Information cohomology

An overview

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Outline

Objectives

1. To present a generalized geometry (in the sense of Grothendieck, Verdier...) adapted to stochastic models.

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2. To link algebraic ("axiomatic") and combinatorial properties of information functions.

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1. To present a generalized geometry (in the sense of Grothendieck, Verdier...) adapted to stochastic models. We shall see that there is an appropriate notion of "localization" (marginalizations) and a non-trivial cohomology theory.
2. To link algebraic ("axiomatic") and combinatorial properties of information functions.
3. To generalize some operational/"practical" properties (e.g. AEP) using this viewpoint.

3. If a choice be broken down into two successive choices, the original H should be the weighted sum of the individual values of H. The meaning of this is illustrated in Fig. 6. At the left we have three possibilities $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{3}$, $p_3 = \frac{1}{6}$. On the right we first choose between two possibilities each with probability $\frac{1}{2}$, and if the second occurs make another choice with probabilities $\frac{2}{3}$, $\frac{1}{3}$. The final results have the same probabilities as before. We require, in this special case, that

$$H(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{6}) = H(\tfrac{1}{2}, \tfrac{1}{2}) + \tfrac{1}{2}H(\tfrac{2}{3}, \tfrac{1}{3})$$

The coefficient $\frac{1}{2}$ is because this second choice only occurs half the time.

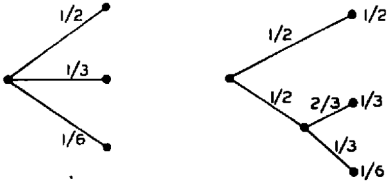

Fig. 6—Decomposition of a choice from three possibilities.

Some motivations

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The same algebraic construction fit recurrence properties like

$$\binom{n}{k_1, k_2, k_3} = \binom{n}{(k_1+k_2), k_3}\binom{k_1+k_2}{k_1, k_2}$$

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Both formulae are linked, since

$$\binom{n}{p_1 n, \ldots, p_s n} = \exp(nH(p_1, \ldots, p_s) + o(n)),$$

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is equivalent to

$$\exp(nH(p_1, p_2, p_3) + o(n)) = \exp\left(n\left\{H(p_1 + p_2, p_3) + (p_1 + p_2)H(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2})\right\} + o(n)\right).$$

Outline

Simplicial information structures

Consider a collection of random variables $\{X_i\}_{i\in I}$ (I countable), each X_i takes values in E_i.

Let $\Delta(I)$ be the category whose objects are subsets of I and whose arrows are opposite to inclusions (then I initial object, $\emptyset$ terminal).

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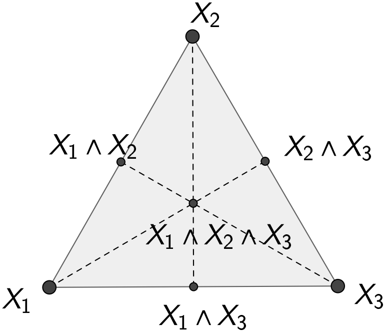
Definition

A **simplicial information structure** is a couple $(\mathcal{S}, E)$ such that

1. the category $\mathcal{S}$ of "variables" is isomorphic to a simplicial subcomplex K of $\Delta(I)$. Let us denote its objects X_S, with $S \subset I$. It is a poset such that, for every diagram $X_T \leftarrow X_S \rightarrow X_{T'}$, the meet $X_T \wedge X_{T'}$ (that equals $X_{T\cup T'}$) also belongs to $\mathcal{S}$ (think in term of faces of K...).
2. a functor $E : \mathcal{S} \rightarrow \mathcal{Sets}$ such that $E(X_\emptyset) = \{*\}$ and $E_S = \prod_{i\in S} E_i$, for a given collection $\{E_i\}_{i\in I}$. The arrows are the natural projections.

Examples: graphical models (Ising model, Markov fields, Bayesian networks...)


X_2
$X_1 \wedge X_2$
$X_2 \wedge X_3$
$X_1 \wedge X_2 \wedge X_3$
X_1
X_3
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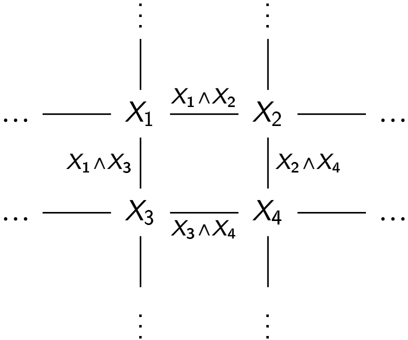
X_1
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Information topos

The category of covariant functors $[\mathscr{S}, \mathscr{S}ets]$ as well as that of contravariant functors $[\mathscr{S}^{op}, \mathscr{S}ets]$ are important in applications. (For example, probabilities are a covariant functor; the probabilitic functionals—like entropy—are contravariant.)

