Sheaves on graphs and their homological invariants

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Outline

A **category C** consists of **objects** $a, b, c...$ and **arrows** (morphisms) $f, g, h....$ Each arrow f has a domain $(\operatorname{dom} f)$ and a codomain $(\operatorname{cod} f)$, both are objects. Moreover, for each object a there is a distinguished arrow 1_a, called identity, and for each pair of arrows (f, g) such that $\operatorname{dom}(g) = \operatorname{cod}(f)$, there is another arrow $g \circ f : \operatorname{dom} f \to \operatorname{cod} g$ called their **composition**. The operation of composition is supposed to be associative, and identities to act as neutral elements under composition.[1]

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A **functor** $T : \mathbf{C} \to \mathbf{D}$ assign to each object a of $\mathbf{C}$ an object $T(a)$ of $\mathbf{D}$, and to each morphism f of $\mathbf{C}$ a morphism $T(f)$ of $\mathbf{D}$, in such a way that

$$T(1_a) = 1_{Ta}, \quad T(g \circ f) = Tg \circ Tf.$$

A **subfunctor** S of $T : \mathbf{C} \to \mathbf{Sets}$ associates to every $c \in \mathrm{Ob}\,\mathbf{C}$ a subset $S(c)$ of $T(c)$ and to every arrow $f : c \to c'$ the restriction of $T(f)$ to $S(c)$.

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6. A poset is a set E with a binary relation $\leq$ that is reflexive, transitive and anti-symmetric. Every poset can be seen as a category, whose objects are the elements of E; there is an arrow $e \to e'$ iff $e \leq e'$. A functor between posets is a monotone map.

Natural transformations

Given two functors $S, T : \mathbf{C} \to \mathbf{D}$, a natural transformation $\tau : S \to T$ is a rule that assigns to each object c of $\mathbf{C}$ an arrow $\tau_c : Sc \to Tc$ of $\mathbf{D}$, so that $f : c \to c'$ in $\mathbf{C}$ yields the commutative diagram

$$\begin{array}{ccc} Sc & \xrightarrow{\tau_c} & Tc \\ \downarrow{\scriptstyle Sf} & & \downarrow{\scriptstyle Tf} \\ Sc' & \xrightarrow{\tau_{c'}} & Tc' \end{array} .$$

Limits

A diagram of shape $\mathbf{J}$ in $\mathbf{C}$ is a functor $F : \mathbf{J} \to \mathbf{C}$. The category $\mathbf{J}$ is thought here as an index category, usually finite.

A **cone** to the diagram $F : \mathbf{J} \to \mathbf{C}$ is an object N of $\mathbf{C}$ and a natural transformation $\psi : \underline{N} \to F$, where $\underline{N}$ is a constant functor of value N.

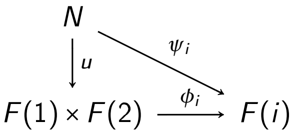
A **limit** of the diagram F is a universal cone (L, ϕ): this means that for any other cone (N, ψ) of F, there exists a *unique* arrow $u : N \to L$ such that, for every $X \in \mathrm{Ob}\,\mathbf{C}$, $\psi_X = \phi_X \circ u$.

A limit is unique *up to unique isomorphism*: if (L_1, ψ_1) and (L_2, ψ_2) are limits of F, the universal property gives unique maps $u : L_1 \to L_2$ and $v : L_2 \to L_1$ that are inverse to each other.

Colimits are defined similarly, as universal cocones $F \to \underline{N}$.

Limits: Examples

For instance, if **J** is a category with two objects (say 1 and 2) and their identity morphisms, then $F : \mathbf{J} \to \mathbf{Sets}$ is defined simply by a pair of sets, $F(1)$ and $F(2)$. A cone is a pair of maps $\psi_1 : N \to F(1)$ and $\psi : N \to F(2)$, and a limit is precisely the cartesian product $F(1) \times F(2)$, with the canonical projections $\phi_i : F(1) \times F(2) \to F(i)$. The universality means that there is a unique map $u : N \to F(1) \times F(2)$ such that



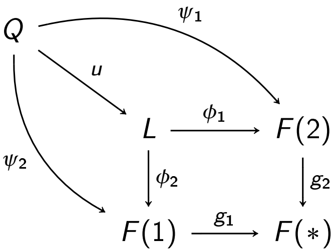
commutes for each i.

Limits: Examples

Similarly, one obtains fiber products as universal cones of **J**-diagrams, when **J** is the category with objects $*$, 1, and 2, and non-identity arrows

$$1 \longrightarrow * \longleftarrow 2.$$

The universal property looks like



(The components ϕ_* and ψ_* are omitted, because they can be deduced from the other arrows.)

Limits and colimits

Other examples of limits: terminal objects ($\mathbf{J}$ empty), equalizers ($\mathbf{J} = (* \rightrightarrows *)$), kernels (an equalizer with one of the non-identity arrows mapping to a zero map), etc.

