ON THE VECTOR BUNDLES OVER RATIONALLY CONNECTED VARIETIES

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Abstract. Let $X$ be a rationally connected smooth projective variety defined over $\mathbb{C}$ and $E \to X$ a vector bundle such that for every morphism $\gamma : \mathbb{C}P^1 \to X$, the pullback $\gamma^*E$ is trivial. We prove that $E$ is trivial. Using this we show that if $\gamma^*E$ is isomorphic to $L(\gamma)^{\oplus r}$ for all $\gamma$ of the above type, where $L(\gamma) \to \mathbb{C}P^1$ is some line bundle, then there is a line bundle $\zeta$ over $X$ such that $E = \zeta^{\oplus r}$.

Résumé. Soit $X$ une variété rationnellement connexe sur $\mathbb{C}$ et soit $E \to X$ un fibré vectoriel tel que, pour tout morphisme $\gamma : \mathbb{C}P^1 \to X$, le fibré $\gamma^*E$ est trivial. Nous montrons que $E$ est trivial. Nous en déduisons que si, pour tout $\gamma$ comme avant, $\gamma^*E$ est isomorphe à $L(\gamma)^{\oplus r}$, où $L(\gamma) \to \mathbb{C}P^1$ est un fibré en droites, alors il existe un fibré en droites $\zeta$ sur $X$ et un isomorphisme $E \cong \zeta^{\oplus r}$.

1. Introduction

Let $E$ be a holomorphic vector bundle over a connected complex projective manifold $X$. If for every pair of the form $(C, \gamma)$, where $C$ is a compact connected Riemann surface, and $\gamma : C \to X$ a holomorphic map, the pullback $\gamma^*E$ is semistable, then it is known that $E$ is semistable, and $c_i(\text{End}(E)) = 0$ for all $i \geq 1$ [BB, pp. 3–4, Theorem 1.2]. Our aim here is to show that if $X$ is rationally connected, then the above conclusion remains valid even if we insert in the condition that $C$ is a rational curve. We recall that a complex projective variety $X$ is said to be rationally connected if any two points of $X$ can be joined by an irreducible rational curve on $X$; see [KMM, Theorem 2.1] for equivalent conditions. We prove the following theorem:

Theorem 1.1. Let $E$ be a vector bundle of rank $r$ over a rationally connected smooth projective variety $X$ defined over $\mathbb{C}$ such that for every morphism

$$\gamma : \mathbb{C}P^1 \to X,$$

the pullback $\gamma^*E$ is isomorphic to $L(\gamma)^{\oplus r}$ for some line bundle $L(\gamma) \to \mathbb{C}P^1$. Then there is a line bundle $\zeta$ over $X$ such that $E = \zeta^{\oplus r}$.

In [AW] this was proved under the extra assumption that $\text{Pic}(X) = \mathbb{Z}$ (see [AW, p. 211, Proposition 1.2]).

Theorem 1.1 is deduced from the following proposition (see Proposition 2.1):

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Proposition 1.2. Let $X$ be as in Theorem 1.1. Let $E \to X$ be a vector bundle such that for every morphism $\gamma : \mathbb{CP}^1 \to X$, the pullback $\gamma^*E$ is trivial. Then $E$ itself is trivial.

The condition in Theorem 1.1 that $\gamma^*E$ is of the form $L(\gamma)^{\oplus r}$ can be replaced by an equivalent condition which says that $\gamma^*E$ is semistable (see Corollary 2.3).

2. Criterion for triviality

Let $X$ be a rationally connected smooth projective variety defined over $\mathbb{C}$. Let $E \to X$ be a vector bundle.

Proposition 2.1. Assume that for every morphism $\gamma : \mathbb{CP}^1 \to X$ the vector bundle $\gamma^*E \to \mathbb{CP}^1$ is trivial. Then $E$ itself is trivial.

Proof. Let $x \in X$ be a closed point. There is a smooth family of rational curves on $X$

\[
\begin{array}{c}
\begin{array}{c}
Z \xrightarrow{\varphi} X \\
\downarrow \sigma \downarrow f \\
T
\end{array}
\end{array}
\]

where

1. $T$ is open in $\text{Mor}(\mathbb{CP}^1, X; (0 : 1) \to x)$ (hence $T$ is quasiprojective),
2. $f \circ \sigma = \text{Id}_T$,
3. $\varphi$ is dominant, and
4. $\varphi(\sigma(t)) = x$ for all $t \in T$.

