HOMFLY categories, defect skein theory and Turaev coproduct
(DRAFT)

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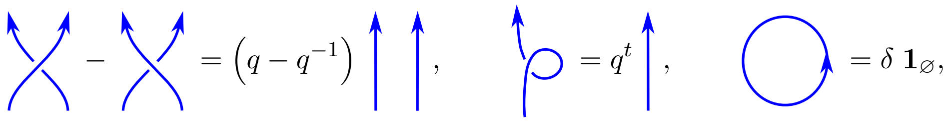
Abstract. We define a HOMFLY versions of the categories $\mathrm{Rep}_q\mathrm{L}$ and $\mathrm{Rep}_q\mathrm{P}$ of quantum representations of a Levi and a parabolic subgroup $\mathrm{L} \subseteq \mathrm{P} \subseteq \mathrm{GL}_{m+n}$ and we construct central algebra and centered bimodule structures on these categories. These structures serve as the algebraic ingredients for constructing a skein theory on manifolds with surface and line defects. Finally, we recover the Turaev coproduct on the HOMFLY skein algebra as a particular instance of this theory. In particular, this coproduct is compatible with cutting/gluing surfaces.

Contents

To do

Overview of the main results. (Provisional) The HOMFLY category $\mathrm{Rep}_q\mathrm{GL}_t$ ([De07, Bru17]) is a diagrammatic category whose objects are configurations of directed points and whose morphisms are linear combinations of framed oriented tangles, modulo the HOMFLY skein relations



where q^t is a formal parameter and $\mathbf{1}_\varnothing$ is the empty diagram. This category is a universal version of the category of quantum representations of GL_N in the sense that $\mathrm{Rep}_q\mathrm{GL}_N$ can be recovered as a quotient of the specialization of (a certain completion of) $\mathrm{Rep}_q\mathrm{GL}_t$ at $t = N$. Our first contribution is to define a HOMFLY version of the category $\mathrm{Rep}_q\mathrm{P}_t$ of quantum representations of a parabolic subgroup $\mathrm{P} \subseteq \mathrm{GL}_{m+n}$ (see Section 2). We define it by considering a certain subcategory of the rigid category underlying $\mathrm{Rep}_q\mathrm{L}_t = \mathrm{Rep}_q\mathrm{GL}_t \boxtimes \mathrm{Rep}_q\mathrm{GL}_t$, the HOMFLY analogue of $\mathrm{Rep}_q\mathrm{L}$ for the Levi subgroup $\mathrm{L} = \mathrm{GL}_m \times \mathrm{GL}_n$ of GL_{m+n}. Together with $\mathrm{Rep}_q\mathrm{P}_t$, we construct four functors

$$\begin{tikzcd}
 & \mathrm{Rep}_q\mathrm{P}_t \arrow[dr, "j_t^*"] & \\
\mathrm{Rep}_q\mathrm{GL}_t \arrow[ur, "\iota_t^*"] \arrow[rr, "\mathrm{res}_t"] & & \mathrm{Rep}_q\mathrm{GL}_t \boxtimes \mathrm{Rep}_q\mathrm{GL}_t \arrow[ul, bend right, "\pi_t^*"']
\end{tikzcd},$$

interpolating the ordinary restriction functors between the corresponding categories of representations. In Section 3, we prove the following (see Theorem 3.4):

Theorem. The functor $\iota_t^* \boxtimes \pi_t^*$ lifts to a braided monoidal functor

$$\mathrm{Rep}_q\mathrm{GL}_t \boxtimes (\overline{\mathrm{Rep}_q\mathrm{L}_t}) \;\to\; Z(\mathrm{Rep}_q\mathrm{P}_t),$$

where $\overline{(-)}$ stands for the opposite braided category and $Z(\mathrm{Rep}_q\mathrm{P}_t)$ is the Drinfeld center of $\mathrm{Rep}_q\mathrm{P}_t$. In particular, $\mathrm{Rep}_q\mathrm{P}_t$ is a $(\mathrm{Rep}_q\mathrm{GL}_t, \mathrm{Rep}_q\mathrm{L}_t)$-central algebra. ■

In [BJ25], Brown and Jordan construct skein modules in 3-dimensional manifolds with surface defects. Skein relations around the defects are induced by the algebraic data of a central algebra. In Sections 4 and 5 we extend this framework by allowing a certain type of line defect decorated with $\mathrm{Rep}_q\mathrm{L}_t$. In particular, in Section 4, we construct planar theories on framed surfaces with codimension 1 defects. The local model around these line defects is induced by a $(\mathrm{Rep}_q\mathrm{P}_t, \mathrm{Rep}_q\mathrm{L}_t)$-centred bimodule structure on $\mathrm{Rep}_q\mathrm{L}_t$ (see Section 3.2). Incorporating the central algebra structures, in Section 5 we generalise this to a 3-dimensional theory, with both surface and line defects. Defects of codimension 1 are decorated with central algebras and defects of codimension 2 with centred bimodules (see Theorem 5.4):

Theorem. There is a 2-dimensional stratified framed TQFT

$$\mathcal{Z} \colon \mathrm{Bord}_2^{\mathrm{bip,mkd}} \to \textsc{Bimod}$$

assigning $\mathrm{Rep}_q\mathrm{L}_t$ (resp. $\mathrm{Rep}_q\mathrm{P}_t$) to an interval labelled with L_t (resp. P_t). To a bipartite framed surface S, it assigns a bimodule functor whose components are vector spaces of stratified graphs in $S \times I$, modulo local relations. ■

The HOMFLY skein algebra $\mathrm{Sk}_{\mathrm{GL}_t}(S)$ of a framed surface S is the algebra of endomorphisms of the empty object in the skein category $\mathrm{SkCat}_{\mathrm{GL}_t}(S)$. In [Tur91], Turaev showed the existence of a coproduct

$$\Delta\colon \mathrm{Sk}_{\mathrm{GL}_t}(S) \to \mathrm{Sk}_{\mathrm{GL}_t}(S) \otimes \mathrm{Sk}_{\mathrm{GL}_t}(S)$$

by extending Jaeger's formula [Jae89] to arbitrary surfaces. In Section 6, we give a new definition using the formalism of skein theory with defects. In particular, we show the following:

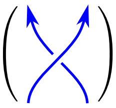
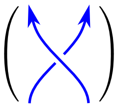
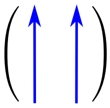
Theorem. Let S be a framed surface. Then, a stratification on S can be chosen so that the vector space $\mathcal{A}(S)(\varnothing)$ is isomorphic to $\mathrm{Sk}_{\mathrm{L}_t}(S) \cong \mathrm{Sk}_{\mathrm{GL}_t}(S) \otimes \mathrm{Sk}_{\mathrm{GL}_t}(S)$. Under this identification, the algebra morphism $\mathrm{Sk}_{\mathrm{GL}_t}(S) \to \mathrm{Sk}_{\mathrm{GL}_t}(S) \otimes \mathrm{Sk}_{\mathrm{GL}_t}(S)$ induced by acting over P_t-region of the empty diagram is Turaev's coproduct.

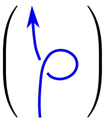
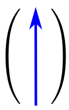
A crucial consequence of this description is that the coproduct is compatible in a very non-obvious way with gluing and cutting surfaces.

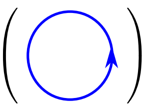
1 Background

1.1 Jaeger's formula and Turaev's coproduct. The HOMFLY polynomial is a two-variable polynomial invariant of framed oriented links that encompasses both the Jones and the Alexander polynomial. It can be defined in terms of skein relations:

Proposition 1.1. [FYH+85, PT87] *There exists a unique invariant $H_{a,z}$ of framed oriented links with values in $\mathbb{Z}[a^{\pm}, z^{\pm}]$ satisfying the following properties:*

(i) $H_{a,z}$  $- H_{a,z}$  $= zH_{a,z}$  ;

(ii) $H_{a,z}$  $= aH_{a,z}$  ;

(iii) $H_{a,z}$  $= z^{-1}(a - a^{-1})$.