Examples of colimits: initial objects, coequalizers, cokernels, etc.

Outline

Category of sheaves

Let $\mathbf{C}$ be a category.

The category of presheaves on $\mathbf{C}$, denoted $\widehat{\mathbf{C}}$ or $\mathrm{PSh}(\mathbf{C})$, is the category $[\mathbf{C}^{op}, \mathbf{Sets}]$ of functors from $\mathbf{C}^{op}$ to $\mathbf{Sets}$. The morphisms are natural transformations i.e. an arrow from a sheaf $\mathscr{A}$ to a sheaf $\mathscr{B}$ is a collection of functions $\{\tau_c : \mathscr{A}(c) \to \mathscr{B}(c)\}_{c \in \mathrm{Ob}\,\mathbf{C}}$, called **components**, such that for each $f : c \to c'$ in $\mathbf{C}$,

$$\begin{array}{ccc} \mathscr{A}(c') & \xrightarrow{\tau_{c'}} & \mathscr{B}(c') \\ \Big\downarrow \scriptstyle{\mathscr{A}(f)} & & \Big\downarrow \scriptstyle{\mathscr{B}(f)} \\ \mathscr{A}(c) & \xrightarrow{\tau_c} & \mathscr{B}(c) \end{array} .$$

Similarly, the category of sheaves of $\mathbb{F}$-vector spaces, denoted $\widehat{\mathbf{G}}_{\mathbb{F}}$ or $\mathbf{Mod}(\mathbb{F})$, is the category of functors $[\mathbf{C}^{op}, \mathbf{Vect}_{\mathbb{F}}]$. The morphisms are natural transformations whose components τ_c are linear maps.

Theorem

Let $\mathbf{C}$ *be a category. All limits and colimits exist in the category* $\widehat{\mathbf{C}}$*. Moreover, for each* $c \in \operatorname{Ob} C$*, the evaluation functor* $\mathrm{ev}_c : \widehat{\mathbf{C}} \to \mathbf{Sets}$, $\mathscr{A} \to \mathscr{A}(c)$ *commutes with limits and colimits.*

In other words, limits and colimits can be computed "object-wise". See [5, Tag 00VB].

For instance, the product of two sheaves $\mathscr{A}, \mathscr{B}$ in $\widehat{\mathbf{C}}$ is the sheaf that associates to $c \in \operatorname{Ob} \mathbf{C}$ the set $\mathscr{A}(c) \times \mathscr{B}(c)$, and to each arrow $f : c \to c'$ in $\mathbf{G}$ the map

$$\mathscr{A}(f) \times \mathscr{B}(f) : \mathscr{A}(c') \times \mathscr{B}(c') \to \mathscr{A}(c) \times \mathscr{B}(c).$$

Similarly, the "abelian" constructions are performed object-wise. For example, given a morphism $\tau : \mathscr{A} \to \mathscr{B}$ in $\widehat{\mathbf{C}}_{\mathbb{F}}$, its kernel is the presheaf that associates to each $c \in \operatorname{Ob} \mathbf{C}$ the vector space $\ker(\tau_c : \mathscr{A}(c) \to \mathscr{B}(c))$.

Yoneda embedding

Given a category $\mathbf{C}$. We suppose that for every $c, c' \in \operatorname{Ob}\mathbf{C}$, $\operatorname{Hom}(c, c')$ is a set.

Given $c \in \operatorname{Ob}\mathbf{C}$, the functor $\hbar_c : \mathbf{C}^{op} \to \mathbf{Sets}$ that associates to every $d \in \operatorname{Ob}\mathbf{C}$ the set $\hbar_c(d) = \operatorname{Hom}(d, c)$ and to every arrow $f : d \to d'$ the map $\hbar_c(f) : \operatorname{Hom}(d', c) \to \operatorname{Hom}(d, c), \phi \mapsto \phi \circ f$ is called the **presheaf represented by** c.

Proposition

Let $\mathbf{C}$ be a category, $\mathcal{F}$ a presheaf on $\mathbf{C}$, and c an object of $\mathbf{C}$. There exists an isomorphism, functorial in c and $\mathcal{F}$,

$$\iota : \operatorname{Hom}_{\widehat{\mathbf{C}}}(\hbar_c, \mathcal{F}) \xrightarrow{\sim} \mathcal{F}(c). \tag{1}$$

In particular, if $\mathcal{F} = \hbar_{c'}$, there is a bijection $\operatorname{Hom}_{\mathbf{C}}(c, c') = \operatorname{Hom}_{\widehat{\mathbf{C}}}(\hbar_c, \hbar_{c'})$: in other words, $\hbar : \mathbf{C} \to \widehat{\mathbf{C}}$ is a fully faithful functor.

Outline

Directed graphs

Definition

A **directed graph** (digraph) is a 4-tuple $G = (V_G, E_G, t_G, h_G)$, where V_G and E_G are sets (respectively, the vertexes and edges of the digraph), and $h_G : E_G \to V_G$ and $t_G : E_G \to V_G$ are functions that assign to each edge a “head” and a “tail”, respectively.

