

III. Diagrammatics for Soergel categories

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SC_1 : category of Bott-Samelson bimodules

* Objects : $w = s_{i_1} \cdots s_{i_k} \rightsquigarrow BS(w) = B_{i_1} \otimes_R \cdots \otimes_R B_{i_k}$

* Morphisms : $\text{Hom}_{SC_1}(B, B') = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{R\text{-mod}_{\mathbb{Z}}-R}(B, B'(n))$

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SB : category of Soergel bimodules

$$SC_1 \xrightarrow[\text{grading shifts}]{\text{additive envelope}} SB_2 \xrightarrow{\text{Karoubi envelope}} SB$$

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SB : category of Soergel bimodules

$$SC_1 \xrightarrow[\text{grading shifts}]{\text{additive envelope}} SB_2 \xrightarrow{\text{Karoubi envelope}} SB$$

→ All the information contained in SB can be recovered from SB_1

→ Objects in SB_1 combinatorial in nature

→ we can capture this describing morphisms by planar graphs

III. Diagrammatics for Soergel categories

DB_1 : diagrammatic category

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B_1

s_1

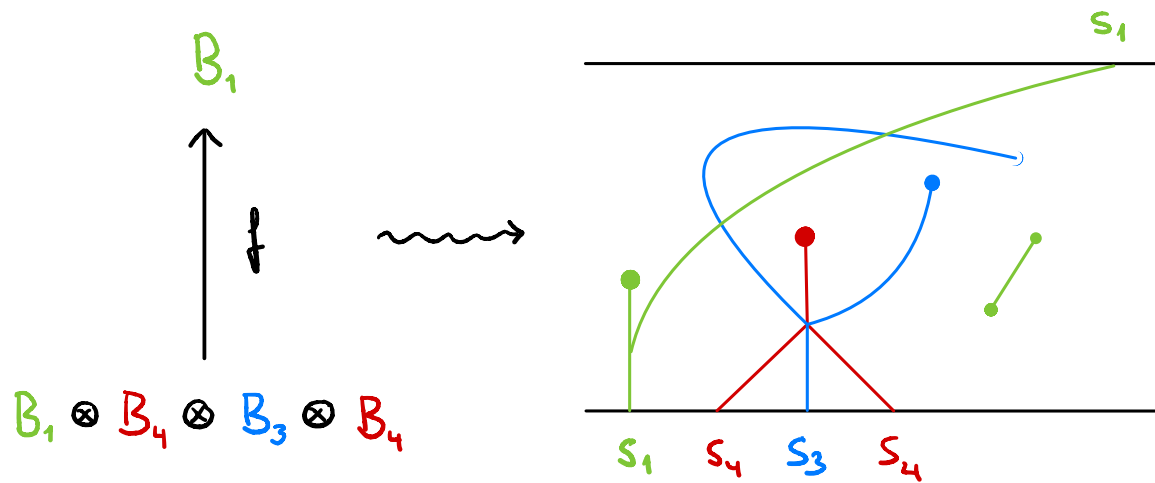
DB_1 : diagrammatic category

* Objects: words $w = s_{i_1} \cdots s_{i_n}$

$B_1 \otimes B_4 \otimes B_3 \otimes B_4$

$s_1 \quad s_4 \quad s_3 \quad s_4$

III. Diagrammatics for Soergel categories

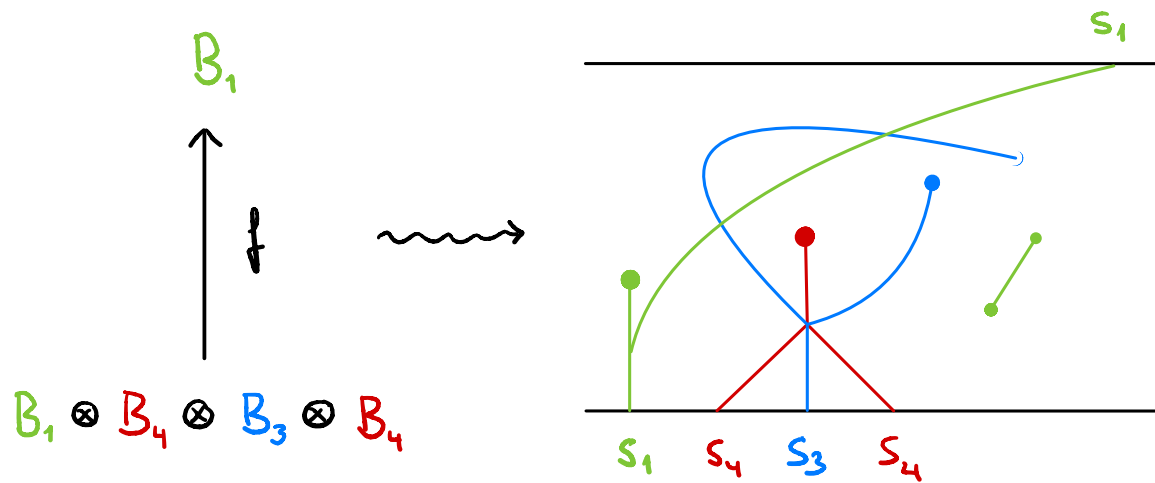


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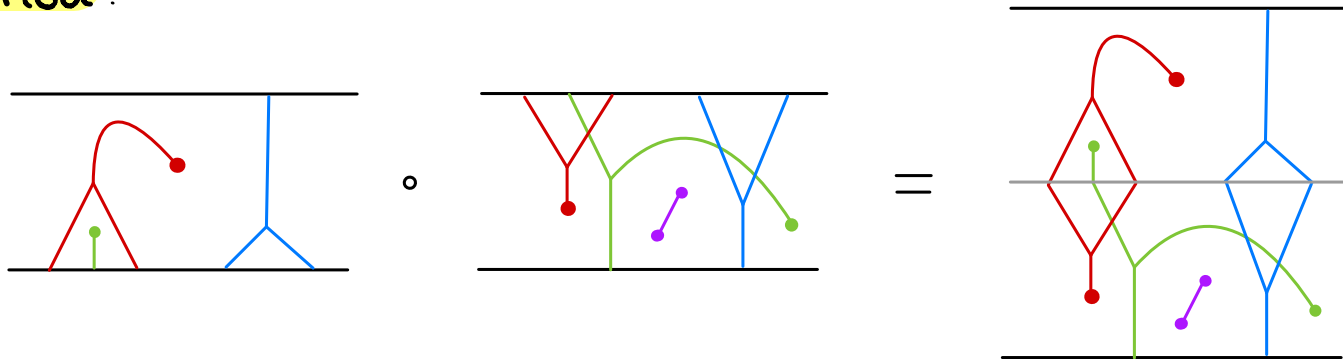


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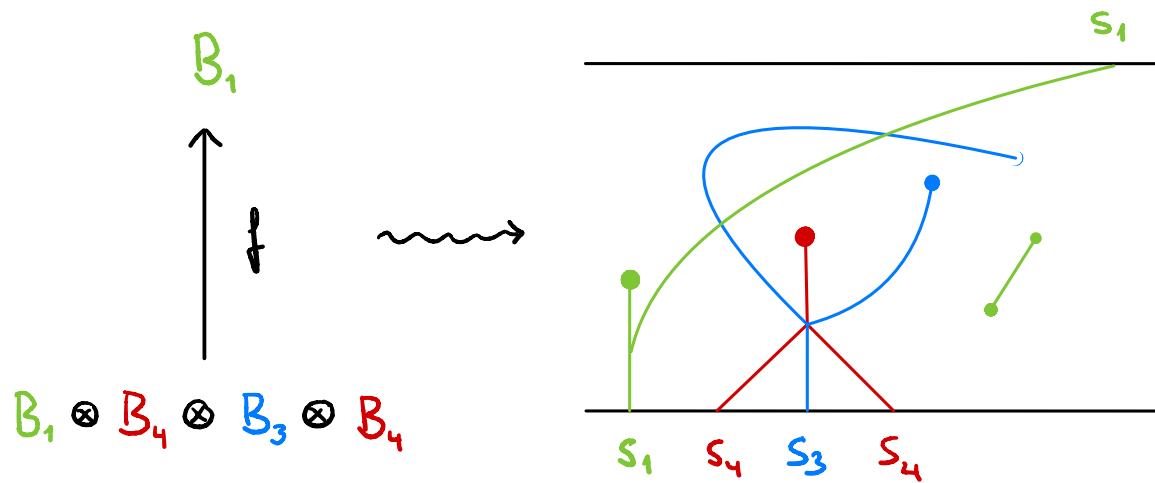
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• Composition:



