreover, flat descent [EGA IV₂, 2.7.1, p.29] assures that $\mathcal{U}_S \times^\rho F \rightarrow S$ is proper, so that $\mathcal{U}_S \times^\rho F \in \mathbf{StrP}(S)$.

Remark 11. Of course, it is not without restrictions that the quotient $(\mathcal{U}_S \times F)/\Pi(S)$ is a scheme. Two important instance in which this occurs are the case where F is affine and where there exists a finite dimensional $\Pi(S)$ -module V and a $\Pi(S)$ -equivariant immersion $F \rightarrow \mathbb{P}(V)$. The first class of examples follows directly from [TDTE 190–19, Theorem 2], while the second from [BLR90, Theorem 7, p.138].

Our goal is to introduce some elementary constructions which will allow us to identify some objects of $\mathbf{StrP}(S)$ arising from \mathcal{U}_S . For this matter, let

$$(9) \quad \mathcal{R}_S = \left\{ \begin{array}{l} \text{couples } (F, \rho : \Pi(S) \rightarrow \mathbf{Aut}_k(F)) \text{ consisting of a proper } k\text{-scheme } F \\ \text{and a left action } \rho \text{ of } \Pi(S) \text{ on } F \text{ such that } \mathcal{U}_S \times^\rho F \text{ is a scheme} \end{array} \right\}.$$

Together with the obvious arrows, \mathcal{R}_S is a category and $(F, \rho) \mapsto \mathcal{U}_S \times^\rho F$ is a functor to $\mathbf{StrP}(S)$. Objects of $\mathbf{StrP}(S)$ stemming from \mathcal{R}_S are named “stratified proper schemes coming from the fundamental group scheme.”

6.3. Local triviality of proper stratified schemes. The following result is reminiscent of the fact that coherent modules with stratifications are locally free [BO78, 2.16].

Lemma 12. *Let $(\pi : E \rightarrow S, \{\theta_\nu\})$ be a stratified scheme and assume that π is proper. Let F stand for the fibre of π above s_0 . Then, there exists an fppf covering $S' \rightarrow S$ and an isomorphism*

$$S' \times_S E \xrightarrow{\cong} S' \times F.$$

In particular, π is flat.

Proof. Let \widehat{E}_{s_0} be the formal fibre of π at s_0 . Using the stratification, we can easily construct an isomorphism of formal schemes over $\mathrm{Spf} \widehat{\mathcal{O}}_{s_0}$

$$\widehat{E}_{s_0} \simeq \mathrm{Spf} \widehat{\mathcal{O}}_{s_0} \times F.$$

As π is proper, we can apply Grothendieck's algebraization theorem [EGA III₁, 5.4.1, p.156] to obtain an isomorphism of $\widehat{\mathcal{O}}_{s_0}$ -schemes

$$\mathrm{Spec} \widehat{\mathcal{O}}_{s_0} \times_S E \xrightarrow{\simeq} \mathrm{Spec} \widehat{\mathcal{O}}_{s_0} \times F.$$

Using [EGA IV₃, 8.8.2.5, p.32] and the fact that $\widehat{\mathcal{O}}_{s_0}$ is a direct limit of flat \mathcal{O}_{s_0} -algebras, it follows that there exists a flat morphism of finite type $U \rightarrow S$ whose image contains s_0 and an isomorphism of U -schemes

$$E \times_S U \xrightarrow{\simeq} U \times F.$$

Moreover, since $U \rightarrow S$ is open, for each k -point s on the image of U , we have $F \simeq \pi^{-1}(s)$. Consequently, for any $s \in S(k)$ we have $\pi^{-1}(s) \simeq F$. \square

Remark 13. The reader must have realized that in the preceding proof we used that $\widehat{\mathcal{O}}_{S,s_0}$ is a direct limit of flat \mathcal{O}_{S,s_0} -algebras of finite type. This result is an immediate consequence of a much more powerful one [T93]. We would be put in a hard position were we to demonstrate this weaker version from first principles.

6.4. Proper stratified schemes obtained from \mathcal{U}_S .

Proposition 14. *Let $(\pi : E \rightarrow S, \{\theta_\nu\})$ be an object of $\mathbf{StrP}(S)$. Put $F := \pi^{-1}(s_0)$ and assume that $\mathbf{Aut}_k(F)$ is affine. Then there exists a representation $\rho : \Pi(S) \rightarrow \mathbf{Aut}_k(F)$ and an isomorphism in $\mathbf{StrP}(S)$ between E and $\mathcal{U}_S \times^\rho F$. Said differently, E comes from the fundamental group scheme (see page 14).*

The proof will rely on a concept arising from Differential Geometry. Define the functor of frames

$$\mathbf{I} : (\mathbf{Sch}/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}, \quad (U \rightarrow S) \longmapsto \mathrm{Isom}_U(F \times U, E \times_S U).$$

To make notation more suggestive, given $U \in \mathbf{Sch}/S$ and $a \in \mathrm{Hom}_S(U, \mathbf{I})$, we write

$$\varphi_a : F \times U \xrightarrow{\simeq} E \times_S U$$

to signify the obtained isomorphism of U -schemes. We shall find convenient to think of \mathbf{I} as a functor $\mathbf{Sch}^{\mathrm{op}} \rightarrow \mathbf{Set}$ together with a morphism $\psi : \mathbf{I} \rightarrow S$, so that for each $a \in \mathbf{I}(U) = \mathrm{Hom}_k(U, \mathbf{I})$, we have

$$\varphi_a : F \times U \xrightarrow{\simeq} E \times_{S, \psi(a)} U.$$

It is also worth noting that \mathbf{I} is a sheaf for the fpqc topology on \mathbf{Sch} [TDTE, I, p. 19] and that it possesses an action of the group scheme $\mathbf{Aut}_k(F)$ on its right. Explicitly, given $U \in \mathbf{Sch}$, an U -automorphism $\gamma : F \times U \rightarrow F \times U$, and $a \in \mathbf{I}(U)$, we define an element $a * \gamma$ by means of the isomorphism

$$(10) \quad \varphi_{a * \gamma} : F \times U \xrightarrow{\gamma} F \times U \xrightarrow{\varphi_a} E \times_{S, \psi a} U.$$

Before going any further, we remark that nothing guarantees that \mathbf{I} is not the empty functor; it is by means of Lemma 12 that we avoid this.

To prove Proposition 14, we will show that E is obtained from \mathbf{I} , and that \mathbf{I} is a scheme induced from \mathcal{U}_S .

We begin with a sequence of exercises in descent theory and non-abelian cohomology. Let $S' \rightarrow S$ be the morphism mentioned in the statement of Lemma 12. Then we have isomorphisms of functors

$$\begin{aligned} \mathbf{I} \times_S S' &\simeq \mathbf{Isom}_{S'}(F \times S', E \times_S S') \\ &\simeq \mathbf{Isom}_{S'}(F \times S', F \times S') \\ &= \mathbf{Aut}_k(F) \times S'. \end{aligned}$$

Moreover, letting $\mathbf{Aut}_k(F)$ act on itself on the right, these isomorphisms are all $\mathbf{Aut}_k(F)$ -equivariant. As $\mathbf{Aut}_k(F)$ is affine by assumption, descent theory tells us that \mathbf{I} is scheme, affine over S . From the isomorphism $\mathbf{I} \times_S S' \simeq \mathbf{Aut}_k(F) \times S'$, we conclude that $\psi : \mathbf{I} \rightarrow S$ is an $\mathbf{Aut}_k(F)$ -torsor: ψ is faithfully flat and affine due to 2.5.1, p. 22 and 2.7.1, p.29 of [EGA IV₂], and $\mathbf{I} \times_S \mathbf{I} \simeq \mathbf{I} \times \mathbf{Aut}_k(F)$.

Let $q : F \times \mathbf{I} \rightarrow E$ be the composition of the “universal” isomorphism $\varphi_{\text{univ}} : F \times \mathbf{I} \xrightarrow{\cong} E \times_S \mathbf{I}$ with the projection $E \times_S \mathbf{I} \rightarrow E$. If U is a k -scheme and $a : U \rightarrow \mathbf{I}$ is a morphism, then φ_a is just $(q \circ (\text{id}_F \times a), \text{pr}_U) : F \times U \rightarrow E \times_S U$. Therefore, for each $(y, a) \in (F \times \mathbf{I})(U)$, we have

$$(11) \quad \varphi_a(y, \text{id}_U) = (q(y, a), \text{id}_U).$$

Note that, in addition, $\pi(q(y, a)) = \psi(a)$.

Lemma 15. (a) *Let $\mathbf{Aut}_k(F)$ act on $F \times \mathbf{I}$ on the right by $(y, a) \cdot \gamma = (\gamma^{-1}y, a * \gamma)$. Then, for each $U \in \mathbf{Sch}$ and each $(y, a) \in (F \times \mathbf{I})(U)$, we have*

$$q[(y, a) \cdot \gamma] = q(y, a).$$

(b) *The morphism $q : F \times \mathbf{I} \rightarrow E$ induces an isomorphism*

$$\bar{q} : (F \times \mathbf{I})/\mathbf{Aut}_k(F) \xrightarrow{\cong} E.$$

Proof. (a) This follows directly from equation (10) defining the right action of $\mathbf{Aut}_k(F)$ on \mathbf{I} and from the above characterization of q .

(b) Endow $F \times \mathbf{I}$ with the action envisaged in (a) and $E \times_S \mathbf{I}$ with the action on \mathbf{I} . It follows that $\varphi_{\text{univ}} = (q, \text{pr}_{\mathbf{I}})$ is $\mathbf{Aut}_k(F)$ -equivariant. We then obtain an isomorphism of quotient sheaves

$$\bar{\varphi}_{\text{univ}} : (F \times \mathbf{I})/\mathbf{Aut}_k(F) \xrightarrow{\cong} (E \times_S \mathbf{I})/\mathbf{Aut}_k(F).$$

Due to the fact that formation of the sheaf associated to a presheaf commutes with fibre products [DG, III, §1, no.1.1, p.287] and the fact that $\mathbf{I} \rightarrow S$ is a $\mathbf{Aut}_k(F)$ -torsor, the quotient on the right hand side above is simply E . Now it is immediate that \bar{q} equals $\bar{\varphi}_{\text{univ}}$. \square

We now show how to obtain a stratification on \mathbf{I} starting from that on E . Fix a groupoid of $\mathbf{Set}^{\mathbf{Sch}^{\text{op}}}$

$$\mathfrak{P} \begin{array}{c} \xrightarrow{p_0} \\ \xrightarrow{p_1} \end{array} S, \quad \delta : \mathfrak{P} \times_{p_0, p_1} \mathfrak{P} \longrightarrow \mathfrak{P}, \quad \varepsilon : S \longrightarrow \mathfrak{P}, \quad s : \mathfrak{P} \longrightarrow \mathfrak{P},$$

and admit the existence of an action of \mathfrak{P} on the left of $\pi : E \rightarrow S$:

$$\theta : E \times_{\pi, S, p_0} \mathfrak{P} \xrightarrow{\sim} \mathfrak{P} \times_{p_1, S, \pi} E.$$

(See Definition 8.) Let $U \in \mathbf{Sch}$, $a \in \mathbf{I}(U)$, and $\partial \in \mathfrak{P}(U)$ be such that $p_0(\partial) = \psi(a)$ in $S(U)$. By means of φ_a and the isomorphism θ_∂ of eq. (4), we produce another isomorphism of U -schemes

$$(12) \quad F \times U \xrightarrow{\varphi_a} E \times_{S, \psi_a} U = E \times_{S, p_0 \partial} U \xrightarrow{\theta_\partial} E \times_{S, p_1 \partial} U.$$

Hence, we arrive at an element

$$\partial \bullet a \in \text{Isom}_U(F \times U, E \times_{S, p_1(\partial)} U) = \mathbf{I} \left(U \xrightarrow{p_1 \partial} S \right)$$

characterized by

$$\varphi_{\partial \bullet a} = \text{composition in (12)}.$$

Note that

$$(13) \quad \psi(\partial \bullet a) = p_1(\partial).$$

Lemma 16. *The association*

$$(\partial, a) \in (\mathfrak{P} \times_{p_0, S, \psi} \mathbf{I})(U) \longrightarrow \mathbf{I}(U), \quad (\partial, a) \longmapsto \partial \bullet a$$

defines an action of \mathfrak{P} on $\psi : \mathbf{I} \rightarrow S$.

Proof. Let $\partial' : U \rightarrow \mathfrak{P}$ be another arrow and assume that $p_1(\partial) = p_0(\partial')$. Since $\theta_{\partial'} \circ \theta_\partial = \theta_{\partial' \partial}$, we conclude that $\partial' \bullet (\partial \bullet a) = \partial' \partial \bullet a$, so that A3 of §5.1 holds. The verification of A2 is trivial. Finally, that the arrow constructed on the statement is a natural transformation of functors is immediate. \square

Lemma 17. (a) *The actions of \mathfrak{P} and of $\mathbf{Aut}_k(F)$ on \mathbf{I} commute. More precisely, let $U \in \mathbf{Sch}$, $\partial \in \mathfrak{P}(U)$, $\gamma \in \mathbf{Aut}_k(F)(U)$, and $a \in \mathbf{I}(U)$. Assume that $p_0(\partial) = \psi(a)$. Then*

$$\partial \bullet (a * \gamma) = (\partial \bullet a) * \gamma.$$

(b) *Endow $F \times \mathbf{I}$ with the left action of \mathfrak{P} via \mathbf{I} . Then q is equivariant.*

Proof. (a) It is immediate to see that the composition

$$F \times U \xrightarrow{\gamma} F \times U \xrightarrow{\varphi_a} E \times_{S, \psi_a} U \xrightarrow{=} E \times_{S, p_0 \partial} U \xrightarrow{\theta_\partial} E \times_{S, p_1 \partial} U$$

defines $\varphi_{\partial \bullet (a * \gamma)}$ and $\varphi_{(\partial \bullet a) * \gamma}$, see equations (12) and (10).

(b) This is again a triviality. Let $U \in \mathbf{Sch}$, $\partial \in \mathfrak{P}(U)$, $\gamma \in \mathbf{Aut}_k(F)(U)$, $y \in F(U)$, and $a \in \mathbf{I}(U)$. Assume that $p_0(\partial) = \psi(a)$ and recall that in this case $\psi(\partial \bullet a) = p_1(\partial)$. For every $(e, \text{id}_U) \in (E \times_{S, p_0 \partial} U)(U)$ we have by definition $\theta_\partial(e, \text{id}_U) = (\partial \cdot e, \text{id}_U)$, see eq. (4). Then, by definition of q , we have

$$\begin{aligned} (\partial \cdot q(y, a), \text{id}_U) &= \theta_\partial(q(y, a), \text{id}_U) \\ &= \theta_\partial(\varphi_a(y, \text{id}_U)). \end{aligned}$$

By definition of $\partial \bullet a$, the last term above is $\varphi_{\partial \bullet a}(y, \text{id}_U)$. By eq. (11), this is just $(q(y, \partial \bullet a), \text{id}_U)$. \square

Corollary 18. *The left action of \mathfrak{P} on $F \times \mathbf{I}$ considered in Lemma 17 descends to a left action of \mathfrak{P} on the quotient S -scheme $(F \times \mathbf{I})/\mathbf{Aut}_k(F)$. For this action, the isomorphism of S -schemes \bar{q} , defined in Lemma 15, is equivariant. \square*

We now assume that $\mathfrak{P} = \widehat{P}_S$ is the formal diagonal groupoid of S (§5.2); it acts on the left of $\pi : E \rightarrow S$ by means of its stratification, see Lemma 9. Then, $\psi : \mathbf{I} \rightarrow S$ becomes a stratified affine scheme over S , so that the quasi-coherent \mathcal{O}_S -algebra $\mathcal{O}_{\mathbf{I}}$ associated to it becomes a stratified \mathcal{O}_S -module. In order to relate $\mathcal{O}_{\mathbf{I}}$ with the universal torsor \mathcal{U}_S , we need to show that the \mathcal{D}_S -module $\mathcal{O}_{\mathbf{I}}$ is a direct limit of elements in $\mathcal{D}_S\text{-mod}$. This will be hidden in the fact that *every* representation of an affine group scheme is a direct limit of finite dimensional ones.

Endow $\mathbf{I} \times_S \mathbf{I}$ with the obvious stratification and $\mathbf{I} \times \mathbf{Aut}_k(F)$ with the stratification coming from \mathbf{I} . It is straightforward to see that the canonical isomorphism

$$\mathbf{I} \times \mathbf{Aut}_k(F) \xrightarrow{\cong} \mathbf{I} \times_S \mathbf{I}, \quad (a, \gamma) \longmapsto (a, a * \gamma)$$

is an isomorphism of stratified S -schemes. Such properties characterize what is sometimes called a *stratified torsor*. If we go by the terminology of [dS07, 2.3.2(a)], this says that $\mathcal{O}_{\mathbf{I}}$ is a torsor algebra of the category $\mathcal{D}_S\text{-Mod}$ (but not necessarily of $\text{Ind}(\mathcal{D}_S\text{-mod})!$). We are now capable of constructing a tensor functor

$$\mathcal{L}_{\mathbf{I}} : \left\{ \begin{array}{c} \text{Finite dimensional} \\ \text{representations of } \mathbf{Aut}_k(F) \end{array} \right\} \longrightarrow \mathcal{D}_S\text{-mod}$$

by imitating [J87, Part I, 5.8], or [dS07, p.701].

Lemma 19. *The stratified torsor $\psi : \mathbf{I} \rightarrow S$ is of the form $\mathcal{U}_S \times^\rho \mathbf{Aut}_k(F)$ for a certain morphism $\rho : \Pi(S) \rightarrow \mathbf{Aut}_k(F)$.*

Proof. We begin by constructing ρ . Since $\psi^{-1}(s_0)$ possesses a k -rational point (recall that $F := \pi^{-1}(s_0)$), the composition of $\mathcal{L}_{\mathbf{I}}$ with $s_0^* : \mathcal{D}_S\text{-mod} \rightarrow k\text{-mod}$ is naturally equivalent to the forgetful functor. By definition of $\Pi(S)$, it follows that s_0^* factors through the category of finite dimensional representations of $\Pi(S)$. Hence, Tannakian reconstruction of homomorphisms [DM82, 2.9, p.130] allows us to say that $s_0^* \circ \mathcal{L}_{\mathbf{I}}$ is the restriction functor between representation categories associated to some homomorphism ρ . Now the proof can be finished following [Nor76, §2] or [dS07, 2.3.2]. \square

Proof of Proposition 14. Just put together Lemma 19 and Corollary 18. \square

7. INFINITESIMAL EQUIVALENCE RELATIONS AND STRATIFICATIONS

We fix a noetherian separated scheme B . The category of separated noetherian B -schemes will be denoted by \mathbf{Sch}/B . That of noetherian formal separated B -schemes, by \mathbf{FSch}/B . The category \mathbf{Sch}/B is considered as a full subcategory of \mathbf{FSch}/B following [EGA I, 10.4.8, p.186]. More importantly, [EGA I, 10.6.2, p.189] and [EGA I, 10.6.10, p.192] show that \mathbf{Sch}/B is a dense subcategory [Mac98, Ch X, §6, 245ff] of \mathbf{FSch}/B , so that the restriction of the Yoneda functor $\mathbf{FSch}/B \rightarrow \mathbf{Set}^{(\mathbf{FSch}/B)^{\text{op}}}$ to \mathbf{Sch}/B is full and faithful [Mac98, Ch X, §6, Prp. 2, p. 246]. To ease notation we let

\mathbf{Sch}/B be the scenario of all our constructions and arrows. In compliance with this, we usually omit references to B , e.g. $\times = \times_B$, $\Omega_Z^1 = \Omega_{Z/B}^1$, etc. Let Z be a separated B -scheme of finite type.

7.1. Notations concerning the sheaves of principal parts. Let P_Z^μ stand for the μ -th infinitesimal neighbourhood of the diagonal in $Z \times Z (= Z \times_B Z)$; this is just $\text{Spec } \mathcal{P}_Z^\mu$, or $(|Z|, \mathcal{P}_Z^\mu)$, where \mathcal{P} is as in [EGA IV₄, 16.3.1, p.14]. Moreover, following [EGA IV₄, 16.3.5, p.15], \mathcal{P}_Z^μ is always considered as an \mathcal{O}_Z -algebra or module, except in explicit admonishment, via “the first projection”

$$\mathcal{O}_Z \longrightarrow \mathcal{P}_Z^\mu, \quad a \longmapsto a \otimes 1.$$

The structure associated to the “second projection” will be denoted by

$$\tau_Z^\mu : \mathcal{O}_Z \longrightarrow \mathcal{P}_Z^\mu,$$

or anything resembling such a notation. This homomorphism should be thought as associating to a function in \mathcal{O}_Z its “Taylor series”, or truncated “Taylor series.” For convenience, given $a \in \mathcal{O}_Z$, we write

$$(14) \quad d^\mu(a) = \tau^\mu(a) - a.$$

Remark 20. This notation differs from that in [EGA IV₄, 16.3.5, p.15].

Instead of working with the schemes P_Z^μ , we can use the formal completion \widehat{P}_Z of $Z \times Z$ along the diagonal. This is a formal noetherian scheme [EGA I, §10]. For purely algebraic purposes, it is interesting to regard \widehat{P}_Z as

$$(15) \quad \begin{array}{l} \text{the topological space } |Z| \text{ endowed} \\ \text{with the sheaf of rings } \widehat{\mathcal{P}}_Z = \varprojlim_{\mu} \mathcal{P}_Z^\mu. \end{array}$$

As for its truncated version, $\widehat{\mathcal{P}}_Z$ (in [EGA IV₄, 16.3.1, p.14], this is denoted by $\mathcal{P}_{Z/B}^\infty$) will be regarded as an \mathcal{O}_Z -algebra via $a \mapsto a \widehat{\otimes} 1$, and the “Taylor series” will be denoted by τ_Z or simply τ . The ideal of $Z \hookrightarrow \widehat{P}_Z$ will be denoted by \mathfrak{D}_Z . We also write

$$d(a) = \tau(a) - a,$$

which is an element of \mathfrak{D}_Z . It should be remarked that this notation is not in conflict with the usual notation for a differential of a function, since $d(a)$ induces, upon passage to $\mathfrak{D}_Z/\mathfrak{D}_Z^2 \simeq \Omega_Z^1$ this differential.

7.2. Basic definitions.

Definition 21. Let $(u_0, u_1) : U \rightarrow Z \times Z$ be two U -points of Z . We say that u_0 is infinitesimally close to u_1 , notation: $u_0 \equiv_\infty u_1$, if $u_{0,\text{red}} = u_{1,\text{red}}$. This defines an equivalence relation in Z [DG, p. 292], whose graph will be denoted by $(Z/B)_\infty$, or Z_∞ .

As on a noetherian scheme U the nilradical Nil_U is nilpotent, the functor $Z_\infty : (\mathbf{Sch}/B)^{\text{op}} \rightarrow \mathbf{Set}$ is represented by the formal scheme \widehat{P}_Z .

Definition 22. (a) An **infinitesimal equivalence relation** (i.e.r.) is a sub-presheaf Φ of $Z \times Z : (\mathbf{Sch}/B)^{\text{op}} \rightarrow \mathbf{Set}$ which is an equivalence relation [DG, p. 292] and which coincides with $\Delta_Z \subseteq Z \times Z$ when evaluated at reduced schemes in \mathbf{Sch}/B . (In other words, an infinitesimal equivalence relation is an equivalence relation in Z which is weaker than Z_∞ .)

(b) Let Φ and Φ' be equivalence relations in Z and Z' respectively. A morphism $h : Z \rightarrow Z'$ **interweaves** Φ and Φ' if, for every $U \in \mathbf{Sch}/B$, the map $h(U) \times h(U)$ sends $\Phi(U)$ into $\Phi'(U)$. (In [SGA3 IV, 3.1.4] such an arrow is called “compatible.”)