The digraphs are finite i.e. V_G and E_G are finite sets.

Definition

A **morphism of digraphs** $\mu : G \to K$ is a pair of maps $(\mu_V : V_G \to V_K, \mu_E : E_G \to E_K)$ that commute with the head and tail maps:

$$\begin{array}{ccc} E_G & \xrightarrow{t_G} & V_K \\ \downarrow{\scriptstyle \mu_E} & & \downarrow{\scriptstyle \mu_V} \\ E_K & \xrightarrow{t_K} & V_K \end{array} \qquad \begin{array}{ccc} E_G & \xrightarrow{h_G} & V_K \\ \downarrow{\scriptstyle \mu_E} & & \downarrow{\scriptstyle \mu_V} \\ E_K & \xrightarrow{h_K} & V_K \end{array}$$

The category of digraphs

The category **Digraphs** has a terminal object: the category $\Delta_\mathbf{0}$ with one object and one morphism (the identity).

It also has fiber products: given maps $\mu_1 : G_1 \to G$ and $\mu_2 : G_2 \to G$, their fiber product $K = G_1 \times_G G_2$ is defined by

$$V_K = \{(v_1, v_2) \in V_{G_1} \times V_{G_2} \,|\, \mu_1(v_1) = \mu_2(v_2)\},$$
$$E_K = \{(e_1, e_2) \in E_{G_1} \times E_{G_2} \,|\, \mu_1(e_1) = \mu_2(e_2)\},$$
$$t_K = (t_{G_1}, t_{G_2}), \quad h_K = (h_{G_1}, h_{G_2}).$$

Sheaves

Each digraph $G = (V, E, t, h)$ can be seen as a category $\mathbf{G}$, with objects $V \cup E$, and arrows $t(e) \to e$ and $h(e) \to e$ for each $e \in E$, in addition to the identities.

Remark that a morphism of digraphs $\mu : G \to K$ gives a functor $\mu : \mathbf{G} \to \mathbf{K}$

Definition

A **sheaf** of sets (resp. of $\mathbb{F}$-vector spaces) on G is an object of $\widehat{\mathbf{G}}$ (resp. $\widehat{\mathbf{G}}_{\mathbb{F}}$).[a]

[a] If the category $\mathbf{G}$ is equipped with the trivial Grothendieck topology, every presheaf on $\mathbf{G}$ is a sheaf according to the general definition [1, Def. 2.1].

In other words, $\mathcal{F}$ consists of

1. sets $\mathcal{F}(o)$ (called **values**) for each element $o \in V \cup E = \mathrm{Ob}\,\mathbf{G}$;
2. maps $\mathcal{F}(t, e) : \mathcal{F}(e) \to \mathcal{F}(te)$ and $\mathcal{F}(h, e) : \mathcal{F}(e) \to \mathcal{F}(he)$ (called **restriction maps**), for each $e \in E$.

Grothendieck's operations

Let $\varphi : \mathbf{G} \to \mathsf{K}$ be a functor.

Given a $\mathscr{B}$ on K, the sheaf $\varphi^*\mathscr{B} := \mathscr{B} \circ \varphi : \mathbf{G}^{op} \to \mathbf{Sets}$ is called its pullback. It maps $g \in \operatorname{Ob}\mathbf{G}$ to $\mathscr{B}(\varphi(g))$.

The morphism $\varphi^* : \widehat{\mathsf{K}} \to \widehat{\mathbf{G}}$ has a left adjoint $\varphi_! : \widehat{\mathbf{G}} \to \widehat{\mathsf{K}}$, which means that, for all $\mathscr{A} \in \widehat{\mathbf{G}}$ and $\mathscr{B} \in \widehat{\mathsf{K}}$

$$\operatorname{Hom}_{\widehat{\mathsf{K}}}(\varphi_!\mathscr{A}, B) \cong \operatorname{Hom}_{\widehat{\mathbf{G}}}(\mathscr{A}, \varphi^*\mathscr{B}).$$

Similarly, there is a right adjoint φ_* to φ^*. Therefore, φ^* commutes with limits and colimits (e.g. $\varphi^*(\mathscr{A} \times \mathscr{B}) = \varphi^*\mathscr{A} \times \varphi^*\mathscr{B}$, etc.); $\varphi_!$ commutes with colimits (a.k.a. inductive limits) and φ_* with limits (a.k.a. projective limits). These are general properties of adjoints.[2]

[2] The general construction of these functors is the subject of [1, Sec. I.5] (in french); the particular case of graphs is treated in [2, Sec. 1.4].

What is $\varphi_! : \widehat{\mathbf{G}} \to \widehat{\mathbf{K}}$ in the case of graphs?