This invariant is called the HOMFLY polynomial. ∎

Given an integer $N \in \mathbb{N}$, the specialisation of the HOMFLY polynomial at $z = q - q^{-1}$ and $a = q^N$ is a Laurent polynomial in one variable q, that we denote by $P_N(D)$, for any link diagram D. Jaeger constructed in [Jae89] a recursive formula allowing to compute $P_{m+n}(D)$ in terms of $P_m(D_1)$ and $P_n(D_2)$, where D_1 and D_2 are certain subdiagrams of D. Concretely, a link diagram is a 4-valent graph with two distinguished classes of vertices: positive and negative crossings. Given a vertex v, we denote by a (resp. b) the upper (resp. lower) incoming edge, and by c (resp. d) the upper (resp. lower) outcoming edge, as shown in the following pictures:

Definition 1.2. Let E_D be the set of edges of D. An *admissible labelling* is a map $f : E_D \to \{1, 2\}$ such that, at any vertex v, either $f(a) = f(c)$ and $f(b) = f(d)$, or $f(a) = f(d) > f(b) = f(c)$. In the second case, we say that v is a *cutting vertex* for f.

Given an admissible labelling f and a vertex v, we set

$$\langle v \mid f \rangle \coloneqq \begin{cases} \operatorname{sgn}(v)(q - q^{-1}), & \text{if } v \text{ is a cutting vertex,} \\ 1, & \text{otherwise,} \end{cases}$$

where $\operatorname{sgn}(v)$ is 1 if v is a positive crossing and -1 otherwise. We define the *interaction of D with f* as

$$\langle D \mid f \rangle \coloneqq \prod_{v} \langle v \mid f \rangle ,$$

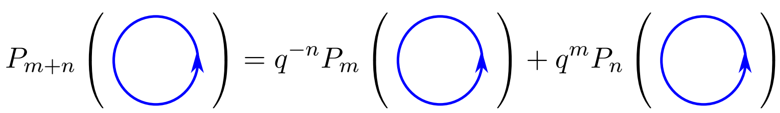
where the product runs over the set of vertices of D.

Proposition 1.3. ([Jae89, Proposition 1]) *Let $m, n \in \mathbb{N}$. Then, for any link diagram D,*

$$P_{m+n}(D) = \sum_{f \text{ admissible}} \langle D \mid f \rangle\, q^{-n \cdot r(D_{f,1})} q^{m \cdot r(D_{f,2})} P_m(D_{f,1}) P_n(D_{f,2}), \tag{1.1}$$

where $D_{f,i} \coloneqq f^{-1}(i)$ and $r(D_{f,i})$ is the rotation number of $D_{f,i}$, for $i \in \{1, 2\}$. ∎

Example 1.4. For the trivial knot, we have

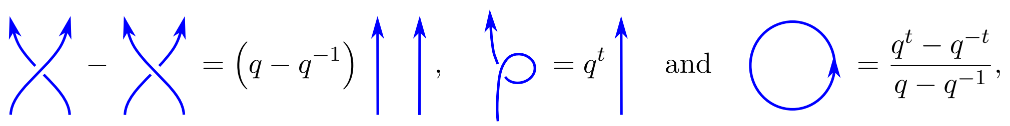


$$= q^{-n}[m]_q + q^m [n]_q = [m+n]_q.$$

◊

Extending Jaeger's composition formula, Turaev constructed in [Tur91] a coproduct for the HOMFLY skein algebra of an arbitrary framed surface S. Let us briefly recall this construction.

Definition 1.5. Let S be an oriented surface and $\Bbbk$ a commutative ring. We define the *HOMFLY skein algebra of S* as the $\Bbbk$-module $\mathrm{Sk}_{\mathrm{GL}_t}(S)$ spanned by isotopy classes of framed oriented links in $S \times [0, 1]$ modulo the local relations



with parameters $q, q^t \in \Bbbk^\times$. Multiplication is given by vertically stacking two copies of $S \times [0, 1]$ and retracting $S \times [0, 2]$ to $S \times [0, 1]$.

Again, any framed oriented link in $S \times [0, 1]$ can be represented by a link diagram (with blackboard framing) on S, and we can define admissible labellings as in the previous subsection.

Theorem 1.6. ([Tur91, Theorem 9.2]) *Let D be a link diagram on a framed surface S. Then, the formula*

$$\Delta(D) = \sum_{f \text{ admissible}} \langle D | f \rangle \left(\left(q^{-t}\right)^{r(D_{f,1})} D_{f,1} \right) \otimes \left(\left(q^{t}\right)^{r(D_{f,2})} D_{f,2} \right),$$

defines a coassociative coproduct on $\mathrm{Sk}_{\mathrm{GL}_t}(S)$. The pair $(\mathrm{Sk}_{\mathrm{GL}_t}(S), \Delta)$ is a bialgebra. ∎

1.2 A quick review of rigid categories. We recall in this section some facts about rigid categories that will be useful throughout the paper. Recall that a *monoidal category* is a category $\mathcal{C}$ endowed with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is associative and unital up to given natural isomorphisms. Every monoidal category is equivalent to a *strict* monoidal one, so we will assume that $\otimes$ is associative and unital on the nose.

Definition 1.7. A monoidal category $\mathcal{C}$ is *right rigid* if every object X has a right dual X^*, i.e., there are morphisms

$$\mathrm{ev}_X : X \otimes X^* \to \mathbf{1} \qquad \text{and} \qquad \mathrm{coev}_X : \mathbf{1} \to X^* \otimes X$$

satisfying the usual *zig-zags identities.* Similarly, we say that $\mathcal{C}$ is *left rigid* if every object X has a left dual ${}^*X$ and we say that it is *rigid* if it is both left and right rigid.

Duals are unique up to canonical isomorphism: if $(X^*, \mathrm{ev}_X, \mathrm{coev}_X)$ and $(Y, \widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X)$ are right duals of X, then

$$\varphi :\ Y \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_Y} X^* \otimes X \otimes Y \xrightarrow{\mathrm{id}_{X^*} \otimes \widetilde{\mathrm{ev}}_X} X^* \tag{1.2}$$

is a isomorphism between them and the (co)evaluations are related by

$$\widetilde{\mathrm{ev}}_X = \mathrm{ev}_X \circ (\mathrm{id}_X \otimes \varphi), \quad \widetilde{\mathrm{coev}}_X = (\varphi^{-1} \otimes \mathrm{id}_X) \circ \mathrm{coev}_X.$$

The same holds for left dualities. When the category $\mathcal{C}$ is rigid, we will always assume that distinguished left a right duals $\{X^*, \mathrm{ev}_X, \mathrm{coev}_X\}$ and $\{{}^*X, \mathrm{ev}'_X, \mathrm{coev}'_X\}$ has been fixed for every object. We thus have left and right duality functors

$${}^*(-), (-)^* : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$$

that, in general, do not coincide.

