III. Diagrammatics for Soergel categories

III. Diagrammatics for Soergel categories

$\mathcal{SC}_1$: category of Bott-Samelson bimodules

* <u>Objects</u> : $w = s_{i_1} \cdots s_{i_k} \rightsquigarrow BS(w) = B_{i_1} \otimes_R \cdots \otimes_R B_{i_k}$

* <u>Morphisms</u> : $\mathrm{Hom}_{\mathcal{SC}_1}(B, B') = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{R\text{-}\mathrm{mod}_{\mathbb{Z}}\text{-}R}(B, B'(n))$

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$\mathcal{SC}$: category of Soergel bimodules

$$\mathcal{SC}_1 \xrightarrow[\text{grading shifts}]{\text{additive envelope}} \mathcal{SC}_2 \xrightarrow{\text{Karoubi envelope}} \mathcal{SC}$$

III. Diagrammatics for Soergel categories

$\mathcal{SC}_1$: category of Bott-Samelson bimodules

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$\mathcal{SB}$: category of Soergel bimodules

$$\mathcal{SC}_1 \xrightarrow[\text{grading shifts}]{\text{additive envelope}} \mathcal{SB}_2 \xrightarrow{\text{Karoubi envelope}} \mathcal{SB}$$

→ All the information contained in $\mathcal{SB}$ can be recovered from $\mathcal{SB}_1$

→ Objects in $\mathcal{SB}_1$ combinatorial in nature

$\rightsquigarrow$ we can capture this describing morphisms by planar graphs

III. Diagrammatics for Soergel categories

$\mathcal{DC}_1$: diagrammatic category

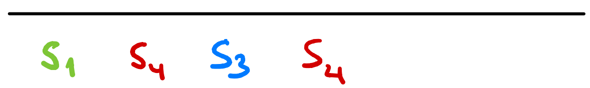
III. Diagrammatics for Soergel categories

B_1

$\mathcal{DB}_1$: diagrammatic category

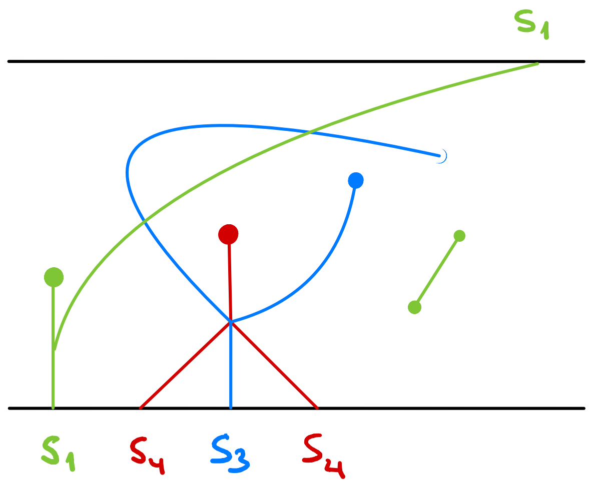
* <u>Objects</u> : words $w = s_{i_1} \cdots s_{i_k}$

$B_1 \otimes B_4 \otimes B_3 \otimes B_4$



III. Diagrammatics for Soergel categories

$$B_1 \otimes B_4 \otimes B_3 \otimes B_4 \xrightarrow{f} B_1 \quad \rightsquigarrow$$



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III. Diagrammatics for Soergel categories

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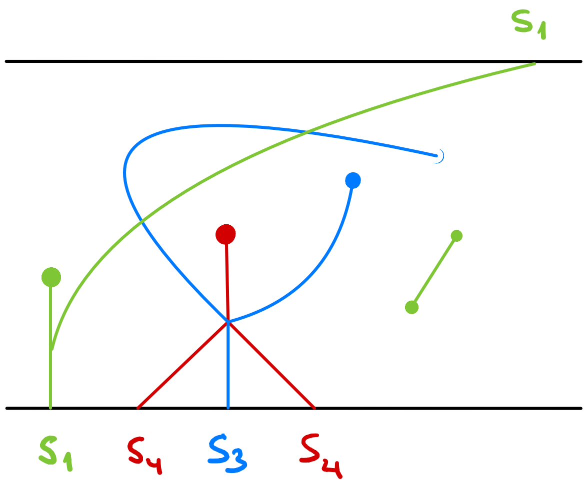

$\mathcal{DB}_1$: diagrammatic category

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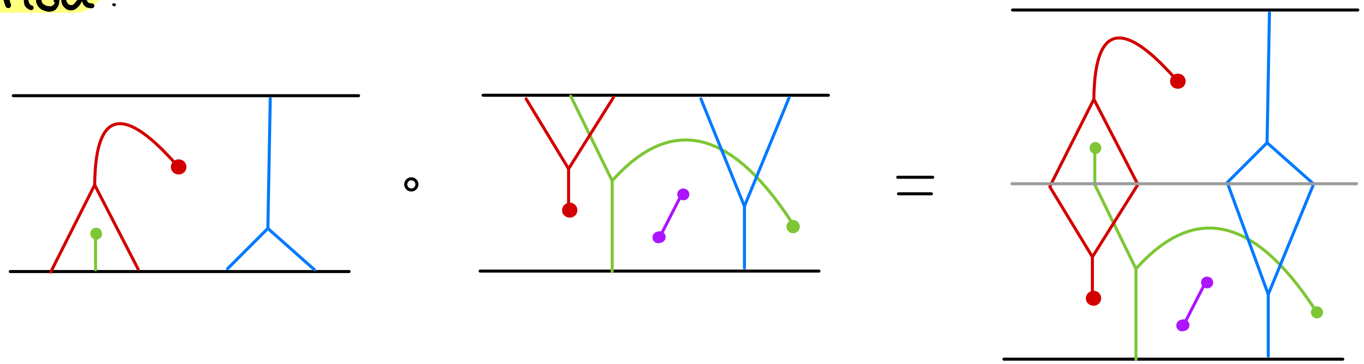
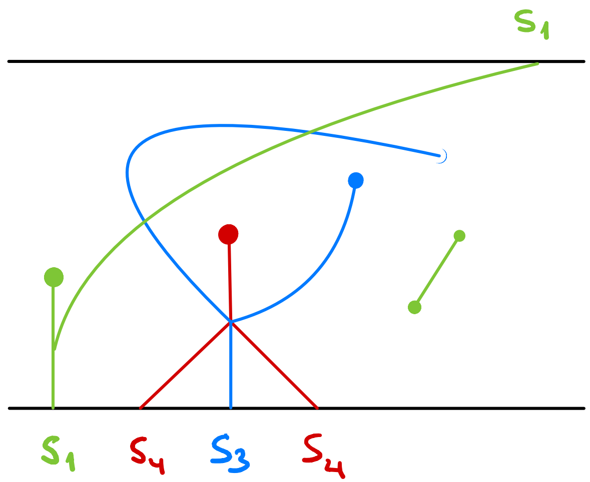
• Composition :