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1. Geometric intuitions
2. Logical theories
3. Universal constructions

Information cohomology: Definition

Remark that we can associate to each X the set $\mathscr{S}_X := \{Y : X \to Y\}$, that is a monoid under the multiplication $XY := X \wedge Y$ (i.e. $X_T X_{T'} = X_{T \cup T'}$, the joint variable). Each arrow $X \to Y$ induces an inclusion $\mathscr{S}_Y \to \mathscr{S}_X$, which defines a particular presheaf (contravariant functor). Let $\mathscr{A}$ denote the presheaf $X \mapsto \mathbb{R}[\mathscr{S}_X]$ (formal linear combinations); it is a presheaf of algebras.

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The category $\mathcal{Mod}(\mathcal{A})$ of $\mathcal{A}$-modules is an abelian category (i.e. it behaves like the category of modules over a ring), which implies that we can do homological algebra with it. For example, we can introduce the derived functors of $\text{Hom}_{\mathcal{A}}(\mathbb{R}, -)$, denoted $\text{Ext}^\bullet(\mathbb{R}, -)$.

Definition

The **information cohomology** with coefficients in an $\mathcal{A}$-module M is

$$H^\bullet(\mathcal{S}, M) := \text{Ext}^\bullet(\mathbb{R}, M).$$

Information cohomology: Computation

This is computable because there is a projective resolution $0 \leftarrow \mathbb{R} \leftarrow B_0 \leftarrow B_1 \leftarrow B_2 \leftarrow \cdots$ (via the bar construction), such that:

- $B_0(X) := \mathcal{A}_X$-module freely generated by the symbol $[]$,
- $B_1(X) := \mathcal{A}_X$-module freely generated by the symbols $[Y]$ with $Y \in \mathcal{S}_X$, etc.

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Therefore information cohomology can be computed as the cohomology of the differential complex $(\mathrm{Hom}_{\mathcal{A}}(B_\bullet, F), \delta)$, with

$$\delta f[X_1|...|X_{n+1}] = X_1.f[X_2|...|X_{n+1}] + \sum_{k=1}^{n} (-1)^k f[X_1|...|X_k X_{k+1}|...|X_n] + (-1)^{n+1} f[X_1|...|X_n] \quad (1)$$

for any $f \in \mathrm{Hom}_{\mathcal{A}}(B_n, F)$. Notation: $f[X_1|...|X_{n+1}] := f([X_1|...|X_{n+1}])$.

Let $Q : \mathcal{S} \to \mathcal{S}ets$ be a functor that associates to each X the set $Q(X)$ of probas on E_X. Every arrow $X \to Y$ is related to a surjection $\pi : E_X \to E_Y$ that induces a *marginalization* $\pi_* := Q(\pi) : Q(X) \to Q(Y)$ given by $\pi_* p(y) = \sum_{x \in \pi^{-1}(y)} p(x)$.

(Whenever $\mathcal{S} \neq \Delta(I)$, a section $q \in \mathrm{Hom}_{[\mathcal{S}, \mathcal{S}ets]}(*, Q)$ is just a coherent collection of probabilities, called "pseudo-marginal" in the literature.)

Information cohomology: Probabilistic case

Let $Q : \mathcal{S} \to \mathcal{S}ets$ be a functor that associates to each X the set $Q(X)$ of probas on E_X. Every arrow $X \to Y$ is related to a surjection $\pi : E_X \to E_Y$ that induces a *marginalization* $\pi_* := Q(\pi) : Q(X) \to Q(Y)$ given by $\pi_* p(y) = \sum_{x \in \pi^{-1}(y)} p(x)$.

(Whenever $\mathcal{S} \neq \Delta(I)$, a section $q \in \mathrm{Hom}_{[\mathcal{S},\mathcal{S}ets]}(*, Q)$ is just a coherent collection of probabilities, called "pseudo-marginal" in the literature.)

Let $F(X)$ be the additive abelian group of measurable real-valued functions on $Q(X)$, and $F(\pi) : F(Y) \to F(X)$ (contravariant) such that $F(\pi)(\phi) = \phi \circ \pi_*$.