If $(F, J, J_0) : \mathcal{C} \to \mathcal{D}$ is a monoidal functor between rigid categories, then

$$\mathrm{ev}^F_X : F(X) \otimes F(X^*) \xrightarrow{J_{X,X^*}} F(X \otimes X^*) \xrightarrow{F(\mathrm{ev}_X)} F(\mathbf{1}) \xrightarrow{J_0} \mathbf{1}$$

and

$$\mathrm{coev}^F_X : \mathbf{1} \xrightarrow{J_0^{-1}} F(\mathbf{1}) \xrightarrow{F(\mathrm{coev}_X)} F(X^* \otimes X) \xrightarrow{J^{-1}_{X^*,X}} F(X^*) \otimes F(X)$$

exhibit $F(X^*)$ as a right dual of $F(X)$ (and the analogue is true for left duals). If $F(X)^*$ is the distinguished dual of $F(X)$ in $\mathcal{D}$, the isomorphism

$$\varphi_X : F(X^*) \to F(X)^* \tag{1.3}$$

from (1.2) is natural in X.

Definition 1.8. A *pivotal category* is a right rigid category $\mathcal{C}$ (with distinguished duality) endowed with a monoidal natural isomorphism

$$\iota_X : X^{**} \to X.$$

A monoidal functor $F : \mathcal{C} \to \mathcal{D}$ between pivotal categories is *pivotal* if the isomorphism

$$\eta_X : F(X)^{**} \xrightarrow{\varphi_X^*} F(X^*)^* \xrightarrow{\varphi_{X^*}^{-1}} F(X^{**}) \xrightarrow{F(\iota^{\mathcal{C}}_X)} F(X) \tag{1.4}$$

coincides with the pivotal structure of $\mathcal{D}$.

Using the pivotal structure, we can exhibit the distinguished right duality of $\mathcal{C}$ as a left duality. Namely, the morphisms

$$\mathrm{ev}'_X : X^* \otimes X \xrightarrow{\mathrm{id}_{X^*} \otimes \iota_X^{-1}} X^* \otimes X^{**} \xrightarrow{\mathrm{ev}_{X^*}} \mathbf{1}$$

and

$$\mathrm{coev}'_X : \mathbf{1} \xrightarrow{\mathrm{coev}_{X^*}} X^{**} \otimes X^* \xrightarrow{\iota_X \otimes \mathrm{id}_{X^*}} X \otimes X^*$$

satisfy the zig-zag identities. Therefore, for this choice of distinguished left duality, we get a canonical monoidal isomorphism $(-)^* \cong {}^*(-)$.

Recall that a *braided* monoidal category $\mathcal{C}$ is a monoidal category endowed with a natural isomorphism

$$c_{X \otimes Y} : X \otimes Y \to Y \otimes X$$

satisfying the well-known hexagon axioms.

Definition 1.9. A *balanced category* is a braided monoidal category $\mathcal{C}$ endowed with a natural isomorphism

$$\theta_X : X \to X,$$

called *twist*, such that

$$\theta_{X \otimes Y} = c_{Y,X} \circ c_{X,Y} \circ (\theta_X \otimes \theta_Y).$$

We say that $\mathcal{C}$ is *ribbon* if it is rigid and, moreover,

$$\theta_{X^*} = \theta_X^*,$$

for all $X \in \mathcal{C}$.

Proposition 1.10. [HPT16, Corollary A.3] *Let $\mathcal{C}$ be a rigid braided category. Then, there is a one-to-one correspondence between pivotal structures and twists on $\mathcal{C}$. There are exactly two ways of establishing this correspondence.* ■

In particular, if $\mathcal{C}$ is a braided pivotal category, there are two choices of twists given by

$$\theta_X^{(1)} = (\mathrm{id}_X \otimes \mathrm{ev}_{X^*}) \circ (c_{X^{**},X} \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{X^{**}} \otimes \mathrm{coev}_X) \circ \iota_X$$

and

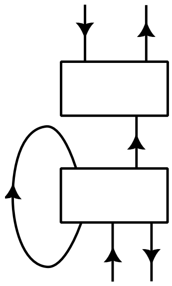
$$\theta_X^{(2)} = \iota_X^{-1} \circ (\mathrm{ev}_X \otimes \mathrm{id}_{X^{**}}) \circ (\mathrm{id}_{X^*} \otimes c_{X^{**},X}) \circ (\mathrm{coev}_{X^*} \otimes \mathrm{id}_X).$$

Proposition 1.11. [HPT16, Proposition A.4] *Let $\mathcal{C}$ be a braided pivotal category. Then, $(\mathcal{C}, \theta^{(i)})$ is ribbon for either $i = 1, 2$ if, and only if, $\theta^{(1)} = \theta^{(2)}$.* ■

1.3 Graphical languages for monoidal categories. The definition of the HOMFLY category $\mathrm{Rep}_q\mathrm{P}_t$ and its central algebra structure (cf. sections 2 and 3) will rely on the graphical calculus for monoidal categories, that we briefly review here. See [Sel09] for an extended survey and [HPT16, Section 2.1] for a nice overview.

⋆ **Rigid categories.** For a rigid monoidal category $\mathcal{C}$, we consider directed "vertical" planar graphs Γ embedded in $\mathbb{R} \times [0, 1]$ with endpoints lying in $\mathbb{R} \times \{0, 1\}$. By vertical we mean that edges never run horizontally. Local extrema (i.e. cups and caps) will be considered as vertices.

The rest of vertices are coupons, with incident edges attached to either the top or the bottom face. For instance,



is a planar graph of this type. We decorate edges of the graph with pairs (C, n), where $C \in \mathcal{C}$ and $n \in \mathbb{Z}$, according to the following rule: if an edge is directed upwards, then n will be even; and n will be odd if it is oriented downwards. To make this consistent, n has to change at local maxima and minima: turning counterclockwise increases n by 1, and turning clockwise decreases n by 1. Interpreting (C, n) as $C^{\overset{\times n}{*\cdots*}}$ and $(C, -n)$ by ${}^{\overset{\times n}{*\cdots*}}C$, for every $n \in \mathbb{N}$, the set of edges incident to the bottom and top faces of a coupon determine two objects $s, t \in \mathcal{C}$. The coupon will thus be decorated with a morphism $f : s \to t$. We consider two planar graphs to be equivalent if they are related by a *rectilinear isotopy* of the plane: this is an isotopy of $\mathbb{R} \times [0, 1]$ that do not rotate coupons.

Definition 1.12. The category $\mathrm{Rig}(\mathcal{C})$ has:

- objects: finite sequences of pairs (C, n), with $C \in \mathcal{C}$ and $n \in \mathbb{Z}$;
- morphisms: equivalence classes of decorated planar graphs as above, whose source and target are determined by the decoration of the bottom and top endpoints, respectively. Composition is given by vertically stacking diagrams.

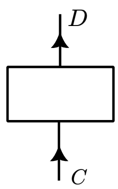
This category has a monoidal structure given by concatenation of objects and horizontal stacking of morphisms.

The category $\mathrm{Rig}(\mathcal{C})$ is the *free rigid category* on $\mathcal{C}$: it comes equipped with a monoidal evaluation functor

$$\mathrm{ev}_{\mathcal{C}} : \mathrm{Rig}(\mathcal{C}) \to \mathcal{C}$$

sending (C, n) to $C^{\overset{\times n}{*\cdots*}}$ and $(C, -n)$ to ${}^{\overset{\times n}{*\cdots*}}C$, for every $n \in \mathbb{N}$. A coupon decorated with f is mapped to f itself, and the image of cups and caps are the distinguished evaluation and coevaluation morphisms of $\mathcal{C}$.