III. Diagrammatics for Soergel categories

$$\begin{tikzcd} B_1 \\ B_1 \otimes B_4 \otimes B_3 \otimes B_4 \arrow[u, "f"] \end{tikzcd} \rightsquigarrow$$



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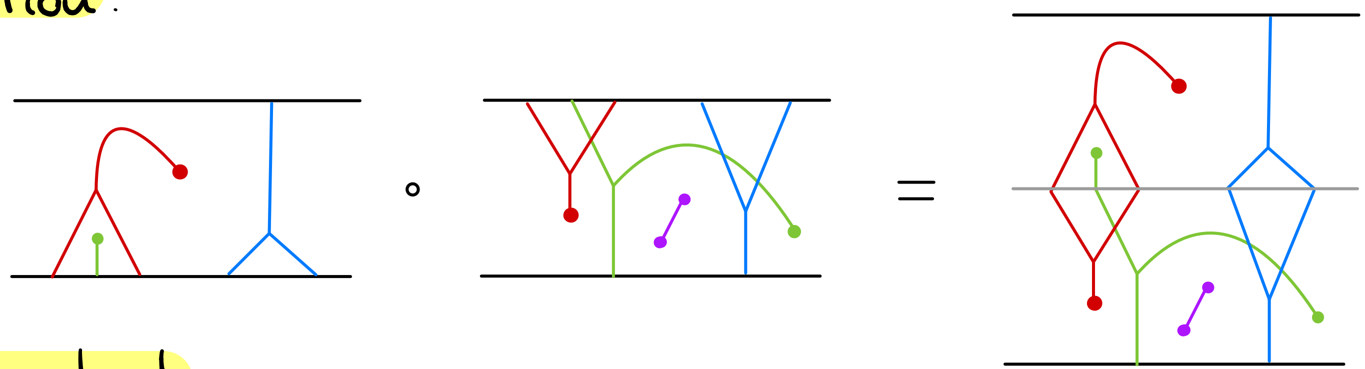
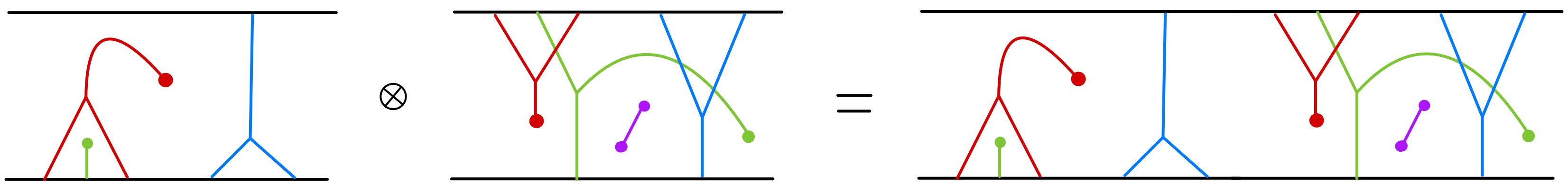
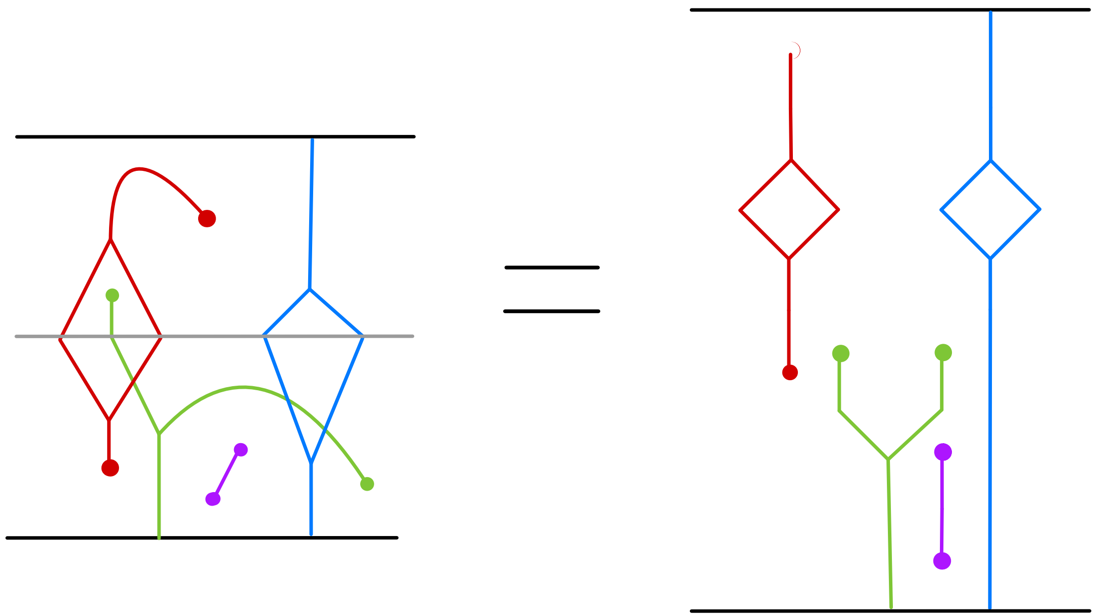


- Tensor product :



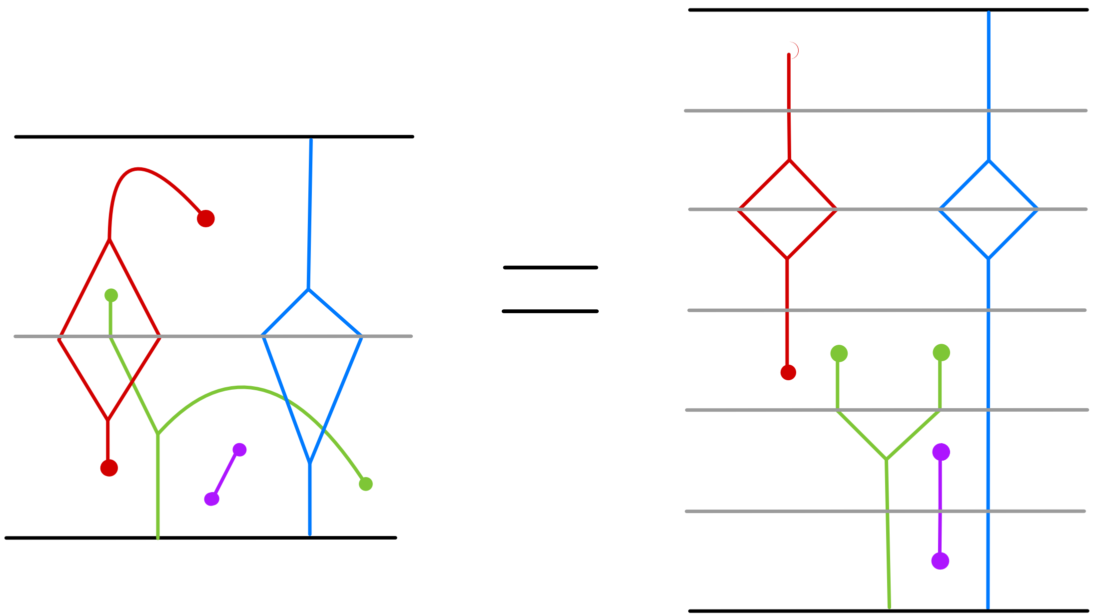
III. Diagrammatics for Soergel categories