III. Diagrammatics for Soergel categories

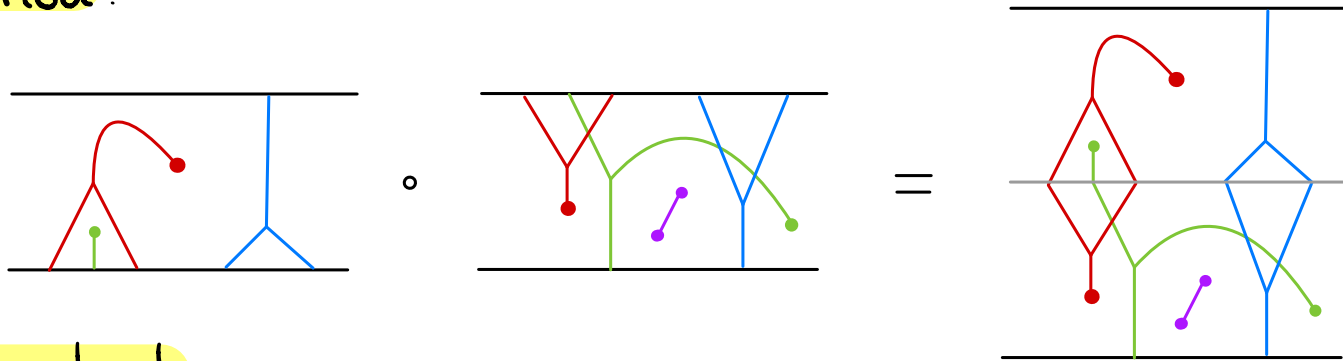


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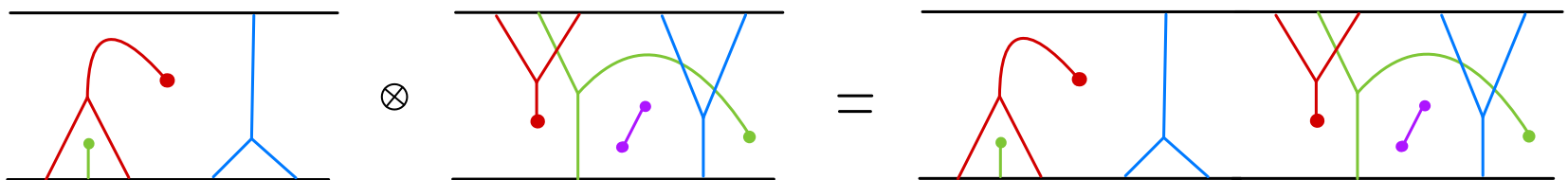
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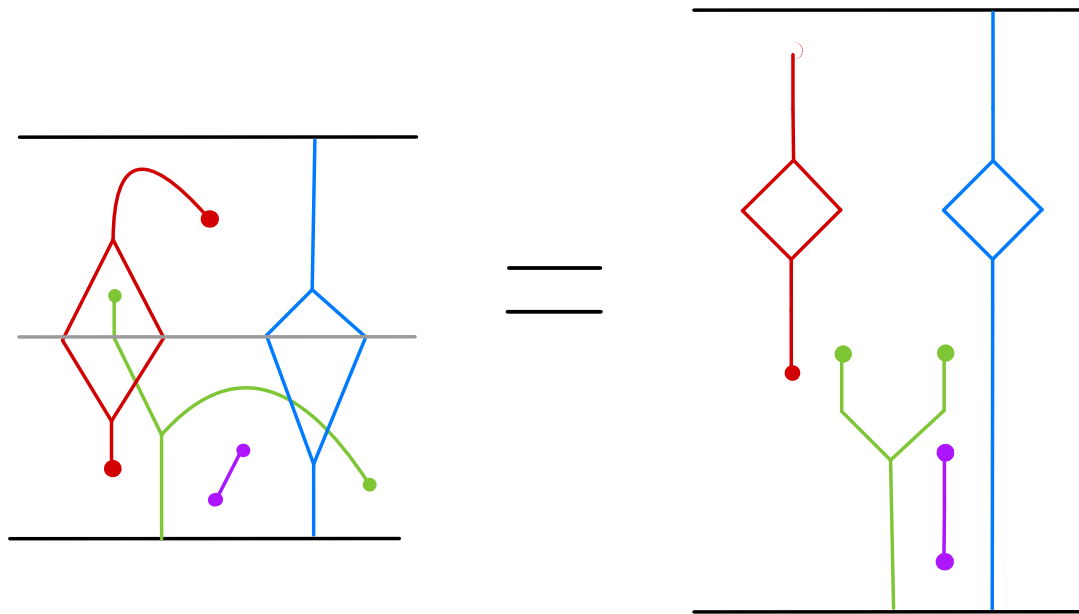


• Tensor product:



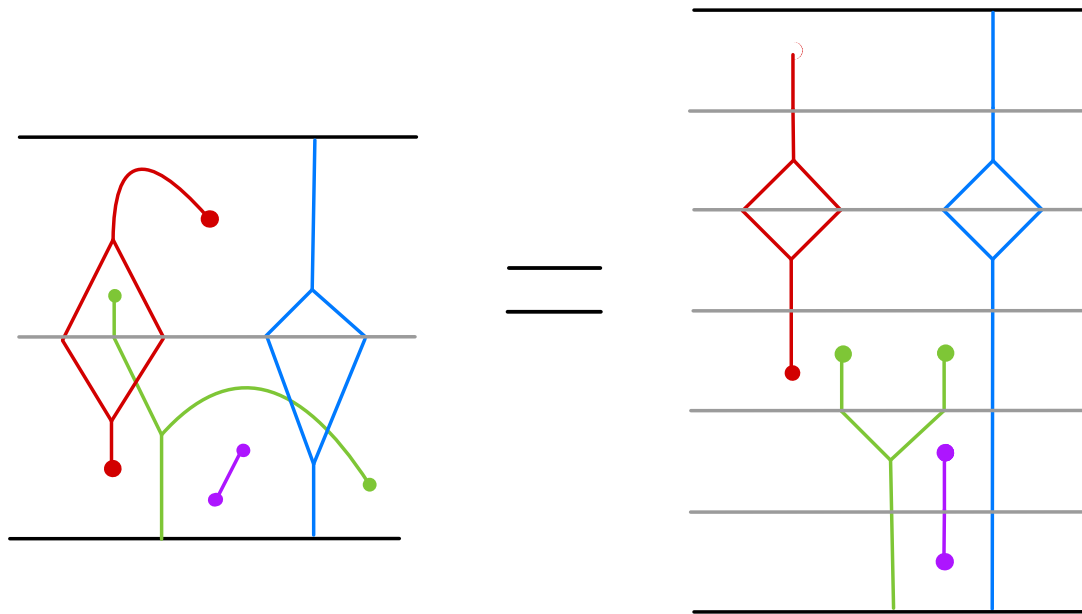
III. Diagrammatics for Soergel categories