$\star$ **Pivotal categories.** As explained in the previous section, pivotal structures yields canonical monoidal isomorphims $(-)^* \cong {}^*(-)$ and $(-)^{**} \cong (-)$. Since multiple duals are canonically identified, a graphical calculus for pivotal categories can be obtained by dropping the integer labels from the graphical calculus for rigid categories. Namely, we consider the same class of graphs as in the previous section, labelled as follows. Edges are coloured just with objects of $\mathcal{C}$. For every coupon, interpreting $(C, \uparrow)$ as C and $(C, \downarrow)$ as C^*, incident edges determine again two objects $s, t \in \mathcal{C}$. The coupon is thus labelled with a morphism $f : \tilde{s} \to \tilde{t}$, where $\tilde{s}, \tilde{t}$ are objects canonically identified with s, t via the pivotal structure. Two graphs are equivalent if they are related by a planar isotopy. In particular, we allow now isotopies rotating coupons by 2π. Note that this yields a more general class of graphs as before, since, for instance,

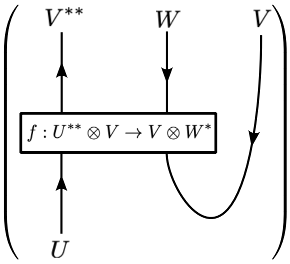


may be decorated by either a morphism of the form $f : C \to D$ or $f : C \to D^{**}$ (among many other possibilities).

As in the previous case, we get a category $\mathrm{Piv}(\mathcal{C})$ whose morphisms are planar graphs, together with an evaluation functor

$$\mathrm{ev}_{\mathcal{C}} : \mathrm{Piv}(\mathcal{C}) \to \mathcal{C},$$

which is the *free pivotal category* on $\mathcal{C}$ (the pivotal structure being trivial). For instance,

$\mathrm{ev}_{\mathcal{C}}$ 

$$= \left(\iota_V^{-1} \otimes \mathrm{id}_W \otimes \mathrm{id}_V\right) \circ (f \otimes \mathrm{id}_{V^*}) \circ \left(\iota_U^{-1} \otimes \mathrm{coev}'_V\right),$$

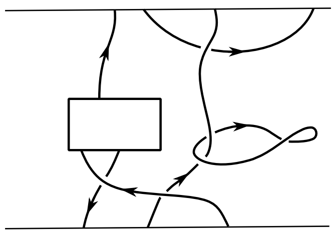
where ι is the pivotal structure of $\mathcal{C}$.

$\star$ **Braided pivotal categories.** The graphical calculus for braided pivotal categories is supported by three-dimensional diagrams:

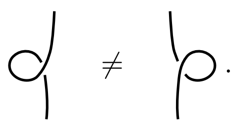
Definition 1.13. Let $\mathcal{C}$ be a braided pivotal category. The category $\mathrm{BrPiv}(\mathcal{C})$ has:

- objects: finite sequences of pairs (C, ε), with $C \in \mathcal{C}$ and $\varepsilon \in \{\uparrow, \downarrow\}$;
- morphisms: equivalence classes of directed graphs embedded in $\mathbb{R}^2 \times [0,1]$ with endpoints lying in $\mathbb{R} \times \{0\} \times \{0,1\}$ whose vertices are coupons decorated following the same rules as for pivotal categories.

The category $\mathrm{BrPiv}(\mathcal{C})$ is itself braided pivotal, with trivial pivotal structure and braiding given by crossing strands. Graphs will be represented by diagrams obtained by projecting into $\mathbb{R} \times [0,1]$ and keeping track of the relative position of the strands in crossings. For example,



is a diagram representing such a three-dimensional graph. Two graphs are equivalent if they are related by *regular isotopy*, i.e., their diagrams can be obtained from each other by applying a finite sequence of the Reidemeister moves II and III and a planar isotopy. In particular,



(1.5)

The two diagrams in (1.5) represent graphically the two families of twists turning $\mathrm{BrPiv}(\mathcal{C})$ into a balanced category (cf. Proposition 1.10).

The category $\mathrm{BrPiv}(\mathcal{C})$ is the *free braided category* on $\mathcal{C}$. Again, we have an evaluation functor

$$\mathrm{ev}_{\mathcal{C}} : \mathrm{BrPiv}(\mathcal{C}) \to \mathcal{C}$$

mapping a positive crossing to the braiding.

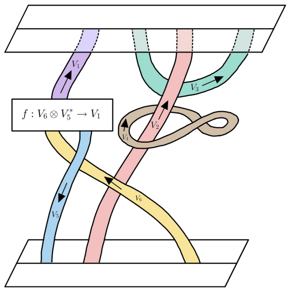


FIGURE 1: Coloured ribbon graph.

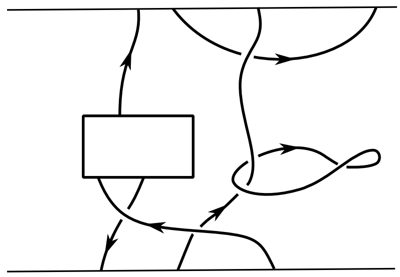


FIGURE 2: Ribbon diagram.

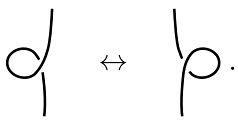
⋆ **Ribbon categories.** The inequality in (1.5) suggests that braided pivotal categories are still not very natural from a topological point of view. Ribbon categories are defined to be the class of braided pivotal categories in which the two terms coincide. The graphical calculus of ribbon categories supported by so-called ribbon graphs.

A *ribbon graph* is a three-dimensional graph whose edges are thick ribbons. More precisely, it is a compact oriented surface Ω embedded in $\mathbb{R}^2 \times [0,1]$ which decomposes into the following elementary pieces:

1. *ribbons*, i.e., homeomorphic copies of the square $[0,1] \times [0,1]$ whose core is directed;
2. *annuli*, i.e., homeomorphic copies of the cilinder $\mathbb{S}^1 \times [0,1]$ whose core is also directed;
3. *coupons*, with distinguished top and bottom bases.

Ribbons can meet coupons just on their distinguished bases. On the other hand, the choice of an orientation for Ω determines a "preferred side". We demand the free bases of ribbons (those not meeting any coupon) to meet the planes $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$ orthogonally at $\mathbb{R} \times \{0\} \times \{0\}$ and $\mathbb{R} \times \{0\} \times \{1\}$, respectively, and in such a way that the preferred side of Ω is turned up. We colour ribbons, coupons and annuli following the same rules as for pivotal categories.

Ribbon graphs will be considered up to isotopy: an *isotopy of ribbon graphs* is an isotopy of $\mathbb{R}^2 \times [0,1]$ which is the identity on $\mathbb{R}^2 \times \{0,1\}$ and preserves the decomposition into ribbons, annuli and coupons. By applying such an isotopy, any ribbon graph is equivalent to a graph whose ribbons, coupons and annuli are everywhere parallel to the vertical plane. Such a graph can be represented by a *ribbon diagram*, obtained by projecting ribbons and annuli onto their cores and projecting the resulting graph into $\mathbb{R} \times [0,1]$ (see Figure 2). Two ribbon diagrams represent the same ribbon graph if the can be obtained from each other by a finite sequence of Reidemeister moves II and III, planar isotopy and movements of the form



Definition 1.14. [Tur94, Section I.2] Let $\mathcal{C}$ be a ribbon category. The category $\mathrm{Rib}(\mathcal{C})$ has:

- objects: finite sequences of pairs (C, ε), with $C \in \mathcal{C}$ and $\varepsilon \in \{\uparrow, \downarrow\}$;
- morphisms: isotopy classes of directed ribbon graphs (defined below) embedded in $\mathbb{R}^2 \times [0,1]$ and coloured with $\mathcal{C}$ in the usual way.

This is a ribbon category, with twist given by .

As in the previous cases, $\mathrm{Rib}(\mathcal{C})$ is the *free ribbon category* on $\mathcal{C}$, and we have an evaluation functor

$$\mathrm{ev}_{\mathcal{C}} : \mathrm{Rib}(\mathcal{C}) \to \mathcal{C},$$

known *Reshetikhin–Turaev evaluation functor*.

