Information cohomology: An example involving topological spaces

Juan Pablo Vigneaux

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Let $(X, \mathfrak{T})$ be a topological space, and $\mathbf{S}$ a category whose objects are the open sets of X and whose arrows are opposite to inclusions, satisfying the following property: whenever $U, V \in \operatorname{Ob}\mathbf{S}$, then $U \cup V \in \operatorname{Ob}\mathbf{S}$ too. The union $U \cup V$ is a *product* in the category $\mathbf{S}$. Such category is a poset that satisfies the axioms of an information structure. We denote by $\mathcal{S}(U)$ the monoid $\{ V \mid U \to V \}$ with the union as product, and by $\mathcal{S}$ the corresponding presheaf.

Let us introduce a covariant functor $C : \mathbf{S} \to \mathbf{Sets}$ that associates to every $U \in \operatorname{Ob}\mathbf{S}$, the set $C(U)$ of continuous functions on U (alternatively: continuous of compact support, simple, etc.), and to every arrow $\pi : U \to V$ the usual restriction of functions (we denote $C(\pi)(f)$ by $\pi_* f$ or $f|_V$).

Let $\mathcal{F}(U)$ be the measurable functionals $\phi : C(U) \to \mathbb{R}$. For every arrow $\pi : U \to V$, there is an map $\mathcal{F}(\pi) : \mathcal{F}(V) \to \mathcal{F}(U)$, $\phi \mapsto \phi \circ C(\pi)$, that turns $\mathcal{F}$ into a presheaf.

The monoid $\mathcal{S}(U)$ acts on $\mathcal{F}(U)$ as follows: for every $V \in \mathcal{S}(U)$, $\phi \in \mathcal{F}(U)$ and $f \in C(U)$,

$$(V.\phi)(f) := \phi(f|_{V^c}) \tag{1}$$

It is easy to verify that this is in fact compatible with the product in the monoid:

$$\begin{aligned} W.(V.\phi)(f) &= (V.\phi)(f|_{W^c}) = \phi((f|_{W^c})|_{V^c}) \\ &= \phi(f|_{W^c \cap V^c}) = \phi(f|_{(W \cup V)^c}) \\ &= (V \cup W.\phi)(f). \end{aligned}$$

Since the definition of information cohomology only depends on the properties of the poset $\mathbf{S}$, it also makes sense in this context. For a 1-cochain $\phi \in \operatorname{Hom}(B_1, \mathcal{F})$, the 1-cocycle condition reads: for all $U \in \mathbf{S}$ and $V, W \in \mathcal{S}(U)$,

$$\phi_U[V \cup W] = V.(\phi_U[W]) + \phi_U[V]. \tag{2}$$

From the naturality of ϕ, we also have *joint locality*: for any W, $\phi_U[W](f) = \phi_W[W](f|_W)$. We write $\phi[U]$ instead of $\phi_U[U]$.

Setting $U = V$, one can conclude as usual that $\phi[U](0) = 0$.

For any $U, V \in \operatorname{Ob}\mathbf{S}$ such that $U \cup V \in \operatorname{Ob}\mathbf{S}$, we can write the symmetric equation: fro all $f \in C(U \cup V)$,

$$\phi[U](f|_U) - \phi[U](f|_{U \setminus V}) = \phi[V](f|_V) - \phi[V](f|_{V \setminus U}). \tag{3}$$

We conclude that both sides equal a "constant" $A = A(f|_{U \cap V})$.

Lemma 1. *Let μ be any measure defined on $(X, \mathfrak{B}(\mathfrak{T}))$. The cochain defined by*

$$\phi[V](f) := \int_V f \,\mathrm{d}\mu = \int_V f|_V \,\mathrm{d}\mu$$

is a 1-cocycle.

Proof. For any $f \in C(V \cup W)$,

$$\phi[V \cup W](f) := \int_{V \cup W} f \, \mathrm{d}\mu = \int_V f|_V \, \mathrm{d}\mu + \int_{V \cup W} f|_{(V \cup W) \setminus V} \, \mathrm{d}\mu \tag{4}$$

$$= \phi[V](f) + \int_W f|_{W \setminus V} \, \mathrm{d}\mu = \phi[V](f) + (V.\phi[W])(f). \tag{5}$$

□

In particular, the evaluation $f \mapsto f(x_0)$, for given $x_0 \in X$, is a 1-cocycle.

However, there are other 1-cocycles: for instance, $f \mapsto f'(x_0)$ is a 1-cocycle (supposing here that C are differentiable functions), as well as $f \mapsto (f(x_0))^2$.

Is it possible to characterize all the 1-cocycles in this general case? And when $\mathcal{F}$ is composed of linear functionals?