Using relations we can **deform** diagrams



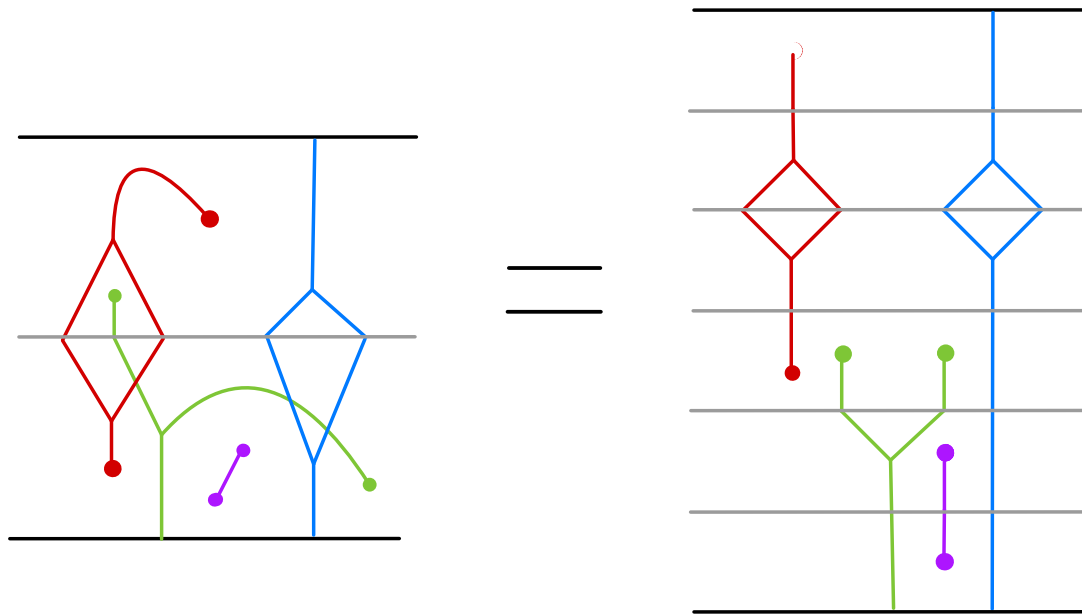
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Using relations we can **deform** diagrams and **decompose** them into some "elementary pieces":



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Using relations we can **deform** diagrams and **decompose** them into some "elementary pieces":



We can describe the diagrammatic category SB by **generators** and **relations**

III. 1. The diagrammatic category DB_1

We construct a category DB_1 of diagrams via generators and local relations by associating a colored graph to each morphism in SB_1 :

$$DB_1 \xrightarrow{F_1} SB_1$$

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We construct a category DB_1 of diagrams via generators and local relations by associating a colored graph to each morphism in SB_1 :

$$\begin{array}{ccc} DB_1 & \xrightarrow{F_1} & SB_1 \\ \oplus, (-)^{sh} \downarrow & & \downarrow \oplus, (-)^{sh} \\ DB_2 & \xrightarrow{F_2} & SB_2 \\ \text{Kar}(-) \downarrow & & \downarrow \text{Kar}(-) \\ DB & \xrightarrow{F} & SB \end{array}$$

F_1 is an equivalence of categories and induces equivalences F_2 and F .

III. 1. The diagrammatic category DB_1

Fix an integer $n \in \mathbb{Z}$

Objects finite sequences of indices taken from $\{1, \dots, n\}$

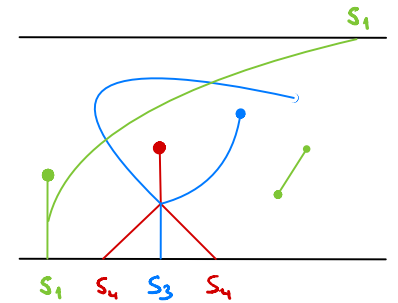
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Fix an integer $n \in \mathbb{Z}$

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→ we represent them by colored points in $\mathbb{R}$

→ color convention: **red** and **blue** represent adjacent indices and **green** is distant to both of them



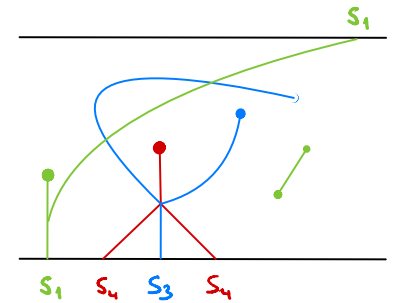
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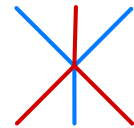
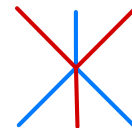
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Generators