The category whose objects are couples (Z, Φ) consisting of a scheme $Z \in \mathbf{Sch}/B$ and an infinitesimal equivalence relation in Z , and whose arrows are interweaving morphisms will be denoted by $\mathbf{IER}(B)$. Analogously, given an i.e.r. Φ in $\mathbf{IER}(B)$, we write \mathbf{IER}/Φ to stand for the subcategory of $\mathbf{IER}(B)$ of objects over Φ , that is, $\mathbf{IER}/\Phi := \mathbf{IER}(B)_{/\Phi}$ in more standard notation.

Note that every Z comes with two i.e.r.: the *trivial* one defined by the diagonal and denoted by Δ_Z , and the *full* one, defined by Z_∞ .

Definition 23 (compare [E87]). Let Φ be an infinitesimal equivalence relation (ier) in Z . We say that Φ is **schematic** if $\Phi : (\mathbf{Sch}/B)^{\text{op}} \rightarrow \mathbf{Set}$ is represented by a formal closed subscheme of \widehat{P}_Z . We will sometimes write \widehat{P}_Φ to denote the formal closed subscheme associated to Φ .

We recall from [EGA I, 10.14] that the data of a closed formal subscheme of \widehat{P}_Z amounts to a sheaf \mathfrak{A} of (coherent) ideals of the sheaf of rings $\widehat{\mathcal{P}}_Z$. The formal scheme associated to \mathfrak{A} will then be the (topologically) ringed space

$$V(\mathfrak{A}) := \left(\text{support of } \widehat{\mathcal{P}}_Z/\mathfrak{A}, \text{ restriction of } \widehat{\mathcal{P}}_Z/\mathfrak{A} \right).$$

By [EGA I, 10.6.2, p.189], $V(\mathfrak{A})$ is the direct limit of the schemes defined by the ideals

$$\frac{\mathfrak{A} + \mathfrak{D}_Z^{\mu+1}}{\mathfrak{D}_Z^{\mu+1}} \subseteq \frac{\widehat{\mathcal{P}}_Z}{\mathfrak{D}_Z^{\mu+1}} = \mathcal{P}_Z^\mu$$

and the morphism $V(\mathfrak{A}) \rightarrow \widehat{P}_Z$ is adic [EGA I, 10.12]. If $\mathfrak{A} \subseteq \mathfrak{D}_Z$ is an ideal, then for $V(\mathfrak{A})$ to define a schematic equivalence relation, we only need that

$$(16) \quad \delta^\#(\mathfrak{A}) \subseteq \mathfrak{A} \widehat{\otimes}_{\mathcal{O}} \widehat{\mathcal{P}}_Z + \widehat{\mathcal{P}}_Z \widehat{\otimes}_{\mathcal{O}} \mathfrak{A}, \quad \text{and} \quad s^\#(\mathfrak{A}) \subseteq \mathfrak{A},$$

where $\delta : \widehat{P}_Z \times_Z \widehat{P}_Z \rightarrow \widehat{P}_Z$ and $s : \widehat{P}_Z \rightarrow \widehat{P}_Z$ are the morphisms of formal schemes considered in §5.2. In particular, the morphisms $\delta^\#$ and $s^\#$ pass to the ring

$$\widehat{\mathcal{P}}_\Phi := \widehat{\mathcal{P}}_Z/\mathfrak{A}.$$

If Φ is a schematic equivalence relation defined by the ideal \mathfrak{A} , we write

$$(17) \quad \mathcal{A}_\mu(\Phi) := \frac{\mathfrak{A} + \mathfrak{D}_Z^{\mu+1}}{\mathfrak{D}_Z^{\mu+1}} \quad \text{and} \quad \mathcal{P}_\Phi^\mu = \mathcal{P}_Z^\mu/\mathcal{A}_\mu(\Phi).$$

In other words,

$$(|Z|, \mathcal{P}_\Phi^\mu) = \widehat{P}_\Phi \cap P_Z^\mu.$$

So, the question of how to find families of ideals $\mathcal{A}_\mu \subseteq \mathcal{P}_Z^\mu$ such that $\varprojlim_\mu \mathcal{A}_\mu$ defines a schematic equivalence relation is at hand. This is treated in the following genetic example.

Example 24 ([Mi87]). Let $B = \text{Spec}(K)$ be the spectrum of a field of characteristic zero, and let Z be a smooth K -scheme. Denote by \mathcal{F} a subbundle of $\mathcal{D}er_{Z/K}$ which is stable under the Lie bracket (an integrable distribution). Fixing a closed point $z \in Z$, we can find an affine open neighbourhood $U = \text{Spec}(A)$ of z , etale coordinates $(x_1, \dots, x_n) : U \rightarrow \mathbb{A}^n$, and *commuting* vector fields D_1, \dots, D_r freely generating $\mathcal{F}|_U$ such that $D_i x_j = \delta_{ij}$ for $1 \leq i, j \leq r$. (See the argument in the proof of Frobenius' Theorem on p.183 of [CLN85].) From this arrangement, we can consider the formal flow of \mathcal{F} , which is the following *homomorphism* of K -algebras

$$\exp(\mathbf{t} \cdot \mathbf{D}) : A \longrightarrow A[[t_1, \dots, t_r]], \quad \psi \longmapsto \sum_{\mathbf{j} \in \mathbb{N}^r} \mathbf{D}^{\mathbf{j}}(\psi) \cdot \frac{\mathbf{t}^{\mathbf{j}}}{\mathbf{j}!}.$$

The formal flow induces a formal action of $\widehat{\mathbb{G}}_{a,K}^r = \text{Spf } K[[\mathbf{t}]]$ on U . Define $\rho : A \otimes_K A \rightarrow A[[\mathbf{t}]]$ by sending $\psi_0 \otimes \psi_1$ to $\psi_0 \cdot \exp(\mathbf{t} \cdot \mathbf{D})(\psi_1)$. Since the kernel of multiplication $A \otimes_K A \rightarrow A$ is generated by elements of the form $1 \otimes \psi_0 - \psi_0 \otimes 1$, see e.g. [BO78, p.2.2], it follows that ρ induces a morphism of A -algebras

$$\hat{\rho} : \widehat{\mathcal{P}}_A \longrightarrow A[[\mathbf{t}]].$$

Moreover, $\hat{\rho}$ is surjective as

$$\exp(\mathbf{t} \cdot \mathbf{D})(x_i) = x_i + t_i, \quad i = 1, \dots, r.$$

Write \mathfrak{A} for $\text{Ker}(\hat{\rho})$. Using the aforementioned action of $\widehat{\mathbb{G}}_{a,K}^r$, it is not difficult to verify that

$$(18) \quad s^\#(\mathfrak{A}) \subseteq \mathfrak{A}, \quad \delta^\#(\mathfrak{A}) \subseteq \mathfrak{A} \widehat{\otimes}_A \widehat{\mathcal{P}}_A + \widehat{\mathcal{P}}_A \widehat{\otimes}_A \mathfrak{A},$$

where δ and s are the morphism used to define the diagonal formal groupoid, see §5.2.

Let

$$\langle -, - \rangle : \mathcal{D}_A^\mu \otimes_A \mathcal{P}_A^\mu \longrightarrow A$$

be the canonical pairing between the module of principal parts and the module of differential operators of order $\leq \mu$ [EGA IV₄, 16.8.4, p.41]. Since $\hat{\rho}$ maps the augmentation ideal $\mathfrak{D}_A \subseteq \widehat{\mathcal{P}}_A$ into (\mathbf{t}) , we obtain a morphism of A -algebras

$$\rho_\mu : \mathcal{P}_A^\mu \longrightarrow A[[\mathbf{t}]]/(\mathbf{t})^{\mu+1}$$

which is none other than

$$\xi \longmapsto \sum_{|\mathbf{i}| \leq \mu} \langle \mathbf{D}^{\mathbf{i}}, \xi \rangle \frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!},$$

see the beginning of [EGA IV₄, 16.8]. Write $\mathfrak{A}_\mu := \text{Ker}(\rho_\mu)$. Since

$$\varprojlim_\mu \rho_\mu : \varprojlim_\mu \mathcal{P}_A^\mu \longrightarrow \varprojlim_\mu A[[\mathbf{t}]]/(\mathbf{t})^{\mu+1}$$

is simply $\hat{\rho}$, we conclude that $\mathfrak{A} = \varprojlim_{\mu} \mathfrak{A}_{\mu}$. Clearly, \mathfrak{A}_{μ} is simply $\Gamma(U, \mathcal{A}_{\mu}(\mathcal{F}))$, where

$$\mathcal{A}_{\mu}(\mathcal{F}) = \left\{ \begin{array}{l} \text{principal parts } \omega \in \mathcal{P}_Z^{\mu} \text{ which are annihilated by all differential} \\ \text{operators of positive degree obtained by composition of fields in } \mathcal{F} \end{array} \right\}.$$

It follows that $\varprojlim_{\mu} \mathcal{A}_{\mu}(\mathcal{F})$ defines a closed formal subscheme of \widehat{P}_Z which is an infinitesimal equivalence relation due to (18).

Definition 25. Let $g : Z \rightarrow S$ be a morphism in \mathbf{Sch}/B and let Φ be an equivalence relation in Z . We say that Φ is **S -linear** if $\Phi \subseteq Z \times_S Z$.

If Φ is arbitrary, then its **S -linearisation** is the equivalence relation $\Phi_{/S} = \Phi \cap (Z \times_S Z)$ in Z , i.e. two U -points (u_0, u_1) of Z are $\Phi_{/S}$ -equivalent if and only if $u_0 \equiv_{\Phi} u_1$ and $g(u_0) = g(u_1)$.

Remark 26. What we called an S -linear equivalence relation is called an equivalence relation compatible with $Z \rightarrow S$ in [SGA3 IV, 3.1.2]. The phrase ‘‘equivalence relation in Z over S ’’ [SGA3 IV, 3.1.3] is also used.

Definition 27. Let (Z, Φ) and (T, Ψ) be infinitesimal equivalence relations. Let $g : Z \rightarrow S$ and $h : T \rightarrow S$ be morphisms. On $Z \times_S T$ we have the i.e.r defined on U -points by the following rule. Two U -points (z_0, t_0) and (z_1, t_1) of $Z \times_S T$ are $\Phi \times_S \Psi$ equivalent if $z_0 \equiv_{\Phi} z_1$ and $t_0 \equiv_{\Psi} t_1$. To simplify notation, once Ψ is the trivial i.e.r. Δ_T , we write $\Phi \times_S T$ instead of $\Phi \times_S \Delta_T$.

7.3. Relation to stratifications.

Definition 28 (Complementarity). Let $g : Z \rightarrow S$ be a morphism and let Φ be an infinitesimal equivalence relation in Z . We say that Φ is complementary to g if

$$\Phi \cap (Z \times_S Z) \subseteq \Delta_Z.$$

In other words, if $z_0 \equiv_{\Phi} z_1$ and $g(z_0) = g(z_1) \Rightarrow z_0 = z_1$.

Here is another way to say that an i.e.r. Φ in Z is complementary to $g : Z \rightarrow S$. Let $p_0 : \Phi \rightarrow Z$ be the composition of the inclusion $\Phi \subseteq Z \times Z$ with the first projection, $\pi_0 : S_{\infty} \rightarrow S$ the first projection, and let $g_{\infty} : Z_{\infty} \rightarrow S_{\infty}$ be the obvious arrow of functors. Then Φ is complementary to g if and only if the arrow

$$(19) \quad \Phi \xrightarrow{(p_0, g_{\infty})} Z \times_{S, \pi_0} S_{\infty}$$

is a monomorphism.

Definition 29 (Total complementarity). Let $g : Z \rightarrow S$ be a morphism of schemes and Φ an infinitesimal equivalence relation in Z . We say that Φ is **totally complementary** to g if the arrow in (19) defines an isomorphism.

Rephrasing Definition 29, we see that the i.e.r. Φ in Z is totally complementary to $g : Z \rightarrow S$ if:

(TC-1) $\Phi_{/S}$ is trivial.

(TC–2) For any given couple $z \in Z(U)$, $s \in S(U)$ satisfying $g(z) \equiv_\infty s$, there exists $\tilde{z} \in Z(U)$ such that $\tilde{z} \equiv_\Phi z$, and $g(\tilde{z}) = s$.

If we think of the condition $g(z) \equiv_\infty s$ as being “there exists a path from $g(z)$ to s ”, and of the condition $\tilde{z} \equiv_\Phi z$ as “there exists a path in Z contained in a leaf of Φ ”, then (TC–2) affirms the possibility of lifting paths “horizontally”. This is reminiscent of what Ehresmann calls in [E50, p. 36] a connection on a fibered space. However, this parallel is simply verbal, as totally complementary infinitesimal equivalence relations are just stratifications (see below).

Remark 30. The role played by the base scheme B goes unmentioned just to simplify notation. However, when studying restrictions of infinitesimal equivalence relations, its utility will become apparent. Whenever necessary, we will say that Φ is *totally complementary to some morphism with respect to the base B* .

To illustrate with a pertinent instance, we consider Φ an i.e.r. which is totally complementary to the B –morphism $g : Z \rightarrow S$. If $f : S \rightarrow T$ is another arrow of B –schemes, then $\Phi_{/T}$ becomes totally complementary to g with respect to the base T .

The connection between this concept and stratifications is obtained as follows. Let $g : Z \rightarrow S$ be endowed with a stratification as explained in §5:

$$\alpha : Z \times_S \widehat{P}_S \longrightarrow Z.$$

(It is useful to remind the reader that we are following Convention 10 when forming fibre products involving principal parts.) We write α_μ for the composition of α with the canonical morphism $Z \times_S P_Z^\mu \rightarrow Z \times_S \widehat{P}_S$. Let $\pi_0 : Z \times Z \rightarrow Z$ be the first projection and

$$\beta_\mu := (\text{pr}_Z, \alpha_\mu).$$

From the definition of an action of \widehat{P}_S , the composition

$$Z \times_S P_S^\mu \xrightarrow{\beta_\mu} Z \times Z \xrightarrow{(\pi_0, g \times g)} Z \times_S (S \times S)$$

is just

$$\text{id}_Z \times_S (\text{natural inclusion}) : Z \times_S P_S^\mu \longrightarrow Z \times_S (S \times S).$$

Since β_μ factors through the closed embedding $P_Z^\mu \rightarrow Z \times Z$, what we just observed implies that β_μ is a section to

$$(\pi_{0\mu}, g_\mu) : P_Z^\mu \longrightarrow Z \times_S P_S^\mu,$$

where we write $(\pi_{0\mu}, g_\mu)$ to denote the restriction of $(\pi_0, g \times g)$ to P_Z^μ . Passing to the limit, we get a morphism

$$\beta : Z \times_S \widehat{P}_S \longrightarrow \widehat{P}_Z$$

which is a section to $(\pi_0, g_\infty) : \widehat{P}_Z \rightarrow Z \times_S \widehat{P}_S$. It follows from the fact that $\beta_0 = \text{id}_Z$ and [EGA I, 10.14.4, p.210] that β is a closed embedding of formal schemes. Write \widehat{P}_Φ for the image of β in \widehat{P}_Z . The reader will have no difficulty in establishing that \widehat{P}_Φ is the graph of an infinitesimal equivalence relation. From the very construction, \widehat{P}_Φ is totally complementary to g .

On the other hand, interpreting \mathbf{FSch}/B as a full subcategory of $\mathbf{Set}^{(\mathbf{Sch}/B)^{\text{op}}}$ (see the beginning of this section) it is not hard to show that if $g : Z \rightarrow S$ is totally complementary to Φ , then Φ actually comes from a stratification. Indeed, given any $U \in \mathbf{Sch}/B$ and any couple $(z, \partial) \in (Z \times_S \widehat{P}_S)(U)$, there exists a unique $(z, \alpha(\partial, z)) \in \Phi(U)$ which is taken by (π_0, g_∞) to (z, ∂) . The association $(\partial, z) \mapsto \alpha(\partial, z)$ defines an action of \widehat{P}_S on $g : Z \rightarrow S$, which is equivalent to a stratification by Lemma 9.

8. SMOOTH SCHEMATIC INFINITESIMAL EQUIVALENCE RELATIONS

In this section, B stands for a noetherian and separated scheme. We accompany the flow of notations introduced in §7.1, so that \mathbf{Sch}/B is the category of separated noetherian B -schemes. We let $g : Z \rightarrow B$ be a separated morphism of finite type. For the sake of graphical simplicity, we will omit notational references to B whenever harmless: in this spirit, $\widehat{\mathcal{P}}_Z = \widehat{\mathcal{P}}_{Z/B}$, $\mathfrak{D}_Z = \mathfrak{D}_{Z/B}$ (see §7.1 for notation), $\Omega_Z^1 = \Omega_{Z/B}^1$, etc.

8.1. Basic objects. Let Φ be a schematic infinitesimal equivalence relation in Z cut out by the ideal \mathfrak{A} of $\widehat{\mathcal{P}}_Z$. The sheaf of rings of the formal scheme $\widehat{\mathcal{P}}_\Phi$ (the underlying space is simply $|Z|$) is denoted by $\widehat{\mathcal{P}}_\Phi$. The “Taylor expansion in the direction of Φ ”,

$$(20) \quad \tau_\Phi : \mathcal{O}_Z \longrightarrow \widehat{\mathcal{P}}_\Phi,$$

will be the evident homomorphism of sheaves of algebras on Z : $\mathcal{O}_Z \xrightarrow{\tau} \widehat{\mathcal{P}}_Z \rightarrow \widehat{\mathcal{P}}_\Phi$, where τ is the morphism of sheaves of rings alluded to in §7.1. In like fashion, if the need presents, we will write

$$(21) \quad d_\Phi(a) := \tau_\Phi(a) - a.$$

Note that τ_Φ is a homomorphism of rings, while d_Φ is additive and verifies

$$(22) \quad d_\Phi(a \cdot b) = a \cdot d_\Phi(b) + b \cdot d_\Phi(a) + d_\Phi(a) \cdot d_\Phi(b).$$

We let \mathfrak{D}_Φ stand for the image of \mathfrak{D}_Z in $\widehat{\mathcal{P}}_\Phi$.

Definition 31. The schematic infinitesimal equivalence relation $\Phi \hookrightarrow Z_\infty$ is **differentially smooth of rank r** if $\mathfrak{D}_\Phi \subseteq \widehat{\mathcal{P}}_\Phi$ is a quasi-regular ideal [EGA IV₄, 16.9.1,p.46] and the locally free $\widehat{\mathcal{P}}_\Phi/\mathfrak{D}_\Phi = \mathcal{O}_Z$ -module, called henceforth the **co-tangent module of Φ** ,

$$\Omega_\Phi := \mathfrak{D}_\Phi/\mathfrak{D}_\Phi^2$$

is of rank r .

One important class of examples of differentially smooth i.e.r.s is given by the totally complementary i.e.r.s of Definition 29. Indeed, if $g : Z \rightarrow S$ is a smooth morphism of smooth schemes, then an infinitesimal equivalence relation Φ on Z which is totally complementary to g must be differentially smooth of rank $\dim S/B$.

Recall that a more direct way to say that an ideal \mathcal{J} of some locally noetherian formal scheme \mathfrak{V} is quasi-regular is to say that the $\mathcal{O}_{\mathfrak{V}}/\mathcal{J}$ -module

$$\mathfrak{G}r_\mu(\mathcal{J}) = \mathcal{J}^\mu/\mathcal{J}^{\mu+1}$$

is locally free of rank $\binom{r-1+\mu}{\mu}$, this can be easily deduced from [EGA IV₄, 16.9.4, p. 47] and Nakayama's Lemma. This allows us to prove the following.

Lemma 32. *If Φ is differentially smooth of rank r in the B -scheme of finite type Z , then for each $B' \in \mathbf{Sch}/B$, the i.e.r. $\Phi \times B'$ is differentially smooth of rank r over B' .*

Proof. Let Φ be a differentially smooth i.e.r. of rank r in Z . We fix some $\mu \in \mathbb{N}$ and let $P := P_\Phi^\mu$; denote by Δ the diagonal embedding $Z \rightarrow P$. It follows that for all $\nu < \mu$, $\mathcal{G}r_\nu(\Delta) = \mathcal{D}_\Phi^\nu/\mathcal{D}_\Phi^{\nu+1}$ is locally free of rank $\binom{r-1+\nu}{\nu}$ over \mathcal{O}_Z . Now, let $u : Z' \rightarrow Z$ be a morphism in \mathbf{Sch}/B and write $P' = Z' \times_Z P$, $\Delta' = \text{id}_{Z'} \times_Z \Delta \circ u$. From [EGA IV₄, 16.2.4,p.12], we conclude that

$$\mathcal{G}r_\nu(\Delta) \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z'} \xrightarrow{\simeq} \mathcal{G}r_\nu(\Delta')$$

for all $\nu < \mu$. From the arbitrariness of μ and the above remark, we derive that the obvious morphism

$$(23) \quad Z' \longrightarrow Z' \times_Z \widehat{P}_\Phi = \varinjlim_{\mu} Z' \times_Z P_\Phi^\mu$$

is a quasi-regular embedding. Now, if $u : Z' \rightarrow Z$ is just the projection from $Z' = Z \times B'$, it is not hard to deduce an isomorphism in $\mathbf{Set}^{(\mathbf{Sch}/B)^{\text{op}}}$ between $Z' \times_Z \widehat{P}_\Phi = B' \times \widehat{P}_\Phi$ and $\Phi \times B'$, under which the closed embedding in (23) corresponds to the diagonal $Z \times B' \rightarrow \Phi \times B'$. □

Whenever studying a differentially smooth i.e.r Φ , the co-tangent module Ω_Φ will play a prominent role. Before proceeding, we wish to attach a more familiar expression to it. Let \mathfrak{A} be the ideal of Φ and denote by $\mathcal{A}_1(\Phi)$ the image of \mathfrak{A} in $\mathcal{P}_Z^1 = \mathcal{O}_Z \oplus \Omega_Z^1$. We have exact sequences

$$(24) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{A}_1(\Phi) & \longrightarrow & \Omega_Z^1 & \longrightarrow & \Omega_Z^1/\mathcal{A}_1(\Phi) \longrightarrow 0 \\ & & \parallel & & \parallel & & \\ 0 & \longrightarrow & (\mathcal{D}_Z^2 + \mathfrak{A})/\mathcal{D}_Z^2 & \longrightarrow & \mathcal{D}_Z/\mathcal{D}_Z^2 & \longrightarrow & \Omega_\Phi \longrightarrow 0, \end{array}$$

which lead to

$$(25) \quad \Omega_\Phi \simeq \Omega_Z^1/\mathcal{A}_1(\Phi).$$

The reason for which we mention that this should be more familiar is the following. Let $T_{\mathcal{F}} \subseteq T_M$ be the tangent bundle of a r dimensional foliation on the differentiable manifold M . Let $A_{\mathcal{F}}$ be the sheaf of C^∞ 1-forms which annihilate the vectors in $T_{\mathcal{F}}$, i.e. the kernel of $T_M^* \rightarrow T_{\mathcal{F}}^*$. This is usually called the conormal bundle of \mathcal{F} , as it can be identified with the dual of the **normal bundle** $N_{\mathcal{F}} = T_M/T_{\mathcal{F}}$, see [MM03, p. 12]. The cotangent bundle of \mathcal{F} , $\Omega_{\mathcal{F}} := T_{\mathcal{F}}^*$, is then isomorphic to $T_M^*/A_{\mathcal{F}}$. In particular, the rank of $\Omega_{\mathcal{F}}$ is the dimension of the leaves of the foliation \mathcal{F} .