Using relations we can deform diagrams



III. Diagrammatics for Soergel categories

Using relations we can deform diagrams and decompose them into some "elementary pieces":



III. Diagrammatics for Soergel categories

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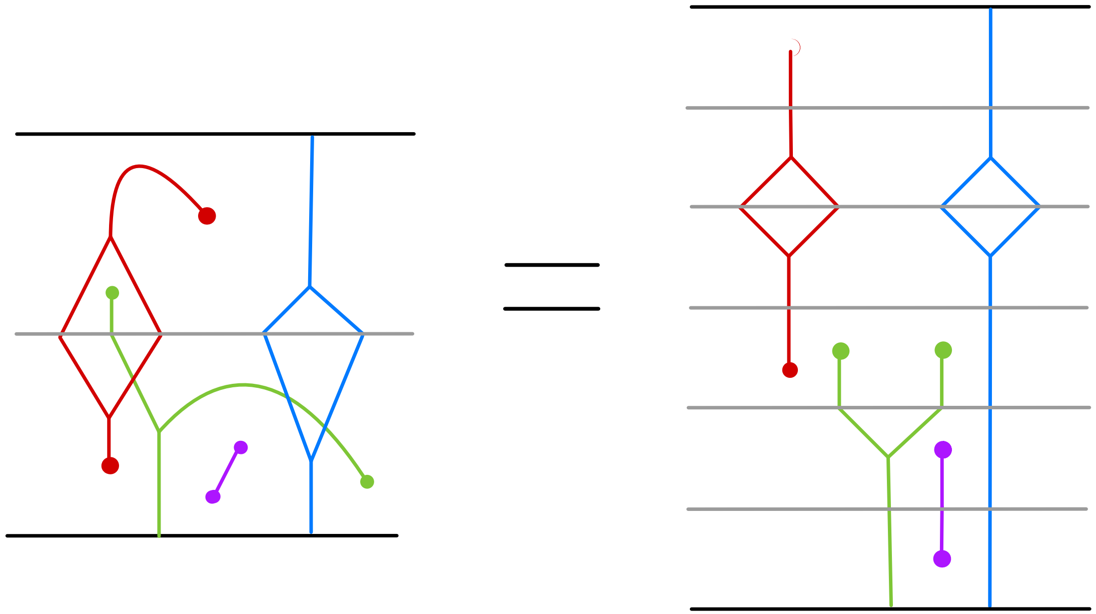

We can describe the diagrammatic category $\mathcal{SC}_1$ by generators and relations

III.1. The diagrammatic category $\mathcal{DC}_1$

We construct a category $\mathcal{DC}_1$ of diagrams via generators and local relations by associating a colored graph to each morphism in $\mathcal{SC}_1$:

$$\mathcal{DC}_1 \xrightarrow{F_1} \mathcal{SC}_1$$

III.1. The diagrammatic category $\mathcal{DC}_1$

We construct a category $\mathcal{DC}_1$ of diagrams via generators and local relations by associating a colored graph to each morphism in $\mathcal{SC}_1$:

$$\begin{tikzcd}
\mathcal{DC}_1 \arrow[r, "F_1"] \arrow[d, "{\oplus,\ (-)^{sh}}"'] & \mathcal{SC}_1 \arrow[d, "{\oplus,\ (-)^{sh}}"] \\
\mathcal{DC}_2 \arrow[r, "F_2"] \arrow[d, "\mathrm{Kar}(-)"'] & \mathcal{SC}_2 \arrow[d, "\mathrm{Kar}(-)"] \\
\mathcal{DC} \arrow[r, "F"] & \mathcal{SC}
\end{tikzcd}$$

F_1 is an equivalence of categories and induces equivalences F_2 and F.

III. 1. The diagrammatic category $\mathcal{DG}_1$

Fix an integer $n \in \mathbb{Z}$

$\boxed{\text{Objects}}$ finite sequences of indices taken from $\{1, \dots, n\}$

III.1. The diagrammatic category $\mathcal{D}\mathcal{G}_1$

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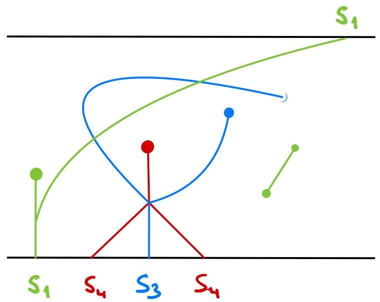
→ we represent them by colored points in $\mathbb{R}$



→ color convention: red and blue represent adjacent indices and green is distant to both of them

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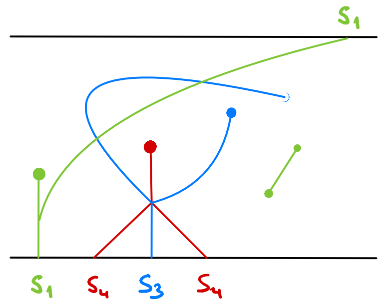

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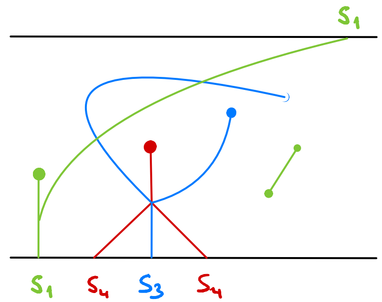

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$\boxed{\text{Generators}}$

$i \to \emptyset$ $\quad$ $\emptyset \to i$ $\quad$ $i \to ii$ $\quad$ $ii \to i$ $\quad$ $ij \to ji$ $\quad$ $i(i+1)i \to (i+1)i(i+1)$ $\quad$ $(i+1)i(i+1) \to i(i+1)i$