Let $\varphi : \mathbf{G} \to \mathbf{K}$ be a morphism of digraphs. For any $\mathscr{A} \in \widehat{\mathbf{G}}$, we take

$$(\varphi_! \mathscr{A})(k) = \bigoplus_{g \in \varphi^{-1}(k)} \mathscr{A}(g).$$

Given a nonidentity arrow $f : v_k \to e_k$ in $\mathbf{K}$, the map $\varphi_! \mathscr{A}(f)$ is the only one that makes the diagram

$$\begin{array}{ccccc} \mathscr{A}(e_g) & & \xrightarrow{\iota_{e_g}} & & \bigoplus_{e \in \varphi^{-1}(e_k)} \mathscr{A}(e) \\ \downarrow & & & & \downarrow \varphi_! \mathscr{A}(f) \\ \mathscr{A}(e_g) & \xrightarrow{\mathscr{A}\xi} & \mathscr{A}(v_g) & \xrightarrow{\iota_{v_g}} & \bigoplus_{v \in \varphi^{-1}(v_k)} \mathscr{A}(v) \end{array} \tag{2}$$

Here $\xi : v_g \to e_g$ is an arrow $\mathbf{G}$ such that $f = \varphi(\xi)$ i.e. a tail map (resp. head map) if f is a tail (resp. head) map.

Remarks on $\varphi_!$

If φ is clear from context, we write $\mathscr{A}_G$ instead of $\varphi_! \mathscr{A}$.

When $\iota : G' \to G$ is an inclusion, then $\underline{\mathbb{F}}_{G'}$ is just the sheaf whose values are $\mathbb{F}$ on G' and 0 elsewhere.

If $\varphi : G \to K$ is a morphism of digraphs and $\mathscr{A} \in \widehat{\mathsf{K}}$, then

$$\mathscr{A}_G := \varphi_! \varphi^* \mathscr{A} = \mathscr{A} \otimes \underline{\mathbb{F}}_G.$$

The tensor product is computed object-wise. If $K' \to G$ is another morphism, then

$$\underline{\mathbb{F}}_K \otimes \underline{\mathbb{F}}_{K'} \simeq \underline{\mathbb{F}}_{K \times_G K'}.$$

If $L \to G$ is an arbitrary morphism of digraphs, then

$$\varphi^* \underline{\mathbb{F}}_L = \underline{\mathbb{F}}_{K \times_G L}.$$

If $\mu : G' \to G''$ is a morphism of graphs "over G" i.e. there is a commutative triangle

$$\begin{array}{ccc} G' & \xrightarrow{\quad\mu\quad} & G'' \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & G & \end{array} ,$$

then there is an induced morphism $\mu_* : \underline{\mathbb{F}}_{G'} \to \underline{\mathbb{F}}_{G''}$ in $\widehat{\mathbf{G}}$, which includes the category of digraphs over G as a subcategory of sheaves over G. *This functor is not full.*

Homology

Let $G = (V, E, h, t)$ be a digraph and $\mathscr{F}$ a sheaf on it. Set

$$\mathscr{F}(E) := \bigoplus_{e \in E} \mathscr{F}(e), \quad \mathscr{F}(V) = \bigoplus_{v \in V} \mathscr{F}(v).$$

Let $d_h : \mathscr{F}(E) \to \mathscr{F}(V)$ map $\phi \in \mathscr{F}(e)$ to $\mathscr{F}(h,e)(\phi) \in \mathscr{F}(he)$. A map d_t is defined similarly. Set $d = d_h - d_t$.

Definition

The **zeroth** and **first homology groups** of $\mathscr{F}$ are respectively

$$H_0(\mathscr{F}) := \operatorname{coker}(d) = \mathscr{F}(V)/\operatorname{im} d, \quad H_1(\mathscr{F}) := \ker(d).$$

The **Betti numbers** are their dimensions, $h_i(\mathscr{F}) = \dim H_i(\mathscr{F})$.

When $\mathscr{F} = \underline{\mathbb{F}}$, the constant sheaf with value $\mathbb{F}$, then d is the usual incidence matrix, and $H_i(G) := H_i(\underline{\mathbb{F}})$ is the usual homology of W seen as a directed CW-complex.

Euler characteristic

$$\chi(\mathscr{F}) := h_0(\mathscr{F}) - h_1(\mathscr{F}) = \dim \mathscr{F}(V) - \dim \mathscr{F}(E).$$

Algebraic graph theory

Given a sheaf $\mathcal{F}$ on a digraph G, suppose that for each $g \in \operatorname{Ob}\mathbf{G}$, $\mathcal{F}(g)$ is equipped with an inner product. Then there are adjoint operators d_h^*, d_t^* and $d^* = d_h^* - d_t^*$ from $\mathcal{F}(V)$ to $\mathcal{F}(E)$. The **laplacians** of $\mathcal{F}$ are

$$\Delta_0 = dd^* : \mathcal{F}(V) \to \mathcal{F}(V), \quad \Delta_1 = d^* d : \mathcal{F}(E) \to \mathcal{F}(E).$$

When $\mathbb{F}$ is of characteristic zero, then the Δ_i are positive semi-definite operators.

