Clifford modules and K-theory

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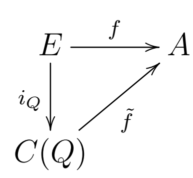
1. Clifford algebras and periodicity

First we stablish well some notation. Let $\mathbb{K}$ be a commutative field (for our applications, it is enough to consider $\mathbb{K}$ as $\mathbb{R}$ or $\mathbb{C}$), and let Q be a quadratic form on a $\mathbb{K}$-module E. Let $T(E) = \bigoplus_{i\geq 0} \bigotimes^i E = \mathbb{K} \oplus E \oplus (E \otimes E) \oplus (E \otimes E \otimes E) \oplus ...$, and define $I(Q)$ as the two sided ideal generated by the elements $x \otimes x - Q(x) \cdot 1$ in $T(E)$. The **Clifford algebra** of Q is defined as the quotient $T(E)/I(Q)$ and denoted $C(Q)$. Note that $E = \bigotimes^1 (E) \hookrightarrow T(E)$, and in turn there is a canonical projection $T(E) \to C(Q)$; the composition of these maps is denoted i_Q and turns out to be injective. The following proposition gives a universal characterization of the Clifford algebra.

Proposition 1.1. *[2, p. 8] Let $f : E \to A$ be a linear map into an associative $\mathbb{K}$-algebra with unit, such that $f(v) \cdot f(v) = Q(v) \cdot 1$, for all $v \in E$. Then f extends uniquely to a $\mathbb{K}$-algebra homomorphism $\tilde{f} : C(Q) \to A$. Furthermore, $C(Q)$ is the unique associative $\mathbb{K}$-algebra with this property, up to isomorphism.*

Proof. Any such f extends uniquely to an algebra homomorphism from $T(E)$ to A; by the condition imposed to f, it vanishes on $I(Q)$, passing to the quotient. A typical category-theoretical argument gives unicity. □

We obtain in this case the following commutative diagram:



We shall limit our analysis to the case $E = \mathbb{R}^k$. We define $Q_k(x_1, , ..., x_k) = -\sum_{i=1}^{n} x_i^2$, and adopt the notation $C_k = C(Q_k)$ and $C'_k = C(-Q_k)$. The following proposition characterize these algebras.

Proposition 1.2. *[1, p. 6] Let* $\{e_i\}_{i=1}^k$ *be the canonical basis of* $\mathbb{R}^k$*. This base generates* C_k *(resp.* C'_k*) multiplicatively and satisfy the relations*

$$e_i^2 = -1 \quad (\text{resp. } e_i^2 = 1), \qquad e_ie_j + e_je_i = 0 \quad \text{for } i \neq j.$$

Then, C_k *(resp.* C'_k*) may be identified with the universal algebra generated over* R *by a unit,* 1*, and the symbols* e_i*,* $i = 1, ...k$ *subject to the relations stated above.*

Some work allow us to conclude that

$$\begin{aligned} C_1 &\cong \mathbb{C}\,, & C'_1 &\cong \mathbb{R} \oplus \mathbb{R} \\ C_2 &\cong \mathbb{H}\,, & C'_2 &\cong M_2(\mathbb{R}). \end{aligned}$$

In fact, Proposition 1.2 says that $C_1 = \mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$. The other isomorphisms follow similarly.

A key fact about the Clifford algebras is presented in the following proposition, that allow us to compute them recursively.

Proposition 1.3. *[1, p.10] There exist isomorphisms*

$$\begin{aligned} C_k \otimes_{\mathbb{R}} C'_2 &\cong C'_{k+2} \\ C'_k \otimes_{\mathbb{R}} C_2 &\cong C_{k+2}. \end{aligned}$$

Proof. [2] Let $e_1, ..., e_{k+2}$ be an orthonormal basis of $\mathbb{R}^{k+2}$ with standard inner product. Denote by $e'_1, ..., e'_n$ the standard generator of C_n, and by e''_1, e''_2 those of C'_2, in the sense of Proposition 1.2. Define $f : \mathbb{R}^{k+2} \to C_k \otimes_{\mathbb{R}} C'_2$, prescribing that

$$f(e_i) = \begin{cases} e'_i \otimes e''_1 e''_2 & \text{for } 1 \leq i \leq n \\ 1 \otimes e''_{i-n} & \text{otherwise} \end{cases}.$$

and extending by linearity. A direct computation shows that $f(e_i)f(e_j) + f(e_j)f(e_i) = 2\delta_{ij} 1 \otimes 1$; for example, given $i, j \in \{1, ..., n\}$ we have

$$f(e_i)f(e_j) + f(e_j)f(e_i) = e'_ie'_j \otimes e''_1e''_2e''_1e''_2 + e'_je'_i \otimes e''_1e''_2e''_1e''_2 = (e'_ie'_j + e'_je'_i) \otimes (-1) = 2\delta_{ij} 1 \otimes 1.$$