For each $Y \in \mathcal{S}_X$ and $\phi \in F(X)$, define

$$(Y.\phi)(P_X) = \sum_{y \in E_Y} P_X(Y = y_i)^\alpha \phi(P_X|_{Y=y_i}). \tag{2}$$

By convention, a summand is simply 0 if $P_X(Y = y_i) = 0$. This turns F into an $\mathcal{A}$-module that we denote F_α.

Computing $H^\bullet(\mathcal{S}, F_\alpha)$

- The 0-cochains are $\phi \in \mathrm{Hom}_{\mathcal{A}}(B_0, F_\alpha)$: this is a collection $\{\phi_X[]\}_{X \in \mathrm{Ob}\, S}$ but functoriality properties mean that for any X, $\phi_X[] = \phi_1[] =$ constant $\in \mathbb{R}$ (say $= K$).
- 0-cocyles: the boundary of ϕ is $(\delta\phi)[Y] = Y.\phi[] - \phi[]$ which evaluated on a proba $P \in Q(X)$ reads

$$\sum_{y \in E_Y} P(y)^\alpha K - K = \begin{cases} 0 & \text{if } \alpha = 1 \\ KS_\alpha[Y] & \text{otherwise} \end{cases}.$$

- The 1-cochains are characterized by collections of functionals $\{\phi[X] : Q(X) \to \mathbb{R}\}_{X \in \mathrm{Ob}\, S}$.
- The 1-cocycles are cochains that satisfy

$$0 = X.\phi[Y] - \phi[XY] + \phi[X]$$

as functions on $Q(XY)$, marginalizations are implicit.

Computing $H^\bullet(\mathcal{S}, F_\alpha)$

When $X \neq Y$, the equation

$$0 = X.\phi[Y] - \phi[XY] + \phi[X]$$

has as unique solution (up to a multiplicative constant) the function

$$S_\alpha[X] = \begin{cases} -\sum_{x\in E_X} P(x)\log P(x) & \text{when } \alpha = 1 \\ \sum_{x\in E_X} P(x)^\alpha - 1 & \text{when } \alpha \neq 1 \end{cases}.$$

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Globally, there number of free constants depends on $\beta_0(K)$, the number of connected components of the simplicial complex K,

$$H^1(\mathcal{S}, F_1) \cong \mathbb{R}^{\beta_0(K)}; \quad H^1(\mathcal{S}, F_\alpha) \cong \mathbb{R}^{\beta_0(K)-1} \text{ when } \alpha \neq 1.$$

Let $C : \mathcal{S} \to \mathcal{S}ets$ be a functor that associates to each X the set $C(X)$ of functions $\nu : E_X \to \mathbb{N}$ such that $\|\nu\| := \sum_{x \in E_X} \nu(x) > 0$. Given $\pi : X \to Y$, the arrow $\pi_* := C(\pi) : C(X) \to C(Y)$ is given by $\pi_* \nu(y) = \sum_{x \in \pi^{-1}(y)} \nu(x)$.

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Let $G(X)$ be the multiplicative abelian group of measurable $(0,\infty)$-valued functions on $C(X)$, and $C(\pi) : C(Y) \to C(X)$ (contravariant) such that $C(\pi)(\phi) = \phi \circ \pi_*$.

For each $Y \in \mathcal{S}_X$ and $\phi \in G(X)$, define

$$(Y.f)(P_X) = \prod_{\substack{y \in E_Y \\ \nu(Y=y) \neq 0}} \phi(\nu|_{Y=y_i}). \tag{3}$$

This turns G into an $\mathcal{A}$-module.

Computing $H^\bullet(\mathcal{S}, F_\alpha)$

- The 0-cochains are $\phi \in \mathrm{Hom}_{\mathcal{A}}(B_0, F_\alpha)$: this is a collection $\{\phi_X[]\}_{X \in \mathrm{Ob}\, S}$ but functoriality properties mean that for any X, $\phi_X[](\nu) = \phi_1[](\pi_{1X*}\nu) =: \varphi(\|\nu\|)$.
- 0-cocyles: the cocycle condition is $1 = (\delta\phi)[Y] = (Y.\phi[])(\phi[])^{-1}$ which evaluated on a counting function $\nu \in C(Y)$ gives

$$\varphi(\|\nu\|) = \varphi(\nu_1)\varphi(\nu_2)\cdots\varphi(\nu_s),$$

  whose only solutions are $\varphi(x) = \exp(kx)$, with $k \in \mathbb{R}$.
- The 1-cochains are characterized by collections of functionals $\{\phi[X] : C(X) \to \mathbb{R}\}_{X \in \mathrm{Ob}\, S}$.
- The 1-cocycles are cochains that satisfy

$$\phi[XY] = (X.\phi[Y])\phi[X]$$

  as functions on $C(XY)$, marginalizations are implicit.