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Fix an integer $n \in \mathbb{Z}$

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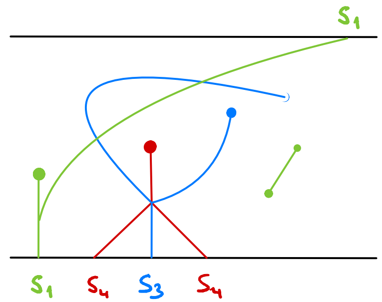

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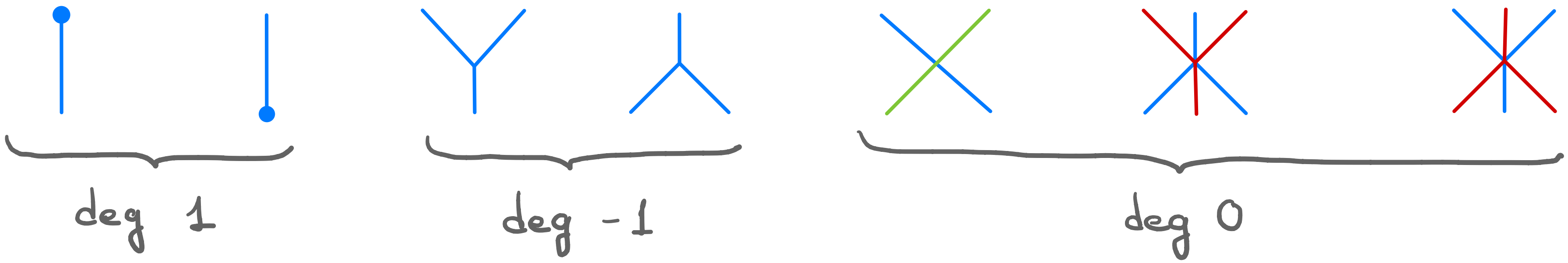
Generators



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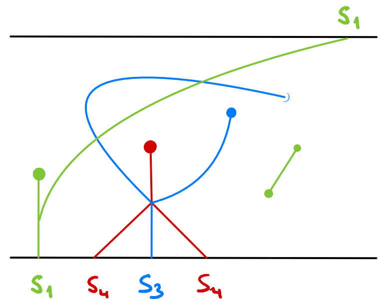

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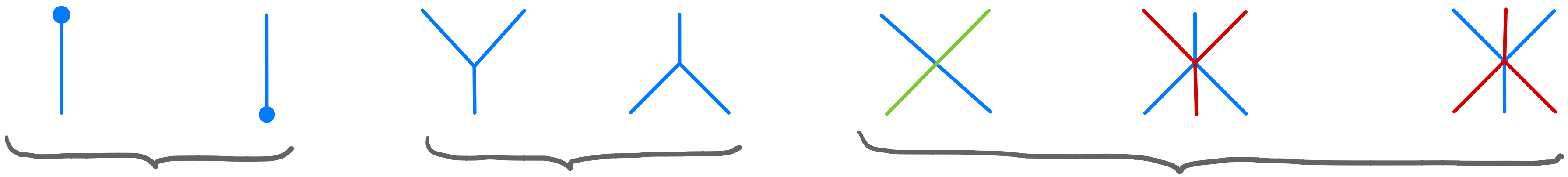


deg 1 deg −1 deg 0

↳ Shorthands:

$\emptyset \to ii$

⇝ deg 0

$ii \to \emptyset$

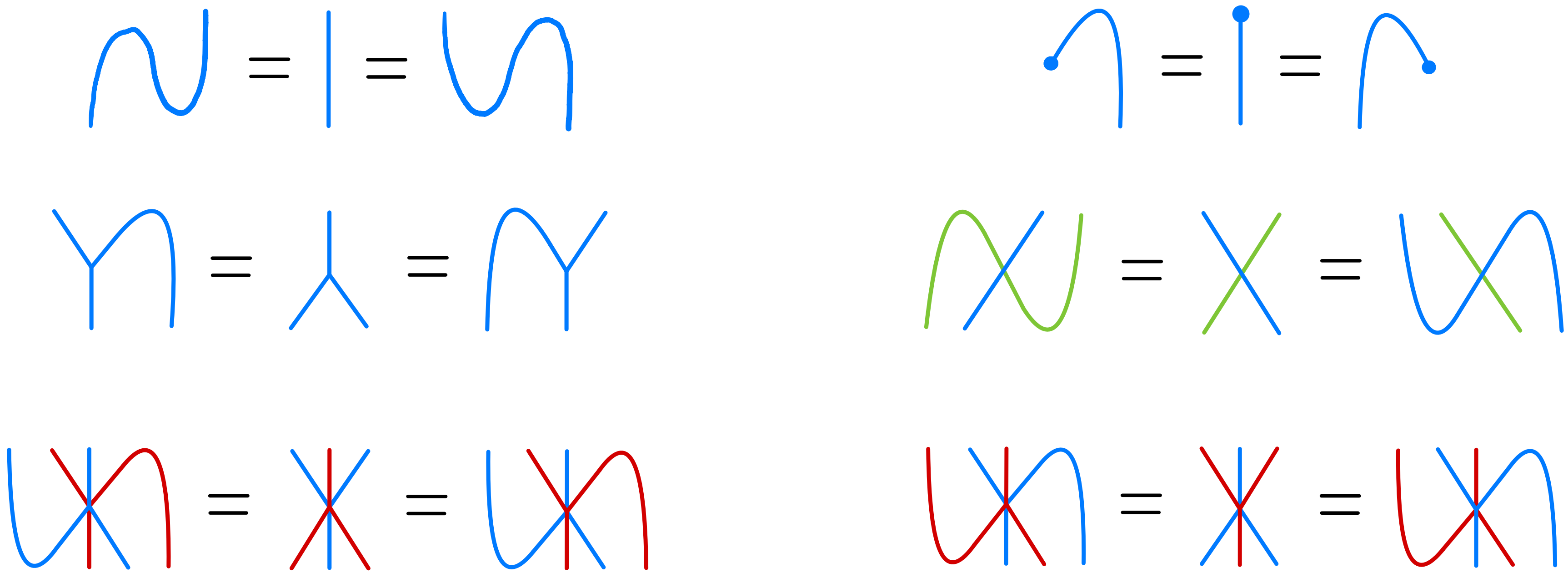
III. 1. The diagrammatic category $\mathcal{DC}_1$