When $\mathcal{F} = \underline{\mathbb{F}}$, with the standard inner products, the laplacians above are the usual laplacians of the graph.

Moreover, one can define the “degree” operator $D_0 = d_h d_h^* + d_t d_t^*$ and the “adjacency” operator $A_0 = d_h d_t^* + d_t d_h^*$ in such a way that $\Delta_0 = D_0 - A_0$, etc.

What are the spectral properties of these matrices?

Theorem

To each short exact sequence $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ *(i.e. such that the kernel of each arrow is the image of the preceding one), there is a long exact sequence of homology groups*

$$0 \to H_1(\mathscr{F}_1) \to H_1(\mathscr{F}_2) \to H_1(\mathscr{F}_3) \to H_0(\mathscr{F}_1) \to H_0(\mathscr{F}_2) \to H_0(\mathscr{F}_3) \to 0.$$

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Definition

A sequence $x_0, \dots, x_n$ of real numbers is **triangular** if for all i, $0 < i < n$,

$$x_i \le x_{i-1} + x_{i+1}.$$

If $A \xrightarrow{f} B \xrightarrow{g} C$ satisfies $\operatorname{im} f = \ker g$, then

$$\dim B = \dim(\ker g) + \dim(\operatorname{im} g) = \dim(\operatorname{im} f) + \dim(\operatorname{im} g) \le \dim A + \dim C.$$

Hence the Betti numbers of a long exact sequence form a triangular sequence.

Quasi-Betti numbers

Definition

Let $\mathbf{G}$ be a digraph, and α_0, α_1 be two functions from $\text{Ob}\,\widehat{\mathbf{G}}$ to $[0,\infty)$. We say that (α_0,α_1) is a **quasi-Betti number pair** if

1. For each $\mathscr{A} \in \widehat{\mathbf{G}}$, $\alpha_0(\mathscr{A}) - \alpha_1(\mathscr{A}) = \chi(\mathscr{A})$.
2. For any $\mathscr{A},\mathscr{B} \in \widehat{\mathbf{G}}$ and $i \in \{1,2\}$,
$$\alpha_i(\mathscr{A}\oplus\mathscr{B}) = \alpha_i(\mathscr{A}) + \alpha_i(\mathscr{B}).$$
3. For any short exact sequence of sheaves on G, $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$, the sequence of integers
$$0, \alpha_1(\mathscr{F}_1), \alpha_1(\mathscr{F}_2), \alpha_1(\mathscr{F}_3), \alpha_0(\mathscr{F}_1), \alpha_0(\mathscr{F}_2), \alpha_0(\mathscr{F}_3), 0$$
is triangular.

We say that α_1 is a "first quasi-Betti number".

Outline

Hanna Neumann's conjecture

The conjecture is a statement about the rank of the intersection $K \cap L$ of two finitely generated subgroups K and L of a free group. (The rank is the smallest cardinality of a generating set.)

In 1954, Howson proved that the intersection of two finitely generated subgroups is always finitely generated. Hanna Neuman proved that

$$\operatorname{rank}(K \cap L) - 1 \leq 2(\operatorname{rank} K - 1)(\operatorname{rank} L - 1),$$

and she also conjectured that one can remove the factor 2 in the bound:

$$\operatorname{rank}(K \cap L) - 1 \leq (\operatorname{rank} K - 1)(\operatorname{rank} L - 1).$$

Later Walter Neumann proposed an stronger conjectural inequality, known as the SHNC.

SHNC: graph-theoretic version

A bicolored digraph is a directed graph G such that each edge is labeled 1 or 2; equivalently, it is a digraph morphism $\nu : G \to B_2$, where B_2 is the graph with one vertex and two loops. It is an étale bigraph if ν is étale: an injection of incoming (resp. outgoing) edges of v into incoming (resp. outgoing) edges of $\nu(v)$.

The SHNC is equivalent to

$$\rho(K \times_{B_2} L) \leq \rho(K)\rho(L)$$

for all étale bigraphs K and L, where ρ denotes the **reduced cyclicity** of a graph,

$$\rho(G) = \sum_{X \in \mathrm{conn}(G)} \max(0, h_1(X) - 1).$$

The sum runs over the connected components of G, and h_1 is its usual homology as a CW-complex (number of independent cycles).

Definition

Let $\mathscr{F}$ be a sheaf on a digraph G, and U a subspace of $\mathscr{F}(V)$. The head/tail neighborhood of U is

$$\Gamma_{\mathrm{ht}}(U) = \bigoplus_{e \in E} \{w \in \mathscr{F}(e) \mid d_h(w), d_t(w) \in U\}.$$

The excess of $\mathscr{F}$ at U is

$$\mathrm{ex}(\mathscr{F}, U) = \dim \Gamma_{ht}(U) - \dim U,$$

and its maximum over all subspaces of $\mathscr{F}(V)$ is the **maximum excess** of $\mathscr{F}$.