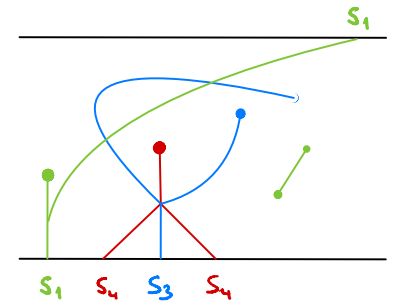
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Generators



$$i \rightarrow \emptyset$$



$$\emptyset \rightarrow i$$



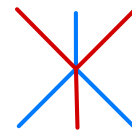
$$i \rightarrow ii$$



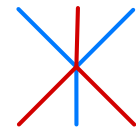
$$ii \rightarrow i$$



$$ij \rightarrow ji$$



$$i(i+1)i \rightarrow (i+1)i(i+1)$$



$$(i+1)i(i+1) \rightarrow i(i+1)i$$

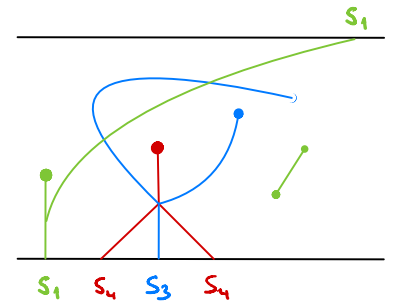
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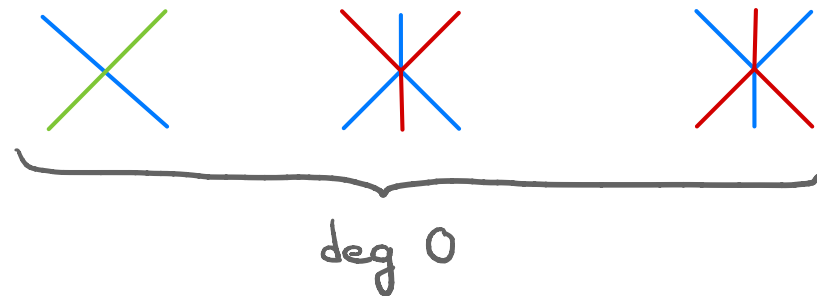
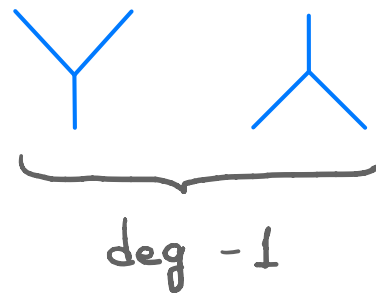
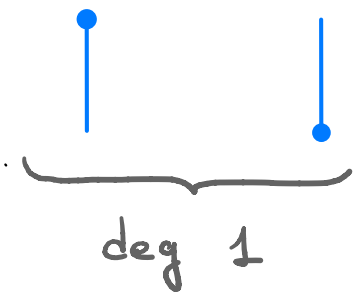
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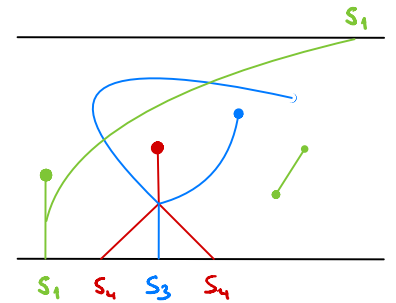
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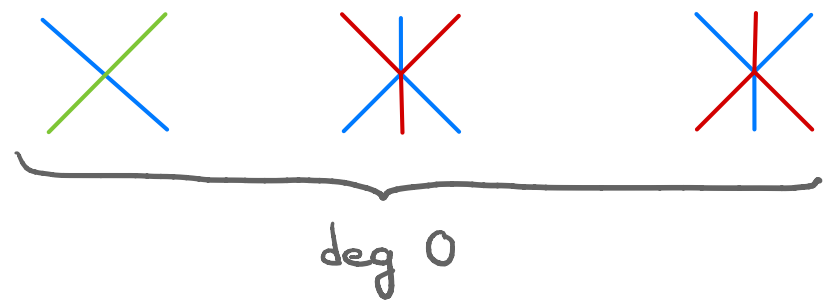
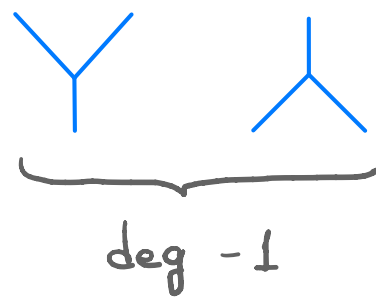
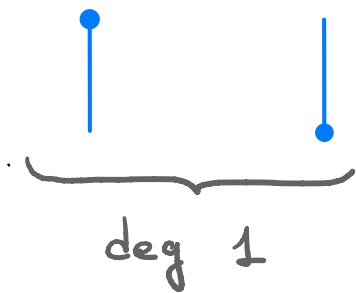
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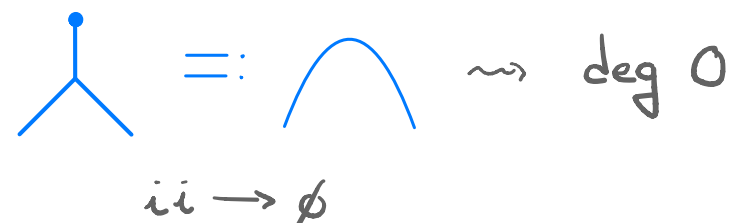
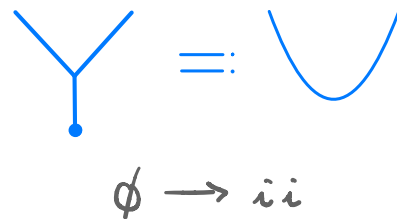
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Generators