1.4 Skein categories. The graphical calculus for ribbon categories extend to arbitrary oriented surfaces, yielding the notion of skein categoy. Skein categories are categorical invariants of oriented surfaces generalising the notion of skein algebras. They were introduced in [Wal06, JF19] and take a ribbon category $\mathcal{C}$ as algebraic ingredient:

Definition 1.15. ([Coo19, Definition 1.3]) Let S be an oriented surface. A *$\mathcal{C}$-coloured ribbon graph in S* is an embedding of a $\mathcal{C}$-coloured ribbon graph into $S \times [0,1]$ such that its free bases are sent to $S \times \{0,1\}$ and the rest lies in $S \times (0,1)$.

Definition 1.16. ([Coo19, Definition 1.5]) Let $\Bbbk$ be a commutative ground ring. The *category $Rib(\mathcal{C}, S)$ of ribbon graphs in S* has:

- objects: finite sets $\left\{x_1^{(C_1,\varepsilon_1)}, \dots, x_m^{(C_m,\varepsilon_m)}\right\}$ of disjoint framed points $x_i \in S$ coloured with pairs (C_i, ε_i), where $C_i \in \mathcal{C}$ and $\varepsilon_i \in \{\uparrow, \downarrow\}$;
- morphisms from $\left\{x_1^{(C_1,\varepsilon_1)}, \dots, x_m^{(C_m,\varepsilon_m)}\right\}$ to $\left\{y_1^{(D_1,\eta_1)}, \dots, y_n^{(D_m,\eta_n)}\right\}$ are $\Bbbk$-linear combinations of isotopy classes of $\mathcal{C}$-coloured ribbon graphs in S whose bottom and top are, respectively, $\left\{x_1^{(C_1,\varepsilon_1)}, \dots, x_m^{(C_m,\varepsilon_m)}\right\}$ and $\left\{y_1^{(D_1,\eta_1)}, \dots, y_n^{(D_m,\eta_n)}\right\}$.

Composition is given by vertically stacking two copies of $S \times [0,1]$ and retracting $S \times [0,2]$ to $S \times [0,1]$. In particular, $\mathrm{Rib}(\mathcal{C}, \mathbb{R}^2) = \mathrm{Rib}(\mathcal{C})$.

Definition 1.17. ([Coo19, Definition 1.9]) Let $\mathcal{C}$ be a ribbon category and S an oriented surface. The *skein category $SkCat_{\mathcal{C}}(S)$* is the quotient of $\mathrm{Rib}(\mathcal{C}, S)$ by the following relations: a morphism $\sum_i \lambda_i \Omega_i \sim 0$ if there exists an orientation preserving embedding $\iota : \mathbb{R}^2 \times [0,1] \hookrightarrow S \times [0,1]$ such that:

1. the intersection of each Ω_i with the boundary of $\iota(\mathbb{R}^2 \times [0,1])$ consist only of transverse ribbons intersecting $\iota(\mathbb{R} \times \{0\} \times \{0\})$ and $\iota(\mathbb{R} \times \{0\} \times \{1\})$;
2. the Ω_i are equal outside $\iota(\mathbb{R}^2 \times [0,1])$;
3. we have
$$\sum_i \lambda_i \mathrm{ev}_{\mathcal{C}} \left(\iota^{-1}\left(\Omega_i \cap \iota(\mathbb{R}^2 \times [0,1])\right)\right) = 0.$$

Note that skein categories are not monoidal in general. When $S = C \times I$ for some one-manifold C, a tensor product can be defined by stacking two copies of $C \times I$. The following follows directly from the definition:

Proposition 1.18. *Let $\mathcal{C}$ be a ribbon category. Then,*

$$SkCat_{\mathcal{C}}(\mathbb{R}^2) \simeq \mathcal{C}$$

as ribbon categories. ■

1.5 Parabolic restriction. Let $\Bbbk$ be a field of characteristic 0 and $q \in \Bbbk^\times$. Fix $N \in \mathbb{N}$ and assume that q is not a root of unity. The algebra $U_q(\mathfrak{gl}_N)$ is defined by generators e_i, f_i, d_j, d_j^{-1}, $i = 1, \dots, N-1$, $j = 1, 2, \dots, N$, and relations

$$d_i d_j = d_j d_i, \quad d_i d_i^{-1} = d_i^{-1} d_i = 1,$$

$$d_i e_j d_i^{-1} = q^{\delta_{ij}-\delta_{i,j+1}} e_j, \quad d_i f_j d_i^{-1} = q^{-\delta_{i,j}+\delta_{i,j+1}} f_j,$$
$$e_i f_j - f_j e_i = \delta_{ij} \frac{d_i d_{i+1}^{-1} - d_i^{-1} d_{i+1}}{q-q^{-1}},$$
$$e_i e_j = e_j e_i, \quad f_i f_j = f_j f_i, \quad |i-j| \geq 2,$$
$$e_i^2 e_{i\pm 1} - (q+q^{-1}) e_i e_{i\pm 1} e_i + e_{i\pm 1} e_i^2 = 0,$$
$$f_i^2 f_{i\pm 1} - (q+q^{-1}) f_i f_{i\pm 1} f_i + f_{i\pm 1} f_i^2 = 0.$$

This algebra becomes a Hopf algebra with the comultiplication $\Delta : U_q(\mathfrak{gl}_N) \to U_q(\mathfrak{gl}_N) \otimes U_q(\mathfrak{gl}_N)$ defined on generators by

$$\Delta(e_i) = d_i^{-1} d_{i+1} \otimes e_i + e_i \otimes 1, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes d_i d_{i+1}^{-1}, \quad \Delta(d_i^{\pm 1}) = d_i^{\pm 1} \otimes d_i^{\pm 1}.$$

The counit $\varepsilon : U_q(\mathfrak{gl}_N) \to \Bbbk$ and the antipode $S : U_q(\mathfrak{gl}_N) \to U_q(\mathfrak{gl}_N)$ are given, respectively, by

$$\varepsilon(e_i) = 0, \quad \varepsilon(f_i) = 0, \quad \varepsilon(d_i) = 1$$

and

$$S(e_i) = -d_i d_{i+1}^{-1} e_i, \quad S(f_i) = -f_i d_i^{-1} d_{i+1}, \quad S(d_i) = d_i^{-1}.$$

Definition 1.19. The *natural representation* $\rho : U_q(\mathfrak{gl}_N) \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^N)$ of $U_q(\mathfrak{gl}_N)$ is defined by

$$\rho(d_i) = qE_{i,i} + \sum_{i \neq j} E_{j,j}, \quad \rho(e_i) = E_{i,i+1}, \quad \rho(f_i) = E_{i+1,i},$$

where $E_{i,j}$ is the $N \times N$ matrix with 1 in the (i,j)-position and 0 elsewhere. We denote this representation by V_N.