Combinatorial 1-cocycles

The solutions to $\phi[XY] = (X.\phi[Y])\phi[X]$ are

$$\phi[Y](v) = \frac{[\|v\|]_D!}{\prod_{y \in E_Y} [v(y)]_D!}$$

where $[0]_D! = 1$ and $[n]_D! = D_n D_{n-1} \cdots D_1 D_0$, for any sequence $\{D_i\}_{i \geq 1}$ such that $D_1 = 1$.

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- When $D_n = n$, we obtain the usual multinomial coefficients, denoted $\binom{\|v\|}{v_1, \ldots, v_s}$.
- When $D_n = \frac{q^n - 1}{q - 1}$, we obtain the q-multinomial coefficients, denoted $\left[{\|v\| \atop v_1, \ldots, v_s}\right]_q$. If q is a prime power, $\left[{n \atop k_1, \ldots, k_s}\right]_q$ counts the number of flags of vector spaces $V_1 \subset V_2 \subset \ldots \subset V_s = \mathbb{F}_q^n$ such that $\dim V_i = \sum_{j=1}^{i} k_j$.
- When D_n is the Fibonacci sequence, we obtain the so-called Fibonimial coefficients.

Asymptotic relation I

Proposition

Let ϕ be a combinatorial 1-cocycle. Suppose that, for every variable X, there exists a measurable function $\psi[X]:\Delta(X)\to\mathbb{R}$ with the following property: for every sequence of counting functions $\{\nu_n\}_{n\geq 1}\subset C_X$ such that

1. $\|\nu_n\|\to\infty$, *and*
2. *for every* $x\in E_X$, $\nu_n(x)/\|\nu_n\|\to p(x)$ *as* $n\to\infty$

the asymptotic formula

$$\phi[X](\nu_n)=\exp(\|\nu_n\|^{\alpha}\psi[X](p)+o(\|\nu_n\|^{\alpha}))$$

holds. Then ψ is a 1-cocycle of type α, i.e. $f\in Z^1(\mathcal{S},F_\alpha)$.

Example:

- 0-cocycles: the exponential $\exp(k\|v\|)$ is a combinatorial 0-cocycle, the constant k is a probabilistic 0-cocycle.
- 1-cocycles:

$$\binom{n}{p_1 n, \ldots, p_s n} = \exp(nH(p_1, \ldots, p_s) + o(n))$$

and we're going to see that

$$\begin{bmatrix} n \\ p_1 n, \ldots, p_s n \end{bmatrix}_q = \exp(n^2 \frac{\ln q}{2} H_2(p_1, \ldots, p_s) + o(n^2)).$$

Given an indeterminate q, the q-integers $\{[n]_q\}_{n\in\mathbb{N}}$ are defined by $[n]_q := (q^n - 1)/(q-1)$, the q-factorials by $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$, and the q-multinomial coefficients are

$$\begin{bmatrix} n \\ k_1, \ldots, k_s \end{bmatrix}_q := \frac{[n]_q!}{[k_1]_q! \cdots [k_s]_q!}, \tag{4}$$

where $k_1, \ldots, k_s \in \mathbb{N}$ are such that $\sum_{i=1}^{s} k_i = n$. When q is a prime power, these coefficients count the number of flags of vector spaces $V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{F}_q^n$ such that $\dim V_i = \sum_{j=1}^{i} k_j$

New example: q-multinomials

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Being 1-cocycles they satisfy multiplicative relations like

$$\begin{bmatrix} n \\ k_1, k_2, k_3 \end{bmatrix}_q = \begin{bmatrix} n \\ (k_1 + k_2), k_3 \end{bmatrix}_q \begin{bmatrix} k_1 + k_2 \\ k_1, k_2 \end{bmatrix}_q.$$

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implies

$$H_2(p_1, p_2, p_3) = H_2(p_1 + p_2, p_3) + (p_1 + p_2)^2 H\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right).$$

Extensions and open problems

There is a quantum version of the theory, where Von Neumann entropy appears as a 0-cochain, whose coboundary is related to Shannon entropy.

I am working now with continuous variables. The cocycle equation gives an algebraic characterization of differential entropy.

The same formalism gives other derived functors. For example, the derived functors of the global sections functor $\Gamma_{\mathcal{S}}(-)$.

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