The orthonormality of the basis implies that $f(v) \cdot f(v) = -Q(v) \cdot 1$ for any $v \in \mathbb{R}^k$. Proposition 1.1 allows us to extend this function to all C'_{k+2}. By definition, there is a basis of $C_k \otimes_{\mathbb{R}} C'_2$ in the image of f, so its extension is surjective. Finally, $\dim C'_{k+2} = 2^{k+2} = 2^k 2^2 = \dim C_k \dim C'_2 = \dim C_k \otimes_{\mathbb{R}} C'_2$, and this gives the injectivity. □

Recall the following isomorphisms (for a proof, see [2, p.26]):

$$\begin{aligned} M_m(\mathbb{R}) \otimes M_n(\mathbb{R}) &\cong M_{mn}(\mathbb{R}), \quad \text{for } n, m \in \mathbb{N}. \\ M_n(\mathbb{R}) \otimes_{\mathbb{R}} K &\cong M_n(K), \quad \text{for } K = \mathbb{C}, \mathbb{H}. \\ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\cong \mathbb{C} \oplus \mathbb{C} \\ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} &\cong M_2(\mathbb{C}) \\ \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} &\cong M_4(\mathbb{R}) \end{aligned}$$

k	C_k	C'_k
1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R}$
2	$\mathbb{H}$	$M_2(\mathbb{R})$
3	$\mathbb{H} \oplus \mathbb{H}$	$M_2(\mathbb{C})$
4	$M_2(\mathbb{H})$	$M_2(\mathbb{H})$
5	$M_4(\mathbb{C})$	$M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$
6	$M_8(\mathbb{R})$	$M_4(\mathbb{H})$
7	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$	$M_8(\mathbb{C})$
8	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})$

Table 1: Clifford algebras.

Proposition 1.4. *For all $k \geq 0$ there are periodicity isomorphisms*

$$C_{k+8} \cong C_k \otimes C_8$$

$$C'_{k+8} \cong C'_k \otimes C'_8$$

and $C_8 \cong C'_8 \cong M_{16}(\mathbb{R})$.

Proof. By repeated application of Proposition 1.3, we obtain that

$$\begin{aligned} C_{k+8} &\cong C_k \otimes C'_2 \otimes C_2 \otimes C'_2 \otimes C_2 \cong C_k \otimes \mathbb{H} \otimes \mathbb{H} \otimes M_2(\mathbb{R}) \otimes M_2(\mathbb{R}) \\ &\cong C_k \otimes M_4(\mathbb{R}) \otimes M_4(\mathbb{R}) \cong C_k \otimes M_{16}(\mathbb{R}). \end{aligned}$$

The other case is completely analogous. □

Given this periodicity, the the algebras C_k and C'_k are completely described by the Table 1.

2. Clifford modules

We turn now to the study of representations of Clifford algebras. As we already know, we can define subgroups of C_k^* (invertible elements in C_k), namely $Pin(k)$ and $Spin(k)$, that are two-fold coverings of $O(k)$ and $SO(k)$ respectively. The study of representations of Clifford algebras is particularly interesting because they give us, by restriction, non trivial representations of $O(k)$ and $SO(k)$.

If A is an algebra and M a (real) left A-module, then $\rho : A \to End(M)$ given by $\rho(a)(x) = a \cdot x$ is a representation of A on M. Conversely, if ρ is any representation of A on a (real) vector space E, then E becomes a (real) left A-module when we define $a \cdot x = \rho(a)(x)$. Thus, there is a one-to-one correspondence between left A-modules and representations of A. We will call a module irreducible if the associated representation is irreducible. As the Clifford algebras have the form $M_{2^k}(\mathbb{K})$, its representation theory is particularly simple and is given by the following theorem.

