Cup products in information cohomology

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1 Preliminaries

Let $(\mathbf{S}, \mathcal{E})$ be a finite information structure, and $\mathcal{Q}$ an adapted functor of probabilities (e.g. the functor that assigns to each X, the set of all probabilities on the finite set $\mathcal{E}_X$, and to each arrow in $\mathbf{S}$ the marginalization operation).

Recall that $\mathcal{F}$ is the presheaf that associates to each object X in the information structure $\mathbf{S}$, the real vector space $\mathcal{F}_X$ of real-valued functions of probabilities in $\mathcal{Q}_X$. It comes with a left action of $\mathcal{S}_X$ such that $(X.f)(P) = \sum_{x \in E_X} P(X = x) f(P|_{X=x})$. On the right, the variables act trivially. The scalars act on the left and right by usual multiplication. Therefore, for every element $a \in \mathcal{A}_X$, we get a linear map $\Lambda(a) : \mathcal{F}_X \to \mathcal{F}_X$.

Let $\mathcal{F} \otimes \mathcal{F}$ be the presheaf that associates to each $X \in \operatorname{Ob} \mathbf{S}$, the real vector space $\mathcal{F}_X \otimes_{\mathbb{R}} \mathcal{F}_X$. The left action of $a \in \mathcal{A}_X$ on $\mathcal{F}_X$ is defined by $\Pi_a = \Lambda_X^{\otimes 2} : \mathcal{F}_X^{\otimes 2} \to \mathcal{F}_X^{\otimes 2}$ i.e. $X.(f \otimes g) = (X.f) \otimes (X.g)$.

The elements of $\mathcal{F}_X \otimes \mathcal{F}_X$ can be seen as real-valued functionals of pairs $(p, q) \in \mathcal{Q}_X^{\otimes 2}$ that are *separable*.

2 Cup product

Let $\phi \in \operatorname{Hom}_{\mathcal{A}}(\mathcal{B}_n, \mathcal{F})$ be an n-cocycle i.e. $\delta\phi = 0$, and $\psi \in \operatorname{Hom}_{\mathcal{A}}(\mathcal{B}_m, \mathcal{F})$ an m-cocycle.

Following the example of group cohomology, one introduces an element $\phi \wedge \psi$ of $\operatorname{Hom}_{\mathcal{A}}(\mathcal{B}_{n+m}, \mathcal{F} \otimes \mathcal{F})$ by the following explicit formula: for each $Z \in \operatorname{Ob} \mathbf{S}$ and $X_1, ..., X_{n+m} \in \mathcal{S}_Z$,

$$(\phi \wedge \psi)_Z[X_1|...|X_{n+m}] = \phi_Z[X_1|...|X_n] \otimes X_1 \cdots X_n.\psi[X_{n+1}, ..., X_{n+m}]. \tag{1}$$

It is easy to verify that

$$d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^n \phi \wedge d\psi \tag{2}$$

For the sake of simplicity, we do the computation for $n = m = 1$; again $Z \in \operatorname{Ob} \mathbf{S}$ and

$X_1, X_2, X_3 \in \mathcal{S}_Z$:

$$\begin{aligned}(d(\phi \wedge \psi))_Z[X_1|X_2|X_3] &\overset{\text{(def. } d)}{=} X_1((\phi \wedge \psi)_Z[X_2|X_3]) - (\phi \wedge \psi)_Z[X_1X_2|X_3] \\ &\quad + (\phi \wedge \psi)_Z[X_1|X_2X_3] - (\phi \wedge \psi)_Z[X_1|X_2] \\ &\overset{\text{(def. } \phi\wedge\psi)}{=} X_1.(\phi_Z[X_2] \otimes X_2.\psi_Z[X_3]) - \phi[X_1X_2] \otimes X_1X_2.\psi_Z[X_3] \\ &\quad + \phi[X_1] \otimes X_1.\psi_Z[X_2X_3] - \phi[X_1] \otimes X_1.\psi_Z[X_2] \\ &= (X_1.\phi_Z[X_2] - \phi[X_1X_2] + \phi[X_1]) \otimes X_1X_2\psi_Z[X_3] \\ &\quad - \phi[X_1] \otimes X_1(X_2.\psi_Z[X_3] - \psi_Z[X_2X_3] + \psi_Z[X_2] \\ &= (d\phi)_Z[X_1|X_2] \otimes X_1X_2.(\psi_Z[X_3]) - \phi_Z[X_1] \otimes X_1.((d\psi)_Z[X_2|X_3]) \\ &= (d\phi \wedge \psi)_Z[X_1|X_2|X_3] - (\phi \wedge d\psi)_Z[X_1|X_2|X_3].\end{aligned}$$

Corollary 1. *The element* $S_1 \wedge S_1 \in \operatorname{Hom}_{\mathcal{A}}(\mathcal{B}_2, \mathcal{F} \otimes \mathcal{F})$*, given by*

$$(S_1 \wedge S_1)_Z[X_1|X_2](p,q) = S_1[X_1](p)(X_1.S_1[X_2](q))$$

for any $Z \in \operatorname{Ob}\mathbf{S}$ *and* $X_1, X_2 \in \mathcal{S}_Z$*, is a* 2*-cocycle in information cohomology.*

As a reference, see any notes on group cohomology. I have used Sharifi's lecture notes, available online.