$\boxed{\text{Relations}}$

III. 1. The diagrammatic category $\mathcal{DG}_1$

Relations

* Isotopy relations



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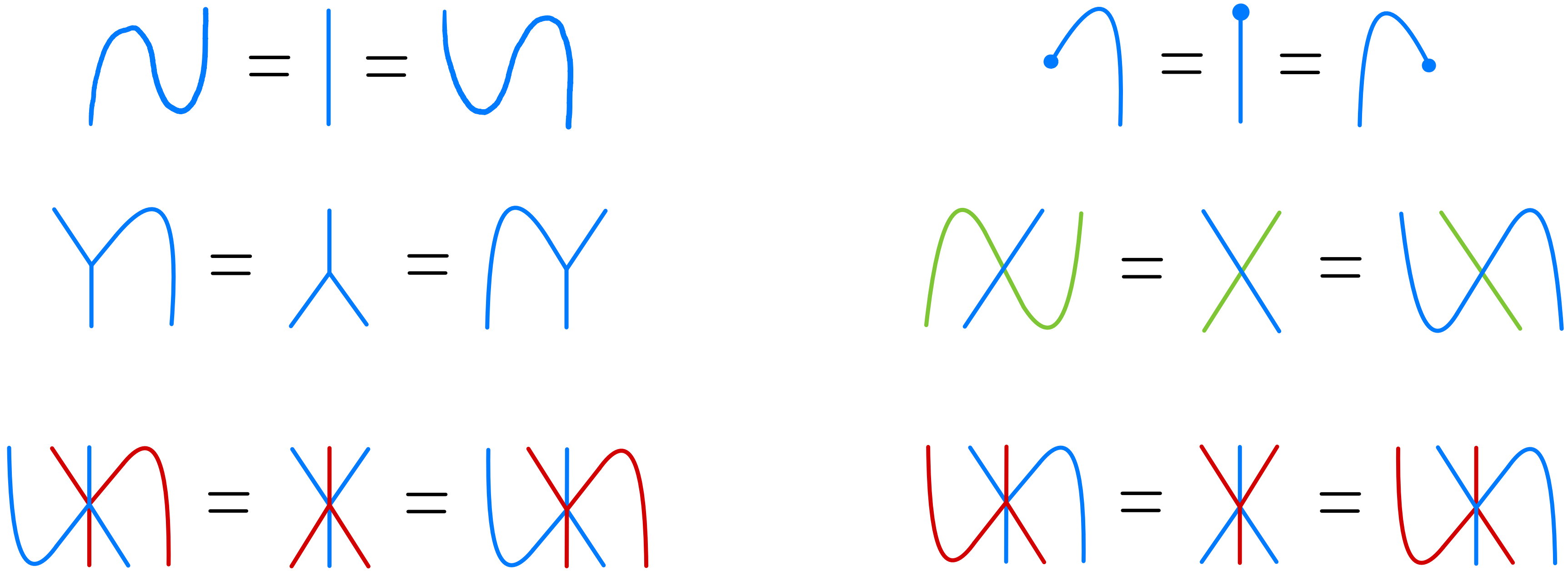
Relations

* Isotopy relations

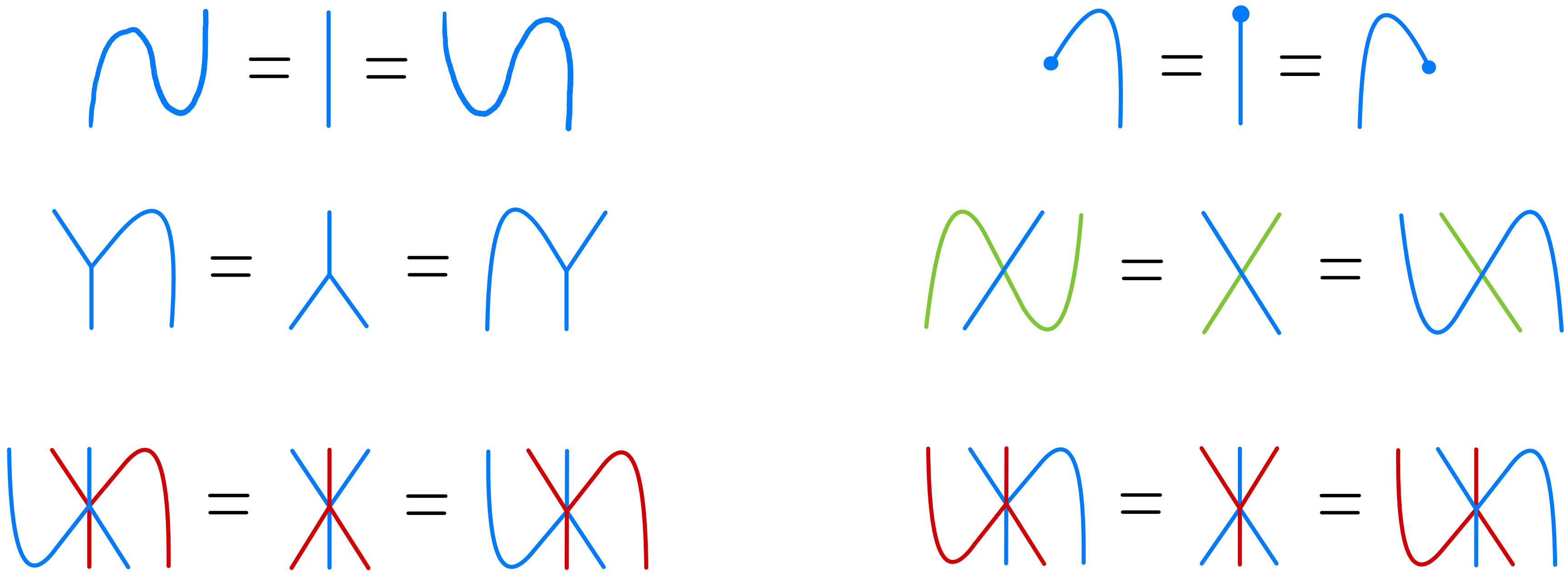


The morphism represented by a particular graph is idependent of the isotopy class of the embedding.