Fix $m, n \in \mathbb{N}$ and consider the subalgebra $U_q(\mathfrak{l})$ of $U_q(\mathfrak{gl}_{m+n})$ generated by $\{d_i, e_j, f_j \mid j \neq m\}$. We may identify $U_q(\mathfrak{l}) \cong U_q(\mathfrak{gl}_m) \otimes U_q(\mathfrak{gl}_n)$ by mapping

$$d_i \mapsto d_i, \qquad e_j, f_j \mapsto \begin{cases} e_j \otimes 1, f_j \otimes 1, & \text{for } 0 \leq j \leq m-1, \\ 1 \otimes e_{j-m}, 1 \otimes f_{j-m}, & \text{for } m+1 \leq j \leq m+n-1, \end{cases}$$

so that $U_q(\mathfrak{l})$ acts on V_m and V_n. Similarly, let $U_q(\mathfrak{p})$ be the subalgebra of $U_q(\mathfrak{gl}_{m+n})$ generated by $\{d_i, e_i, f_j \mid j \neq m\}$. Note that $U_q(\mathfrak{l})$ is a subalgebra of $U_q(\mathfrak{p})$, but also a quotient by the ideal generated by e_m. Restricting along the inclusion and projection morphisms, we get functors

$$\begin{tikzcd}
 & \mathrm{Rep}\ U_q(\mathfrak{p}) \arrow[dr, bend left, "j^*"] & \\
\mathrm{Rep}\ U_q(\mathfrak{gl}_{m+n}) \arrow[ur, "\iota^*"] \arrow[rr, "\mathrm{res}_{m,n}"] & & \mathrm{Rep}\ U_q(\mathfrak{l}). \arrow[ul, "\pi^*"']
\end{tikzcd} \tag{1.6}$$

Note that $\mathrm{res}_{m,n}(V_{m+n}) \simeq V_m \oplus V_n$ in Rep $U_q(\mathfrak{l})$.

For any $N \in \mathbb{N}$, the universal R-matrix of $U_q(\mathfrak{gl}_N)$ is an invertible element $\mathcal{R}$ lying in a completion of $U_q(\mathfrak{gl}_N) \otimes U_q(\mathfrak{gl}_N)$. It induces a braiding on Rep $U_q(\mathfrak{gl}_N)$ given, in Sweedler notation, by

$$c_{V,W}(v,w) \coloneqq \mathcal{R}_{(2)} w \otimes \mathcal{R}_{(1)} v,$$

for $v \in V$ and $w \in W$. Applied to the fundamental representation, the braiding yields an automorphism $\beta_N : V_N \otimes V_N \to V_N \otimes V_N$ that can be written explicitly as

$$\beta_N = q \sum_i (E_{i,i} \otimes E_{i,i}) + \sum_{i \neq j} (E_{i,i} \otimes E_{i,j}) + \left(q - q^{-1}\right) \sum_{i<j} (E_{i,j} \otimes E_{j,i}). \tag{1.7}$$

Taking $N = m+n$ and restricting to $U_q(\mathfrak{l})$, this isomorphism decomposes as

$$\mathrm{res}_{m,n}(\beta_{m+m}) = \left(\begin{array}{c|c|c|c} \beta_m & 0 & 0 & 0 \\ \hline 0 & 0 & \sigma_{m,n} & 0 \\ \hline 0 & \sigma_{n,m} & (q-q^{-1})\,\mathrm{Id}_{m,n} & 0 \\ \hline 0 & 0 & 0 & \beta_n \end{array}\right) : \ (V_m \oplus V_n)^{\otimes 2} \to (V_m \oplus V_n)^{\otimes 2} \qquad (1.8)$$

in Rep $U_q(\mathfrak{l})$, hence the monoidal functor $\mathrm{res}_{m,n}$ is not braided.

On th other hand, it is also well-known that the category Rep $U_q(\mathfrak{p})$ is not braided. Let $W \in \mathrm{Rep}\ U_q(\mathfrak{gl}_{m+n})$ and $V \in \mathrm{Rep}\ U_q(\mathfrak{p})$. The universal R-matrix $\mathcal{R}$ of $U_q(\mathfrak{gl}_n)$ lies in $U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_-) \subseteq U_q(\mathfrak{p}) \otimes U_q(\mathfrak{b}_-)$, where $\mathfrak{b}_\pm$ are the positive and negative Borel subalgebras of $\mathfrak{gl}_{m+n}$. Thus, $\mathcal{R}$ has a well-defined action on $V \otimes \iota^*(W)$ and composing with the flip of tensor factors yields an isomorphism of $V \otimes j^*(W) \to j^*(W) \otimes V$ in Rep $U_q(\mathfrak{p})$ which is natural in V. Similarly, if $U \in \mathrm{Rep}\ U_q(\mathfrak{l})$, the R-matrix of $U_q(\mathfrak{l})$ acts on $V \otimes \pi^*(U)$ providing an isomorphism $V \otimes \pi^*(U) \to \pi^*(U) \otimes V$ natural in V.

Definition 1.20. The restriction functors $\iota^* :\ \mathrm{Rep}\ U_q(\mathfrak{gl}_{m+n}) \to \mathrm{Rep}\ U_q(\mathfrak{p})$ and $\pi^* :\ \mathrm{Rep}\ U_q(\mathfrak{l}) \to \mathrm{Rep}\ U_q(\mathfrak{p})$ lift to a braided functor

$$\iota^* \boxtimes \pi^* \colon \mathrm{Rep}\ U_q(\mathfrak{gl}_{m+n}) \boxtimes \overline{\mathrm{Rep}\ U_q(\mathfrak{l})} \to \mathcal{Z}\left(\mathrm{Rep}\ U_q(\mathfrak{p})\right),$$

where $\overline{\mathrm{Rep}\ U_q(\mathfrak{l})}$ is the opposite of Rep $U_q(\mathfrak{l})$ as braided category and $\mathcal{Z}(\mathrm{Rep}\ U_q(\mathfrak{p}))$ is the Drinfeld center of Rep $U_q(\mathfrak{p})$. We call this functor *parabolic restriction.*

1.6 Categories and bimodules. We recall in this section a few categorical notions that will appear repeatedly in our constructions. We fix a ground ring k and we set Vect for the category k-modules. A *k-linear category* is a category $\mathcal{C}$ enriched over Vect. The category Cat of small k-linear categories and k-linear functors is symmetric monoidal:

Definition 1.21. Given $\mathcal{C}, \mathcal{D} \in$ Cat, we define $\mathcal{C} \boxtimes \mathcal{D}$ as the k-linear category whose

- objects are pairs (c, d) with $c \in \mathcal{C}, d \in \mathcal{D}$;
- morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}\boxtimes\mathcal{D}}((c_1, d_1), (c_2, d_2)) \coloneqq \mathrm{Hom}_{\mathcal{C}}(c_1, c_2) \boxtimes_k \mathrm{Hom}_{\mathcal{D}}(d_1, d_2).$$

Definition 1.22. Let $F \colon C \boxtimes \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ be a k-linear bifunctor and suppose that $\mathcal{D}$ is cocomplete. The *coend of F* is the object

$$\int^{c\in\mathcal{C}} F(c,c) \coloneqq \mathrm{colim}\left(\prod_{f\colon c\to c'} F(c,c') \overset{F(f,c')}{\underset{F(c,f)}{\rightrightarrows}} \prod_{c\in\mathcal{C}} F(c,c) \right) \in \mathcal{D}.$$

If $\mathcal{D} =$ Vect, this is explicitly given by

$$\int^{c\in\mathcal{C}} F(c,c) = \bigoplus_{c\in\mathcal{C}} F(c,c) \,/\!\sim,$$

where we mod out by the image of morphisms of the form

$$F(c,c') \xrightarrow{F(f,c')-F(c,f')} F(c,c) \oplus F(c',c').$$

Definition 1.23. The category Bimod has:

- objects: small k-linear categories;
- morphisms: a morphism from $\mathcal{C}$ to $\mathcal{D}$ is a functor of the form $F : \mathcal{C} \boxtimes \mathcal{D}^{\mathrm{op}} \to \mathrm{VECT}$.