The excess is a supermodular function

$$\mathrm{ex}(U) + \mathrm{ex}(V) \leq \mathrm{ex}(U + V) + \mathrm{ex}(U \cap V),$$

hence the spaces that maximize it form a lattice.

The key fact is m.e.$(\mathbb{F}) = \rho(G)$.

Ideas for the proof: contagious vanishing

Theorem

If α_1 is any first quasi-Betti number for sheaves of $\mathbb{F}$-vector spaces on a graph G, and if $\alpha_1(\mathscr{F}) = 0$ for such certain sheaf $\mathscr{F}$, then for any subgraph G' of G it holds that $\alpha_1(\mathscr{F}_{G'}) = 0$.

Proof.

Consider the short exact sequence

$$0 \to \mathscr{F}_{G'} \to \mathscr{F} \to \mathscr{F}/\mathscr{F}_{G'} \to 0.$$

The triangularity of the sequence $0, \alpha_1(\mathscr{F}_{G'}), \alpha_1(\mathscr{F}), \ldots$ implies the result. □

To establish the SHNC in its graph-theoretic form, one proves first that the maximum excess is a first quasi-Betti number. Then one considers certain exact sequences

$$0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$$

where $\mathscr{F}_1$ is a so-called ρ-kernel. It is proved then that the maximum excess of a generic ρ-kernels vanish, which in turn implies that $\mathrm{m.e.}(\mathscr{F}_2) \leq \mathrm{m.e.}(\mathscr{F}_3)$.

For any subgraph $G' \subset G$, one can prove that tensoring with the sheaf $\underline{\mathbb{F}}_{G'}$ is an exact functor i.e. there are also short exact sequences

$$0 \to \mathscr{F}_1 \otimes \underline{\mathbb{F}}_{G'} \to \mathscr{F}_2 \otimes \underline{\mathbb{F}}_{G'} \to \mathscr{F}_3 \otimes \underline{\mathbb{F}}_{G'} \to 0.$$

In view of the last theorem and the remarks above, one gets the stronger statement $\mathrm{m.e.}((\mathscr{F}_2)_{G'}) \leq \mathrm{m.e.}((\mathscr{F}_3)_{G'})$—recall that $\mathscr{F}_1 \otimes \underline{\mathbb{F}}_{G'} = (\mathscr{F}_1)_{G'}$.

The maximum excess is a first quasi-Betti number?

To prove this, Friedman shows that the maximum excess of a sheaf $\mathscr{F}$ on G can be computed as a *twisted cohomology* of the sheaf $\varphi^*\mathscr{F}$ provided one has a "sufficiently good" covering map $\varphi : G' \to G$.

Twisted cohomology

Let $\mathbb{F}'$ be a field extension of $\mathbb{F}$, and $\psi : E_G \to \mathbb{F}'$ a function. By a twisting of $\mathcal{F} \in \widehat{\mathbf{G}}$ by ψ, we mean a sheaf of $\mathbb{F}'$-vector spaces $\mathcal{F}^{\psi}$ such that $\mathcal{F}^{\psi}(g) = \mathcal{F}(g) \otimes_{\mathbb{F}} \mathbb{F}'$, for each object g, and $\mathcal{F}^{\psi}(h,e) = \mathcal{F}(h,e)$, $\mathcal{F}^{\psi}(t,e) = \psi(e)\mathcal{F}(t,e)$.

In particular, ψ can be seen as $|E_G|$ indeterminates, in which case $\mathbb{F}'$ is taken to be $\mathbb{F}(\psi)$, the field of rational functions in the $\psi(e)$. The differential $d = d_{\mathcal{F}^{\psi}}$ is a morphism of finite dimensional vector spaces over $\mathbb{F}(\psi)$.

Definition

The i-th twisted homology group is $H_i^{\mathrm{twist}}(\mathcal{F})$, for $i = 0, 1$, is respectively the cokernel and kernel of $d_{\mathcal{F}^{\psi}}$.

There is an analogous short/long exact sequences theorem, hence the Betti numbers h_i^{twists} also give a triangular sequence.

$h_1^{twist}(\mathbb{F}) = \rho(G)$. In turn, $h_0^{twist}(\mathbb{F}) = h_1^{twist}(\mathbb{F}) + \chi(\mathbb{F}) = \rho(G) + \chi(G)$ is the number of acyclic components of G.

The fundamental theorem

Theorem

For any sheaf $\mathscr{F}$ on a digraph G, let $\mu : G'' \to G$ be a covering map where the Abelian girth is at least

$$2(\dim \mathscr{F}(V) + \dim \mathscr{F}(E)) + 1.$$

Then

$$h_1^{twist}(\mu^* \mathscr{F}) = \mathrm{m.e.}(\mu^* \mathscr{F}).$$

Recall that if $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$, the same is true for $0 \to \mu^* \mathscr{F}_1 \to \mu^* \mathscr{F}_2 \to \mu^* \mathscr{F}_3 \to 0$.