8.2. Local behaviour and transverse sections. Let Φ be a differentially smooth infinitesimal equivalence relation in Z of rank r , z a point of Z , and $b = g(z)$. We assume that $\mathbf{k}(b) \rightarrow \mathbf{k}(z)$ is bijective. As $\Omega_Z^1(z)$ is generated by elements of the form $d\psi(z)$, the fact that $\mathbf{k}(b) = \mathbf{k}(z)$ allows us to find $x_1, \dots, x_r \in \mathfrak{M}_{Z,z}$ such that the image of $dx_1, \dots, dx_r \in \Omega_{Z,z}^1$ freely generates $\Omega_{\Phi,z}$. As \mathfrak{D}_Φ is a quasi-regular ideal of $\widehat{\mathcal{P}}_\Phi$, an application of [EGA IV₄, 16.9.5,p.47] gives $(dx_1, \dots, dx_r) = \mathfrak{D}_\Phi$ near z , where we use the notation introduced in §7.1. Hence, near z ,

$$(26) \quad \mathcal{O}_Z[[dx_1, \dots, dx_r]] = \widehat{\mathcal{P}}_\Phi,$$

due to [EGA IV₁, 19.5.4, p.91].

Definition 33. Let $x_1, \dots, x_r \in \mathfrak{M}_{Z,z}$ be functions such that the image of the differentials $dx_1, \dots, dx_r \in \Omega_{Z,z}^1$ in $\Omega_{\Phi,z}$ form a basis of the latter. Then the closed subscheme $\Sigma = \{x_1 = \dots = x_r = 0\}$ is called a transverse section to Φ at z .

We observe that there are usually many transverse sections. If $\{x_1 = \dots = x_r = 0\}$ is such a section and $(a_{ij}) \in \mathbf{GL}_r(\mathcal{O}_{Z,z})$, then $\Sigma' = \{x'_1 = \dots = x'_r = 0\}$, where $x'_j := \sum_i a_{ij} \cdot x_i$, is another transverse section. However, finding an element $y \in \mathcal{O}_{Z,z}$ such that dy belongs to the kernel of $\Omega_{Z,z}^1 \rightarrow \Omega_{\Phi,z}$ is not a problem of linear algebra: such functions are *first integrals*.

For further reference we record a direct consequence of the Jacobian criterion [BLR90, §2.2, Proposition 7, p.39].

Lemma 34. *Assume that $Z \rightarrow B$ is smooth of relative dimension n at z . Let Σ be a transverse section to Φ at z . Then Σ is smooth of relative dimension $c = n - r$ at z . \square*

A transverse section induces a decomposition of Ω_Z^1 as follows. Let $\Sigma = \{x_1 = \dots = x_r = 0\}$ be a transverse section at $z \in Z$. If y_1, \dots, y_l are elements of $\mathcal{O}_{Z,z}$ such that

$$d(y_1|_\Sigma), \dots, d(y_l|_\Sigma)$$

generate $\Omega_{\Sigma,z}^1$, then the co-normal exact sequence

$$\mathcal{N}_{\Sigma/Z,z} \rightarrow \Omega_{Z,z}^1 \otimes \mathcal{O}_{\Sigma,z} \rightarrow \Omega_{\Sigma,z}^1 \rightarrow 0$$

shows that $\Omega_{Z,z}^1 \otimes \mathcal{O}_{\Sigma,z}$ is generated by (the images of) dy_1, \dots, dy_l and dx_1, \dots, dx_r . Hence, by Nakayama's Lemma,

$$(27) \quad \Omega_{Z,z}^1 = \sum_{i=1}^r \mathcal{O}_{Z,z} \cdot dx_i + \sum_{j=1}^l \mathcal{O}_{Z,z} \cdot dy_j.$$

8.3. Formal behaviour. Let Φ be a differentially smooth infinitesimal equivalence relation in Z of rank r , z a point of Z whose residue field $\mathbf{k}(z)$ coincides with $\mathbf{k}(g(z))$. We will concentrate on the “formal” behavior of Φ , and therefore assume that $Z = \text{Spec}(\mathcal{O})$, that $B = \text{Spec}(\Lambda)$, and that there exist functions $x_1, \dots, x_r \in \mathcal{O}$ vanishing on z such that dx_1, \dots, dx_r generate \mathfrak{D}_Φ ; the existence of these was clarified in §8.2. In particular, $\Sigma = \{x_1 = \dots = x_r = 0\}$ is a transverse section to Φ (Definition 33) at z .

Recall from §5.2 that we have a morphism of \mathcal{O} -algebras

$$\delta : \widehat{\mathcal{P}}_{\mathcal{O}} \longrightarrow \widehat{\mathcal{P}}_{\mathcal{O}} \widehat{\otimes}_{\mathcal{O}} \widehat{\mathcal{P}}_{\mathcal{O}}$$

which is employed to define the formal diagonal groupoid. By eq. (16), δ passes to the quotient and defines a morphism of \mathcal{O} -algebras $\delta : \widehat{\mathcal{P}}_{\Phi} \rightarrow \widehat{\mathcal{P}}_{\Phi} \widehat{\otimes}_{\mathcal{O}} \widehat{\mathcal{P}}_{\Phi}$ fitting into the commutative diagram

$$(28) \quad \begin{array}{ccc} \mathcal{O} & \xrightarrow{\tau_{\Phi}} & \widehat{\mathcal{P}}_{\Phi} \\ \tau_{\Phi} \downarrow & & \downarrow \delta \\ \widehat{\mathcal{P}}_{\Phi} & \xrightarrow{\mathbf{1} \widehat{\otimes} \text{id}} & \widehat{\mathcal{P}}_{\Phi} \widehat{\otimes}_{\mathcal{O}} \widehat{\mathcal{P}}_{\Phi}. \end{array}$$

(We write $\mathbf{1}$ for the identity of $\widehat{\mathcal{P}}_{\mathcal{O}}$ or $\widehat{\mathcal{P}}_{\Phi}$.) Let $\Lambda[[d_1, \dots, d_r]]$ be the formal Hopf algebra associated to the completion of $\mathbb{G}_{a, \Lambda}^r$ at the identity. Using the identification $\widehat{\mathcal{P}}_{\Phi} = \mathcal{O}[[dx_1, \dots, dx_r]]$ made in (26), we can write

$$\tau_{\Phi} y = \sum_{\alpha \in \mathbb{N}^r} \tau_{\Phi, \alpha}(y) \cdot d\mathbf{x}^{\alpha},$$

where $\tau_{\Phi, \alpha}(y) \in \mathcal{O}$, and $d\mathbf{x}^{\alpha} = (dx_1)^{\alpha_1} \cdots (dx_r)^{\alpha_r}$. In terms of this notation, the composition $(\mathbf{1} \widehat{\otimes} \text{id}) \circ \tau_{\Phi}$ is simply

$$\begin{aligned} y &\longmapsto \mathbf{1} \widehat{\otimes} \sum_{\alpha \in \mathbb{N}^r} \tau_{\Phi, \alpha}(y) \cdot d\mathbf{x}^{\alpha} \\ &= \sum_{\alpha \in \mathbb{N}^r} \tau_{\Phi}(\tau_{\Phi, \alpha}(y)) \widehat{\otimes} d\mathbf{x}^{\alpha}. \end{aligned}$$

This fact and the identities $\delta(dx_i) = \mathbf{1} \widehat{\otimes} dx_i + dx_i \widehat{\otimes} \mathbf{1}$ show that τ_{Φ} defines a co-action, in the category of topological Λ -algebras, of $\Lambda[[d_1, \dots, d_r]]$ on the discrete topological algebra \mathcal{O} . (The cautious reader will remark that we left out the considerations concerning the behaviour of the co-identity; this is readily surpassed using the fact that $\tau(y) \equiv y \pmod{\mathfrak{D}_{\mathcal{O}}}$.) Clearly, this is reminiscent of the proof of Frobenius' Theorem by means of \mathbb{R}^r -actions, see Corollary 1 on p. 181 and the proof on 182ff of [CLN85]. In our setting, a formal analogue of the Frobenius theorem, pointed out by [ESB05], is:

Proposition 35. *Let \mathcal{O}^* stand for the (\mathbf{x}) -adic completion of \mathcal{O} , and $\mathcal{O}(\Sigma)$ for the quotient $\mathcal{O}/(\mathbf{x})$. Then*

$$y \longmapsto \sum_{\alpha} \{\tau_{\Phi, \alpha}(y) \pmod{(\mathbf{x})}\} d_1^{\alpha_1} \cdots d_r^{\alpha_r}$$

induces an isomorphism $\sigma : \mathcal{O}^ \rightarrow \mathcal{O}(\Sigma)[[\mathbf{d}]]$ of adic rings. Moreover, if $\tau_{\Phi}^* : \mathcal{O}^* \rightarrow \mathcal{O}^*[[dx_1, \dots, dx_r]]$ stands for the obvious morphism, then the diagram*

$$\begin{array}{ccc} \mathcal{O}^* & \xrightarrow{\sigma} & \mathcal{O}(\Sigma)[[\mathbf{d}]] \\ \tau_{\Phi}^* \downarrow & & \downarrow d_i \mapsto d_i + dx_i \\ \mathcal{O}^*[[\mathbf{dx}]] & \xrightarrow{\sigma[[\mathbf{dx}]]} & \mathcal{O}(\Sigma)[[\mathbf{d}]] [[\mathbf{dx}]] \end{array}$$

commutes. More geometrically, the formal scheme \widehat{Z}/Σ is $\widehat{\mathbb{G}}_{a,\Lambda}^r$ -equivariantly isomorphic to the product $\Sigma \times \widehat{\mathbb{G}}_{a,\Lambda}^r$.

Proof. The co-action of $\Lambda[\mathbf{d}]$ on \mathcal{O} extends to a co-action on \mathcal{O}^* , since $\tau_\Phi(x_i) = x_i + dx_i$ and the (\mathbf{x}, \mathbf{d}) -adic completion of $\mathcal{O}[\mathbf{d}] = \mathcal{O} \widehat{\otimes} \Lambda[\mathbf{d}]$ is $\mathcal{O}^* \widehat{\otimes} \Lambda[\mathbf{d}]$. Then [ESB05, Lemma 2.1, p.115] can be applied since

$$\Lambda[\mathbf{d}] \longrightarrow \mathcal{O}^*, \quad d_i \longmapsto x_i$$

is equivariant. See also [ESB05, Theorem 2.15]. \square

Remark 36. One should observe, as [ESB05] does, that this technique goes back to Zariski.

Corollary 37. *For all \mathcal{O} -modules in sight, denote with an asterisk the (\mathbf{x}) -adic completion. Let $\tau^* : \mathcal{O}^* \rightarrow \widehat{\mathcal{P}}_{\mathcal{O}}^*$ be the morphism obtained from τ . If $\psi \in \mathcal{O}^*$ is such that $\sigma(\psi) \in \mathcal{O}(\Sigma)$, then $d^*(\psi) := \tau^*(\psi) - \psi \in \mathfrak{A} \cdot \widehat{\mathcal{P}}_{\mathcal{O}}^*$.*

Proof. Use the diagram in the statement of Proposition 35 to obtain $\tau_\Phi^*(\psi) = \psi$. \square

9. TANGENCY AND INVARIANCE OF SUB-SCHEMES

Let B be a noetherian and separated scheme. We work with the category \mathbf{Sch}/B of B -schemes which are noetherian and separated, so that a “morphism”, a “scheme”, etc should always be taken in \mathbf{Sch}/B . Continuing with the line of previous sections, we omit notational references to B as much as possible, e.g. $\times = \times_B$, $\mathcal{P}_Z^\mu = \mathcal{P}_{Z/B}^\mu$, etc. Let Z be a scheme of finite type and Φ an infinitesimal equivalence relation in it.

9.1. Tangency. Let $\theta : Y \rightarrow Z$ be a closed embedding whose image is defined by the sheaf of ideals I . There is a canonical surjective homomorphism of sheaves of \mathcal{O}_Y -algebras, “the restriction of principal parts” [EGA IV₄, 16.4.3.3, p.18]

$$(29) \quad \theta^* \mathcal{P}_Z^\mu \longrightarrow \mathcal{P}_Y^\mu.$$

More precisely, arrow (29) is constructed from the morphism of $\theta^{-1} \mathcal{O}_Z$ -rings

$$\theta^{-1} \mathcal{P}_Z^\mu \longrightarrow \mathcal{P}_Y^\mu,$$

which is used to define the “functoriality” morphism $P_Y^\mu \rightarrow P_Z^\mu$, see [EGA IV₄, 16.2.1, p.9] and [EGA IV₄, 16.4.1.3, p.16]. If $Z = \text{Spec}(\mathcal{O})$, $Y = \text{Spec}(\mathcal{O}_0)$ and $B = \text{Spec}(\Lambda)$, then this is just induced by

$$(30) \quad \mathcal{O}_0 \otimes_{\mathcal{O}} ((\mathcal{O} \otimes_{\Lambda} \mathcal{O})/\mathfrak{D}^{\mu+1}) \rightarrow (\mathcal{O}_0 \otimes_{\Lambda} \mathcal{O}_0)/\mathfrak{D}_0^{\mu+1},$$

where \mathfrak{D} and \mathfrak{D}_0 are the kernels of the corresponding multiplication homomorphism. The kernel of (29) is generated, as a \mathcal{O}_Y -module, by the image of $\tau_Z^\mu(I)$ [EGA IV₄, 16.4.20, p.24].

Definition 38. Assume that Φ is schematic. For each $\mu \in \mathbb{N}$, let $\mathcal{A}_\mu \subseteq \mathcal{P}_Z^\mu$ stand for the sheaf of ideals of $\widehat{P}_\Phi \cap P_Z^\mu$. We say that a closed embedding $\theta : Y \hookrightarrow Z$ is ν -tangent to Φ (respectively, tangent to Φ) relatively to B , or rel. B , if the natural composite

$$\theta^* \mathcal{A}_\mu \longrightarrow \theta^* \mathcal{P}_Z^\mu \longrightarrow \mathcal{P}_Y^\mu$$

is null for each $\mu \leq \nu$ (respectively, for every μ).

A word of caution concerning the base scheme and the term “relatively to B ” is in order, and this is better illustrated by an example. Let k be a field and put $B = \text{Spec } k$. Let $g : \mathbb{A}^2 \rightarrow \mathbb{A}^1$ be the projection on the first coordinate and let Φ be the \mathbb{A}^1 -linear i.e.r. in \mathbb{A}^2 defined by $(\mathbb{A}^2/\mathbb{A}^1)_\infty$: two points $u_0, u_1 : U \rightarrow \mathbb{A}^2$ are Φ -equivalent if and only if $gu_0 = gu_1$ and $u_0|_{U_{\text{red}}} = u_1|_{U_{\text{red}}}$. Therefore, $\mathbb{A}^2 \subseteq \mathbb{A}^2$ is not tangent to Φ *relatively to* B , but is clearly tangent to Φ *relatively to* \mathbb{A}^1 . This is because tangency is a property on the fibres over B , and not an absolute one.

In the special case where $\nu = 1$, so that $\mathcal{P}_Z^\mu = \mathcal{O}_Z \oplus \Omega_Z^1$ [EGA IV₄, 16.3.1, p.14], requiring that $\theta : Y \rightarrow Z$ be 1-tangent to Φ implies that Ω_Y^1 is a quotient of $\Omega_\Phi^1|_Y$, where Ω_Φ is the cotangent sheaf of Φ defined in eq. (25).

An equivalent definition of tangency consists in saying that $Y \subseteq Z$ is tangent to Φ rel. B if for any $U \in \mathbf{Sch}/B$ and any couple of infinitesimally close points $y_0, y_1 \in Y(U)$, we have $y_0 \equiv_\Phi y_1$. Or, in yet another clothing, Y is tangent to Φ if $Y \rightarrow Z$ interweaves (Definition 22) Y_∞ and Φ . These definitions, being completely expressed in terms of functors, apply to any infinitesimal equivalence relation in Z , schematic or not. The reason behind our choice of presenting this concept through Definition 38 will become clear in Section 10.

Remark 39. In the usual setting, a sub-manifold $S \rightarrow M$ is tangent to a foliation \mathcal{F} if, and only if, for each $x \in S$, $T_x S \subseteq T_x \mathcal{F}$. This is possible if and only if for every $\omega \in \text{Ann}(\mathcal{F})$, $\omega|_S = 0$.

9.2. Invariant sub-schemes. We now deal with a neighboring notion. Throughout, $\theta : Y \rightarrow Z$ stands for a closed subscheme defined by the sheaf of ideals $I \subseteq \mathcal{O}_Z$.

Definition 40. We say that $\theta : Y \rightarrow Z$ is Φ -invariant, or Φ -saturated, if, for each $U \in \mathbf{Sch}/B$, $Y(U) \subseteq Z(U)$ is saturated under the equivalence relation $\Phi(U)$.

We now take Φ schematic. Let us assume that $\theta : Y \rightarrow Z$ is Φ -invariant; fix some $\nu \in \mathbb{N}$ and let U be the inverse image of Y by $p_0^\nu : P_\Phi^\nu \rightarrow Z$. Projecting to Y defines a morphism $u_0 : U \rightarrow Y$ and projecting to P_Φ^ν and then composing with $p_1^\nu : P_\Phi^\nu \rightarrow Z$ defines a morphism $u_1 : U \rightarrow Z$. Obviously, $u_0 \equiv_\Phi u_1$ which, by invariance of Y , entails that u_1 factors through $\theta : Y \rightarrow Z$: this shows that $(p_0^\nu)^{-1}(Y) \subseteq (p_1^\nu)^{-1}(Y)$.

On the other hand, assume that for each $\nu \in \mathbb{N}$, the second projection $p_1^\nu : P_\Phi^\nu \rightarrow Z$ restricted to $(p_0^\nu)^{-1}(Y)$ factors through $\theta : Y \rightarrow Z$. Let $u_0 : U \rightarrow Y$ and $u_1 : U \rightarrow Z$ be Φ -close, so that, for some $\nu \in \mathbb{N}$, $(u_0, u_1) : U \rightarrow P_\Phi^\nu$ factors through P_Φ^ν . As u_0 is a point of Y , it follows that (u_0, u_1) factors through $(p_0^\nu)^{-1}(Y)$ so that $u_1 = p_1^\nu \circ (u_0, u_1)$ must equally factor through Y . We state these findings in a more algebraic form, which will suit us later on.

Lemma 41. *We admit that Φ is schematic. Let I be an ideal of \mathcal{O}_Z and let Y be the closed subscheme cut out by I . Then Y is Φ -invariant if and only if*

$$\tau_\Phi(I) \subseteq I \cdot \widehat{\mathcal{P}}_\Phi.$$

□

The reader should note that for any ideal $I \subseteq \mathcal{O}_Z$, the inclusion $\tau_\Phi(I) \subseteq I \cdot \widehat{\mathcal{P}}_\Phi$ implies

$$(31) \quad \tau_\Phi(I) \cdot \widehat{\mathcal{P}}_\Phi = I \cdot \widehat{\mathcal{P}}_\Phi,$$

since the switch automorphism

$$s : \widehat{\mathcal{P}}_\Phi \longrightarrow \widehat{\mathcal{P}}_\Phi, \quad \psi_0 \widehat{\otimes} \psi_1 \longmapsto \psi_1 \widehat{\otimes} \psi_0$$

permutes $\tau_\Phi(I) \cdot \widehat{\mathcal{P}}_\Phi$ and $I \cdot \widehat{\mathcal{P}}_\Phi$.

One of the pleasing characteristics of invariance is that it is a “closed condition.”

Proposition 42. *Let $Y \subseteq Z$ be a closed subscheme of Z cut out by the ideal I . Assume that there exists an open subset $Z^\circ \subseteq Z$ such that $Y^\circ = Y \cap Z^\circ$ is saturated under Φ and schematically dense in Y . Then Y is saturated.*

Proof. This is a local problem, so we assume $Z = \text{Spec}(\mathcal{O})$. Moreover, we can suppose that \mathcal{P}_Φ^μ is free over \mathcal{O} on the basis

$$d\mathbf{x}^\alpha = dx_1^{\alpha_1} \cdots dx_r^{\alpha_r}, \quad \alpha_1 + \cdots + \alpha_r \leq \mu.$$

It follows in particular that $I \cdot \mathcal{P}_\Phi^\mu = \bigoplus_{|\alpha| \leq \mu} I \cdot d\mathbf{x}^\alpha$. Let $\psi \in I$. Then, $\tau_\Phi^\mu(\psi) = \sum_\alpha \psi_\alpha \cdot d\mathbf{x}^\alpha$, and from the hypothesis, $(\psi_\alpha|Y)|Y^\circ = 0$. Since Y° is schematically dense in Y , $\psi_\alpha|Y = 0$. As μ is arbitrary, we arrive at the desired conclusion. \square

9.3. Relation between the two concepts.

Lemma 43. *Let $Y \subseteq Z$ be a closed subscheme of Z which is smooth of relative dimension r over B . Assume that Φ is differentially smooth of rank r . Then Y is tangent rel. B if and only if Y is invariant.*

Proof. This is just a matter of counting ranks, applying Nakayama’s Lemma and using [EGA IV₄, 14.4.20, p. 24], which says that

$$\mathcal{P}_Y^\mu = \frac{\mathcal{P}_Z^\mu}{I \cdot \mathcal{P}_Z^\mu + \tau^\mu(I) \cdot \mathcal{P}_Z^\mu}.$$

\square

It should be remarked that the assumption on smoothness is necessary. Just take the i.e.r. Φ on $\mathbb{A}_\mathbb{C}^2$ defined by $(x, y) \equiv (x', y')$ if and only if $x \equiv_\infty x'$ and $y = y'$ and note that $Y = \{y^2 = 0\}$ is invariant, but not tangent.

9.4. The formal shape of invariant subschemes of maximal codimension. We suppose that $B = \text{Spec}(\Lambda)$, and that Z is the spectrum of a smooth Λ -domain \mathcal{O} of relative dimension n . We let \mathfrak{m} be a closed point of Z whose residue field coincides with that of its image \mathfrak{n} in B . We fix a differentially smooth i.e.r. Φ in Z of rank r .

We now introduce a series of working properties all of which can be guaranteed by passing to a smaller open subset of Z . We assume that Ω_Φ (Definition 31) is freely generated by the image of differentials dx_1, \dots, dx_r , where $x_i \in \mathfrak{m}$ for all i . The associated transverse section (see Definition 33) is

$$\Sigma = \{x_1 = \cdots = x_r = 0\};$$

this is a smooth B -scheme of relative dimension $c = n - r$, by Lemma 34. There is no serious impediment in assuming that

$$\Omega_{\Sigma}^1 = \bigoplus_{i=1}^c \mathcal{O}(\Sigma) \cdot dy_i,$$

with $y_i \in \mathcal{O}(\Sigma)$ vanishing on \mathfrak{m} . So we include this hypothesis. Moreover, we assume that eq. (27) actually gives

$$\Omega_Z^1 = \sum_{i=1}^r \mathcal{O} \cdot dx_i + \sum_{j=1}^c \mathcal{O} \cdot d\tilde{y}_j,$$

where $\tilde{y}_i \in \mathcal{O}$ lifts y_i . Let $y_i^* \in \mathcal{O}^*$ correspond to y_i under the isomorphism

$$\sigma : \mathcal{O}^* \longrightarrow \mathcal{O}(\Sigma)[[\mathbf{d}]], \quad \mathbf{d} = (d_1, \dots, d_r),$$

of Proposition 35.