III. 1. The diagrammatic category $\mathcal{DG}_1$

Relations

* Isotopy relations



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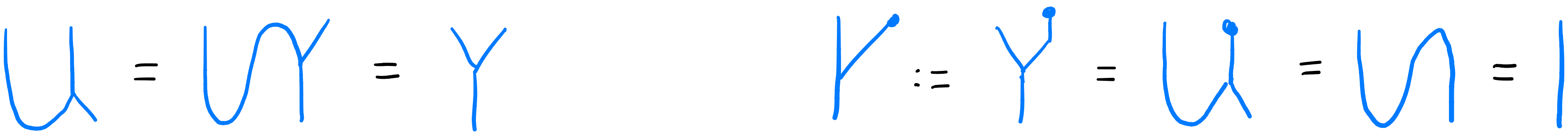
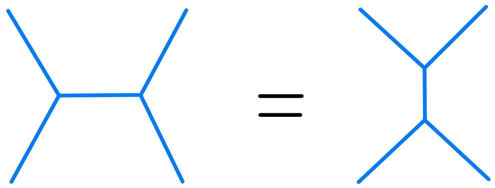
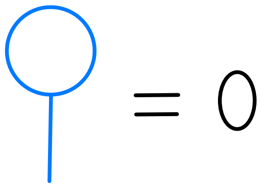


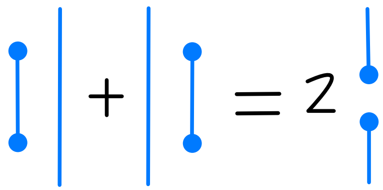
III. 1. The diagrammatic category DG_1

Relations

* One color relations







III. 1. The diagrammatic category $\mathcal{DG}_1$

Relations

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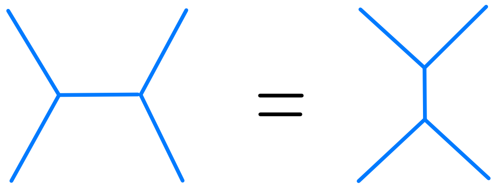
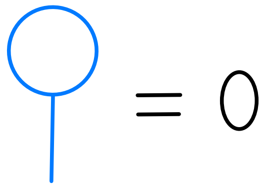
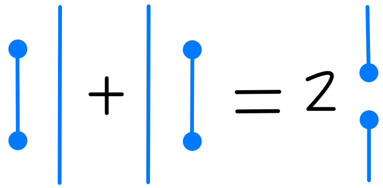
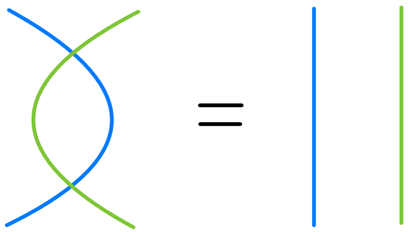
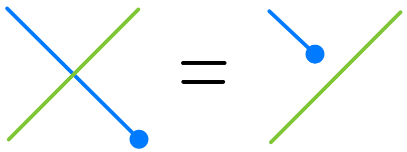


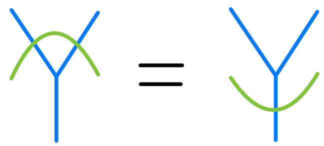


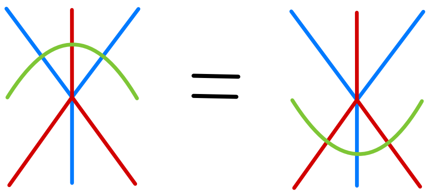


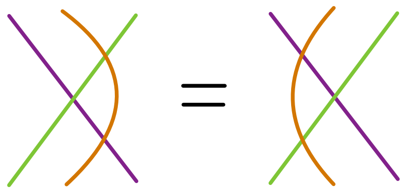
* Distant sliding relations









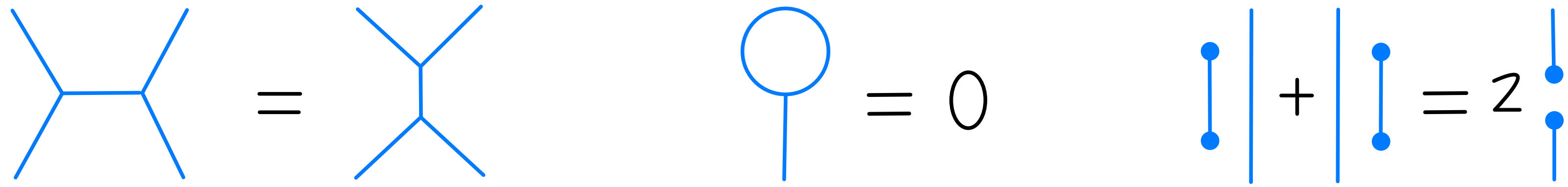


← three mutually distant colors

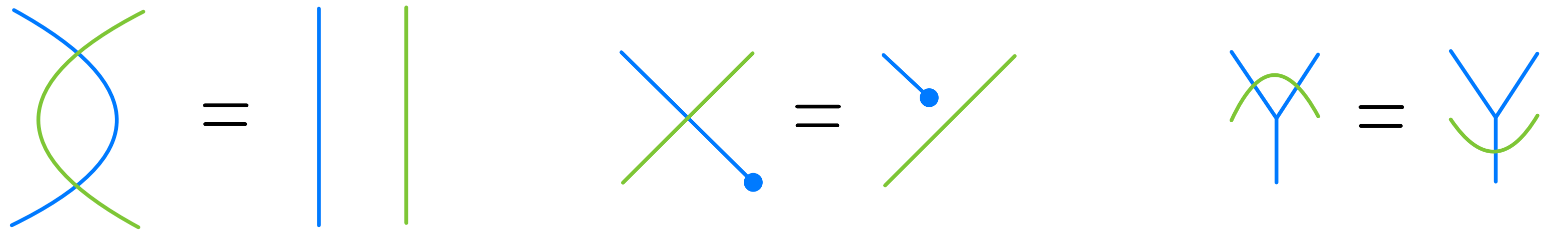
III. 1. The diagrammatic category $\mathcal{DG}_1$

Relations

* One color relations

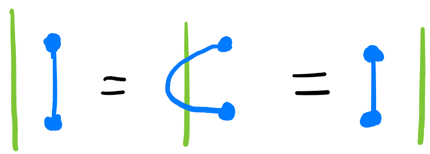


* Distant sliding relations



three mutually distant colors

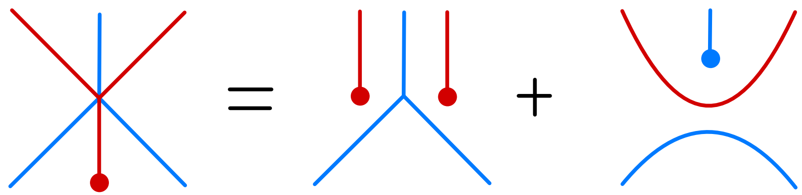
$\Rightarrow$ Distant colors do not interact