(See [Ca, Section 3], [Ko2, Theorem 3].)

Let $\beta := [\varphi(f^{-1}(t))] \in H_2(X, \mathbb{Z})$ be the homology class, where $t \in T(\mathbb{C})$. Let $\overline{\mathcal{M}}_{0,1}(X, \beta)$ be the moduli stack classifying families of stable maps from 1-pointed genus zero curves to $X$ which represent the class $\beta$. (We are following the terminology of [FP].) We know that $\overline{\mathcal{M}}_{0,1}(X, \beta)$ is a proper Deligne–Mumford stack [BM, p. 27, Theorem 3.14].

Let

\[
\rho : T \to \overline{\mathcal{M}}_{0,1}(X, \beta)
\]

be the morphism associated to the family in (2.1).

By “Chow’s Lemma” [LMB, p. 154, Corollaire 16.6.1], there exists a projective $\mathbb{C}$-scheme $Y$ together with a proper surjective morphism $\psi : Y \to \overline{\mathcal{M}}_{0,1}(X, \beta)$. There
exists a Cartesian diagram

\[
\begin{array}{ccc}
T_1 & \xrightarrow{\rho_1} & Y \\
\psi_1 & \downarrow & \psi \\
T & \xrightarrow{\rho} & \mathcal{M}_{0,1}(X, \beta)
\end{array}
\]

where \(T_1\) is a scheme and \(\psi_1\) is proper and surjective. This last assertion is justified by the fact that the diagonal of a Deligne–Mumford stack is schematic [LMB, p. 26, Lemme 4.2] and [LMB, p. 21, Corollaire 3.13]. As \(T\) is separated (it is open in \(\text{Mor}(\mathbb{P}^1, X)\)), we can apply Nagata’s Theorem [Lü, p. 106, Theorem 3.2] to find a proper \(\mathbb{C}\)-scheme \(\overline{T}_1\) and a schematically dense open immersion

\[
i : T_1 \hookrightarrow \overline{T}_1.
\]

Eliminating the “indeterminacy locus” (see e.g. [Lü, pp. 99–100]), we can find a blow-up

\[
\xi : \overline{T} \longrightarrow \overline{T}_1
\]

whose center is disjoint from \(T_1\) and a morphism

\[
\overline{\rho} : \overline{T} \longrightarrow Y
\]

which extends \(\rho_1 : T_1 \longrightarrow Y\). The composition \(\psi \circ \overline{\rho} : \overline{T} \longrightarrow \mathcal{M}_{0,1}(X, \beta)\) represents a family of 1–pointed genus zero stable maps

\[
\begin{array}{ccc}
\overline{Z} & \xrightarrow{\overline{\sigma}} & X \\
\overline{T} & \xleftarrow{\overline{\rho}} & \overline{T}_1
\end{array}
\]

whose pull-back via \(i : T_1 \hookrightarrow \overline{T}\) is the pull-back of the family in (2.1) via \(\psi_1\). Clearly \(\overline{\rho}\) is dominant (hence surjective) and \(\overline{\psi} \circ \overline{\sigma}\) is a constant morphism. Note that, without loss of generality, we can assume \(\overline{T}\) to be reduced.

We recall that the pullback of \(E\) by any map from \(\mathbb{CP}^1\) is trivial. Consequently, for any point \(t \in \overline{T}(\mathbb{C})\), the restriction of

\[
\overline{E} := \overline{\rho}^* E
\]

to the curve \(\overline{f}^{-1}(t)\) — which is a tree of \(\mathbb{CP}^1\) — is trivial. Therefore, \(\overline{E}\) descends to \(\overline{T}\). More precisely, the direct image \(\overline{f}_* \overline{E}\) is a vector bundle on \(\overline{T}\), and the canonical arrow

\[
\overline{f}^* \overline{f}_* \overline{E} \longrightarrow \overline{E}
\]

is an isomorphism [Mu, §5]. The homomorphism in (2.4) is injective because any section of a trivial vector bundle, over a connected projective scheme, that vanishes at one point actually vanishes identically; the homomorphism is surjective also because \(\overline{E}|_{\overline{f}^{-1}(t)}\) is trivial for all \(t\). We also note that the image of (2.4) by \(\overline{\sigma}\) defines an isomorphism between \(\overline{\sigma} \overline{E}\) and \(\overline{f}_* \overline{E}\). Therefore, using (2.4),

\[
(2.5) \quad \overline{f} \overline{\sigma} \overline{E} = \overline{E}.
\]
Now from the condition that $\varphi \circ \sigma$ is a constant map it follows immediately that $\sigma^* \varphi^* E = \sigma^* E$ is a trivial vector bundle. Consequently, using (2.5) we conclude that the vector bundle $\varphi^* E$ is trivial.