→ Shorthands:



III. 1. The diagrammatic category DB_1

Relations

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Relations

* Isotopy relations

$$\cap = | = \cup$$

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The morphism represented by a particular graph is independent of the isotopy class of the embedding.

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Relations

* Isotopy relations

A diagram illustrating the equivalence of three representations of a vertical line. It consists of three blue shapes connected by equals signs: a wavy line, a straight vertical line, and an inverted wavy line.

A diagrammatic equation showing the crossing of two lines. On the left, a red line and a blue line cross, with the red line on top and the blue line on the bottom. This is equal to a diagram where the red line is on the bottom and the blue line is on the top. This is equal to a diagram where the red line is on the top and the blue line is on the bottom, but the lines are now curved to represent a different configuration.

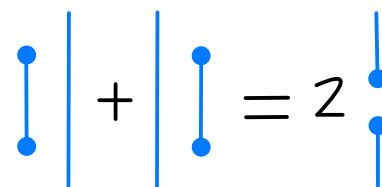
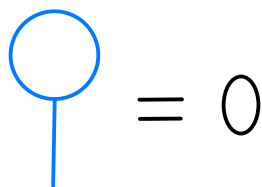
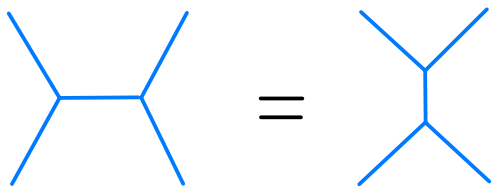
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$$U = W = Y$$

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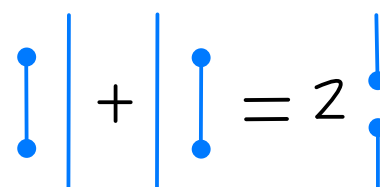
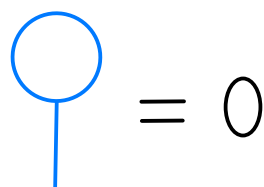
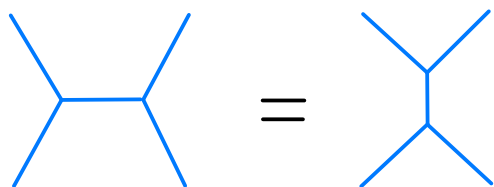
* One color relations



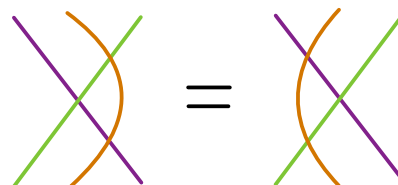
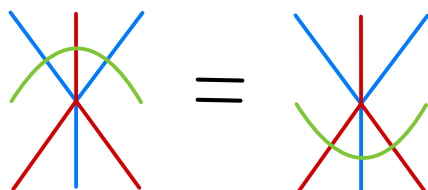
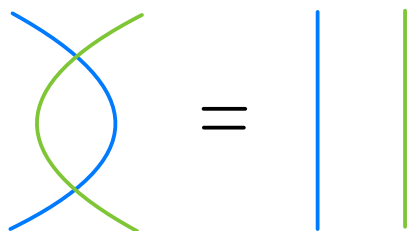
III. 1. The diagrammatic category DB_1

Relations

* One color relations



* Distant sliding relations

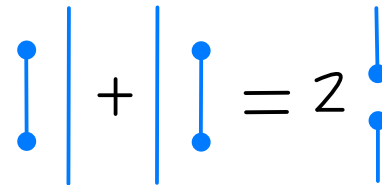
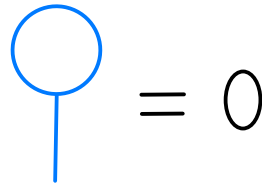
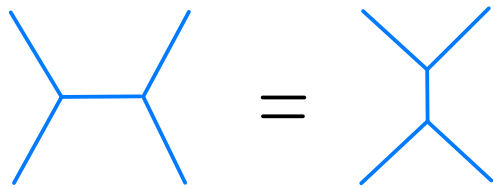