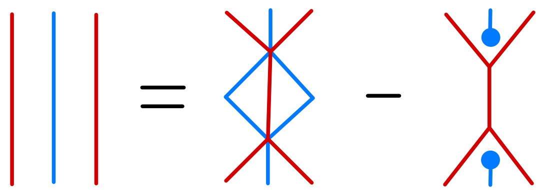


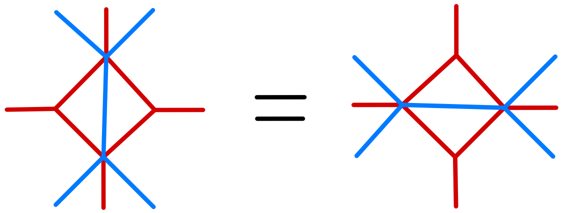
III.1. The diagrammatic category $\mathcal{D}\mathcal{G}_1$

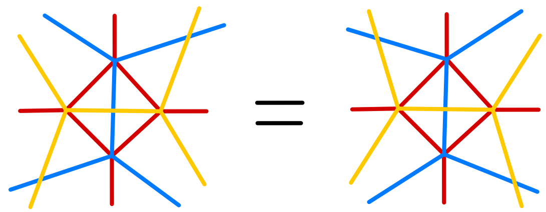
Relations

* Adjacent colors relations

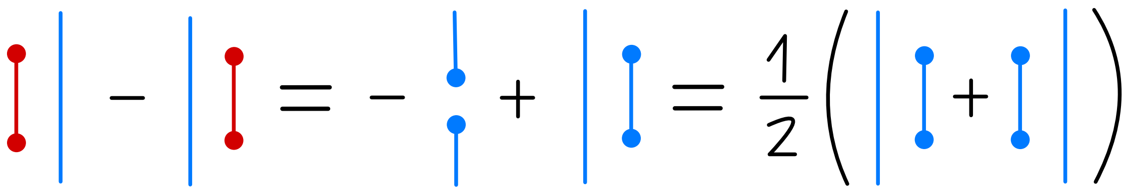








same adjacency as $\{1, 2, 3\}$



III. 2. The functor $F_1 : \mathcal{DC}_1 \longrightarrow \mathcal{SC}_1$

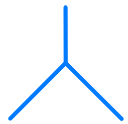
III. 2. The functor $F_1 : \mathcal{DB}_1 \longrightarrow \mathcal{SB}_1$

 $\longmapsto \begin{array}{c} fg \\ \uparrow \\ f \otimes g \end{array}$

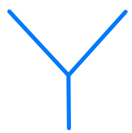
 $\longmapsto \begin{array}{c} x_i \otimes 1 - 1 \otimes x_{i+1} \\ \uparrow \\ 1 \end{array}$

III.2. The functor $F_1 : \mathcal{DB}_1 \longrightarrow \mathcal{SB}_1$

 $\longmapsto$ $\begin{array}{c} fg \\ \uparrow \\ f\otimes g \end{array}$

 $\longmapsto$ $\begin{array}{c} 0 \\ \uparrow \\ f\otimes 1\otimes g \end{array} \qquad \begin{array}{c} f\otimes g \\ \uparrow \\ f\otimes x_i\otimes g \end{array}$

 $\longmapsto$ $\begin{array}{c} x_i\otimes 1 - 1\otimes x_{i+1} \\ \uparrow \\ 1 \end{array}$

 $\longmapsto$ $\begin{array}{c} f\otimes 1\otimes g \\ \uparrow \\ f\otimes g \end{array}$

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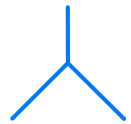 $\longmapsto$ $$\begin{array}{c} 0 \\ \uparrow \\ f\otimes 1\otimes g \end{array} \qquad \begin{array}{c} f\otimes g \\ \uparrow \\ f\otimes x_i \otimes g \end{array}$$

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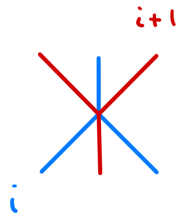
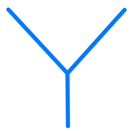
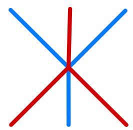
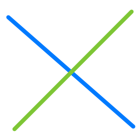


$\longmapsto$ $$\begin{array}{c} 1\otimes 1\otimes 1\otimes 1 \\ \uparrow \\ 1\otimes 1\otimes 1\otimes 1 \end{array} \qquad \begin{array}{c} (x_i+x_{i+1})\otimes 1\otimes 1\otimes 1 - 1\otimes 1\otimes 1\otimes x_{i+2} \\ \uparrow \\ 1\otimes x_i\otimes 1\otimes 1 \end{array}$$

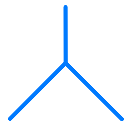
 $\longmapsto$ $$\begin{array}{c} f\otimes 1\otimes g \\ \uparrow \\ f\otimes g \end{array}$$

 $\longmapsto$ $$\begin{array}{c} 1\otimes 1\otimes 1\otimes 1 \\ \uparrow \\ 1\otimes 1\otimes 1\otimes 1 \end{array} \qquad \begin{array}{c} 1\otimes 1\otimes 1\otimes (x_{i+1}+x_{i+2}) - x_i\otimes 1\otimes 1\otimes 1 \\ \uparrow \\ 1\otimes x_{i+2}\otimes 1\otimes 1 \end{array}$$

 $\longmapsto$ $$\begin{array}{c} f\otimes 1\otimes g \\ \uparrow \\ f\otimes 1\otimes g \end{array}$$

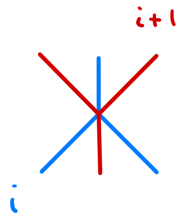
III. 2. The functor $F_1 : \mathcal{DB}_1 \longrightarrow \mathcal{SB}_1$

$$\longmapsto \quad \begin{array}{c} fg \\ \uparrow \\ f\otimes g \end{array}$$

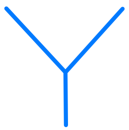


$$\longmapsto \quad \begin{array}{c} 0 \\ \uparrow \\ f\otimes 1\otimes g \end{array} \qquad \begin{array}{c} f\otimes g \\ \uparrow \\ f\otimes x_i\otimes g \end{array}$$

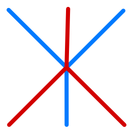
$$\longmapsto \quad \begin{array}{c} x_i\otimes 1 - 1\otimes x_{i+1} \\ \uparrow \\ 1 \end{array}$$