Friedman also proves that $\mathrm{m.e.}(\mu^* \mathscr{F}) = \mathrm{m.e.}(\mathscr{F}) \deg(\mu)$ using Galois theory of graphs.

Since h_1^{twist} is a first Betti number, one gets a triangular sequence involving the $\mu^* \mathscr{F}_i$, and normalization by $\deg(\mu)$ shows that the same holds for the maximum excess of the $\mathscr{F}_i$s.

Outline

Sheaves on topological spaces

Given a topological space (X,τ), let $\mathcal{O}(X)$ be the category whose objects are τ and whose arrows are inclusions.

A **sheaf** on X is a functor $F : \mathcal{O}(X)^{op} \to \mathbf{Sets}$ such that:

1. for any open covering $\{U_i\}_i$ of an open set U, if $f, g \in \mathcal{F}(U)$ are such that $f|_{U_i} = g|_{U_i}$ for each U_i, then $f = g$, and
2. for any open covering $\{U_i\}_i$ of an open set U, if $\{f_i \in \mathcal{F}(U_i\}_i$ is given such that $f_i|_{Ui \cap U_j} = f_j|_{U_i \cap U_j}$ for every pair (i,j), then there is $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$ for each i.

A **subsheaf** of a sheaf $\mathcal{F}$ is a subfunctor of $\mathcal{F}$ that is itself a sheaf.

The full subcategory of $\widehat{\mathcal{O}(X)}$ made of sheaves is denoted $\mathrm{Sh}(X)$ or $\widetilde{\mathcal{O}(X)}$. It has a terminal object, 1, that associates to every open U the singleton $\{*\}$ and to every inclusion the identity map. Remark that 1 is a representable functor, $1 = \hbar_X$.

Inclusion of $\mathcal{O}(X)$ in Sh(X)

From Sh(X) one can recover the lattice $\mathcal{O}(X)$ of open subsets of X as the lattice of *subsheaves* of the terminal sheaf 1.

Indeed, any open set U determines, by the Yoneda embedding, a subfunctor $\hbar_U$ of 1, and it is easy to verify that it is a sheaf. Conversely, if $\mathcal{F} \hookrightarrow 1$ is a monomorphism, then $\mathcal{F} = \hbar_W$, where $W = \bigcup\{U \in \mathcal{O}(X) \mid \mathcal{F}(U) = \{*\}\}$, which is clearly an open set that is mapped by $\mathcal{F}$ to $\{*\}$ by definition of a sheaf.

Thus we can recover X itself provided that each point is determined by its open neighborhoods. For instance, if X is Hausdorff. (The precise condition is being *sober*.)

Graphs as usual topological spaces

Let G be a graph *without self loops*. Then $\mathrm{Top}_G = \{\text{subgraphs of } G\}$ defines a topology on $V_G \sqcup E_G$.

An open set is called irreducible if it cannot be written as a union of its proper open subsets. The irreducible opens of $(V_G \sqcup E_G, \mathrm{Top}_G)$ are the vertexes $\{v\}$ and the sets $\{te, e, he\}$.

If every open can be written as a union of irreducible opens, a sheaf in the usual sense is determined by its values on these irreducibles. So we recover the definition above.

But here is the problem: if G is a category with one vertex v and one loop e, then the resulting topological space has trivial H^1. This is because one only gets *one* arrow from $\{v\}$ to $\{v, e, v\} = \{v, e\}$. This is always the case in topological spaces, because opens form a poset.

Beyond topology

As we saw, a sober topological space X can be recovered from the category $\mathrm{Sh}(X)$. Based on this result, Grothendieck and his school introduced a vast generalization of point-set topology. The idea is to introduce a notion of topology on an arbitrary category $\mathbf{C}$ (nowadays known as *Grothendieck topologies*) and to give a general definition of sheaf in that setting. Of course, the definitions must coincide with the former ones when $\mathbf{C} = \mathcal{O}(X)$.

Beyond topology

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In the topological case, a possible Grothendieck topology associates to every open U of X the set $J(U)$ of all the open coverings of U. Remark that:

1. Every open cover $\{U_i\}$ of U can be pulled-back under an inclusion $\iota : V \to U$ to get an open cover $\{V \cap U_i\}_i$ of V;
2. If each open set of an open cover $\{U_i\}_i$ of U is covered by opens $\{V^i_j\}_j$ (relative to U), then $\{V^i_j\}_{i,j}$ is an open covering of U;
3. For every U, the set $\{U\}$ is an open covering.

Remark that for an irreducible open U, $J(U) = \{U\}$.

Sieves

Given a category **C** and an object U, a **sieve** on U is a subfunctor $\mathcal{S}$ of $\hbar_U$. It generalizes the concept of *open covering*: given an open covering $\{U_i\}$ of an open U, the associated sieve $\mathcal{S} : \mathcal{O}(X)^{op} \to$ **Sets** satisfies $\mathcal{S}(V) = \{*\}$ iff $V \subset U_i$ for some i. (Remark that the sieve determines a subcategory of **C**.)