↪ three mutually distant colors

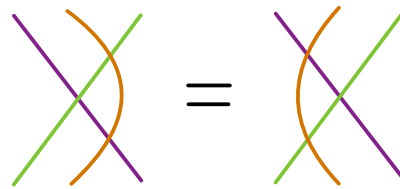
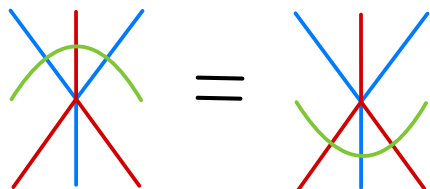
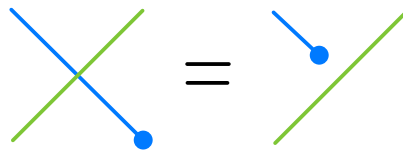
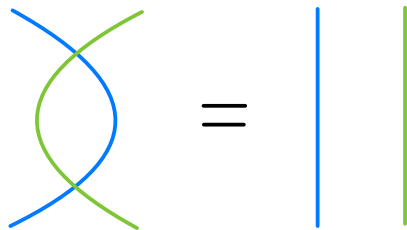
III. 1. The diagrammatic category DB_1

Relations

* One color relations



* Distant sliding relations



↖ three mutually distant colors

⇒ Distant colors do not interact



III.1. The diagrammatic category DB_1

Relations

* Adjacent colors relations

$$\text{Red over Blue (red dot)} = \text{Blue over Red (red dots)} + \text{Red over Blue (blue dot)}$$

$$\text{Red, Blue, Red} = \text{Diamond} - \text{Y-junction (blue dots)}$$

$$\text{Red square with blue crossings} = \text{Blue square with red crossings}$$

$$\text{Complex diagram 1} = \text{Complex diagram 2}$$

$$\text{Red over Blue} - \text{Blue over Red} = - \text{Blue over Blue} + \text{Blue over Blue} = \frac{1}{2} \left(\text{Blue over Blue} + \text{Blue over Blue} \right)$$

↖
same adjacency
as {1, 2, 3}

III. 2. The functor $F_1 : \mathcal{DB}_1 \longrightarrow \mathcal{SL}_1$

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$$\begin{array}{c} \bullet \\ | \\ \hline \end{array} \longmapsto \begin{array}{c} fg \\ \uparrow \\ f \otimes g \end{array}$$

$$\begin{array}{c} \bullet \\ | \\ \hline \end{array} \longmapsto \begin{array}{c} x_i \otimes 1 - 1 \otimes x_{i+1} \\ \uparrow \\ 1 \end{array}$$

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$$\begin{array}{c} \textcolor{red}{i+1} \\ \diagup \quad \diagdown \\ \textcolor{blue}{i} \end{array} \longmapsto \begin{array}{c} 1 \otimes 1 \otimes 1 \otimes 1 \\ \uparrow \\ 1 \otimes 1 \otimes 1 \otimes 1 \end{array} \quad \begin{array}{c} (x_i + x_{i+1}) \otimes 1 \otimes 1 \otimes 1 - 1 \otimes 1 \otimes 1 \otimes x_{i+2} \\ \uparrow \\ 1 \otimes x_i \otimes 1 \otimes 1 \end{array}$$

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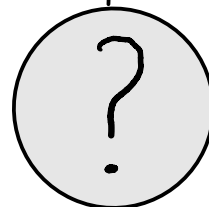
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- * The image of F_1 lies in $\mathcal{SL}_1$
- * F_1 is a functor
- * F_1 is fully faithful

III. 2. The functor $F_1 : \mathcal{DB}_1 \longrightarrow \mathcal{SL}_1$

Proposition Let $\mathcal{F}$ be the family of morphisms of $\mathcal{SL}_1$ consisting of:

- (i) the identity of each object;
- (ii) the generating morphism $R \rightarrow B_i$;
- (iii) isomorphisms yielding the Hecke algebra relations, and the projections to and inclusions from each summand in those relations;
- (iv) the unit and counit of adjunction that make B_i into a self-adjoint bimodule.

Then $\mathcal{SL}_1$ is monoidally generated (over the left action of R) by $\mathcal{F}$

III. 2. The functor $F_1 : \mathcal{DB}_1 \longrightarrow \mathcal{SB}_1$

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- (iii) isomorphisms yielding the Hecke algebra relations, and the projections to and inclusions from each summand in these relations;
- (iv) the unit and counit of adjunction that make B_i into a self-adjoint bimodule.

Then $\mathcal{SB}_1$ is monoidally generated (over the left action of R) by $\mathcal{F}$

Reduction of graphs $\rightsquigarrow$ classification of homomorphisms in $\mathcal{DB}_1$

$\Rightarrow$ morphisms (i) - (iv) are in the image of F_1

$\Rightarrow F_1$ is full

III. 2. The functor $F_1 : \mathcal{DB}_1 \longrightarrow \mathcal{SB}_1$

Proposition For $i \in \{1, \dots, n\}$, the object s_i is self-adjoint in $\mathcal{b}_i$, i.e.,

$$\mathrm{Hom}_{\mathcal{DB}_1}(\underline{w}, \underline{k} s_i) \cong \mathrm{Hom}_{\mathcal{DB}_1}(\underline{w} s_i, \underline{k})$$

$$\forall \underline{w}, \underline{k} \in \langle s_1, \dots, s_n \rangle$$

$$\mathrm{Hom}_{\mathcal{DB}_1}(\underline{w}, s_i \underline{k}) \cong \mathrm{Hom}_{\mathcal{DB}_1}(s_i \underline{w}, \underline{k})$$