Lemma 44. *Let $\mathcal{O}^{\mathfrak{s}}$, respectively $\mathfrak{m}^{\mathfrak{s}}$, denote the \mathfrak{m} -adic completion of \mathcal{O} , respectively the maximal ideal of $\mathcal{O}^{\mathfrak{s}}$. We shall consider $\mathcal{O}^{\mathfrak{s}}$ as an \mathcal{O}^* -algebra in the obvious way.*

- (1) $y_i^* \in \mathfrak{m}^{\mathfrak{s}}$.
- (2) Denote by \mathfrak{n} the image of \mathfrak{m} in $\text{Spec}(\Lambda)$. Let $\mathbf{X} = (X_1, \dots, X_r)$, $\mathbf{Y}^* = (Y_1^*, \dots, Y_c^*)$ be variables. Then the natural morphism $\hat{\Lambda}_{\mathfrak{n}}[[\mathbf{Y}^*, \mathbf{X}]] \rightarrow \mathcal{O}^{\mathfrak{s}}$ sending \mathbf{X} to \mathbf{x} and \mathbf{Y}^* to \mathbf{y}^* is an isomorphism.
- (3) We maintain the setting of the previous item. Let $\tau_{\Phi}^{\mathfrak{s}} : \mathcal{O}^{\mathfrak{s}} \rightarrow \mathcal{O}^{\mathfrak{s}}[[\mathbf{d}\mathbf{x}]]$ be the morphism obtained from τ_{Φ} upon passage to the \mathfrak{m} -adic completion. Then, under the aforementioned isomorphism, $\tau_{\Phi}^{\mathfrak{s}}$ corresponds to

$$\hat{\Lambda}_{\mathfrak{n}}[[\mathbf{Y}^*, \mathbf{X}]] \longrightarrow \hat{\Lambda}_{\mathfrak{n}}[[\mathbf{Y}^*, \mathbf{X}]][[\mathbf{d}\mathbf{x}]], \quad X_i \mapsto X_i + dx_i, \quad Y_i^* \mapsto Y_i^*.$$

Proof. 1. It is enough to show that $y_i^* \in \mathfrak{m} \cdot \mathcal{O}^*$. Since $\tau_{\Phi}(\psi) \equiv \psi \pmod{(\mathbf{d}\mathbf{x})}$ for all $\psi \in \mathcal{O}$, the very definition of σ gives $\sigma(\tilde{y}_i) \equiv y_i \pmod{(\mathbf{d})}$. As $\sigma(x_i) = d_i$, it follows that

$$(32) \quad \tilde{y}_i \equiv y_i^* \pmod{(\mathbf{x})}$$

and this is enough to finish the proof.

2. We are assuming $\mathbf{k}(\mathfrak{n}) \xrightarrow{\cong} \mathbf{k}(\mathfrak{m})$, so, [EGA IV₄, 17.5.3, p.68] guarantees that $\mathcal{O}^{\mathfrak{s}}$ is isomorphic to a power series algebra over $\hat{\Lambda}_{\mathfrak{n}}$ in n variables. Hence, we only need to show that the classes of \mathbf{x}, \mathbf{y}^* generate the radical of $\mathcal{O}^{\mathfrak{s}}/\mathfrak{r}(\hat{\Lambda}_{\mathfrak{n}}) \cdot \mathcal{O}^{\mathfrak{s}}$. (In the sequel, \mathfrak{r} refers to the maximal ideal = radical of a local ring.) From eq. (32), it is sufficient to show that \mathbf{x} and $\tilde{\mathbf{y}}$ generate this radical. Since $\mathcal{O}^{\mathfrak{s}}/\mathfrak{r}(\hat{\Lambda}_{\mathfrak{n}}) \cdot \mathcal{O}^{\mathfrak{s}}$ is simply the completion of the local ring of the fibre of $Z \rightarrow B$ above \mathfrak{n} , $\mathcal{O}_{Z_n, \mathfrak{m}} = \mathcal{O}_{\mathfrak{m}}/\mathfrak{r}(\Lambda_{\mathfrak{n}}) \cdot \mathcal{O}_{\mathfrak{m}}$, the work amounts to showing that \mathbf{x}, \mathbf{y} generate the radical of $\mathcal{O}_{Z_n, \mathfrak{m}}$. This is obvious given (32), the isomorphisms

$$\begin{aligned} \Omega_{\mathcal{O}/\Lambda}^1 \otimes \mathbf{k}(\mathfrak{m}) &\cong \Omega_{Z_n/\mathbf{k}(\mathfrak{n})}^1 \otimes \mathbf{k}(\mathfrak{m}) \\ &\cong \mathfrak{r}(\mathcal{O}_{Z_n, \mathfrak{m}})/\mathfrak{r}(\mathcal{O}_{Z_n, \mathfrak{m}})^2 \end{aligned}$$

and Nakayama's Lemma.

3. We begin by noting that the obtention of $\tau_{\Phi}^{\mathfrak{s}}$ is legitimate. For any $\psi \in \mathcal{O}$, we have $\tau_{\Phi}(\psi) \equiv \psi \pmod{(\mathbf{d}\mathbf{x})}$. Hence, $\tau_{\Phi}(\mathfrak{m}) \subseteq \mathfrak{m} \cdot \mathcal{O}[[\mathbf{d}\mathbf{x}]] + (\mathbf{d}\mathbf{x})$ and this allows us to pass to

completions. The rest follows immediately from the commutative diagram of Proposition 35, since $\sigma(x_i) = d_i$ and $\sigma(y_i^*) = y_i$. \square

We now apply ideas of Seidenberg [Sei67], see Theorem 1 of this reference, to study invariant subscheme of maximal codimension.

Lemma 45. *Let K be a field, $\xi = \xi_1, \dots, \xi_r$, $\xi' = \xi'_1, \dots, \xi'_r$, and $\eta = \eta_1, \dots, \eta_c$ be variables. Denote by α the ring homomorphism*

$$K[[\xi, \eta]] \longrightarrow K[[\xi, \eta, \xi']], \quad \xi_i \mapsto \xi_i + \xi'_i, \quad \eta_i \mapsto \eta_i,$$

and by $\iota : K[[\xi, \eta]] \rightarrow K[[\xi, \eta, \xi']]$ the inclusion. Let \mathfrak{K} be a proper ideal of $K[[\xi, \eta]]$ satisfying ${}^\alpha\mathfrak{K} \subseteq {}^\iota\mathfrak{K}$. Then the following hold.

- (1) $\mathfrak{K} \subseteq (\eta)$.
- (2) Assume that \mathfrak{K} is primary and $\text{ht}(\mathfrak{K}) \geq c$. Then $\sqrt{\mathfrak{K}} = (\eta)$.
- (3) Assume only that $\text{ht}(\mathfrak{K}) \geq c$. Then \mathfrak{K} is (η) -primary.

Proof. 1. Let $\psi_1(\xi, \eta), \dots, \psi_m(\xi, \eta)$ be generators of \mathfrak{K} and let $A_{ij}(\xi, \eta, \xi')$ be such that

$$\psi_i(\xi + \xi', \eta) = \sum_{j=1}^m \psi_j(\xi, \eta) \cdot A_{ij}(\xi, \eta, \xi').$$

Then, making $\xi = \mathbf{0}$ and $\eta = \mathbf{0}$ in the above equation, we have

$$\begin{aligned} \psi_i(\xi', \mathbf{0}) &= \sum_{j=1}^m \psi_j(\mathbf{0}, \mathbf{0}) \cdot A_{ij}(\mathbf{0}, \mathbf{0}, \xi') \\ &= 0. \end{aligned}$$

2. We have $\text{ht}(\eta) \leq c$. From the inclusion $\mathfrak{K} \subseteq (\eta)$, it follows that $\sqrt{\mathfrak{K}} \subseteq (\eta)$; as $\sqrt{\mathfrak{K}}$ is a prime of height at least c , we have $\sqrt{\mathfrak{K}} = (\eta)$.

3. Let \mathfrak{q} be a primary ideal of \mathfrak{K} and denote by \mathfrak{p} its associated prime. Then ${}^\iota\mathfrak{q}$ is a primary ideal with associated prime ${}^\iota\mathfrak{p}$, see [Sei67, Lemma 2, p.24]. Also, it is straightforward to see that the operation of extension $\mathfrak{a} \mapsto {}^\iota\mathfrak{a}$ takes intersections to intersections and is injective. Hence, if $\mathfrak{K} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_e$ is an irredundant primary decomposition of \mathfrak{K} [AM, Ch.4], then ${}^\iota\mathfrak{q}_1 \cap \dots \cap {}^\iota\mathfrak{q}_e$ is an irredundant primary decomposition of ${}^\iota\mathfrak{K}$. As the associated prime ideals are uniquely determined [AM, Theorem 4.5, p.52], it follows that each associated prime of ${}^\iota\mathfrak{K}$ is of the form ${}^\iota\mathfrak{p}$ for some associated prime \mathfrak{p} of \mathfrak{K} .

Let $\lambda : K[[\xi, \eta, \xi']] \rightarrow K[[\xi, \eta, \xi']]$ be the automorphism $(\xi, \eta, \xi') \mapsto (\xi + \xi', \eta, -\xi')$. It follows that $\lambda \circ \iota = \alpha$, $\lambda \circ \alpha = \iota$, and $\lambda \circ \lambda = \text{id}$. Hence, the inclusion ${}^\alpha\mathfrak{K} \subseteq {}^\iota\mathfrak{K}$ is an equality. It follows that if \mathfrak{p} is an associated prime of \mathfrak{K} , then ${}^\alpha\mathfrak{p} = {}^\lambda({}^\iota\mathfrak{p})$ is an associated prime of ${}^\alpha\mathfrak{K} = {}^\iota\mathfrak{K}$. Therefore, there exists another associated prime \mathfrak{l} of \mathfrak{K} for which ${}^\alpha\mathfrak{p} = {}^\iota\mathfrak{l}$. Considering this equality modulo ξ' , we conclude that $\mathfrak{l} = \mathfrak{p}$. Thus, \mathfrak{p} is a prime of height at least c which satisfies ${}^\iota\mathfrak{p} = {}^\alpha\mathfrak{p}$. The previous item shows that $\mathfrak{p} = (\eta)$, and we are done. \square

Corollary 46. *Maintain the assumptions and notations of Lemma 44. Let $L \hookrightarrow Z$ be a closed subscheme passing through \mathfrak{m} and cut out by an ideal $I \subseteq \mathcal{O}$. Assume that L is flat*

over B , of codimension at least c at \mathfrak{m} , and invariant under Φ . Then, if $Z_{\mathfrak{n}}$ stands for the fibre of Z above \mathfrak{n} , we have

$$\sqrt{I \cdot \widehat{\mathcal{O}}_{Z_{\mathfrak{n}}, \mathfrak{m}}} = (\mathbf{y}^*).$$

In particular, if the fibre $L_{\mathfrak{n}}$ is reduced at \mathfrak{m} , then $L \rightarrow B$ is smooth at \mathfrak{m} .

Proof. It is a simple exercise to show that

$$\begin{aligned} \text{ht } I \cdot \widehat{\mathcal{O}}_{Z_{\mathfrak{n}}, \mathfrak{m}} &= \text{ht } I \cdot \widehat{\mathcal{O}}_{Z, \mathfrak{m}} \\ &= \text{ht } I \cdot \mathcal{O}_{Z, \mathfrak{m}}. \end{aligned}$$

(Use Theorem 15.1 on p. 116 and Theorem 17.4 on p. 135 of [Mat89] for the first equality, and [BouAC, VIII.3.4, Cor. 4] for the second.) From this, it follows that $I \cdot \widehat{\mathcal{O}}_{Z_{\mathfrak{n}}, \mathfrak{m}}$ is of height at least c . The first part of the statement now ensues instantaneously from Lemma 44 and Lemma 45.

To prove the final statement, we recall that a Theorem of Zariski and Nagata, see the examples and Theorem 3 of [BouAC, Ch. IX, §4, No.4], guarantees that once $\mathcal{O}_{L_{\mathfrak{n}}, \mathfrak{m}}$ is reduced, then so is $\widehat{\mathcal{O}}_{L_{\mathfrak{n}}, \mathfrak{m}}$. Thus, $\mathbf{k}(\mathfrak{n}) \rightarrow \widehat{\mathcal{O}}_{L_{\mathfrak{n}}, \mathfrak{m}}$ is formally smooth, so that $L_{\mathfrak{n}} \rightarrow \text{Spec}(\mathbf{k}(\mathfrak{n}))$ is smooth at \mathfrak{m} [EGA IV₄, 17.5.3, p.69], so that $L \rightarrow B$ is smooth at \mathfrak{m} [EGA IV₄, 17.5.1, p.67]. \square

10. THE HILBERT SCHEME OF SUB-SCHEMES TANGENT TO AN INFINITESIMAL EQUIVALENCE RELATION

Let B be a noetherian separated scheme. All ensuing constructions are made in \mathbf{Sch}/B , which stands for the category of separated and noetherian B -schemes (not necessarily of finite type). Let

$$\begin{array}{ccc} Z & \xrightarrow{\zeta} & X \\ & \searrow g & \downarrow f \\ & & S \end{array}$$

be a commutative diagram. We suppose that $f : X \rightarrow S$ is smooth of relative dimension r , geometrically connected, and proper. We also suppose that $g : Z \rightarrow S$ is quasi-projective and fix, tacitly, a g -ample invertible sheaf on Z . Under these assumptions, [TDTE, IV, §4 c, p. 19ff] guarantees that

$$\mathbf{Sch}/S \longrightarrow \text{Set}, \quad T \longmapsto \left\{ \begin{array}{l} \text{Closed sub-schemes } Y \hookrightarrow Z \times_S T \text{ flat over } T \\ \text{and isomorphic, via } \zeta \times \text{id}_T, \text{ to } X \times_S T \end{array} \right\}$$

is represented by a scheme

$$\pi_0 : H_0 \longrightarrow S,$$

called the *Weil restriction*. Grothendieck's notation for H_0 is $\Pi_{X/S}(Z/X)$, see [TDTE, II, C2, p.13], [TDTE, IV, §4c], but $\mathfrak{R}_{X/S}(Z)$ can also be found in the literature, e.g. [BLR90, §7.6]. The scheme H_0 is an open sub-scheme of the S -scheme $\text{Hilb}(Z/S)$ and as such is

locally of finite type and separated. In fact, H_0 is an open sub-scheme of the disjoint union

$$\coprod_{\chi} \text{Hilb}^{\chi}(Z/S),$$

where $\chi \in \mathbb{Q}[\lambda]$ runs over all numerical polynomials (those which map \mathbb{Z} into \mathbb{Z}) of degree $r = \dim f$. Given that each $\text{Hilb}^{\chi}(Z/S)$ is a quasi-projective scheme, see [TDTE, IV, §4(a)] or [N03, Theorem 5.20], H_0 is a disjoint union of quasi-projective S -schemes. It is also worth noticing that if ζ is smooth, then so is π_0 ; this is a formal consequence of the “valuative criterion for smoothness”, see [BLR90, §7.6, Proposition 5].

Remark 47. By calling H_0 a scheme, we have deviated from our initial convention: H_0 is not necessarily quasi-compact. This will cause no harm.

Let Φ be an infinitesimal B -linear equivalence relation in Z such that Φ/S is differentially smooth of rank $r = \dim f$. We let

$$\mathcal{A}_{\mu} = \begin{array}{l} \text{sheaf of ideals in } \mathcal{P}_{Z/S}^{\mu} \text{ defining the} \\ \text{closed formal subscheme } \Phi/S \cap P_{Z/S}^{\mu}. \end{array}$$

Let $\mathcal{Y}_0 \rightarrow Z \times_S H_0$ stand for the universal closed subscheme. Composition of the canonical arrow [EGA IV₄, 16.4.3.3, p.18]

$$\mathcal{P}_{Z \times_S H_0/H_0}^{\mu} \Big|_{\mathcal{Y}_0} \longrightarrow \mathcal{P}_{\mathcal{Y}_0/H_0}^{\mu}$$

with the obvious map

$$(\mathcal{A}_{\mu} \times_S H_0) \Big|_{\mathcal{Y}_0} \longrightarrow \left(\mathcal{P}_{Z/S}^{\mu} \times_S H_0 \right) \Big|_{\mathcal{Y}_0} = \mathcal{P}_{Z \times_S H_0/H_0}^{\mu} \Big|_{\mathcal{Y}_0}$$

(see [EGA IV₄, 16.4.5, p.19] to justify the identification made above) defines a homomorphism $\mathcal{O}_{\mathcal{Y}_0}$ -modules

$$(33) \quad w_{\mu} : (\mathcal{A}_{\mu} \times_S H_0) \Big|_{\mathcal{Y}_0} \longrightarrow \mathcal{P}_{\mathcal{Y}_0/H_0}^{\mu}.$$

Lemma 48. *There exists a closed subscheme $H_0^{(\mu)} \subseteq H_0$ having the following property. An arrow $T \rightarrow H_0$ factors uniquely through $H_0^{(\mu)}$ if and only if the morphism obtained from eq. (33)*

$$w_{\mu} \times_{H_0} T : [(\mathcal{A}_{\mu} \times_S H_0) \Big|_{\mathcal{Y}_0}] \times_{H_0} T \longrightarrow \mathcal{P}_{\mathcal{Y}_0/H_0}^{\mu} \times_{H_0} T$$

is null.

Proof. As H_0 -schemes, \mathcal{Y}_0 and $X \times_S H_0$ are isomorphic. Hence, $\mathcal{P}_{\mathcal{Y}_0/H_0}^{\mu}$ is a locally free $\mathcal{O}_{\mathcal{Y}_0}$ -module for any one of its two canonical structures [EGA IV₄, 16.10]. A fortiori, $\mathcal{P}_{\mathcal{Y}_0/H_0}^{\mu}$ is flat over H_0 . From the projectivity of $\mathcal{Y}_0 \rightarrow H_0$, we are in the situation of [N03, Remark 5.9, p.121], and this results in what we have envisaged. More precisely, there exists a linear scheme over H_0 , call it $V \rightarrow H_0$, which represents the functor

$$T \in \mathbf{Sch}/H_0 \longmapsto \text{Hom}_{\mathcal{O}_{\mathcal{Y}_0 \times_{H_0} T}} \left([(\mathcal{A}_{\mu} \times_S H_0) \Big|_{\mathcal{Y}_0}] \times_{H_0} T, \mathcal{P}_{\mathcal{Y}_0/H_0}^{\mu} \times_{H_0} T \right).$$

Since there are at least two homomorphisms of $\mathcal{O}_{\mathcal{Y}_0}$ -modules

$$(\mathcal{A}_{\mu} \times_S H_0) \Big|_{\mathcal{Y}_0} \longrightarrow \mathcal{P}_{\mathcal{Y}_0/H_0}^{\mu}$$

in sight, viz. 0 and w_μ , the morphism $V \rightarrow H_0$ has two sections, and $H_0^{(\mu)}$ is their intersection. \square

From the surjectivity of the natural homomorphisms $\mathcal{A}_{\mu+1} \rightarrow \mathcal{A}_\mu$, it follows that the closed embedding $H_0^{(\mu+1)} \rightarrow H_0$ factors through $H_0^{(\mu)} \rightarrow H_0$.

Definition 49. The S -scheme $\cap_\mu H_0^{(\mu)}$ will be denoted by $H_f(Z, \Phi)$. The structure morphism to S will be denoted by π .

Note that $H_f(Z, \Phi)$ is, as a closed subscheme of H_0 , a disjoint union of quasi-projective schemes over S . Rewriting Lemma 48 in terms of flat families of closed subschemes, $H_f(Z, \Phi)$ represents the functor

$$T \longmapsto \left\{ \begin{array}{l} \text{Closed subschemes } Y \hookrightarrow Z \times_S T \text{ flat over } T \text{ and isomorphic,} \\ \text{via } \zeta \times \text{id}_T, \text{ to } X \times_S T, \text{ such that } \mathcal{A}_\mu \times_S T \rightarrow \mathcal{P}_{Y/T}^\mu \text{ is null for all } \mu \end{array} \right\}$$

Since the image of $\mathcal{A}_\mu \times_S T$ in $\mathcal{P}_{Z/S}^\mu \times_S T \cong \mathcal{P}_{Z \times_S T/T}^\mu$ is just the ideal of $\Phi/S \times_S T$ in $\mathcal{P}_{Z \times_S T/T}^\mu$, $H_f(Z, \Phi)$ represents the functor

$$T \longmapsto \left\{ \begin{array}{l} \text{Closed subschemes } Y \hookrightarrow Z \times_S T \text{ flat over } T, \text{ tangent rel. } T \\ \text{to } \Phi/S \times_S T, \text{ and isomorphic, via } \zeta \times \text{id}_T, \text{ to } X \times_S T \end{array} \right\}.$$

Or, in yet another interpretation, $H_f(Z, \Phi)$ represents the functor which parametrizes sections

$$\sigma : X \times_S T \longrightarrow Z \times_S T$$

to

$$\zeta \times_S T : Z \times_S T \longrightarrow X \times_S T$$

which are tangent to the infinitesimal equivalence relation $\Phi/S \times_S T$. Using Lemma 43, we can also affirm that $H_f(Z, \Phi)$ represents

$$T \longmapsto \left\{ \begin{array}{l} \text{Closed subschemes } Y \hookrightarrow Z \times_S T \text{ flat over } T, \text{ invariant under} \\ \Phi/S \times_S T \text{ and isomorphic, via } \zeta \times \text{id}_T, \text{ to } X \times_S T \end{array} \right\}.$$

With this reformulation, it is easier to establish the following result.

Theorem 50. *Assume that $\zeta : Z \rightarrow X$ is smooth, proper, of relative dimension c , say, and that Φ/S is complementary to ζ . Then the restriction of $\pi : H_f(Z, \Phi) \rightarrow S$ to each one of its connected components is projective.*

Proof. We already know that π is separated and locally of finite type, as $H_f(Z, \Phi)$ is a closed subscheme of the locally quasi-projective scheme H_0 .

Let T be an S -scheme which is the spectrum of a complete discrete valuation ring. The generic, respectively closed, point of T will be denoted by γ , respectively o . We also include the hypothesis that $k := \mathbf{k}(o)$ is algebraically closed. As traditional, we abbreviate the operation of base-changing by decreasing $(*)_\bullet = * \times_S \bullet$.

We consider a closed subscheme $Y_\gamma \subseteq Z_\gamma$ of Z_γ which is saturated under $\Phi/S \times_S \gamma$ and isomorphic, via ζ_γ , to X_γ . In particular, Y_γ is an integral scheme. Let $Y_T \subseteq Z_T$ be a T -flat and closed sub-scheme of Z_T which restricts to Y_γ when intersected with the generic

fibre $Z_\gamma = Z_T \setminus Z_o$ [EGA IV₂, 2.8.5,p.35]. By construction, $Y_\gamma \rightarrow Y_T$ is schematically dense, so that Y_T is integral. From Proposition 42, we know that Y_T is saturated under $\Phi_{/S} \times_S T$. We now prove that $Y_T \rightarrow X_T$ is quasi-finite [EGA II, 6.2,114ff], so that Zariski's Main Theorem [EGA III₁, 4.4.9,p.137] guarantees $Y_T \xrightarrow{\sim} X_T$; the valuative criterion for properness, see [EGA II, 7.3.8, p.144] and [EGA II, 7.3.9i), p. 145], then concludes the proof.

To show that $Y_T \rightarrow X_T$ is quasi-finite, we pick a point $Q \in Y_T$; denote its image in X_T by P . If $Q \in Y_\gamma$, then $\mathcal{O}_{X_T,P} \xrightarrow{\sim} \mathcal{O}_{Y_T,Q}$ by definition of Y_γ . Hence, we can assume that Q , respectively P , lies on the special fibre Y_o , respectively X_o . We assume that Q is closed, or, that $\mathbf{k}(P) = \mathbf{k}(Q)$ is the algebraically closed field k . Since the problem is now local, we write $Z_T = \text{Spec}(\mathcal{O})$, $X_T = \text{Spec}(A)$, and $T = \text{Spec}(\Lambda)$. In order to simplify notation, we let Ψ stand for the i.e.r. $\Phi_{/S} \times_S T$. Recall that due to Lemma 32, Ψ is differentially smooth of rank $r = \text{rel.dim.}(A/\Lambda)$ over T .