Since $\varphi$ is a surjective and proper morphism, and $\varphi^* E$ is trivial, we conclude that the Chern class $c_i(E)$ is numerically equivalent to zero for all $i \geq 1$.

Next we will show that the vector bundle $E$ is semistable.

Let $C \hookrightarrow X$ be a smooth irreducible (proper) curve on $X$, and let $C' \hookrightarrow \overline{Z}$ be an irreducible curve such that $\varphi(C') = C$. (The curve $C'$ can be constructed as the closure of a closed point of the generic fiber of $\varphi^{-1}(C) \to C$.) Since the pull-back of $E|_C$ to $C'$ is trivial, so is the pull-back of $E|_C$ to the normalization of $C'$. Consequently, the vector bundle $E|_C$ is semistable. This allows us to conclude that $E$ is semistable with respect to any chosen polarization on $X$.

Since $E$ is semistable, and both $c_1(E)$ and $c_2(E)$ are numerically equivalent to zero, a theorem of Simpson says that $E$ admits a flat connection (see [Si, p. 40, Corollary 3.10]). On the other hand, $X$ is simply connected because it is rationally connected [Ca, p. 545, Theorem 3.5], [Ko1, p. 362, Proposition 2.3]. Therefore, any flat vector bundle on $X$ is trivial. In particular, the vector bundle $E$ is trivial. □

As before, let $E$ be a vector bundle over the rationally connected variety $X$. Let $r$ be the rank of $E$.

**Theorem 2.2.** Assume that for every morphism 
\[
\gamma : \mathbb{C}P^1 \longrightarrow X, 
\]
there is a line bundle $L(\gamma) \longrightarrow \mathbb{C}P^1$ such that $\gamma^* E = L(\gamma)^{\oplus r}$. Then there is a line bundle $\zeta \longrightarrow X$ such that $E = \zeta^{\oplus r}$.

**Proof.** The above condition on $\gamma^* E$ and Proposition 2.1 ensure that the vector bundle $\text{End}(E)$ is trivial. This implies that, for any $x_0 \in X(\mathbb{C})$, the evaluation map 
\[
H^0(X, \text{End}(E)) \longrightarrow \text{End}_{\mathbb{C}}(E(x_0)) \tag{2.6}
\]
is an isomorphism; let $A : E \longrightarrow E$ be an isomorphism such that all the eigenvalues $\lambda_1, \ldots, \lambda_r$ of $A(x_0)$ are distinct. As the eigenvalues of $A(x)$ are independent of $x \in X$, it follows that $E$ is isomorphic to the direct sum of the line subbundles 
\[
\mathcal{L}_i := \text{kernel}(\lambda_i - A) \subset E,
\]
$1 \leq i \leq r$.

Since the evaluation map in (2.6) is an isomorphism, we have 
\[
\dim H^0(X, \mathcal{L}_i \otimes \mathcal{L}_j^*) \leq 1 \tag{2.7}
\]
for all $i, j \in [1, r]$. Note that if $H^0(X, \mathcal{L}_i \otimes \mathcal{L}_j^*) = 0$ for some $i, j$, then 
\[
\dim H^0(X, \text{End}(E)) < r^2,
\]
which contradicts the fact that $\text{End}(E)$ is trivial. For $s_{ij} \in H^0(X, \mathcal{L}_i \otimes \mathcal{L}_j^*) \setminus \{0\}$, $i, j \in [1, r]$, the composition $s_{ij} \circ s_{ji}$ is an automorphism of $\mathcal{L}_i$, hence each $s_{ij}$ is an isomorphism. This completes the proof of the theorem. \qed

A theorem due to Grothendieck says that any vector bundle over $\mathbb{CP}^1$ decomposes into a direct sum of line bundles [Gr, p. 126, Théorème 2.1]. Therefore, Theorem 2.2 has the following corollary:

**Corollary 2.3.** If for every morphism $\gamma : \mathbb{CP}^1 \to X$, the vector bundle $\gamma^*E$ is semistable, then there is a line bundle $\zeta \to X$ such that $E = \zeta^{\oplus r}$.

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