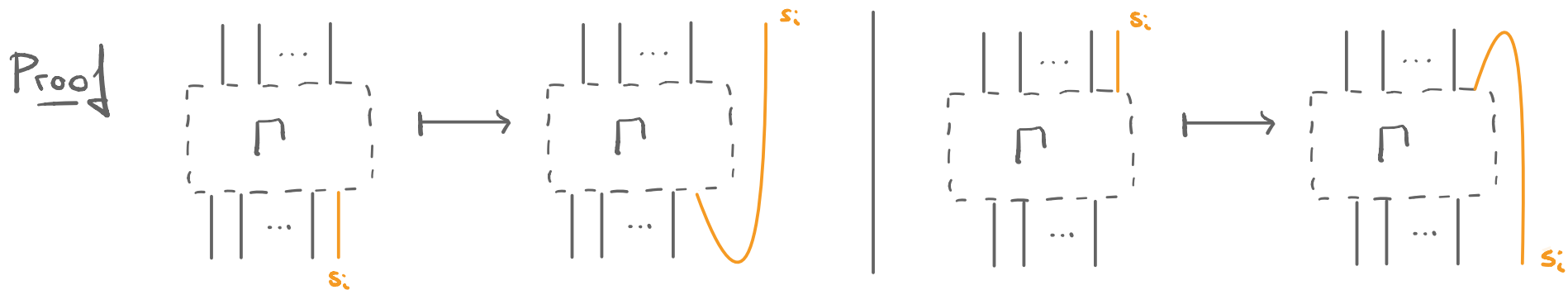
III. 2. The functor $F_1 : \mathcal{DB}_1 \longrightarrow \mathcal{SL}_1$

Proposition For $i \in \{1, \dots, n\}$, the object s_i is self-adjoint in $\mathcal{b}_i$, i.e.,

$$\text{Hom}_{\mathcal{DB}_1}(\underline{w}, \underline{k} s_i) \cong \text{Hom}_{\mathcal{DB}_1}(\underline{w} s_i, \underline{k})$$

$$\forall \underline{w}, \underline{k} \in \langle s_1, \dots, s_n \rangle$$

$$\text{Hom}_{\mathcal{DB}_1}(\underline{w}, s_i \underline{k}) \cong \text{Hom}_{\mathcal{DB}_1}(s_i \underline{w}, \underline{k})$$



$$(b_{\underline{w}}, b_{\underline{k}}) = \text{gr } K \text{ Hom}_{\mathcal{DB}_1}(\underline{w}, \underline{k})$$

- * each $\mathcal{b}_i$ is self-adjoint
- * compatible with relations



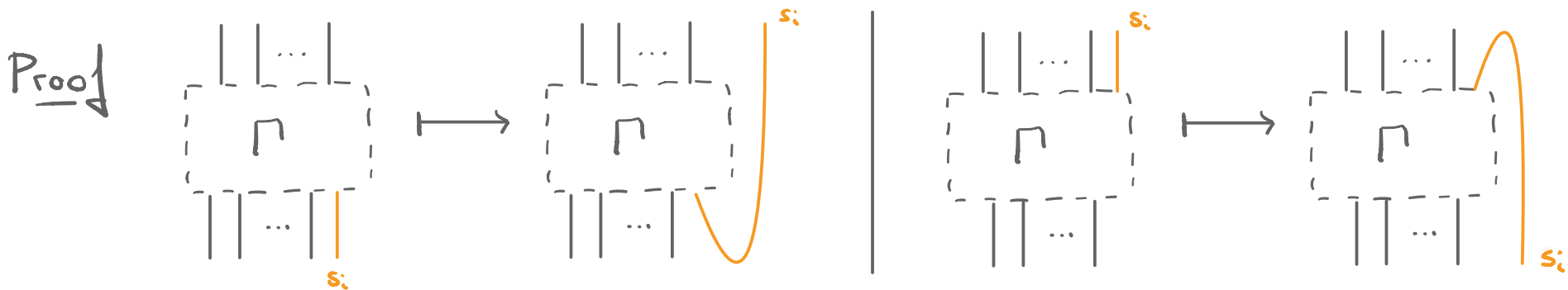
III. 2. The functor $F_1: \mathcal{DB}_1 \longrightarrow \mathcal{SB}_1$

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$$(b_{\underline{w}}, b_{\underline{k}}) = \text{grK Hom}_{\mathcal{DB}_1}(\underline{w}, \underline{k}) \quad \left| \begin{array}{l} * \text{ each } \mathcal{b}_i \text{ is self-adjoint} \\ * \text{ compatible with relations} \end{array} \right.$$

$\rightsquigarrow$ it induces the standard form on $\mathcal{H}$

$$\Rightarrow \text{grK Hom}_{\mathcal{DB}_1}(\underline{w}, \underline{k}) = \text{grK Hom}_{\mathcal{SB}_1}(BS(\underline{w}), BS(\underline{k})) \Rightarrow F_1 \text{ is faithful}$$