Since $\Lambda \rightarrow A$ is smooth of relative dimension r , it is possible to find x_1, \dots, x_r such that the augmentation ideal $\mathfrak{D}_{A/\Lambda}$ of $\widehat{\mathcal{P}}_{A/\Lambda}$ is, near $P \in \text{Spec}(A)$, generated by dx_1, \dots, dx_r . As the ideal of $\widehat{\mathcal{P}}_{Z_T/X_T}$ in $\widehat{\mathcal{P}}_{Z_T/T}$ is generated by the image of the diagonal ideal $\mathfrak{D}_{A/\Lambda} \subseteq \widehat{\mathcal{P}}_{A/\Lambda}$ under the obvious morphism of topological rings

$$\widehat{\mathcal{P}}_{A/\Lambda} \longrightarrow \widehat{\mathcal{P}}_{\mathcal{O}/\Lambda},$$

the fact that $\Psi_{/X_T}$ is trivial means that

$$\mathfrak{D}_{A/\Lambda} \cdot \widehat{\mathcal{P}}_\Psi = \mathfrak{D}_\Psi.$$

Since Ψ is differentially smooth of relative dimension r , $\Omega_\Psi = \mathfrak{D}_\Psi / \mathfrak{D}_\Psi^2$ is then freely generated by the elements dx_1, \dots, dx_r . Due to $\mathbf{k}(P) = k$, we can modify the x_i to comply with the condition $x_i(P) = 0$. A fortiori,

$$\{x_1 = \dots = x_r = 0\} \hookrightarrow Z_T$$

is a transverse section at Q (Definition 33). Due to the smoothness assumption made on ζ , we can apply [EGA IV₄, 17.5.3,p.68], just as we did in Lemma 44, to identify $\widehat{\mathcal{O}}_{Z_T,Q}$ with $\Lambda[\![\boldsymbol{\xi}, \boldsymbol{\eta}]\!]$, where $\boldsymbol{\xi} = (\xi_1, \dots, \xi_r)$ and $\boldsymbol{\eta} = (\eta_1, \dots, \eta_c)$. Moreover, under this identification, ξ_i corresponds to x_i .

Let $I \subseteq \mathcal{O}$ be the ideal of Y_T in Z_T . As Y_T is invariant under Ψ , flat over T and of codimension at least c at Q , we can use Corollary 46 and to conclude that

$$(34) \quad \sqrt{I \cdot \widehat{\mathcal{O}}_{Z_o,Q}} = (\boldsymbol{\eta}).$$

An important consequence of this equality is that for some integer N , η_i^N belongs to $(\mathfrak{r}(\Lambda), I) \cdot \widehat{\mathcal{O}}_{Z_T,Q}$ for all i .

Since the residue field of X_T at P , the image of Q , coincides with the residue field of Λ , an application of [EGA IV₄, 17.5.3, p.68] shows that it is possible to identify $\widehat{\mathcal{O}}_{X_T,P}$ with $\Lambda[\![\boldsymbol{\xi}]\!]$, where $\boldsymbol{\xi} = (\xi_1, \dots, \xi_r)$. Moreover the natural local morphism of complete local rings

$$(\Lambda[\![\boldsymbol{\xi}]\!] =) \widehat{\mathcal{O}}_{X_T,P} \longrightarrow \widehat{\mathcal{O}}_{Z_T,Q} (= \Lambda[\![\boldsymbol{\xi}, \boldsymbol{\eta}]\!])$$

becomes identified with the obvious inclusion of $\Lambda[[\xi]]$ in $\Lambda[[\xi, \eta]]$. Hence, the radical of the local noetherian ring (the fibre local ring)

$$\frac{\widehat{\mathcal{O}}_{Z_T, Q}}{\mathfrak{r}(\widehat{\mathcal{O}}_{X_T, P}) \cdot \widehat{\mathcal{O}}_{Z_T, Q} + I \cdot \widehat{\mathcal{O}}_{Z_T, Q}} = \frac{\widehat{\mathcal{O}}_{Z_o, Q}}{\mathfrak{r}(\widehat{\mathcal{O}}_{X_o, P}) \cdot \widehat{\mathcal{O}}_{Z_o, Q} + I \cdot \widehat{\mathcal{O}}_{Z_o, Q}}$$

is generated by *nilpotent* elements. Therefore, this is a local Artin k -algebra of residue field k , and consequently a k -algebra is of finite dimension. This means, by definition, that

$$\widehat{\mathcal{O}}_{X_T, P} \longrightarrow \widehat{\mathcal{O}}_{Z_T, P}/I \cdot \widehat{\mathcal{O}}_{Z_T, P} = \widehat{\mathcal{O}}_{Y_T, Q}$$

is quasi-finite [EGA I₀, 7.4,68ff], and therefore,

$$\mathcal{O}_{X_T, P} \longrightarrow \mathcal{O}_{Y_T, Q}$$

is quasi-finite. In conclusion, we showed that $Y_T \rightarrow X_T$ is quasi-finite at any closed point. As the set of such points is open in Y_T (a consequence of the Main Theorem, see [EGA III₁, 4.4.10, p.137]) we are done. \square

11. CLOSED SUBSCHEMES EQUIVALENT UNDER AN INFINITESIMAL EQUIVALENCE RELATION

Let B be a noetherian separated scheme. By \mathbf{Sch}/B we understand the category of noetherian separated B -schemes and in what follows, except for explicit admonishment, the words “morphism”, “scheme”, etc should always be in this category. We endow \mathbf{Sch}/B with the Zariski topology, so that in the sequel the word “covering” will have that meaning. Finally, we fix $Z \in \mathbf{Sch}/B$ which is furthermore of finite type over B .

Let $Y_0 \subseteq Z$ and $Y_1 \subseteq Z$ be closed subschemes. For the sake of discussion, we will say that Y_0 is *infinitesimally close* to Y_1 , and write $Y_0 \equiv_\infty Y_1$, if they underlie the same topological space. Enlarging this definition, we say that two closed embeddings $\theta_0 : Y_0 \rightarrow Z$ and $\theta_1 : Y_1 \rightarrow Z$ are infinitesimally close if the associated closed subschemes [EGA I, 4.2.1, p.122] are infinitesimally close. A more explicit way to say this is as follows: There exists a reduced scheme R together with nilpotent closed embeddings $\omega_0 : R \rightarrow Y_0$ and $\omega_1 : R \rightarrow Y_1$ such that $\theta_0 \circ \omega_0 = \theta_1 \circ \omega_1$.

Let Φ be an infinitesimal (B -linear) equivalence relation in Z .

Definition 51. Let $Y_0, Y_1 \subseteq Z$ be closed subschemes. We say that $Y_0 \leq_\Phi Y_1$ if, for each $U \in \mathbf{Sch}/B$ and each $y_0 \in Y_0(U)$, there exists a covering $\tilde{U} \rightarrow U$ and a point $\tilde{y}_1 \in Y_1(\tilde{U})$ which is Φ -equivalent to $y_0|_{\tilde{U}} \in Y_0(\tilde{U})$. (Put in words, every $y_0 \in Y_0$ is locally equivalent to some y_1 in Y_1 .) We say that Y_0 and Y_1 are Φ -equivalent, and write $Y_0 \equiv Y_1 \pmod{\Phi}$, if $Y_0 \leq_\Phi Y_1$ and $Y_1 \leq_\Phi Y_0$.

Obviously, the notion of Φ -equivalence is an equivalence relation. Moreover, it is an infinitesimal equivalence relation in the following sense. Let \mathbf{Hilb} stand for the scheme whose U -points correspond to the closed subschemes of $Z \times U$ which are proper and flat over U (we assume that it exists). Fix some $u : U \rightarrow \mathbf{Hilb}$, and write Y for the closed subscheme of $Z \times U$ corresponding to u . If $\omega : U_{\text{red}} \rightarrow U$ is the canonical nilpotent embedding, then $u \circ \omega : U_{\text{red}} \rightarrow \mathbf{Hilb}$ defines the closed subscheme $Y' = (\text{id}_Z \times \omega)^{-1}(Y)$

of $Z \times U_{\text{red}}$. If assumptions are made to guarantee that Y' is reduced, we derive that $u \circ \omega$ actually corresponds to Y_{red} , which is a closed subscheme of $Z \times U_{\text{red}}$. In this setting, the following Lemma, whose proof is omitted, explains the reason why the notion of Φ -equivalence describes an i.e.r.

Lemma 52. *If $Y_0 \equiv Y_1 \pmod{\Phi}$, then $Y_{0,\text{red}} = Y_{1,\text{red}}$.* □

We now bring in the situation of Section 10 and let $H_f(Z, \Phi)$ be the scheme of Definition 49. Note that $H_f(Z, \Phi)$ is a B -scheme which comes with a natural morphism π to S . (Of course, this just means that $H_f(Z, \Phi)$ is an S -scheme, but we want to emphasize that the morphism π is to be regarded as an extra structure.) Consequently, for each morphism of B -schemes $u : U \rightarrow H_f(Z, \Phi)$, we obtain a closed subscheme $Y(u)$ of $Z \times_{S, \pi u} U$. In the following definition, we identify, for any $u \in \text{Hom}_B(U, H_f(Z, \Phi))$, the scheme $Z \times_{S, \pi u} U$ as a closed subscheme of $Z \times U$.

Definition 53. Let u_0 and u_1 be morphisms of B -schemes from U to $H_f(Z, \Phi)$. Let $Y_0 \hookrightarrow Z \times_{S, \pi u_0} U$ and $Y_1 \hookrightarrow Z \times_{S, \pi u_1} U$ represent u_0 and u_1 respectively. We say that $u_0 \equiv u_1 \pmod{H_\Phi}$, or $u_0 \equiv u_1 \pmod{\Phi}$ if confusions can be avoided, if, as closed subschemes of $Z \times U$, $Y_0 \equiv Y_1 \pmod{\Phi \times U}$. That is, the following two conditions must hold : (i) for each morphism of U -schemes $t_0 : T \rightarrow Y_0$, there exists a covering $c : \tilde{T} \rightarrow T$ and a U -morphism $\tilde{t}_1 : \tilde{T} \rightarrow Y_1$ such that $\tilde{t}_1 \equiv t_0 c \pmod{\Phi \times U}$, and (ii) same as (i) but with the roles of Y_0 and Y_1 interchanged.

A direct application of Lemma 52, the fact that X is smooth over S and the discussion preceding it give:

Lemma 54. *The equivalence relation H_Φ is infinitesimal.* □

Remark 55. Let U be a B -scheme and $Y_0, Y_1 \subseteq Z \times U$ be closed subschemes. Assume that $Y_0 \leq_{\Phi \times U} Y_1$ and let $t_0 : T \rightarrow Y_0$ be a U -morphism. Then there exists a covering $c : \tilde{T} \rightarrow T$ and a morphism $\tilde{t}_1 : \tilde{T} \rightarrow Y_1$ of U -schemes such that

$$(35) \quad \tilde{t}_1 \equiv t_0 c \pmod{\Phi \times U}.$$

However, there is no need to assume that \tilde{t}_1 is a morphism of U -schemes in the presence of (35): *Any* morphism of B -schemes $\tilde{t}_1 : \tilde{T} \rightarrow Y_1$ satisfying (35) will, by definition of $\Phi \times U$, be a morphism of U -schemes. This means that in analyzing the equivalence relation H_Φ we have one less fact to check.

12. INVESTIGATION OF THE ADJOINTNESS PROPERTY OF $H_f(Z, \Phi)$

We keep the assumptions and notations made on Section 10: B is a noetherian and separated scheme, \mathbf{Sch}/B which is the stage of our constructions is the category of noetherian separated schemes; $f : X \rightarrow S$, $\zeta : Z \rightarrow X$ are morphisms such that f is smooth of relative dimension r , geometrically connected and proper; $g := f \circ \zeta$ is quasi-projective; Φ is an i.e.r. such that Φ/S is differentially smooth (Definition 31) of rank r . We also admit that the infinitesimal equivalence relation Φ is schematic. The S -scheme $H_f(Z, \Phi)$ (Definition 49), will be abbreviated to H .

12.1. The evaluation morphism. By definition of H , there exists a morphism

$$\Sigma = (\text{ev}, \text{pr}_H) : X \times_{S,\pi} H \longrightarrow Z \times_{S,\pi} H$$

which is a section to

$$\zeta \times_S \text{id}_H : Z \times_{S,\pi} H \longrightarrow X \times_{S,\pi} H.$$

The above defined morphism

$$(36) \quad \text{ev} : X \times_S H \longrightarrow Z,$$

is henceforth called the “evaluation” morphism.

Proposition 56. *The evaluation morphism (36) interweaves the infinitesimal equivalence relations $X_\infty \times_S H_\Phi$ and Φ .*

Proof. Let $U \in \mathbf{Sch}/B$ and consider (x_0, u_0) and (x_1, u_1) U -points of $X \times_S H$. Put

$$z_i = \text{ev}(x_i, u_i).$$

We suppose that $x_0 \equiv x_1 \pmod{X_\infty}$ and that $u_0 \equiv u_1 \pmod{H_\Phi}$ and set out to show that $(z_0, \text{id}_U) \equiv (z_1, \text{id}_U) \pmod{\Phi \times U}$. Ensuing from $(x_0, u_0) \equiv_\infty (x_1, u_1)$ is the fact that z_0 and z_1 are infinitesimally close. Consequently, the two U -points (z_0, id_U) and (z_1, id_U) of $Z \times U$ are $(Z \times U/U)_\infty$ -close.

The existence of u_i defines, via Σ , a section

$$\sigma_i : X \times_{S,\pi u_i} U \longrightarrow Z \times_{S,\pi u_i} U$$

to $Z \times_{S,\pi u_i} U \rightarrow X \times_{S,\pi u_i} U$. By construction of Σ , it is clear that

$$\sigma_i = (\text{ev} \circ (\text{id}_X \times u_i), \text{pr}_U).$$

In particular, using

$$z_i = \text{ev} \circ (\text{id}_X \times u_i) \circ (x_i, \text{id}_U) : U \longrightarrow Z,$$

we conclude that

$$(z_i, \text{id}_U) = \sigma_i \circ (x_i, \text{id}_U).$$

Let Y_0 and Y_1 be the closed subschemes of $Z \times U$ associated to σ_0 and σ_1 respectively. Due to $u_0 \equiv u_1 \pmod{H_\Phi}$, we have $Y_0 \equiv Y_1 \pmod{\Phi \times U}$ so that it is possible to find a covering $c : \tilde{U} \rightarrow U$ and a point $\tilde{v}_1 \in \text{Hom}_B(\tilde{U}, Y_1)$ such that

$$(37) \quad \tilde{v}_1 \equiv (z_0, \text{id}_U) \circ c \pmod{\Phi \times U}.$$

A fortiori,

$$\tilde{v}_1 \equiv (z_0, \text{id}_U) \circ c \pmod{(Z \times U/U)_\infty},$$

as $\Phi \times U \subseteq (Z \times U/U)_\infty$. Employing the equivalence

$$(z_0, \text{id}_U) \equiv (z_1, \text{id}_U) \pmod{(Z \times U/U)_\infty},$$

it follows that

$$\tilde{v}_1 \equiv (z_1, \text{id}_U) \circ c \pmod{(Z \times U/U)_\infty}.$$

As Y_1 is tangent to $\Phi \times U$ and both \tilde{v}_1 and (z_1, id_U) are points of Y_1 , we obtain

$$(38) \quad \tilde{v}_1 \equiv (z_1, \text{id}_U) \circ c \pmod{\Phi \times U}.$$

Using (37) and (38), we conclude that

$$(39) \quad (z_0, \text{id}_U) \circ c \equiv (z_1, \text{id}_U) \circ c \pmod{\Phi \times U}.$$

Due to the fact that $\Phi \times U$ is a sheaf, (39) guarantees the envisaged conclusion. \square

12.2. Adjointness of $H_f(Z, \Phi)$. We remind the reader that H stands for the S -scheme $\pi : H_f(Z, \Phi) \rightarrow S$, and H_Φ for the infinitesimal equivalence relation in H introduced by Definition 53.

Let $(T, \Psi) \rightarrow (H, H_\Phi)$ be a morphism of \mathbf{IER}/S_∞ . (For notations, see §7.2.) As ev (see §12.1) interweaves $H_\Phi \times_S X_\infty$ and Φ (Proposition 56), the composition

$$(T \times_S X, \Psi \times_S X_\infty) \longrightarrow (H \times_S X, H_\Phi \times_S X_\infty) \xrightarrow{\text{ev}} (Z, \Phi)$$

interweaves $\Psi \times_S X_\infty$ and Φ . We obtain a map

$$(40) \quad \text{Hom}_{S_\infty}(\Psi, H_\Phi) \longrightarrow \text{Hom}_{X_\infty}(\Psi \times_S X_\infty, \Phi),$$

which fits into the commutative diagram

$$(41) \quad \begin{array}{ccc} \text{Hom}_{S_\infty}(\Psi, H_\Phi) & \longrightarrow & \text{Hom}_{X_\infty}(\Psi \times_S X_\infty, \Phi) \\ \downarrow & & \downarrow \\ \text{Hom}_S(T, H) & \longrightarrow & \text{Hom}_X(T \times_S X, Z). \end{array}$$

From the fact that H is a closed sub-scheme of H_0 and

$$\text{Hom}_S(T, H_0) \simeq \text{Hom}_X(T \times_S X, Z),$$

by definition of H_0 (see p. 33), we conclude that arrow (40) is injective.

Let

$$\sigma : T \times_S X \longrightarrow Z$$

be an X -morphism which interweaves $\Psi \times_S X_\infty$ and Φ . In particular,

$$(\text{pr}_T, \sigma) : T \times_S X \longrightarrow T \times_S Z$$

interweaves $T \times_S (X/S)_\infty (\subseteq \Psi \times_S X_\infty)$ and $T \times_S \Phi/S$, which means that (pr_T, σ) is tangent rel. T to $T \times_S \Phi/S$. This guarantees the existence of an arrow

$$(42) \quad T \longrightarrow H$$

of S -schemes which is mapped to σ by the bottom horizontal arrow of (41).

Proposition 57. *The above generated arrow $T \rightarrow H$ interweaves Ψ and H_Φ . That is, the arrow $\sigma \in \text{Hom}_{X_\infty}(\Psi \times_S X_\infty, \Phi)$ belongs to the image of $\text{Hom}_{S_\infty}(\Psi, H_\Phi)$.*

Proof. Let $t_0, t_1 : U \rightarrow T$ be Ψ -equivalent points and write $s_i : U \rightarrow S$ to signify the morphisms induced by $T \rightarrow S$. Using (42) we obtain two U -points of H and these are defined by the closed embeddings

$$(\text{pr}_U, \sigma \circ (t_i \times \text{id}_X)) : U \times_{s_i, S} X \longrightarrow U \times Z.$$

Their images will be denoted by Y_i . (Note that Y_i is in fact a closed sub-scheme of $U \times_{s_i, S} Z$.) Our goal is to establish that Y_0 and Y_1 are $U \times \Phi$ -equivalent.

Let $\omega : U_{\text{red}} \hookrightarrow U$ be the reduced sub-scheme. Since $t_0 \equiv_{\infty} t_1$, we know that $s_0\omega = s_1\omega$; we are then in a position to consider the following *nilpotent embeddings*

$$\omega_i = \omega \times \text{id}_X : U_{\text{red}} \times_S X \longrightarrow U \times_{s_i, S} X.$$

To simplify notation, let us write Y_i instead of $U \times_{s_i, S} X$ in the following three paragraphs.

Let $\rho : \tilde{Y}_0 \rightarrow Y_0$ be a Zariski covering of Y_0 with \tilde{Y}_0 *affine*. We then arrive at a commutative diagram

$$(43) \quad \begin{array}{ccccc} Y_1 & \xleftarrow{\omega_1} & U_{\text{red}} \times_S X & \xleftarrow{\lambda} & \tilde{Y}_{0, \text{red}} \\ \downarrow \pi_1 & & \downarrow \omega_0 & \square & \downarrow \tilde{\omega}_0 \\ U & \xleftarrow{\pi_0} & Y_0 & \xleftarrow{\rho} & \tilde{Y}_0, \end{array}$$

where π_1 is the natural projection, the rightmost square is cartesian and the fibre product is identified with $\tilde{Y}_{0, \text{red}}$ because $U_{\text{red}} \times_S X$ is reduced. As Y_1 is formally smooth over U , we obtain an arrow $\psi : \tilde{Y}_0 \rightarrow Y_1$ rendering commutative the diagram

$$(44) \quad \begin{array}{ccc} Y_1 & \xleftarrow{\omega_1 \lambda} & \tilde{Y}_{0, \text{red}} \\ \downarrow \pi_1 & \dashrightarrow \psi & \downarrow \tilde{\omega}_0 \\ U & \xleftarrow{\pi_0 \rho} & \tilde{Y}_0. \end{array}$$

Let $V \in \mathbf{Sch}/B$, and consider an arbitrary V -point $w = (v, x_0) : V \rightarrow Y_0 = U \times_{s_0, S} X$. Let

$$(45) \quad \begin{array}{ccc} Y_0 & \xleftarrow{\rho} & \tilde{Y}_0 \\ w \uparrow & \square & \uparrow \tilde{w} \\ V & \xleftarrow{c} & \tilde{V} \end{array}$$

be the cartesian diagram deduced from it. Using ψ , we produce a \tilde{V} -point

$$\psi \tilde{w} : \tilde{V} \longrightarrow Y_1.$$

Due to commutativity of diagrams (44) and (45), $\psi \tilde{w} = (vc, \tilde{x}_1)$ for some \tilde{V} -point \tilde{x}_1 of X .

Since $t_0, t_1 : U \rightarrow T$ are Ψ -equivalent, so are $t_0(vc)$ and $t_1(vc)$. As $\tilde{V}_{\text{red}} \rightarrow \tilde{V} \rightarrow \tilde{Y}_0$ factors through $\tilde{\omega}_0 : \tilde{Y}_{0, \text{red}} \rightarrow \tilde{Y}_0$ and $\text{pr}_X \circ \omega_0 = \text{pr}_X \circ \omega_1$, it is not hard to conclude that x_0c and \tilde{x}_1 are infinitesimally close. Hence the \tilde{V} -points $(t_0(vc), x_0c)$ and $(t_1(vc), \tilde{x}_1)$ of $T \times_S X$ are $\Psi \times_S X_{\infty}$ -equivalent. By the hypothesis on σ , the \tilde{V} -points $\sigma(t_0(vc), x_0c)$ and $\sigma(t_1(vc), \tilde{x}_1)$ of Z are Φ -equivalent.

To review, we have started with an arbitrary $(v, z_0) \in U(V) \times Z(V)$ (here $z_0 = \sigma \circ (t_0 \times \text{id}_X)(v, x_0)$) belonging to Y_0 and obtained a Zariski covering $c : \tilde{V} \rightarrow V$ and a \tilde{V} -point (vc, \tilde{x}_1) of $U \times Z$ belonging to Y_1 (here $\tilde{x}_1 = \sigma \circ (t_1 \times \text{id}_X)(vc, \tilde{x}_1)$) and $U \times \Phi$ -equivalent to (vc, z_0c) . This shows that $Y_0 \leq_{U \times \Phi} Y_1$, and as the case $Y_1 \leq_{U \times \Phi} Y_0$ is completely analogous, we conclude that Y_0 and Y_1 are $U \times \Phi$ -equivalent. \square

Corollary 58. *The morphism*

$$\begin{aligned} \mathrm{Hom}_{S_\infty}(\Psi, H_\Phi) &\longrightarrow \mathrm{Hom}_{X_\infty}(\Psi \times_S X_\infty, \Phi), \\ (t : T \rightarrow H) &\longmapsto (\mathrm{ev} \circ (t \times_S \mathrm{id}_X) : T \times_S X \rightarrow Z) \end{aligned}$$

is a bijection. □

12.3. Compatibility of H with base change. We consider a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{\beta} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{\alpha} & S \end{array}$$

in \mathbf{Sch}/B . Denote $Z \times_X X'$ by Z' and $\Phi \times_X X'_\infty$ by Φ' . (Note that $\Phi'_{/S'} = \Phi_{/S} \times_S S'$, which is differentially smooth of rank r .) Our goal is to compare the i.e.r. $H_{\Phi'}$ in $H_{f'}(Z', \Phi')$ with the i.e.r. $H_\Phi \times_S S'_\infty$ in $H_f(Z, \Phi) \times_S S'$. In order to do this, one possible approach is to use the well-known fact that “right adjoints commute with limits”, but due to technical inconveniences we opt to convey a direct verification. In one way or another, this is a straightforward task.