Theorem 2.1. *[2, p. 31] Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and consider the ring $M_n(\mathbb{K})$ of $n \times n$ matrices as al algebra over $\mathbb{R}$. Then the natural representation ρ of $M_n(\mathbb{K})$ on the vector space K^n is, up to equivalence, the only irreducible real (ungraded) representation on $M_n(\mathbb{K})$.*

The algebra $M_n(\mathbb{K}) \oplus M_n(\mathbb{K})$ has exactly two equivalence classes of irreducible real (ungraded) representations. They are given by

$$\rho_1(\phi_1, \phi_2) \equiv \rho(\phi_1) \quad \text{and} \quad \rho_2(\phi_1, \phi_2) \equiv \rho(\phi_2)$$

acting on $\mathbb{K}^n$.

The proof of this theorem is based in the fact that the algebras $M_n(\mathbb{K})$ are simple, and simple algebras just have one irreducible representation up to equivalence.

Now, following [1] we define $M(C_k)$ as the free abelian group generated by the classes of irreducible $\mathbb{Z}_2$-graded C_k-modules, and $N(C_k^0)$ as the corresponding group generated by *ungraded* C_k^0-modules. As we know well the ungraded modules, the first step is to reduce $M(C_k)$ to this case.

Proposition 2.1. *[1, p. 12] Let $R : M \mapsto M^0$ be the functor which assigns to a graded C_k-module $M = M^0 \oplus M^1$ the C_k^0-module M^0 Then R induces isomorphisms*

$$M(C_k) \cong N(C_k^0)$$

Proof. If M^0 is a C_k^0-module, let $S(M^0) = C_k \otimes_{C_k^0} M^0$. The action of C_k on C_k defines $S(M^0)$ as a graded C_k-module. Moreover, it is possible to show that $R \circ S$ and $S \circ R$ are isomorphic to the identity. □

Proposition 2.2. *[1, p. 12] Let $\phi : \mathbb{R}^k \to C_{k+1}^0$ be defined by $\phi(e_i) = e_i e_{k+1}$, for $i = 1, ..., k$. Then ϕ extends to yield an isomorphism $C_k \cong C_{k+1}^0$.*

In virtue of these two propositions, we may write down the groups $M(C_k)$ quite explicitly, as $N(C_{k-1})$. The results are presented in Table 2. Let $i : C_k \to C_{k+1}$ the inclusion that extends the standard inclusion $i : \mathbb{R}^k \to \mathbb{R}^{k+1}$, let $i_k^* : M(C_{k+1}) \to M(C_k)$ be the induced restriction and set $A_k = \text{coker}(i_k^* : M(C_{k+1}) \to M(C_k))$. The computation of these groups is treated in [1]. We just state their definition to quote an important result at the end.

3. Construction of vector fields on spheres

Let S^n denote the unitary sphere in $\mathbb{R}^{n+1}$. Consider the following problem: what is the maximal number of linearly independent vector fields that can be defined over S^n? If n is even, an elementary topological argument gives the answer: zero.[1] For n odd, this problem turns out to be very difficult. A good first step is to construct some.

[1] Basically, assuming the existence of such a vector field allows the construction of a homotopy between the identity and the antipodal map, and those have different degrees in the even case.

k	C_k	$M(C_k)$	A_k
1	$\mathbb{C}$	$\mathbb{Z}$	$\mathbb{Z}_2$
2	$\mathbb{H}$	$\mathbb{Z}$	$\mathbb{Z}_2$
3	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{Z}$	0
4	$M_2(\mathbb{H})$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$
5	$M_4(\mathbb{C})$	$\mathbb{Z}$	0
6	$M_8(\mathbb{R})$	$\mathbb{Z}$	0
7	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$	$\mathbb{Z}$	0
8	$M_{16}(\mathbb{R})$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$

Table 2: Clifford modules.

Suppose that $\mathbb{R}^{n+1}$ has the structure of a C_k-module, for some $k \in \mathbb{N}$. Then, a Clifford multiplication on $\mathbb{R}^{n+1}$ is defined and restricts to a continuous map $\cdot : R^k \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. For our construction bellow, this multiplication is required to be compatible with the inner product in $\mathbb{R}^{n+1}$; more precisely, we require the quadratic form Q on $\mathbb{R}^{n+1}$ to be invariant under Clifford multiplication. The following Lemma ensures that we can restrict ourselves to this case. For the statement, recall Proposition 1.2: if $\{e_i\}$ is a base of $\mathbb{R}^k$, then the monomials $\mathcal{M}_k = \{\prod_{i \in \mathcal{I}} e_i : \mathcal{I} \subset \{1, ..., k\}\}$ form a basis of C_k as an $\mathbb{R}$ vector space.