Definition ([1, Ex. II, Def. 1.1] or [5, Definition 00Z4])

A topology on **C** associates to every $U \in \mathrm{Ob}\,\mathbf{C}$ a set $J(U)$ of sieves on U such that:

1. For every morphism $f : \hbar_U \to \hbar_V$ and every element $\mathcal{S} \in J(U)$, the pullback $\mathcal{S}' \times_{\hbar_U} \hbar_V$ belongs to $J(V)$;
2. For all $U \in \mathrm{Ob}\,\mathbf{C}$ and all sieves $\mathcal{S}, \mathcal{S}'$ on U, if $\mathcal{S} \in J(U)$ and for all $(f : V \to U) \in \mathcal{S}(V)$ the pullback $\mathcal{S}' \times_{\hbar_U} \hbar_V \in J(V)$, then $S' \in J(U)$;[a]
3. For every $U \in \mathrm{Ob}\,\mathbf{C}$, the maximal sieve $\hbar_U$ belongs to $J(U)$.

[a] The morphism $\widetilde{f} : \hbar_V \to \hbar_U$ is the image of f under the Yoneda embedding.

In particular, $J(U) = \{\hbar_U\}$ defines a topology called chaotic or *grossière*.

Definition ([1, Ex. 2, Def. 2.1] or [5, Definition 00Z8])

Let $(\mathbf{C}, J)$ be a Grothendieck topology. A presheaf $\mathcal{F}$ is separable (resp. a **sheaf**) if for every object U of $\mathbf{C}$ and every sieve $\mathcal{S} \in J(U)$, the map

$$\mathrm{Hom}_{\widehat{\mathbf{C}}}(\mathcal{h}_U, \mathcal{F}) \to \mathrm{Hom}_{\widehat{\mathbf{C}}}(\mathcal{S}, \mathcal{F})$$

given by precomposition with $\mathcal{S} \hookrightarrow \mathcal{h}_U$ is an injection (resp. bijection).

If J is the *grossière* topology, then every presheaf is a sheaf.

Definition ([1, Ex. 4, Def. 1.1])

A category $\mathbf{T}$ is called a **topos** if it is equivalent to the category of sheaves on a Grothendieck topology $(\mathbf{C}, J)$.

Back to graphs

The sheaves on **G** according to Friedman's definition are precisely the sheaves on **G** equipped with the *grossière* topology.

Then every object of **G** is *gross* or irreducible. The sieve associated to a vertex $\{v\}$, seen as a subcategory of **G**, only contains $\{v\}$, but the sieve $\hbar_e$ associated to an edge e also contains the head and tail of e

So one might say that irreducible opens are either a vertex or an edge with its endpoints (which could also be a loop). A general subgraph is a colimit of representable sheaves $\hbar_X$.

Outline

Homology

It is certainly very difficult to say what *is* homology or cohomology, since it appears under many different flavors in many different contexts.

A traditional algebraic viewpoint, also introduced by Grothendieck in [3], regards (co)homology as a measure of the inexactness of a functor.

Exactness

A category is *abelian* if the usual operations common to the categories of abelian groups and modules have a meaning (addition of morphisms, kernels, cokernels, etc.). A functor between abelian categories is *exact* if it maps short exact sequences to exact sequences. Many functors are *not* exact. For instance, if $0 \to G_1 \to G_2 \to G_3 \to 0$ is an short exact sequence of abelian groups and G is an arbitrary abelian group, one only has

$$0 \to \mathrm{Hom}(G_3, G) \to \mathrm{Hom}(G_2, G) \to \mathrm{Hom}(G_1, G).$$

The derived functors of $\mathrm{Hom}(-, G)$, called $\{\mathrm{Ext}^i(-, G)\}_{i \geq 1}$, allow us to continue such exact sequence, in principle indefinitely to the right

$$0 \to \mathrm{Hom}(G_3, G) \to \mathrm{Hom}(G_2, G) \to \mathrm{Hom}(G_1, G) \to \mathrm{Ext}^1(G_3, G) \to \mathrm{Ext}^1(G_2, G) \to \mathrm{Ext}^1(G_1, G) \to \mathrm{Ext}^2(G_3, G) \to \cdots \quad (3)$$

The category $\widehat{\mathbf{G}}_{\mathbb{F}}$ of sheaves of $\mathbb{F}$-vector spaces on $\mathbf{G}$ is abelian; as we saw, the abelian operations are performed "object-wise".

Friedman's homology of a sheaf $\mathscr{F}$ in $\widehat{\mathbf{G}}_{\mathbb{F}}$ is

$$H_i(\mathbf{G},\mathscr{F}) := (\mathrm{Ext}^i(\mathscr{F},\underline{\mathbb{F}}))^*,$$

where $*$ denotes duality.

An injective resolution of $\underline{\mathbb{F}}$ gives the explicit formulae that we used above.

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