If $\theta : T \rightarrow S'$ is a morphism, we note that

$$\rho : (\mathrm{pr}_T, \beta \circ \mathrm{pr}_{X'}) : T \times_{S'} X' \longrightarrow T \times_S X$$

is an isomorphism (the structure of S -scheme on T is the obvious one); this permits the definition of the canonical bijection

$$R : \mathrm{Hom}_X(T \times_S X, Z) \longrightarrow \mathrm{Hom}_{X'}(T \times_{S'} X', Z'), \quad \varphi \longmapsto (\varphi \circ \rho, \mathrm{pr}_{X'}).$$

This is functorial in $T \in \mathbf{Sch}/S'$. Let Ψ be an infinitesimal equivalence relation in T and assume that $\varphi \in \mathrm{Hom}_X(T \times_S X, Z)$ interweaves $\Psi \times_S X_\infty$ and Φ . Then, for a variable scheme U and two $\Psi \times_{S'} X'_\infty$ -equivalent U -points (t_0, x'_0) and (t_1, x'_1) of $T \times_{S'} X'$, it follows that

$$R(\varphi)(t_0, x'_0) = (\varphi(t_0, \beta(x'_0)), x'_0) \quad \text{and} \quad R(\varphi)(t_1, x'_1) = (\varphi(t_1, \beta(x'_1)), x'_1)$$

are Φ' -equivalent U -points of Z' : therefore, $R(\varphi)$ interweaves $\Psi \times_{S'} X'_\infty$ and Φ' . Conversely, assume that $R(\varphi)$ interweaves $\Psi \times_{S'} X'_\infty$ and Φ' . Let (t_0, x_0) and (t_1, x_1) be any two given $\Psi \times_S X_\infty$ -equivalent U -points of $T \times_S X$ corresponding to U -points $(t_0, [x_0, \theta(t_0)])$ and $(t_1, [x_1, \theta(t_1)])$ of $T \times_{S'} X'$. It is obvious that

$$(t_0, [x_0, \theta(t_0)]) \equiv (t_1, [x_1, \theta(t_1)]) \quad \text{mod } \Psi \times_{S'} X'_\infty,$$

so that

$$(\varphi(t_0, x_0), [x_0, \theta(t_0)]) \equiv (\varphi(t_1, x_1), [x_1, \theta(t_1)]) \quad \text{mod } \Phi',$$

and a fortiori, $\varphi(t_0, x_0) \equiv_{\Phi} \varphi(t_1, x_1)$. Thus we have shown that the dotted arrow in

$$\begin{array}{ccc}
 \mathrm{Hom}_{S_{\infty}}(\Psi, H_{\Phi}) & \xrightarrow{\sim} & \mathrm{Hom}_{X_{\infty}}(\Psi \times_S X_{\infty}, \Phi) \\
 \downarrow & & \downarrow \\
 \mathrm{Hom}_S(T, H) & \longrightarrow & \mathrm{Hom}_X(T \times_S X, Z) \\
 & & \downarrow \sim \\
 \mathrm{Hom}_{S'}(T, H_{f'}(Z', \Phi')) & \longrightarrow & \mathrm{Hom}_{X'}(T \times_{S'} X', Z') \\
 \uparrow & & \uparrow \\
 \mathrm{Hom}_{S'_{\infty}}(\Psi, H_{\Phi'}) & \xrightarrow{\sim} & \mathrm{Hom}_{X'_{\infty}}(\Psi \times_{S'} X'_{\infty}, \Phi')
 \end{array}$$

is bijective. Such a fact, when allied to the Yoneda Lemma applied to \mathbf{IER}/S'_{∞} and to the obvious isomorphism in $\mathbf{Set}^{(\mathbf{IER}/S'_{\infty})^{\mathrm{op}}}$

$$\mathrm{Hom}_{S_{\infty}}(\bullet, H_{\Phi}) \xrightarrow{\cong} \mathrm{Hom}_{S'_{\infty}}(\bullet, H_{\Phi} \times_S S'_{\infty}),$$

ensures the existence of an isomorphism of infinitesimal equivalence relations $H_{\Phi} \times_S S'_{\infty} \cong H_{\Phi'}$. These findings are collected in the following.

Corollary 59. *Let $S' \rightarrow S$ be a morphism of \mathbf{Sch}/B . Denote by X' the S' -scheme $X \times_S S'$, by Z' the X' -scheme $Z \times_X X'$, and by Φ' the infinitesimal equivalence relation $\Phi \times_X X'_{\infty}$ in Z' . Then, there exists an isomorphism of infinitesimal equivalence relations over S'_{∞} between $H_{\Phi} \times_S S'_{\infty}$ and $H_{\Phi'}$.*

An afterthought. The application of the Yoneda Lemma employed above can profit from further remarks, since (H, H_{Φ}) is not really an object of \mathbf{IER}/S_{∞} . For (H, H_{Φ}) to belong to $\mathbf{IER}(B)$, according to the convention of section 7.2, the scheme H should be quasi-compact. With this in mind, we introduce \mathbf{Sch}'/B , the category of locally noetherian separated schemes whose connected components are quasi-compact. Accompanying \mathbf{Sch}'/B is the category $\mathbf{IER}'(B)$, whose definition is obtained by interchanging \mathbf{Sch}/B and \mathbf{Sch}'/B in Definition 22. But this is not yet what we need, and we introduce $c\mathbf{IER}(B)$, the full subcategory of $\mathbf{IER}(B)$ consisting of those equivalence relations which respect disjoint unions. We define $c\mathbf{IER}'(B)$ analogously. It is then easily proved that the evident functor $c\mathbf{IER}(B) \rightarrow c\mathbf{IER}'(B)$ is full and dense (in the sense of [Mac98, p.246]). From that, the dual of [Mac98, Proposition 2, p. 246] gives us the necessary Yoneda Lemma. That (H, H_{Φ}) belongs in fact to $c\mathbf{IER}'(B)$ yields directly by using that Φ , being schematic, belongs to $c\mathbf{IER}(B)$.

13. CONDITIONS FOR THE EQUIVALENCE RELATION IN $H_f(Z, \Phi)$ TO COME FROM A STRATIFICATION

Let B be a noetherian separated base scheme, \mathbf{Sch}/B be the category of noetherian separated B -schemes. All arrows and objects are in \mathbf{Sch}/B . Let

$$Z \xrightarrow{\zeta} X \xrightarrow{f} S$$

be smooth morphisms of smooth schemes. Denote the composition $f \circ \zeta$ by g and the relative dimension of f by r . Let Φ be a B -linear i.e.r. in Z which is totally complementary to ζ (Definition 29), i.e. it comes from a B -linear stratification of ζ . (In particular, Φ is differentially smooth of rank $\dim X/B$.) Our goal here is to derive Corollary 69 below. This corollary is a translation of the following.

Proposition 60 (Parallel transport). *Let $u, u' : U \rightarrow S$ be infinitesimally close. Let $Y \subseteq Z \times_{S,u} U$ be a closed subscheme which is tangent to $\Phi_{/g} \times_{S,u} U$ rel. U and which is sent isomorphically, via $\zeta \times_{S,u} \text{id}_U$, to $X \times_{S,u} U$. Then there exists a unique closed subscheme $Y' \subseteq Z \times_{S,u'} U$ which is*

- (1) $\Phi \times U$ -equivalent to Y ,
- (2) tangent to $\Phi_{/g} \times_{S,u'} U$ rel. U , and
- (3) sent isomorphically, via $\zeta \times_{S,u'} \text{id}_U$, to $X \times_{S,u'} U$.

We begin by showing that property (2) ensues from (1). Firstly, it is sufficient to prove that Y' is tangent to $\Phi \times U$ rel. U , since Y' is a closed subscheme of $Z \times_{S,u'} U$. Secondly, we have the following.

Lemma 61. *Let $Y \subseteq Z \times U$ and $Y' \subseteq Z \times U$ be closed subschemes which are equivalent modulo $\Phi \times U$. Assume that Y is tangent to $\Phi \times U$ rel. U . Then Y' is tangent to $\Phi \times U$ rel. U .*

Proof. We write $Z_U = Z \times U$ and $\Phi_U = \Phi \times U$. Let $t'_0, t'_1 \in \text{Hom}_U(T, Y)$ be infinitesimally close. Since Y and Y' are equivalent modulo Φ_U , there exists $c : \tilde{T} \rightarrow T$ a covering and \tilde{T} -points $\tilde{t}_0, \tilde{t}_1 \in \text{Hom}_U(\tilde{T}, Y')$ such that $t'_i \circ c \equiv \tilde{t}_i \pmod{\Phi_U}$. Since Φ_U is infinitesimal, \tilde{t}_0 and \tilde{t}_1 are infinitesimally close points in Y . Due to the tangency assumption made on Y , $\tilde{t}_0 \equiv \tilde{t}_1 \pmod{\Phi_U}$. Hence, $t'_0 \circ c \equiv t'_1 \circ c \pmod{\Phi_U}$. Using the fact that Φ_U is a sheaf, it follows that $t'_0 \equiv t'_1 \pmod{\Phi_U}$. \square

Continuing with the peeling off of minor assertions made in Proposition 60, we now show that “uniqueness” also comes without much effort. Since Φ is totally complementary to ζ (Definition 29), $\Phi_{/g}$ is also totally complementary to ζ with respect to S (see Remark 30), so $\Phi_{/g}$ is differentially smooth of rank $\dim X/S = r$ over S . Now let Y' be as in Proposition 60. From conditions (2) and (3) and Lemma 43, it follows that Y' is *invariant* under $\Phi_{/g} \times_{S,u'} U$. Consequently, Y' is also invariant under $\Phi_{/g} \times U$, as a simple manipulation demonstrates. Such an Y' must be unique:

Lemma 62. *Let $j^* : Y^* \rightarrow Z \times U$ and $j' : Y' \rightarrow Z \times U$ be closed subschemes which are invariant under $(\Phi \times U)_{/g \times \text{id}_U}$, and $\Phi \times U$ -equivalent. Let $\bar{u} : U \rightarrow S \times U$ be an U -morphism and assume that $(g \times_U) \circ j^*$ and $(g \times \text{id}_U) \circ j'$ factor through $\bar{u} : U \rightarrow S \times U$. (That is, they are both on the same “fibre” above $S \times U$.) Then $Y^* = Y'$.*

Proof. To ease notation, we write $(\bullet)_U = \bullet \times U$. We want to prove that the two subfunctors Y^* and Y' are identical. Let $t' : T \rightarrow Y' \in \text{Hom}_U(T, Y')$. Since Y^* and Y' are Φ_U -equivalent, there exists a covering $c : \tilde{T} \rightarrow T$ together with $\tilde{t}^* \in \text{Hom}_U(\tilde{T}, Y^*)$ such

that $j't'c \equiv j^*\tilde{t}^* \pmod{\Phi_U}$. Using the commutative diagrams (unnamed arrows are the canonical ones)

$$\begin{array}{ccccc}
 \tilde{T} & \xrightarrow{c} & T & \xrightarrow{t'} & Y' & \xrightarrow{j'} & Z_U & \xrightarrow{g_U} & S_U \\
 & \searrow & & \searrow & \downarrow & & \nearrow & & \\
 & & & & U & & & & \\
 & & & & & & \bar{u} & &
 \end{array}
 \qquad
 \begin{array}{ccccc}
 \tilde{T} & \xrightarrow{\tilde{t}^*} & Y^* & \xrightarrow{j^*} & Z_U \\
 & \searrow & \downarrow & & \downarrow g_U \\
 & & U & \xrightarrow{\bar{u}} & S_U
 \end{array}$$

it follows that $j't'c \equiv j^*\tilde{t}^* \pmod{(\Phi_U)_{/g_U}}$. Since Y^* is invariant under $(\Phi_U)_{/g_U}$, $j't'c = j^*\tilde{t}_1^*$ for some $\tilde{t}_1^* \in \text{Hom}_U(\tilde{T}, Y^*)$. By descent, $\tilde{t}_1^* = t_1^*c$ with $t_1^* \in \text{Hom}_U(T, Y^*)$. We have proved that $j'(\text{Hom}_U(T, Y')) \subseteq j^*(\text{Hom}_U(T, Y^*))$. Applying the same argument with the role of Y^* and Y' interchanged, we arrive at $Y^* = Y'$. \square

The conclusion of Proposition 60 can now be achieved by something more concrete.

Proposition 63. *Assume that $B = U = \text{Spec}(C)$, $S = \text{Spec}(\Lambda)$, $X = \text{Spec}(R)$, and $Z = \text{Spec}(\mathcal{O})$. Assume the existence of etale coordinates $\mathbf{x} = (x_1, \dots, x_\ell)$, $\mathbf{y} = (y_1, \dots, y_r)$, and $\mathbf{z} = (z_1, \dots, z_n)$ for Λ/C , R/Λ , and \mathcal{O}/R respectively. Suppose that the ideal \mathfrak{A} of Φ in $\widehat{\mathcal{P}}_{\mathcal{O}/C}$ is generated by n elements F_1, \dots, F_n .*

Let $u, u' : \Lambda \rightarrow C$ be infinitesimally close C -morphisms and write $\mathfrak{u} = \text{Ker}(u)$ and $\mathfrak{u}' = \text{Ker}(u')$. Let $\sigma : \mathcal{O} \rightarrow R/\mathfrak{u}R$ be a C -morphism enjoying the following properties:

- (a) σ annihilates \mathfrak{u} ;
- (b) σ induces a section to $R/\mathfrak{u}R \rightarrow \mathcal{O}/\mathfrak{u}\mathcal{O}$;
- (c) the closed subscheme $\{\text{Ker}(\sigma) = 0\} \subseteq \text{Spec}(\mathcal{O})$ is tangent to the i.e.r. Φ/Λ rel. C .

Then there exists a unique morphism

$$T(\sigma, u') : \mathcal{O} \longrightarrow R/\mathfrak{u}'R$$

annihilating \mathfrak{u}' and inducing a section to $R/\mathfrak{u}'R \rightarrow \mathcal{O}/\mathfrak{u}'\mathcal{O}$ such that:

- (i) $\{\text{Ker}(T(\sigma, u')) = 0\}$ is Φ -equivalent to $\{\text{Ker}(\sigma) = 0\}$, and
- (ii) $\{\text{Ker}(T(\sigma, u')) = 0\}$ is tangent to Φ/Λ rel. C .

$$\begin{array}{ccc}
 \begin{array}{c} \mathcal{O} \\ (z_1, \dots, z_n) \mid \\ R \\ \mid (y_1, \dots, y_r) \\ \Lambda \\ (x_1, \dots, x_\ell) \mid \\ C \end{array} & \begin{array}{c} \Lambda \\ \uparrow \\ C \\ \downarrow \\ C \end{array} & \begin{array}{ccc} \mathcal{O}/\mathfrak{u}\mathcal{O} & \longleftarrow & \mathcal{O} \\ & \searrow & \downarrow \sigma \\ & & R/\mathfrak{u}R \end{array}
 \end{array}$$

As Lemma 62 and the discussion preceding it clarify, uniqueness of $T(\sigma, u')$ ensues from (i) and (ii). Also, as Lemma 61 and the discussion preceding it elucidate, condition (ii) is guaranteed by (i). Finally, the hypothesis made on the number of generators of \mathfrak{A} is not seriously restrictive as $\widehat{P}_\Phi \rightarrow \widehat{P}_{Z/B}$ is a quasi-regular immersion of codimension n , due to total complementarity of Φ and [EGA IV₄, 19.1.5, p.186]. We are then left with the construction of $T(\sigma, u')$, which is the subject of the following lines.

Write $\mathbf{c} = u(\mathbf{x})$, resp. $\mathbf{c}' = u'(\mathbf{x})$, so that $\mathbf{x} - \mathbf{c} \in \mathfrak{u}^{\times \ell}$, resp. $\mathbf{x} - \mathbf{c}' \in \mathfrak{u}'^{\times \ell}$. Fix an arbitrary morphism of noetherian C -algebras $\varphi : \mathcal{O} \rightarrow A$ which factors through σ and put $\varphi = \overline{\varphi} \circ \sigma$. Note that $\varphi(\mathbf{x}) = \mathbf{c} \cdot 1_A$. Define

$$(46) \quad C[\mathbf{x}, \mathbf{y}, \mathbf{z}] \longrightarrow A, \quad \begin{array}{l} \mathbf{x} \mapsto \mathbf{c}' \\ \mathbf{y} \mapsto \varphi(\mathbf{y}) \\ \mathbf{z} \mapsto \varphi(\mathbf{z}) + \alpha \end{array}$$

where $\alpha \in A^{\times n}$ is nilpotent and will be adjusted later. Since $C[\mathbf{x}, \mathbf{y}, \mathbf{z}] \rightarrow \mathcal{O}$ is etale, we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\rho_A \circ \varphi} & A_{\text{red}} \\ \uparrow & \searrow \varphi' & \uparrow \rho_A \\ C[\mathbf{x}, \mathbf{y}, \mathbf{z}] & \xrightarrow{\text{above defined}} & A \end{array}$$

where $\rho_A : A \rightarrow A_{\text{red}}$ is the obvious map.

Lemma 64. *The above constructed morphism φ' enjoys the following properties.*

- (1) φ and φ' are infinitesimally close.
- (2) $\varphi'(\mathbf{x}) = \mathbf{c}' \cdot 1_A$
- (3) For each $\lambda \in \Lambda$, $\varphi'(\lambda) = u'(\lambda) \cdot 1_A$.

Proof. Only (3) requires justification. The desired conclusion will be attained once shown that $\rho_A \circ \varphi'(\lambda) = u'(\lambda) \cdot 1_{A_{\text{red}}}$ for all $\lambda \in \Lambda$, i.e. $\varphi' : \Lambda \rightarrow A$ and $u' : \Lambda \rightarrow A$ are infinitesimally close. For φ' and u' send \mathbf{x} to \mathbf{c}' , so that the aforementioned equality leads to a commutative diagram

$$(47) \quad \begin{array}{ccc} \Lambda & \xrightarrow{\rho_A \circ \varphi'} & A_{\text{red}} \\ \uparrow & \searrow & \uparrow \rho_A \\ C[\mathbf{x}] & \xrightarrow{\mathbf{x} \mapsto \mathbf{c}'} & A \end{array}$$

where the dotted arrow can be either φ' or u' ; as $C[\mathbf{x}] \rightarrow \Lambda$ is etale, we arrive at the sought conclusion.

Now

$$\begin{aligned} \rho_A\{\varphi'(\lambda)\} &= \rho_A\{\varphi(\lambda)\} && \text{(definition of } \varphi') \\ &= \rho_A\{\overline{\varphi} \circ \sigma(\lambda)\} && \text{(definition of } \varphi) \\ &= \rho_A\{\overline{\varphi}(u(\lambda))\} && (\lambda - u(\lambda) \in \mathfrak{u}\mathcal{O} \subseteq \text{Ker}(\sigma)) \\ &= \rho_A\{\overline{\varphi}(u'(\lambda))\} && (u'(\lambda) - u(\lambda) \text{ is nilpotent}) \\ &= \rho_A\{u'(\lambda) \cdot 1_A\} && (\overline{\varphi} \text{ is } C\text{-linear}) \\ &= u'(\lambda) \cdot 1_{A_{\text{red}}}. \end{aligned}$$

□

Corollary 65. *The ideal $\mathfrak{u}'\mathcal{O}$ is annihilated by φ' .*

□

Let us now assume

$$\varphi : \mathcal{O} \longrightarrow A = \sigma : \mathcal{O} \longrightarrow R/\mathfrak{u}R.$$

Accordingly,

$$\sigma' : \mathcal{O} \longrightarrow R/\mathfrak{u}R = \varphi' : \mathcal{O} \longrightarrow A.$$

We remind the reader that the definition of σ' depends upon a parameter $\alpha \in \text{Nil}(R/\mathfrak{u}R)^{\times n}$ which we will have the liberty to adjust further down.