Lemma 3.1. *[3, p. 24] Let* $(\mathbb{R}^{n+1}, Q)$ *be a quadratic space, then*

$$Q'(x) := \frac{1}{2^k} \sum_{n \in \mathcal{M}_k} Q(m \cdot x),$$

defines a quadratic form on $\mathbb{R}^{n+1}$ *that is invariant under Clifford multiplication, namely* $Q'(v \cdot x) = Q'(x)$ *if* $\|v\| = 1$. *If* Q *is invariant, then* $Q' = Q$.

Proof. Just note that it is sufficient to check the invariance for $v = e_i$; the action of e_i just permute the terms $m \cdot x$, up to signs. □

Proposition 3.1. *[3, p. 24] Suppose that* $\mathbb{R}^{n+1}$ *admits the structure of a* C_k*-module. Then there exist* k *pointwise linearly independent vector fields on* S^n. *In fact, there exists a monomorphism of vector bundles:*

$$\mathbb{R}^k \times S^n \to TS^n.$$

Proof. We assume that the inner product on $\mathbb{R}^{n+1}$ is invariant under Clifford multiplication. So, if $v \in \mathbb{R}^k$ has norm one,

$$\langle v \cdot x, v \cdot y \rangle = \langle x, y \rangle.$$

The required morphism of vector bundles is given by

$$\begin{aligned} \mathbb{R}^k \times S^n &\to R^{n+1} \times S^n \\ (v, x) &\mapsto (v \cdot x, x). \end{aligned}$$

k	0	1	2	3	4	5	6	7
C_k	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$M_8(\mathbb{R})$	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$
b_k	1	2	4	4	8	8	8	8

Table 3: Dimension b_k of the irreducible C_k-modules.

We consider just the case $\|v\| = 1$. Then, by the invariance of the product:

$$\langle v \cdot x, x \rangle = \langle v \cdot v \cdot x, v \cdot x \rangle = -\langle x, v \cdot x \rangle.$$

Therefore $\langle v \cdot x, x \rangle = 0$, and the morphism takes values in TS^n (this is naturally identified with the sub-bundle of the trivial bundle $\mathbb{R}^{n+1} \times S^n$ given by pairs (v, s) such that $\langle v, s \rangle = 0$). □

At this point, a natural question arises: what is the maximal number of independent vector fields that can be constructed by this method? Or equivalently: what is the maximal k suck that $\mathbb{R}^{n+1}$ has the structure of a C_k-module? We denote this maximal k as $\rho(n+1)$. By the previous proposition, we can construct $\rho(n+1)$ pointwise linearly independent vector fields on S^n.

In 2 we determined the dimensions of the *irreducible* C_k-modules, that we denote now by b_k. Table 3 shows the first values; subsequent values are determined by the relation $b_{k+8} = 16b_k$, derived from Proposition 1.4.

In this terms, $\mathbb{R}^{n+1}$ has the structure of a C_k-module if and only if $n+1 \equiv 0 \mod b_k$. As the b^k are all powers of 2, it is convenient to write $n + 1 = (\text{odd number}) \cdot 2^{4a+b}$, where $a, b \in \mathbb{N}$ and $0 \leq b \leq 3$. At this point, it is straightforward to verify the following statement.

Theorem 3.1 (Radon, Hurwitz)**.** *On the sphere S^n there exists k pointwise linearly independent vector fields where $k = 8a + 2^b - 1$.*

Proof. As $b_{k+8} = 16b_k$, it is enough to consider the case $a = 0$. But then the result can be directly verified on the previous table. Note that when $n + 1$ is odd, $b = 0$ and we have no fields (as expected). □

Quite remarkably, this turns out to be also the maximal number.

Theorem 3.2 (Adams)**.** *[2, p. 46] The number of vector fields constructed on S^n in Proposition 3.1 is the largest possible number of linearly independent vector fields that exist on S^n.*

This important theorem was established by Adams using K-theory. The theorems presented here are a first suggestion of a very close relation between C_k-modules and the topological K-theory of vector bundles, that we review in the following section.