The composition of $F : \mathcal{C} \boxtimes \mathcal{D}^{\mathrm{op}} \to \mathrm{VECT}_k$ and $G : \mathcal{D} \boxtimes \mathcal{E}^{\mathrm{op}} \to \mathrm{VECT}_k$ is given by the coend

$$(G \circ F)(c, e) \coloneqq \int^{d \in \mathcal{D}} F(c, d) \otimes_k G(d, e).$$

The identity of $\mathcal{C}$ is the bimodule $\mathrm{Hom}_{\mathcal{C}}(-,-)$. The tensor product of k-linear categories endows BIMOD with the structure of a symmetric monoidal category.

Remark 1.24. The category CAT embeds into Bimod. Indeed, given a k-linear functor $F\colon \mathcal{C} \to \mathcal{D}$, we get a bimodule

$$F^* \colon \mathcal{C} \boxtimes \mathcal{D}^{\mathrm{op}} \to \mathrm{VECT}, \quad c \boxtimes d \mapsto \mathrm{Hom}_{\mathcal{D}}(d, F(c)).$$

$\Diamond$

1.7 Stratified spaces. In sections 4 and 5, we define a skein theory on 3-manifolds with line and surface *defects*. Here, we introduce the terminology and conventions for stratified spaces that will be used later.

Definition 1.25. (i) A *stratified space* is a vector space X endowed with a continuous map

$$\phi\colon X \to P,$$

where P is a poset endowed with the topology whose open sets are generated by $P_{>p} \coloneqq \{x \in P \mid x > p\}$, for $p \in P$. For each $p \in P$, the subset $X_p \coloneqq \phi^{-1}(p)$ is a *stratum*.

(ii) A *morphism of stratified spaces* between (X, ϕ) and (Y, ψ) consists of a pair of continuous maps $f\colon X \to Y$, $F\colon P \to Q$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & Y \arrow[d, "\psi"] \\ P \arrow[r, "F"] & Q \end{tikzcd}$$

is commutative. We say that it is a *stratified embedding* if f and $f|_{X_p}$ are embeddings for every $p \in P$.

(iii) A *stratified isotopy* between two stratified embeddings $(f, F), (g, G)\colon X \to Y$ is a morphism $(h, H)\colon X \times I \to Y$ of stratified spaces such that $h(-, 0) = f$ and $h(-, 1) = g$. Here, the stratification on $X \times I$ is given by $\varphi(x, t) = \phi(x)$, where ϕ is the stratification on X. In particular, a *stratified isotopy of X* is a continuous family of maps $f_t\colon X \to X$ such that

$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[dr, "\phi"'] & X \arrow[d, "\phi"] \\ & P \end{tikzcd}$$

commutes for every $t \in [0, 1]$ and $f_0 = \mathrm{id}_X$.

We will typically with stratified three-manifolds and we will say that X_p is a *line defect* when $\dim(X_p) = 1$ and that it is a *surface defect* when $\dim(X_p) = 2$.

Definition 1.26. A *bipartite surface* is an oriented surface S endowed with a stratification

$$\phi\colon S \to (A \geq B \leq C)$$

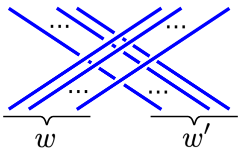
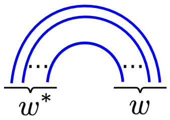
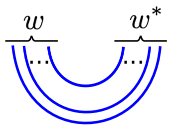
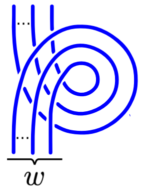
such that $S_B = \phi^{-1}(B)$ is a smooth curve that forms the common boundary between the open strata S_A and S_C.

Throughout the rest of the paper, we fix the ground ring to be

$$\Bbbk = \mathbb{C}(q)[q^{\pm t}, \delta] \Big/ \langle (q - q^{-1})\,\delta = q^t - q^{-t} \rangle .$$

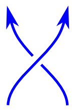
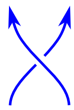
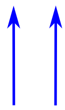
In this section, we recall the definition of the HOMFLY category $\mathrm{Rep}_q\mathrm{GL}_t$ ([Bru17, De07]) and its two-coloured version, $\mathrm{Rep}_q\mathrm{L}_t$. Next, we define a HOMFLY version of the category $\mathrm{Rep}_q\mathrm{P}_t$ of quantum representations for a parabolic subgroup $\mathrm{P} \subseteq \mathrm{GL}_{m+n}$.

2.1 The HOMFLY category $\mathrm{Rep}_q\mathrm{GL}_t$. One of our main objects of study will be the HOMFLY category $\mathrm{Rep}_q\mathrm{GL}_t$, that we introduce following [Bru17]. Let $\mathcal{T}$ be the category of 1-coloured framed oriented tangles, whose objects are finite words over $\{\uparrow, \downarrow\}$ and whose morphisms are $\Bbbk$-linear combinations of isotopy classes of framed oriented *tangles* (i.e. ribbon graphs without coupons) in $\mathbb{R}^2 \times [0, 1]$. This category has a ribbon structure. The tensor product is given by concatenation of objects and by horizontally stacking morphisms. The right dual of an object $w = w_1 \cdots w_k$ is the object $w^* := w_k^* \cdots w_1^*$, where $(-)^*$ interchanges $\uparrow$ and $\downarrow$. The braiding, the evaluation and coevaluation, and the twist are represented by the diagrams

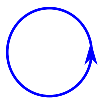
, , , ,

where the directions are induced by the depicted words. Exchanging w and w^* in the evaluation and coevaluation maps, we can exhibit w^* as a left dual of w, hence $\mathcal{T}$ has a strictly pivotal structure.

Definition 2.1. The category $\mathrm{Rep}_q\mathrm{GL}_t$ is the quotient of $\mathcal{T}$ by the *HOMFLY skein* (2.1), the *twist* (2.2) and the *dimension* (2.3) relations:

 $-$  $= (q - q^{-1})$ , (2.1)

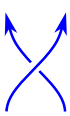
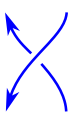
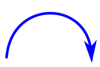
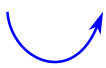
 $= q^t$ , (2.2)

 $= \delta\, \mathbf{1}_\varnothing$, (2.3)

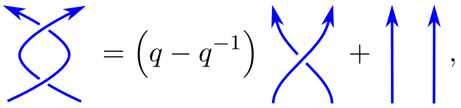
where $\mathbf{1}_\varnothing$ is the empty diagram.

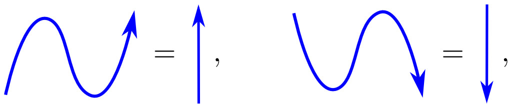
This category inherits a ribbon structure from $\mathcal{T}$ and it has a nice presentation by generators and relations:

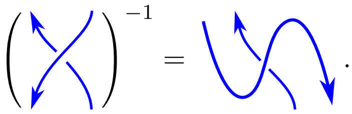
Proposition 2.2. ([Bru17, Theorem 1.1]) *$\mathrm{Rep}_q\mathrm{GL}_t$ is generated, as a $\Bbbk$-linear strict monoidal category, by the objects $\uparrow$ and $\downarrow$ and the morphisms*

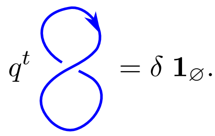
, , , ,

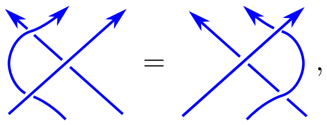
subject to the relations

 (2.4)

 (2.5)

 (2.6)

 (2.7)

 (2.8)

■

Remark 2.3. The HOMFLY category $\mathrm{Rep}_q\mathrm{GL}_t$ interpolates between the categories $\mathrm{Rep}_q\mathrm{GL}_N$ of locally finite representations of $U_q(\mathfrak{gl}_N)$. More precisely, for any integer $N \in \mathbb{N}$, consider the evaluation map

$$\varphi_N : \Bbbk \to \mathbb{C}(q), \quad q^t \mapsto q^N, \quad \delta \mapsto \frac{q^N - q^{-N}}{q - q^{-1}}.$$