Lemma 66. *The restriction of σ' to R induces an isomorphism of C -algebras $R/\mathfrak{u}'R \rightarrow R/\mathfrak{u}R$.*

Proof. Contrary to $\sigma|_R$, $\sigma'|_R : R \rightarrow R/\mathfrak{u}R$ does *not* need to be R -linear. But it is certainly C -linear and satisfies

$$\begin{aligned} \rho_{R/\mathfrak{u}R} \circ \sigma'(r) &= \rho_{R/\mathfrak{u}R} \circ \sigma(r) && \text{(by construction of } \sigma') \\ &= r \cdot 1_{(R/\mathfrak{u}R)_{\text{red}}} && \text{(by definition of } \sigma) \end{aligned}$$

for all $r \in R$. In particular, $\sigma' : R \rightarrow R/\mathfrak{u}R$ is surjective modulo $\text{Nil}(R/\mathfrak{u}R)$. As $C \rightarrow R/R\mathfrak{u}$ is smooth, it follows that $\text{Nil}(C) \cdot (R/\mathfrak{u}R) = \text{Nil}(R/\mathfrak{u}R)$, from which we deduce that

$$R/\mathfrak{u}'R \xrightarrow{\sigma'} R/\mathfrak{u}R \longrightarrow (R/\mathfrak{u}R) \otimes_C C/\text{Nil}(C)$$

is surjective. We now show that $\sigma' : R/\mathfrak{u}'R \rightarrow R/\mathfrak{u}R$ is an isomorphism when tensored over C with $C/\text{Nil}(C)$. This will show that σ' is an isomorphism by a well-known result [S68, Lemma 3.3, p.216]. (We were unable to find a more convenient reference.) Let $r \in R$. If $\sigma'(r)$ belongs to

$$\text{Nil}(C) \cdot (R/\mathfrak{u}R) = \text{Nil}(R/\mathfrak{u}R),$$

then $\rho_{R/\mathfrak{u}R}(\sigma'(r)) = 0$, which gives $r \in \sqrt{\mathfrak{u}R}$. As u and u' are infinitesimally close, we have $\sqrt{u} = \sqrt{u'}$, which entails $\sqrt{u}R = \sqrt{u'}R$ (due to smoothness of $\Lambda \rightarrow R$). Thus, $r \in \sqrt{u'R}$, so that $r + u'R \in \text{Nil}(R/\mathfrak{u}'R)$, which equals $\text{Nil}(C) \cdot (R/\mathfrak{u}'R)$. We are done. \square

Write $\mathbf{F} = (F_1, \dots, F_n)$ and interpret them as elements in $\mathcal{O}[\mathbf{dx}, \mathbf{dy}, \mathbf{dz}] = \widehat{\mathcal{P}}_{\mathcal{O}/C}$ (see eq. (26) and the references nearby). Let \mathbf{F}^σ stand for the list obtained by mapping each F_i to its image in $R/\mathfrak{u}R[\mathbf{dx}, \mathbf{dy}, \mathbf{dz}]$. Clearly:

Lemma 67. *The $R/\mathfrak{u}R$ -points of $\text{Spec}(\mathcal{O})$ defined by σ and σ' are Φ -close if and only if*

$$(48) \quad \mathbf{F}^\sigma(\mathbf{c}' - \mathbf{c}, \mathbf{0}, \alpha) = \mathbf{0},$$

where $\alpha \in \text{Nil}(R/\mathfrak{u}R)^{\times n}$ is the parameter introduced in eq. (46). \square

Lemma 68. *There exists a unique $\alpha \in \text{Nil}(R/R\mathfrak{u})^{\times n}$ satisfying eq. (48).*

Proof. This is a consequence of Hensel's Lemma. Write $\mathbf{F}(\mathbf{dx}, \mathbf{0}, \mathbf{dz}) = \sum_{i,j} \mathbf{F}_{ij} \cdot \mathbf{dx}^i \mathbf{dz}^j$. Fix a positive integer μ such that $\text{Nil}(R/R\mathfrak{u})^{\mu+1} = 0$, and define

$$\mathbf{G} := \sum_{|i|,|j| \leq \mu} \sigma(\mathbf{F}_{ij})(\mathbf{c}' - \mathbf{c})^i \cdot \mathbf{W}^j \in R/R\mathfrak{u}[W_1, \dots, W_n]^{\times n}.$$

Clearly, for each $\mathbf{w} \in \text{Nil}(R/R\mathbf{u})^{\times n}$ we have $\mathbf{F}^\sigma(\mathbf{c}' - \mathbf{c}, \mathbf{0}, \mathbf{w}) = \mathbf{G}(\mathbf{w})$. As the i.e.r. Φ is complementary to $\text{Spec}(\mathcal{O}) \rightarrow \text{Spec}(R)$, Φ/R is the trivial i.e.r., so

$$(\mathbf{d}\mathbf{x}, \mathbf{d}\mathbf{y}, \mathbf{d}\mathbf{z}) = (\mathbf{d}\mathbf{x}, \mathbf{d}\mathbf{y}, \mathbf{F}).$$

Regarding $\mathbf{c}' - \mathbf{c}$ and \mathbf{G} as line vectors, we then have

$$(49) \quad W_i \equiv (\mathbf{c}' - \mathbf{c}) \cdot \mathbf{L}_i + \mathbf{G} \cdot \mathbf{M}_i \pmod{(W_1, \dots, W_n)^{\mu+1}}, \quad (1 \leq i \leq n),$$

where \mathbf{L}_i , respectively \mathbf{M}_i , is an $\ell \times 1$, respectively $n \times 1$, matrix with coefficients in $R/\mathbf{u}R[\mathbf{W}]$. Due to the fact that $\mathbf{F}_{00} = \mathbf{0}$, we have

$$\mathbf{G}(\mathbf{0}) = \sum_{0 < |i| \leq \mu} \sigma(\mathbf{F}_{ij}) \cdot (\mathbf{c}' - \mathbf{c})^i \in \text{Nil}(R/R\mathbf{u})^{\times n}.$$

Taking partial derivatives in eq. (49) and making $\mathbf{W} = \mathbf{0}$, we obtain:

$$\frac{\partial \mathbf{G}}{\partial W_j}(\mathbf{0}) \cdot \mathbf{M}_i(\mathbf{0}) = \underbrace{\delta_{ij} - (\mathbf{c}' - \mathbf{c}) \cdot \frac{\partial \mathbf{L}_i}{\partial W_j}(\mathbf{0})}_{\text{nilpotent in } R/\mathbf{u}R} - \underbrace{\mathbf{G}(\mathbf{0}) \cdot \frac{\partial \mathbf{M}_i}{\partial W_j}(\mathbf{0})}_{\text{nilpotent in } R/\mathbf{u}R}.$$

Consequently,

$${}^t(\text{Jac}(\mathbf{G})(\mathbf{0})) \cdot \mathbf{M}_i(\mathbf{0}) = \begin{bmatrix} \delta_{i1} \\ \vdots \\ \delta_{in} \end{bmatrix} + \text{Nilpotents},$$

which forces $\text{Jac}(\mathbf{G})(\mathbf{0})$ to be in $\text{GL}_n(R/\mathbf{u}R)$. By Hensel's Lemma [BouAC, III§4,no.6,Cor.2, p.271], there exists a unique $\boldsymbol{\alpha} \in \text{Nil}(R/\mathbf{u}R)^{\times n}$ such that $\mathbf{G}(\boldsymbol{\alpha}) = \mathbf{0}$, which gives $\mathbf{F}^\sigma(\mathbf{c}' - \mathbf{c}, \mathbf{0}, \boldsymbol{\alpha}) = \mathbf{0}$, and we are done. \square

We now let $\sigma' : \mathcal{O} \rightarrow R/\mathbf{u}R$ be the morphism defined by means of the element $\boldsymbol{\alpha} \in \text{Nil}(R/\mathbf{u}R)^{\times n}$ obtained in Lemma 68. Since the $R/\mathbf{u}R$ -points of $\text{Spec}(\mathcal{O})$ defined by σ and σ' are Φ -close, it follows that the closed subschemes $\{\text{Ker}(\sigma) = 0\}$ and $\{\text{Ker}(\sigma') = 0\}$ are Φ -equivalent. Let

$$T(\sigma, u') := (\text{inverse of isomorphism in Lemma 66}) \circ \sigma'.$$

Then $\text{Ker}(T(\sigma, u')) = \text{Ker}(\sigma')$, so that $\{\text{Ker}(T(\sigma, u')) = 0\}$ and $\{\text{Ker}(\sigma) = 0\}$ are Φ -equivalent. Furthermore, since σ' annihilates $\mathbf{u}'\mathcal{O}$, the very definition of the isomorphism in Lemma 66 shows that $T(\sigma, u')$ is a section to $R/\mathbf{u}'R \rightarrow \mathcal{O}/\mathbf{u}'\mathcal{O}$.

We have therefore proved the existence of the morphism $T(\sigma, u')$ complying with the two conditions mentioned in Proposition 63. (Recall that the tangency property follows from the other properties, as explained in p. 44.) With this, we have concluded the proof of Proposition 63, and arrive at our goal.

Corollary 69. *Assume in addition that $f : X \rightarrow S$, resp. $g : Z \rightarrow S$, is proper and geometrically connected, resp. quasi-projective. Let $\pi : H_f(Z, \Phi) \rightarrow S$ be the S -scheme of Definition 49 and endow it with the i.e.r. found in Definition 53. Then H_Φ is totally complementary (Definition 29 and following lines) to $\pi : H_f(Z, \Phi) \rightarrow S$. \square*

14. DESCRIPTION OF $H_f(Z, \Phi)$ IN THE PARTICULAR CASE WHERE $S = \text{Spec } k$ AND Z STEMS FROM THE FUNDAMENTAL GROUP SCHEME.

Let k be an algebraically closed field. The category **Sch** is the category of all noetherian separated k -schemes and in the following, all schemes and morphisms thereof are to be taken in **Sch**. Let $X \in \mathbf{Sch}$ be of finite type.

14.1. Simple stratified schemes. Let

$$\{\alpha_\mu : P_X^\mu \times_X E \longrightarrow E : \mu \in \mathbb{N}\}$$

define a k -linear stratification of the X -scheme $\theta : E \rightarrow X$ (see Section 5). Write α for the morphism of formal schemes $\widehat{P}_X \times_X E \rightarrow E$ induced from the system (α_μ) .

Recall that a closed subscheme $Y \subseteq E$ is *invariant or saturated* if α sends $Y \times_X \widehat{P}_X$ into Y .

Definition 70. Under the above notations, we say that E is a simple stratified X -scheme if the only non-empty invariant closed subscheme is E itself.

If θ is affine and X is smooth, then invariance of Y means invariance of the ideal $\mathfrak{J}_Y \subseteq \mathcal{O}_E$ by $\mathcal{D}_{X/k}$. We also note that if α is simple, then E must be connected, since $|\alpha|$ is just the identity on the underlying topological spaces.

Proposition 71 (Universal simplicity). *Let X be smooth and irreducible and θ be affine. Suppose that the stratification α is simple. Fix a scheme of finite type T and let $j : Y \rightarrow E \times T$ be a non-empty closed subscheme. Assume that*

- (i) $\alpha_\mu \times \text{id}_T : (E \times_X P_X^\mu) \times T \rightarrow E \times T$ preserves Y for each μ , and
- (ii) $Y \rightarrow T$ has a section $\sigma : T \rightarrow Y$.

Then $Y = E \times T$.

Proof. If $U \rightarrow T$ is a morphism of schemes, then $Y \times_T U \subseteq E \times U$ is also invariant and $Y \times_T U \rightarrow U$ has a section. By Lemma 72 below, it is therefore sufficient to treat the case where $T = \text{Spec } A$ is the spectrum of a local Artin k -algebra. We proceed by induction on $\dim_k A$ to show that j is an isomorphism. If $\dim_k A = 1$, there is nothing to be done. Let $\varepsilon \in \mathfrak{r}(A) \setminus \{0\}$ be such that $\varepsilon \cdot \mathfrak{r}(A) = 0$ and denote by A_1 the quotient $A/\varepsilon A$. Now $Y \otimes_A A_1 \subseteq E \otimes_k A_1$ is invariant and hence must equal $E \otimes_k A_1$. Let $\mathfrak{J} \subseteq \mathcal{O}_E \otimes_k A$ be the ideal sheaf of Y ; since $Y \otimes_A A_1 = E \otimes_k A_1$, it follows that $\mathfrak{J} \subseteq \mathcal{O}_E \otimes_k \varepsilon A$. Define a sheaf of ideals of \mathcal{O}_E by

$$\mathfrak{J}' := \{\psi \in \mathcal{O}_E : (\psi \otimes 1) \cdot (1 \otimes \varepsilon) \in \mathfrak{J}\}.$$

This is clearly coherent (as a kernel between a morphism of coherent \mathcal{O}_E -modules). Using the fact that \mathcal{P}_X^μ is locally free, it is easy to verify that the closed subscheme of E cut out by \mathfrak{J}' is invariant, and therefore, either $\mathfrak{J}' = \mathcal{O}_E$ or (0) . Since $\mathfrak{J} \neq (0)$ and $\varepsilon A \simeq k$, it is not hard to see that $\mathfrak{J}' \neq (0)$. Thus, 1 is a global section of $\mathfrak{J}' \Rightarrow 1 \otimes \varepsilon$ is a global section of \mathfrak{J} . This is a contradiction with the fact that $Y \rightarrow T$ is dominant. \square

Lemma 72. *Let $f : M \rightarrow T$ be a morphism of algebraic k -schemes and let $j : N \hookrightarrow M$ be a closed sub-scheme. Then j is an isomorphism if for each spectrum D of a local Artin*

k -algebra of residue field k and each morphism $\tau : D \rightarrow T$, the closed embedding $j \times_T \text{id}_D$ is an isomorphism. \square

An application of Proposition 71 which will be useful later is as follows.

Lemma 73. *Let X be smooth and connected. Denote by \mathcal{U}_X the universal torsor and by $\mathcal{U}_X[\bullet]$ the functor*

$$\left\{ \begin{array}{l} \text{Actions } \rho \text{ of } \Pi(X) \text{ on the left of an algebraic} \\ \text{scheme } F \text{ such that } \mathcal{U}_X \times^\rho F \text{ is a scheme} \end{array} \right\} \longrightarrow \{\text{stratified } X\text{-schemes}\}.$$

Then $\mathcal{U}_X[\bullet]$ is fully faithful.

Proof. This is easy, but tedious. Denote by \mathcal{S}_X the domain of $\mathcal{U}_X[\bullet]$. For each $F \in \mathcal{S}_X$, let χ_F stand for the natural projection from $\mathcal{U}_X \times F$ to the quotient $\mathcal{U}_X[F]$. We know [J87, Part1, 5.14,p.88] that

$$(50) \quad (\text{pr}, \chi_F) : \mathcal{U}_X \times F \longrightarrow \mathcal{U}_X \times_X \mathcal{U}_X[F]$$

is an isomorphism of \mathcal{U}_X -schemes. In particular, $\mathcal{U}_X[\bullet]$ is faithful. Endowing $\mathcal{U}_X \times F$ with its standard action of $\Pi(X)$ and $\mathcal{U}_X \times_X \mathcal{U}_X[F]$ with the action on the first factor only, it follows that (50) is $\Pi(X)$ -equivariant. Furthermore, endowing $\mathcal{U}_X \times_X \mathcal{U}_X[F]$ with the *product stratification*, it follows that (pr, χ_F) is an isomorphism of stratified X -schemes.

We now deal with fullness. Let $F, F' \in \mathcal{S}_X$ (recall that they are separated) and let $\varphi : \mathcal{U}_X[F] \rightarrow \mathcal{U}_X[F']$ be a morphism of *stratified* X -schemes. Using eq. (50), we obtain a morphism of \mathcal{U}_X -schemes

$$\bar{\varphi} = (\text{pr}, \theta) : \mathcal{U}_X \times F \longrightarrow \mathcal{U}_X \times F'.$$

It is not hard to see that $\bar{\varphi}$ respects the usual stratifications.

Recall that \mathcal{U}_X is a projective limit of stratified torsors

$$\mathcal{U}_X = \varprojlim_{\lambda} \mathcal{U}_{\lambda},$$

where each $\mathcal{U}_{\lambda} \rightarrow X$ is a torsor under an *algebraic* group scheme. In fact, \mathcal{U}_{λ} corresponds to some affine *quotient* $\Pi(X) \rightarrow \Pi_{\lambda}$, so that any closed non-empty subscheme of \mathcal{U}_{λ} which is invariant under the stratification corresponds to a closed subgroup scheme of Π_{λ} which is invariant under $\Pi(X)$; consequently the stratification of \mathcal{U}_{λ} is simple.

From [EGA IV₃, 8.8.2,p.28] there exists an index λ and a morphism of \mathcal{U}_{λ} -schemes

$$\bar{\varphi}_{\lambda} = (\text{pr}, \theta_{\lambda}) : \mathcal{U}_{\lambda} \times F \longrightarrow \mathcal{U}_{\lambda} \times F'$$

such that

$$\bar{\varphi}_{\lambda} \times_{\mathcal{U}_{\lambda}} \text{id}_{\mathcal{U}_X} = \bar{\varphi}.$$

In particular, we have a commutative diagram

$$(51) \quad \begin{array}{ccc} \mathcal{U}_X \times F & \xrightarrow{\bar{\varphi}} & \mathcal{U}_X \times F' \\ \downarrow & & \downarrow \\ \mathcal{U}_{\lambda} \times F & \xrightarrow{\bar{\varphi}_{\lambda}} & \mathcal{U}_{\lambda} \times F' \end{array}$$

where the vertical arrows respect stratifications and are faithfully flat and affine morphisms. Using this diagram, one is able to verify that $\overline{\varphi}_\lambda$ is a morphism of stratified X -schemes.

Let u_0 stand for a k -point of \mathcal{U}_λ above x_0 and let Y stand for the closed subscheme of $\mathcal{U}_\lambda \times F$ where θ_λ coincides with $(u, y) \mapsto \theta_\lambda(u_0, y)$ (recall that all schemes in sight are separated). Since $\overline{\varphi}_\lambda$ is a morphism of stratified schemes, it follows that Y fulfills conditions (i) and (ii) of Proposition 71, so that $Y = \mathcal{U}_\lambda \times F$. Let $a : F \rightarrow F'$ be the morphism $y \mapsto \theta_\lambda(u_0, y)$; we just proved that $\overline{\varphi}_\lambda = \text{id}_{\mathcal{U}_\lambda} \times a$. It is now easily verified that a induces φ . □

14.2. Determination of $H(Z, \Phi)$. Let us take X smooth and projective. We give ourselves a projective scheme F , an algebraic group G , a principal G -bundle $\theta : E \rightarrow X$, and an action $G \rightarrow \mathbf{Aut}_k(F)$. Moreover, we assume that E is a stratified principal G -bundle and let Γ stand for the accompanying i.e.r. Note that, in this case, the equivalence classes of Γ are preserved by the action of G . Lastly, we take for granted that the source of the associated fibration [J87, §5, Part I]

$$\zeta : E \times^G F \longrightarrow X$$

is represented by a quasi-projective X -scheme Z . The induced i.e.r. in Z , $(\Gamma \times F)/G$, will be denoted by Φ .

Proposition 74. *Assume that the stratification on E is simple. Then $H(Z, \Phi)$ (see Definition 49) is isomorphic to F^G [J87, Part I, 2.6, 27ff].*

Before proceeding to the proof, we describe how such a Z should come about. Let G stand for a *quotient* of the fundamental stratified group scheme $\Pi(X)$ (see §6.1) of X at some k -rational point of X . We define E as being the stratified torsor associated to G , i.e. $E = \mathcal{U}_X \times^{\Pi(X)} G$, see again §6.1. (This also equals the quotient of \mathcal{U}_X by the kernel of $\Pi(X) \rightarrow G$.) It is not hard to note that this stratification is simple as an invariant closed subscheme gives rise to an invariant ideal of \mathcal{O}_E ; see also the justification of simplicity offered for “ \mathcal{U}_λ ” on p.50. We then chose a representation $\rho : G \rightarrow \mathbf{PGL}_m$ leaving invariant a subscheme F of \mathbb{P}^{m-1} and projective descent [BLR90, Thm. 7, §6.1, p.138] assures that $Z = E \times^G F$ is a scheme.

Proof of Proposition 74. We start by defining a natural transformation

$$\omega : F^G \longrightarrow H(Z, \Phi).$$

For that, in all that follows, we let G act on the right of $E \times F$ in the standard fashion “ $(p, y) \cdot g = (pg, g^{-1}y)$ ” and trivially on parameter schemes T .

Given a parameter scheme $T \in \mathbf{Sch}$ and $a : T \rightarrow F^G$, we note that $\text{id}_E \times a : E \times T \rightarrow E \times F$ is G -equivariant and interweaves $\Gamma \times T$ and $\Gamma \times F$. Consequently, using that quotients commute with products [DG, III, §1.1.8, p.287], we can define a morphism of X -schemes

$$\omega_T(a) : X \times T \longrightarrow Z$$

interweaving $\Gamma/G \times T$, which is just $X_\infty \times T$ as Γ is totally complementary to $\theta : E \rightarrow X$, and $\Phi = (\Gamma \times F)/G$. Clearly ω_T respects compositions and the natural transformation ω is constructed.

If $\chi : E \times F \rightarrow Z$ denotes the quotient morphism, the arrow

$$(52) \quad (\text{pr}_E, \chi) : E \times F \longrightarrow E \times_X Z$$

is a G -equivariant isomorphism [J87, Part I, 5.14, p.88] of E -schemes, which results on the fact that $a : T \rightarrow F^G$ can be reobtained from $\omega(a)$. Hence, $\omega_T : F^G(T) \rightarrow H(Z, \Phi)(T)$ is always injective.

We now aim at showing that ω_T is surjective. Let $T \in \mathbf{Sch}$ and

$$\sigma : X \times T \longrightarrow Z$$

be a morphism of X -schemes (a family of sections to ζ). By means of σ and the isomorphism in (52), we derive a G -equivariant morphism

$$h_\sigma : E \times T \longrightarrow F$$

rendering

$$(53) \quad \begin{array}{ccc} E \times T & \xrightarrow{(\text{pr}_E, h_\sigma)} & E \times F \\ & \searrow \sigma \circ (\theta \times \text{id}_T) & \downarrow \chi \\ & & Z \end{array}$$

commutative. At this point we wish to show that h_σ is “constant on certain equivalence classes of $\Gamma \times T$.”

Lemma 75. *Fix $U \in \mathbf{Sch}$ and let $(p_0, t), (p_1, t) \in E(U) \times T(U)$ be $\Gamma \times T$ -equivalent. Assume that the U -points*

$$\chi(p_0, h_\sigma(p_0, t)), \quad \text{and} \quad \chi(p_1, h_\sigma(p_1, t))$$

of Z are Φ -equivalent. Then $h_\sigma(p_0, t) = h_\sigma(p_1, t)$.

Proof. Let us write $h = h_\sigma$ to simplify. The fact that $\chi(p_0, h(p_0, t))$ and $\chi(p_1, h(p_1, t))$ are equivalent modulo Φ means that we can find an fppf covering $c : U' \rightarrow U$ and an element $g' \in G(U')$ such that

$$(p_0c, h(p_0, t)c) \cdot g' \equiv (p_1c, h(p_1, t)c) \pmod{\Gamma \times F}.$$

Hence, $(p_0c) \cdot g' \equiv_\Gamma p_1c$; since $p_0 \equiv_\Gamma p_1$, we obtain that

$$\begin{aligned} (p_1c) \cdot g' &\equiv_\Gamma (p_0c) \cdot g' \\ &\equiv_\Gamma p_1c. \end{aligned}$$

Since Γ is complementary to $\theta : E \rightarrow X$ (Definition 28) and $\theta(p_1c) = \theta(p_1c \cdot g')$, we arrive at $(p_1c) \cdot g' = p_1c$, which gives $g' = \text{id}$. Therefore, $h(p_0, t)c = h(p_1, t)c$ and hence $h(p_0, t) = h(p_1, t)$ by descent. \square

We now assume that T is algebraic and that σ interweaves $X_\infty \times T$ and Φ , so that σ corresponds to a T -point of $H(Z, \Phi)$. Let $p_0 : \text{Spec}(k) \rightarrow E$ be a point of E ; by means of p_0 , define a closed subscheme Y of $E \times T$ having as U -points the set

$$Y(U) = \{(p, t) \in E(U) \times T(U) : h_\sigma(p, t) = h_\sigma(p_0, t)\}.$$

This Y enjoys the following properties:

- (a) Y is invariant under $\Gamma \times T$ due to the interweaving property of σ , diagram (53), and Lemma 75;
- (b) the morphism $(p_0, \text{id}_T) : T \rightarrow Y$ is a section to $\text{pr}_T : Y \rightarrow T$.

As the stratification on E is simple, Proposition 71 lets us conclude that $Y = E \times T$. Hence, for each $U \in \mathbf{Sch}$, $p \in E(U)$ and $t \in T(U)$, we have $h_\sigma(p, t) = h_\sigma(p_0, t)$, which means that, writing

$$a_\sigma := h_\sigma|_{\{p_0\} \times T},$$

we have

$$h_\sigma = a_\sigma \circ \text{pr}_T.$$

From the fact that h_σ is G -equivariant (recall that T has the trivial G -action) we conclude that a_σ takes its values in F^G .

We hence conclude that $\sigma = \omega_T(a_\sigma)$, which shows that ω_T is surjective for $T \in \mathbf{Sch}$ algebraic. A standard limit argument [EGA IV₃, 8.8.2,p.28] then shows that ω_T is unrestrictedly surjective, and the proof of Proposition 74 is achieved. \square

15. PROOF OF THE MAIN RESULT

We now prove the main result, Theorem 1, part (2). Notations and conventions are that of the statement. We parsimoniously omit references to base points from now on.

We wish to deal with some preparatory material on Category Theory. We oppose the arrows in the definition given on p. 55 of [Mac98].

Definition 76. Let $F : A \rightarrow B$ be a functor, $b \in B$. An universal arrow from F to b is a pair (b_*, u) consisting of an object b_* of A and an arrow $u : Fb_* \rightarrow b$ such that for any $\alpha : Fa \rightarrow b$, there exists a unique $\tilde{\alpha} : a \rightarrow b_*$ rendering commutative the diagram

$$\begin{array}{ccc} Fa & \xrightarrow{\alpha} & b \\ & \searrow & \uparrow u \\ & F\tilde{\alpha} & Fb_* \end{array}$$

commutes. An element b_* is universal from F to b if there exists a universal arrow from F to b of the form (u, b) .