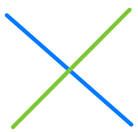
$$\longmapsto \quad \begin{array}{c} 1\otimes 1\otimes 1\otimes 1 \\ \uparrow \\ 1\otimes 1\otimes 1\otimes 1 \end{array} \qquad \begin{array}{c} (x_i+x_{i+1})\otimes 1\otimes 1\otimes 1 - 1\otimes 1\otimes 1\otimes x_{i+2} \\ \uparrow \\ 1\otimes x_i\otimes 1\otimes 1 \end{array}$$



$$\longmapsto \quad \begin{array}{c} f\otimes 1\otimes g \\ \uparrow \\ f\otimes g \end{array}$$



$$\longmapsto \quad \begin{array}{c} 1\otimes 1\otimes 1\otimes 1 \\ \uparrow \\ 1\otimes 1\otimes 1\otimes 1 \end{array} \qquad \begin{array}{c} 1\otimes 1\otimes 1\otimes (x_{i+1}+x_{i+2}) - x_i\otimes 1\otimes 1\otimes 1 \\ \uparrow \\ 1\otimes x_{i+2}\otimes 1\otimes 1 \end{array}$$



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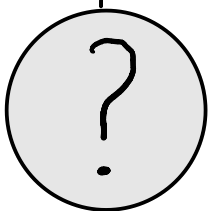

* The image of F_1 lies in $\mathcal{SB}_1$
* F_1 is a functor
* F_1 is fully faithful

III. 2. The functor $F_1 : \mathcal{DB}_1 \longrightarrow \mathcal{SB}_1$

Proposition Let $\mathcal{F}$ be the family of morphisms of $\mathcal{SB}_1$ consisting of:

(i) the identity of each object;

(ii) the generating morphism $R \to B_i$;

(iii) isomorphisms yielding the Hecke algebra relations, and the projections to and inclusions from each summand in those relations;

(iv) the unit and counit of adjunction that make B_i into a self-adjoint bimodule.

Then $\mathcal{SC}_1$ is monoidally generated (over the left action of R) by $\mathcal{F}$

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Reduction of graphs $\rightsquigarrow$ classification of homomorphisms in $\mathcal{DB}_1$

$\Rightarrow$ morphisms (i) - (iv) are in the image of F_1

$\Rightarrow$ F_1 is full

III. 2. The functor $F_1 : \mathcal{DB}_1 \longrightarrow \mathcal{SB}_1$

Proposition For $i \in \{1, \dots, n\}$, the object s_i is self-adjoint in $\mathcal{B}_i$, i.e.,

$$\mathrm{Hom}_{\mathcal{DB}_1}(\underline{w}, \underline{K} s_i) \cong \mathrm{Hom}_{\mathcal{DB}_1}(\underline{w} s_i, \underline{K})$$

$$\mathrm{Hom}_{\mathcal{DB}_1}(\underline{w}, s_i \underline{K}) \cong \mathrm{Hom}_{\mathcal{DB}_1}(s_i \underline{w}, \underline{K}) \qquad \forall\, \underline{w}, \underline{K} \in \langle s_1, \dots, s_n \rangle$$

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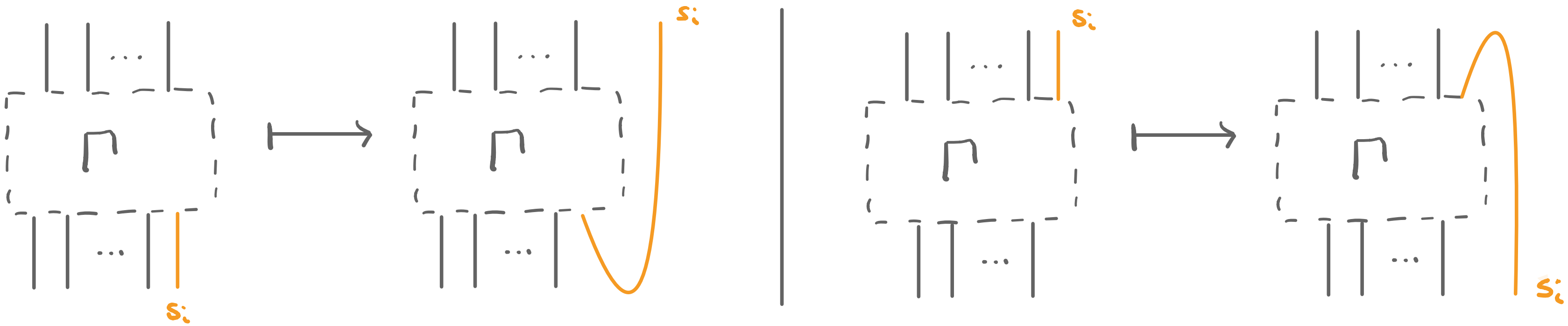
Proof



$\boxtimes$

$$(b_{\underline{w}}, b_{\underline{x}}) = \mathrm{grk}\, \mathrm{Hom}_{\mathcal{DB}_1}(\underline{w}, \underline{x})$$

* each b_i is self-adjoint
* compatible with relations

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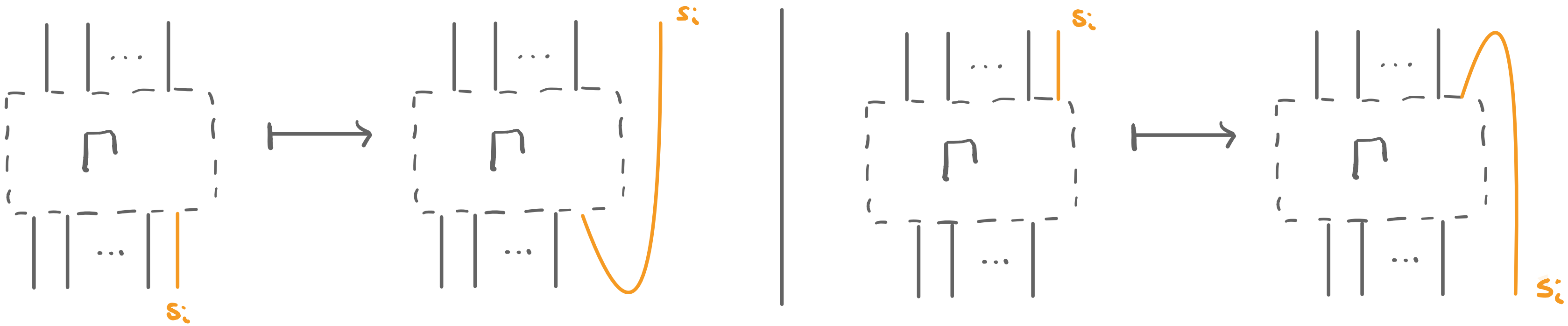

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* each b_i is self-adjoint
* compatible with relations

$\rightsquigarrow$ it induces the standard form on H

$\Rightarrow \mathrm{grk}\, \mathrm{Hom}_{\mathcal{DB}_1}(\underline{w}, \underline{x}) = \mathrm{grk}\, \mathrm{Hom}_{\mathcal{SB}_1}(BS(\underline{w}), BS(\underline{x})) \Rightarrow F_1$ is faithful