4. K-theory in a nutshell

Let X be a compact vector space. We denote by $V(X)$ the set of all isomorphism classes of $\mathbb{K}$ vector bundles over X. The set $V(X)$ is an abelian semigroup under direct sum of vector bundles. We can turn $V(X)$ into a group following a famous general construction recognized by Grothendieck: let $F(X)$ be the free abelian group generated by the elements of $V(X)$, and $E(X)$ its subgroup generated by elements of the form $[V] + [W] - ([V] \oplus [W])$, where $+$ denotes the addition in $F(X)$ and $\oplus$ the addition in $V(X)$. The K**-group** of X is defined to be the quotient

$$K^0(X) = F(X)/E(X).$$

It is clear that $K^0(X)$ is an abelian group. Its elements are called virtual bundles [2, p. 59]. This method generalizes the construction of $\mathbb{Z}$ from $\mathbb{N}$. The group $K^0(X)$ is denoted $KO^0(X)$ when $\mathbb{K} = \mathbb{R}$ and $KU^0(X)$ when $\mathbb{K} = \mathbb{C}$.

When V and W are bundles over spaces X and Y, respectively, then $V \otimes W$ is a bundle on $X \times Y$. If $X = Y$, we can use the diagonal map $\Delta : X \to X \times X$ to define an interior tensor product on $K(X)$:

$$[V] \cdot [W] = [\Delta^*(V \otimes W)].$$

This turns out to be well defined and gives $K(X)$ a ring structure.

Suppose now that $f : X \to Y$ is a continuous, then we have a map $f^* : V(Y) \to V(X)$ given by the induced bundle construction. Moreover, this is a semigroup homomorphism, that descends to a homomorphism of groups $f^* : K(X) \to K(Y)$. Hence, K is a contravariant functor from the category of topological spaces to that of abelian groups.

The K-groups allow us to construct a **generalized cohomology theory**, satisfying the axioms of Eilenberg-Steenrod. In this context, we must consider pointed topological spaces.

For a pointed space (X, p_X), define the **reduced** K**-ring** $\tilde{K}^0(X)$ as the kernel of the natural projection $K^0(X) \to K^0(p_X) \cong \mathbb{Z}$. We have then the following split short exact sequence:

$$0 \to \tilde{K}^0(X) \to K^0(X) \to K^0(p_X) \to 0.$$

The relative K-groups are defined as $K^0(X, Y) := \tilde{K}^0(X/Y)$, based at the class of Y. By convention $X/\emptyset = X \sqcup (\text{point})$, where $\sqcup$ denote disjoint union. Introduce the **wedge** and the **smash product** of two spaces (X, p_x) and (Y, p_Y), given respectively by

$$X \vee Y := (X \times p_Y) \cup (p_x \times Y) \subset X \times Y,$$
$$X \wedge Y := X \times Y / X \vee Y.$$

Define also the **(reduced) suspension** $\Sigma(X) := S^1 \wedge X$ and denote Σ^i its i-th iterate.

With all these tools, we are ready now to define the other K-groups. For X, Y compact pointed spaces, with $Y \subset X$, set for $i \geq 0$

$$\tilde{K}^{-i}(X) := \tilde{K}^0(\Sigma^i X),$$
$$K^{-i}(X,Y) := \tilde{K}^{-i}(X/Y) = \tilde{K}^0(\Sigma^i(X/Y)).$$

These group fit into the exact sequence

$$... \to \tilde{K}^{-(i+1)}(Y) \to K^{-i}(X,Y) \to \tilde{K}^i(X) \to \tilde{K}^i(Y) \to ... \to \tilde{K}^0(X) \to \tilde{K}^0(Y).$$

Quite remarkably, the ring structure over $K^0(X)$ extends to a graded ring structure over $K^{-*}(X)$. Periodicity also appears in this context.

Theorem 4.1 (Bott periodicity). *Let X be a compact Hausdorff space. Then the map*

$$\mu_x : KO^{-i}(X) \to KO^{-1-8}(X)$$

given by module multiplication by $x \in KO^{-8}$(point), is an isomorphism for all $i \geq 0$.

Remember that the groups A_k introduced in 2 were periodic, too. Moreover:

Theorem 4.2 (Atiyah-Bott-Shapiro isomorphism). *[1, p. 28] There is a (graded) ring isomorphism between A_* and $\sum_{k\geq 0} KO^{-k}$(point).*

This isomorphism could give a proof of Bott periodicity, but in fact the later is used [1] to establish the result. Anyway, the theorem shows a very intimate relation between C_k-modules and K-theory, as suggested in the previous section.

References

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