Another way to define the previous concept is the following [Mac98, III.§2,p.59]. If $u : Fb_* \rightarrow b$ is universal from F to b , then

$$\eta_{(u, b_*)} : \text{Hom}_A(-, b_*) \longrightarrow \text{Hom}_B(F(-), b), \quad (x \xrightarrow{\beta} b_*) \longmapsto (Fx \xrightarrow{F\beta} Fb_* \xrightarrow{u} b)$$

is a natural isomorphism of contravariant functors. Conversely, any natural isomorphism between these two functors is of the form $\eta_{(u, b_*)}$ for a unique universal arrow $u : Fb_* \rightarrow b$.

Lemma 77. *Let*

$$\begin{array}{ccc} B & \xrightarrow{U} & B' \\ F \uparrow & & \uparrow F' \\ A & \xrightarrow{V} & A' \end{array}$$

be a commutative diagram between categories (up to natural isomorphism). Assume that U and V are fully faithful. Let $b \in B$ and let $b_* \in A$ be such that $V(b_*)$ is universal from F' to Ub . Then b_* is universal from F to b .

Proof. We have natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_A(-, b_*) &\xrightarrow{\sim} \mathrm{Hom}_{A'}(V(-), V(b_*)) \\ &\xrightarrow{\sim} \mathrm{Hom}_{B'}(F'V(-), U(b)) \\ &\xrightarrow{\sim} \mathrm{Hom}_{B'}(UF(-), U(b)) \\ &\xrightarrow{\sim} \mathrm{Hom}_B(F(-), b). \end{aligned}$$

□

This lemma will be applied to the categories and functors introduced of §6.2. Recall from that section that \mathcal{R}_X stands for the category of proper schemes F endowed with a left action of $\Pi(X)$ such that $\mathcal{U}_X \times^{\Pi(X)} F$ is an X -scheme. We then have functors

$$\mathcal{U}_X[\bullet] := \mathcal{U}_X \times^{\Pi(X)} (\bullet) \quad \text{and} \quad \mathcal{U}_S[\bullet] := \mathcal{U}_S \times^{\Pi(S)} (\bullet).$$

(See §6.2 for more details concerning them.) Using the isomorphism of stratified pre-sheaves

$$(\mathcal{U}_X \times F) / \Pi(X) \xrightarrow{\sim} \{[\mathcal{U}_X \times \Pi(S)] / \Pi(X) \times F\} / \Pi(S),$$

the isomorphism of stratified schemes $\mathcal{U}_X \times^{\Pi(X)} \Pi(S) \simeq \mathcal{U}_S \times_S X$ [Nor76, Proposition 2.9(c)], the isomorphism of stratified pre-sheaves

$$\{(\mathcal{U}_S \times_S X) \times F\} / \Pi(S) \xrightarrow{\sim} \{(\mathcal{U}_S \times F) / \Pi(S)\} \times_S X,$$

and the fact that “taking the associated sheaf commutes with fibre products” (see p. 14 for the references to SGA4), it is possible to verify that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{R}_X & \xrightarrow{\mathcal{U}_X[\bullet]} & \mathbf{StrP}(X) \\ \text{restriction}=\mathbf{R} \uparrow & \Downarrow & \uparrow f^* \\ \mathcal{R}_S & \xrightarrow{\mathcal{U}_S[\bullet]} & \mathbf{StrP}(S). \end{array}$$

Let $\tilde{\rho} : \Pi(X) \rightarrow \mathbf{GL}(V)$ be a finite dimensional representation of $\Pi(X)$ and let $\rho : \Pi(X) \rightarrow \mathbf{PGL}(V)$ be the induced projective representation. Let

$$\pi : H = H_f(\mathcal{U}_X[\mathbb{P}(V), \rho]) \longrightarrow S$$

denote the S -scheme of Definition 49. We know from Corollary 69 and Theorem 50 that H is a stratified scheme over S whose connected components are projective. (There is no

reason for π to be quasi-compact a priori.) Consequently, any connected component is a stratified proper scheme over S and, as such, is mapped *onto* S , see Lemma 12.

We also know that $\pi^{-1}(s_0) \simeq \mathbb{P}(V)^{\text{Im}(a)}$: apply Corollary 59, the commutativity of the above diagram in the case $X_0 \rightarrow X$, and Proposition 74, respectively. As $\mathbb{P}(V)^{\text{Im}(a)}$ has only finitely many connected components, we conclude that H has only finitely many connected components. We have deduced that π is proper.

As the automorphism group scheme of $\mathbb{P}(V)^{\text{Im}(a)}$ is affine (see Proposition 81), we obtain a homomorphism $\tau : \Pi(S) \rightarrow \mathbf{Aut}_k(\mathbb{P}(V)^{\text{Im}(a)})$ and an isomorphism in $\mathbf{StrS}(S)$

$$\mathcal{U}_S[\mathbb{P}(V)^{\text{Im}(a)}, \tau] \simeq H.$$

(See Proposition 14.) Since H is universal from f^* to $\mathcal{U}_X[\mathbb{P}(V), \rho]$ (see Corollary 58) and both $\mathcal{U}_S[\bullet]$, $\mathcal{U}_X[\bullet]$ are fully faithful (Lemma 73), Lemma 77 allows us to say that $\mathbb{P}(V)^{\text{Im}(a)}$, with the aforementioned action of $\Pi(S)$, is universal from \mathbf{R} to $(\mathbb{P}(V), \rho)$. Using that $\Pi(X)/\text{Ker}(b) \simeq \Pi(S)$, we deduce that $\mathbb{P}(V)^{\text{Ker}(b)}$ is universal from \mathbf{R} to $(\mathbb{P}(V), \rho)$. Conclusion: the $\Pi(S)$ -schemes $\mathbb{P}(V)^{\text{Im}(a)}$ and $\mathbb{P}(V)^{\text{Ker}(b)}$ are isomorphic. A fortiori, the *reduced k -schemes* underlying $\mathbb{P}(V)^{\text{Ker}(b)}$ and $\mathbb{P}(V)^{\text{Im}(a)}$ are isomorphic, so that the closed embedding

$$\mathbb{P}(V)^{\text{Ker}(b)} \longrightarrow \mathbb{P}(V)^{\text{Im}(a)}$$

must induce a bijection on k -points. This means that Property 4.2 holds, and we conclude that $\text{Im}(a) = \text{Ker}(b)$ using Lemma 4 and then Lemma 3.

Remark 78. The cautious reader must have realized

- i) that the fact that $\mathcal{U}_X[\mathbb{P}(V), \rho]$ is a scheme and $\mathcal{U}_X[\mathbb{P}(V), \rho] \rightarrow X$ is projective and smooth needs justification.
- ii) That $F := \mathbb{P}(V)^{\text{Ker}(b)}$ is not necessarily an element of \mathcal{R}_S : for that, it is needed that $\mathcal{U}_S \times^{\Pi(S)} F$ be a scheme. (See equation (9) for the definition.)

In the ensuing lines we justify these claims.

(i) Pick an algebraic quotient G of $\Pi(X)$ such that $\Pi(X) \rightarrow \mathbf{GL}(V)$ factors through G . Then

$$\begin{aligned} \mathcal{U}_X \times^{\Pi(X)} \mathbb{P}(V) &\simeq \left(\mathcal{U}_X \times^{\Pi(X)} G \right) \times^G \mathbb{P}(V) \\ &\simeq P \times^G \mathbb{P}(V), \end{aligned}$$

and $P \rightarrow X$ is a G -torsor. It follows from [BLR90, Theorem 7, p.138] that $P \times^G \mathbb{P}(V)$ is a scheme and that the obvious morphism to X is quasi-projective. That it is proper, respectively smooth, follows from [EGA IV₂, 2.7.1, p.29], respectively [EGA IV₄, 17.7.3, p.72].

(ii) Pick an algebraic quotient G of $\Pi(X)$ such that $\Pi(X) \rightarrow \mathbf{GL}(V)$ factors through G . Let $K \trianglelefteq \Pi(X)$ be the kernel of $\Pi(X) \rightarrow G$. Let L be the image of K in $\Pi(S)$. As b is a quotient morphism, $L \trianglelefteq \Pi(S)$ and $H := \Pi(S)/L$ is a quotient of G . It is then straightforward to show that L acts trivially on F . Hence,

$$\begin{aligned} \mathcal{U}_S \times^{\Pi(S)} F &\simeq \left(\mathcal{U}_S \times^{\Pi(S)} H \right) \times^H F \\ &\simeq P \times^H F, \end{aligned}$$

where $P \rightarrow S$ is an H -torsor. (This detour intends simply to avoid analysis of the morphism $\Pi(S) \rightarrow \mathbf{Aut}_k(F)$.) As H and P are reduced and k is perfect, $(P \times F)_{\text{red}} = P \times F_{\text{red}}$ is invariant under H . We are then in a position to apply [DG, III, §2, 7.1, p.318], which guarantees that $(P \times F)/H$ is representable if $(P \times F_{\text{red}})/H$ is likewise. Since $\text{Ker}(b)$ acts on $\mathbb{A}(V)$, the scheme F_{red} is a disjoint union of projective spaces. Consequently, the action of H on F_{red} can be linearized by using the anti-canonical bundle [GIT, Ch. 1, §3]; projective fpqc descent [BLR90, Theorem 7, p.138] assures representability of $(P \times F_{\text{red}})/H$.

16. DIGRESSION ON SPECIFIC AUTOMORPHISM GROUP SCHEMES

Our goal here is to study a simple instance where automorphism group schemes are *affine*. These are tailored to fit some technical requisites found in studying the schemes $\mathbb{P}(V)^G$ appearing in §4.2. The most relevant result here is Proposition 81. Indeed, if G is an affine group scheme (over k) and V is a finite dimensional representation of it, then

$$(\mathbb{P}(V)^G)_{\text{red}} = \coprod_{\lambda \text{ a character of } G} H_\lambda,$$

where

$$H_\lambda = \mathbb{P}(\{v \in V : G \text{ acts on } v \text{ through } \lambda\}).$$

Note that, already in the case where \mathbb{G}_a acts on \mathbb{P}^1 by “translations”, the fixed point scheme is not reduced.

Throughout, **Sch** stands for the category of schemes over an algebraically closed field k , where all morphisms and constructions take place. The full subcategory of all reduced schemes is denoted by **RSch**.

The reader should bear in mind that for a proper scheme M , the functor of its automorphisms

$$\mathbf{Aut}_k(M) : \mathbf{Sch}^o \longrightarrow \mathbf{Grp}, \quad T \longmapsto \mathbf{Aut}_T(M \times T)$$

is represented by a group scheme which is locally of finite type. For a proof under the extra assumption of projectivity, see [N03, p. 133]. The more general case is in [MO67, Theorem 3.7, p.17].

Two paragraphs concerning some basic tools employed in this section are at hand. Firstly, we state two results which will be much used: Lemma 79 and Lemma 80. Proofs are in [EGA I, 5.1.10, p.131] and [SGA3 VI_A, 2.4.1, p.304] respectively.

Lemma 79. *Let W be a scheme of finite type over k . Then W is affine if and only if W_{red} is likewise.* □

Lemma 80. *Let G be a group scheme locally of finite type over k . If G is connected, then G is of finite type.* □

Secondly, we discuss the *neutral component* of a group scheme. Given a group scheme G locally of finite type over k , we let G^0 stand for the connected component of the identity of the topological space underlying G . Since G is locally noetherian, G^0 is an open subset of G [EGA I, 6.1.9]. As $G^0 \times G^0$ is connected, we obtain a group scheme structure on G^0 which turns the inclusion into a morphism of group schemes. The group scheme G^0

is called *the neutral component of G* . Due to Lemma 80, G^0 is of finite type over k . For more on this, the reader should consult [SGA3, VI_A, §2].

Proposition 81. *Let M be projective. Assume that, for each connected component C of M_{red} , $\mathbf{Aut}_k(C)$ is affine and its Néron-Severi group*

$$\text{NS}(C) = \text{Pic}(C)/\text{Pic}^0(C)$$

has rank one (as a finitely generated abelian group). Then $\mathbf{Aut}_k(M)$ is affine.

The reader should bear in mind that NS is of finite type, cf. Exposé XIII, p. 35 of [SGA6]. The proof of Proposition 81 is built by the following results.

Lemma 82. *Let W be projective and connected. Let W' stand for the disjoint union of n copies of W .*

- (a) *Let \mathfrak{S}_n be the constant group scheme associate to the symmetric group on n letters. Then there exists a morphism of group schemes*

$$\sigma : \mathbf{Aut}_k(W') \longrightarrow \mathfrak{S}_n$$

whose kernel is $\mathbf{Aut}_k(W)^{\times n}$.

- (b) *In particular, if $\mathbf{Aut}_k(W)$ is affine, then $\mathbf{Aut}_k(W')$ is likewise.*

Proof. Firstly, we label W_1, \dots, W_n the connected components of W' . Let $T \in \mathbf{Sch}$ be *connected* and consider an automorphism of T -schemes $\varphi : W' \times T \rightarrow W' \times T$. It is clear that $W' \times T$ is the disjoint union of the $W_i \times T$, and that each one of these is connected. Then φ induces a permutation $\sigma(\varphi)$ of the set $\{1, \dots, n\}$. It is easily verified that if $U \rightarrow T$ is a morphism from a connected scheme, then $\sigma(\varphi \times_T U) = \sigma(\varphi)$. This constructs the morphism of group functors σ . The kernel is easily determined and we conclude our proof of (a). From this it follows that

$$(\mathbf{Aut}_k(W)^{\times n})^0 \xrightarrow{\simeq} (\mathbf{Aut}_k(W'))^0.$$

Since each connected component of $\mathbf{Aut}_k(W')$ is a scheme isomorphic to $\mathbf{Aut}_k(W)^0$, it follows that $\mathbf{Aut}_k(W')$ is a disjoint union of affine schemes. Moreover, since σ is surjective — in fact there exists a section $\mathfrak{S}_n \rightarrow \mathbf{Aut}_k(W')$ — we conclude that $\mathbf{Aut}_k(W')$ is a disjoint union of a *finite* number of quasi-compact spaces. Consequently $\mathbf{Aut}_k(W')$ is quasi-compact, so that the number of its connected components is finite: a finite disjoint union of finitely many affine schemes is affine. \square

The proof of the following result is quite easy and we leave it to the reader.

Corollary 83. *Let M be a projective scheme. If $\mathbf{Aut}_k(V)$ is affine for each connected component V of M , then $\mathbf{Aut}_k(M)$ is affine. \square*

One important ingredient in verifying that an automorphism group scheme is affine relies on verifying that there are *only finitely many* connected components. Here is our result in this direction.

Lemma 84. *Let M be a projective scheme. Assume that its Néron-Severi group*

$$\mathrm{NS}(M) = \mathrm{Pic}(M)/\mathrm{Pic}^0(M)$$

has rank one (as a finitely generated abelian group). Then $\mathbf{Aut}_k(M)$ is of finite type.

Proof. Given a very ample invertible sheaf \mathcal{L} on M , we obtain an open immersion [N03, Exercise, p.133] $\gamma : \mathbf{Aut}_k(M) \rightarrow \mathbf{Hilb}_{M \times M}$ defined by associating to each automorphism its graph. (We use the very ample sheaf $\mathrm{pr}_1^* \mathcal{L} \otimes \mathrm{pr}_2^* \mathcal{L}$ on $M \times M$.) Write, as usual, $\mathbf{Hilb}_{M \times M}^P$ for the open subset of $\mathbf{Hilb}_{M \times M}$ associated to the polynomial $P \in \mathbb{Q}[m]$. If a is a k -point of $\mathbf{Aut}_k(M)$, its image in $\mathbf{Hilb}_{M \times M}$ lies in $\mathbf{Hilb}_{M \times M}^P$, where $P(m) = \chi(\mathcal{L}^m \otimes a^* \mathcal{L}^m)$ for large m . From this, it follows that, if $a^* \mathcal{L}$ is τ -equivalent to \mathcal{L} (see p.29 of SGA6, Exposé XIII) for every $a \in \mathbf{Aut}_k(M)(k)$, then γ sends $\mathbf{Aut}_k(M)$ into $\mathbf{Hilb}_{M \times M}^P$, where $P(m) = \chi(\mathcal{L}^{2m})$ for large m (as explained in Theorem 4.6 of SGA6, Exposé XIII). In particular, $\mathbf{Aut}_k(M)$ is quasi-projective.

Let \mathbf{Pic}_M^λ be a connected component of \mathbf{Pic}_M which generates the free part of $\mathrm{NS}(M)$ and which has a very ample class \mathcal{L} . We conclude that for each $a \in \mathbf{Aut}_k(M)(k)$, $a^* \mathcal{L}$ is τ -equivalent to \mathcal{L} . \square

Let us now fix a connected and projective scheme V . Denote the nilradical [EGA I, 5.1] of V by \mathcal{N} . For each μ , let V_μ stand for the closed subscheme of V cut out by \mathcal{N}^μ .

Given any reduced k -scheme T , the closed sub-scheme $V_{\mathrm{red}} \times T$ of $V \times T$ is reduced [EGA IV₂, 4.6.5, p. 69] and has the same underlying topological space as $V \times T$. Consequently, the nilradical of $\mathcal{O}_{V \times T}$ is the pull-back of \mathcal{N} by the obvious projection: $\mathcal{N} \times T$. Thus, to each $\varphi \in \mathrm{Aut}_T(V \times T)$, the associated isomorphism of rings

$$\varphi^\# : \mathcal{O}_{V \times T, \varphi(x)} \longrightarrow \mathcal{O}_{V \times T, x}$$

maps $(\mathcal{N} \times T)_{\varphi(x)}$ onto $(\mathcal{N} \times T)_x$, and we can associate an automorphism of $V_\mu \times T$ to φ . In this way, we are able to define a morphism ρ_μ from the functor

$$\mathbf{A}_V : \mathbf{RSch}^o \longrightarrow \mathrm{Grp}, \quad T \longmapsto \mathrm{Aut}_T(V \times T)$$

to the functor

$$\mathbf{A}_{V_\mu} : \mathbf{RSch}^o \longrightarrow \mathrm{Grp}, \quad T \longmapsto \mathrm{Aut}_T(V_\mu \times T).$$

Lemma 85. *Let μ be such that \mathcal{N}^μ is of square zero. Then the functors*

$$\mathbf{RSch}^o \longrightarrow \mathbf{Set}, \quad T \longmapsto \mathrm{Ker} \rho_\mu(T)$$

and

$$\mathbf{RSch}^o \longrightarrow \mathbf{Set}, \quad T \longmapsto \mathrm{Der}_{\mathcal{O}_T}(\mathcal{O}_{V \times T}, \mathcal{N}^\mu \times T).$$

are isomorphic. (Here $\mathcal{N}^\mu \times T$ is the pull-back of \mathcal{N}^μ by the obvious projection.)

Proof. The proof is in [SGA1, III, 5.1]. Let T be reduced and let

$$\varphi : V \times T \longrightarrow V \times T$$

be an isomorphism of T -schemes inducing the identity on $V_\mu \times T$. Then, φ is determined by an automorphism of \mathcal{O}_T -algebras

$$\varphi^\# : \mathcal{O}_{V \times T} \longrightarrow \mathcal{O}_{V \times T}$$

such that

$$\varphi^\#(a) \equiv a \pmod{\mathcal{N}^\mu \times T}, \quad \forall a \in \mathcal{O}_{V \times T}.$$

Thus, φ is determined by the element

$$\varphi^\# - \text{id}_{\mathcal{O}_{V \times T}} \in \text{Der}_{\mathcal{O}_T}(\mathcal{O}_{V \times T}, \mathcal{N}^\mu \times T).$$

Conversely, any element δ of the above group defines a morphism $\text{id} + \delta : \mathcal{O}_{V \times T} \rightarrow \mathcal{O}_{V \times T}$ of \mathcal{O}_T -algebras. Since this morphism is an isomorphism (the identity) modulo a nilpotent ideal, it follows that $\text{id} + \delta$ is an automorphism. \square

Let μ be as in the Lemma 85. Note that the functors \mathbf{A}_V and \mathbf{A}_{V_μ} are represented by $\mathbf{Aut}_k(V)_{\text{red}}$ and $\mathbf{Aut}_k(V_\mu)_{\text{red}}$; as the category \mathbf{RSch} possesses direct products, $\mathbf{Aut}_k(V)_{\text{red}}$ and $\mathbf{Aut}_k(V_\mu)_{\text{red}}$ are naturally group objects in \mathbf{RSch} and ρ_μ is a morphism of group objects. So we have a morphism of *group schemes*

$$\rho_\mu : \mathbf{Aut}_k(V)_{\text{red}} \longrightarrow \mathbf{Aut}_k(V_\mu)_{\text{red}}.$$

By [N03, Theorem 5.8] together with Lemma 85 and the equality

$$\text{Der}_{\mathcal{O}_T}(\mathcal{O}_{V \times T}, \mathcal{N}^\mu \times T) = \text{Hom}_{V \times T} \left(\Omega_{V \times T/T}^1, \mathcal{N}^\mu \times T \right),$$

the functor

$$\mathbf{RSch}^o \longrightarrow \mathbf{Set}, \quad T \longmapsto \text{Ker } \rho_\mu(T)$$

is represented by an affine space \mathbb{A}^ℓ . (We are not affirming that it is represented by \mathbb{G}_a^ℓ !) Hence, $\text{Ker}(\rho_\mu)_{\text{red}}$ is affine and connected.

Lemma 86. *We maintain the above notations. If $(\mathbf{Aut}_k(V_\mu)_{\text{red}})^0$ is affine, then $\mathbf{Aut}_k(V)^0$ is likewise.*

Proof. Let $G = \mathbf{Aut}_k(V)_{\text{red}}$ and $H = \mathbf{Aut}_k(V_\mu)_{\text{red}}$. Let K stand for the kernel of $\rho_\mu : G \rightarrow H$; above we showed that K_{red} is connected and affine. From Lemma 79 and Lemma 80, K is affine, connected and of finite type. Due to the connectedness of K , the arrow $K \hookrightarrow G$ factors through G^0 .

According to [SGA3, VI_A, 3.2], the quotient of the fppf sheaf G^0 by K is representable by a group scheme G^0/K . Moreover, $\mathfrak{p} : G^0 \rightarrow G^0/K$ is faithfully flat and of finite presentation [DG, III,§3,2.5,p.328]. As K is affine, [DG, III,§3,2.6,p.329] shows that \mathfrak{p} is affine. Now we only need to show that G^0/K is affine.

Since G^0 is of finite type and \mathfrak{p} is affine and faithfully flat, [BouAC, I,§3,no.6, Prp. 11] guarantees that G^0/K is locally of finite type. Since G^0 is quasi-compact and \mathfrak{p} is surjective, G^0/K is quasi-compact. As the induced morphism $G^0/K \rightarrow H^0$ is a monomorphism, it is a closed embedding by [DG, II,§5,5.1p.249]. It follows that G^0/K is affine. The proof that G^0 is affine is then achieved.

Since the topological spaces underlying G^0 and $\mathbf{Aut}_k(V)^0$ coincide, it follows from Lemma 79 and Lemma 80 that $\mathbf{Aut}_k(V)^0$ is affine. \square

Proof of Proposition 81. Let V be a connected component of M . Using Theorem 1.1 of Exposé XII of [SGA6] (see also Lemma 4.3 of Exposé XIII of SGA6), we know that

$$\mathrm{NS}(V_{\mathrm{red}})/(\mathrm{torsion}) \simeq \mathbb{Z} \implies \mathrm{NS}(W)/(\mathrm{torsion}) \simeq \mathbb{Z}$$

for *any* nilpotent immersion $W \rightarrow V$. Using this fact, induction, Lemma 84, Lemma 86 and the affinity of $\mathbf{Aut}_k(V_{\mathrm{red}})$, we conclude that $\mathbf{Aut}_k(V)$ is affine. Corollary 83 now ends the proof. \square

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