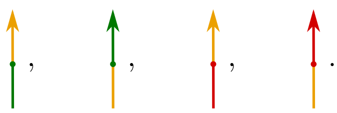
There exists a φ_N-linear ribbon functor

$$\mathrm{ev}_N^{\mathrm{G}} : \mathrm{Rep}_q\mathrm{GL}_t \to \mathrm{Rep}_q\mathrm{GL}_N$$

sending the object $\uparrow$ to the natural representation V_N. The category $\mathrm{Rep}_q\mathrm{GL}_N$ can be recovered as the ind-completion of the quotient of $\mathrm{Rep}_q\mathrm{GL}_t$ by the ideal of negligible morphisms (see [Bru17, Theorem 1.3] for a precise statement). ◊

2.2 The HOMFLY category $\mathrm{Rep}_q\mathrm{L}_t$. Next, we describe diagrammatically the *two-coloured HOMFLY category* $\mathrm{Rep}_q\mathrm{L}_t$, interpolating the categories $\mathrm{Rep}_q\mathrm{L}$ of locally finite representations of $U_q(\mathfrak{l})$. In the same lines as for the construction of $\mathrm{Rep}_q\mathrm{GL}_t$, let $\mathcal{T}$(🟧, 🟩, 🟥) be the category of *three-coloured tangles with defects*, whose

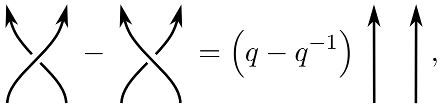
- objects are finite sequences of oriented points coloured with {🟧, 🟩, 🟥};
- morphisms are $\Bbbk \otimes_{\mathbb{C}(q)} \Bbbk$-linear combinations of isotopy classes of coloured framed oriented tangles tangles with 0-dimensional defects of the form



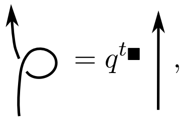
This category is again ribbon, with ribbon structure defined as in the previous section.

Definition 2.4. The category $\mathrm{Rep}_q\mathrm{L}_t$ is the quotient of $\mathcal{T}$(🟧, 🟩, 🟥) by

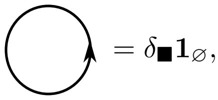
- the one-coloured skein, twist and dimension relations:



(2.9)



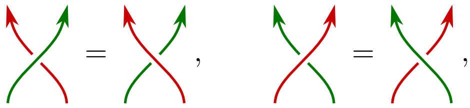
(2.10)



(2.11)

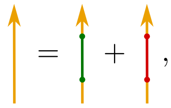
with all the strands coloured in the same colour $\blacksquare \in \{\blacksquare = 1, \blacksquare = 2\}$ and $q^{t_1} := q^t \otimes 1$, $q^{t_2} := 1 \otimes q^t$, $\delta_1 := \delta \otimes 1$ and $\delta_2 := 1 \otimes \delta$;

- the two-coloured crossing relations:



(2.12)

- the relations



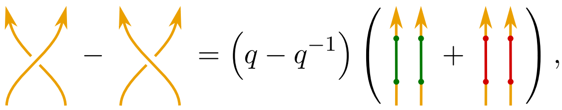
(2.13)

(2.14)

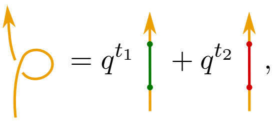
Proposition 2.5. *$\mathrm{Rep}_q L_t$ inherits a ribbon structure from $\mathcal{T}(\blacksquare, \blacksquare, \blacksquare)$.* ■

Given relations (2.12), we no longer need to distinguish between positive and negative two-coloured crossings. Furthermore, relations (2.13) and (2.14) imply that $\uparrow = \uparrow \oplus \uparrow$. This identity justifies the reference to $\mathrm{Rep}_q L_t$ as "two-coloured".

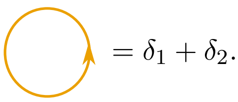
Lemma 2.6. *The following relations hold in $\mathrm{Rep}_q L_t$:*



(2.15)

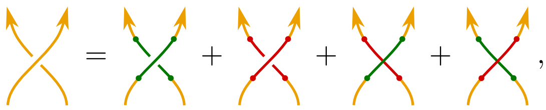


(2.16)



(2.17)

Proof. Applying (2.13), we can write



(2.18)

and the same is true for the negative crossing. The first relation of the statement thus follows from the green and red skein relations. Likewise, the blue twist and dimension relations are an easy consequence of the green and red ones. □

Remark 2.7. Again, $\mathrm{Rep}_q\mathrm{L}_t$ is universal among the categories $\mathrm{Rep}_q\mathrm{L}$. Specialising t_1 and t_2 at integers values $m, n \in \mathbb{N}$, we get an evaluation morphism

$$\varphi_{m,n} : \Bbbk \otimes_{\mathbb{C}(q)} \Bbbk \to \mathbb{C}(q),$$

and there is a canonical evaluation functor

$$\mathrm{ev}^{\mathrm{L}}_{m,n} : \mathrm{Rep}_q\mathrm{L}_t \to \mathrm{Rep}_q\, U_q(\mathfrak{l})$$

mapping $\uparrow$ and $\uparrow$ to the natural representations of $U_q(\mathfrak{gl}_m)$ and $U_q(\mathfrak{gl}_n)$, respectively, and $\uparrow$ to their direct sum. $\Diamond$

Proposition 2.8. *There is a monoidal equivalence of categories*

$$\mathit{Rep}_q\mathit{L}_t \cong \mathit{Rep}_q\mathit{GL}_t \boxtimes \mathit{Rep}_q\mathit{GL}_t.$$

Proof. We define an equivalence $\Phi : \mathrm{Rep}_q\mathrm{L}_t \to \mathrm{Rep}_q\mathrm{GL}_t \boxtimes \mathrm{Rep}_q\mathrm{GL}_t$ as follows. Since $\uparrow = \uparrow \oplus \uparrow$ in $\mathrm{Rep}_q\mathrm{L}_t$, we just have to define Φ on green and red objects and extend by cocontinuity. Moreover, relations (2.13) and (2.14) imply that any morphism between two configurations of green and red points may be represented by a linear combination of ribbon diagrams containing no orange component. For a word w over $\{\uparrow, \downarrow, \uparrow, \downarrow\}$, we set $\boldsymbol{w_1}$ and $\boldsymbol{w_2}$ for the subwords of green and red symbols, respectively. Similarly, if f is a tangle diagram in $\mathrm{Rep}_q\mathrm{L}_t$, let $\boldsymbol{f_1}$ (resp. $\boldsymbol{f_2}$) be the tangle diagrams obtained by removing all the red (resp. green) strands in f. We define Φ by $w \mapsto (\boldsymbol{w_1}, \boldsymbol{w_2})$ on objects and by $f \mapsto \boldsymbol{f_1} \otimes \boldsymbol{f_2}$ on generating morphisms. This functor is clearly surjective on objects and fully-faithful (this follows easily from the fact that we do not distinguish between a positive and a negative crossing when strands have different colours). Moreover, it preserves the tensor product, so it yields an equivalence of monoidal categories. A quasi-inverse is given by the functor $\Psi : \mathrm{Rep}_q\mathrm{GL}_t \boxtimes \mathrm{Rep}_q\mathrm{GL}_t \to \mathrm{Rep}_q\mathrm{L}_t$ defined by $(\boldsymbol{w}_1, \boldsymbol{w}_2) \mapsto \boldsymbol{w}_1\boldsymbol{w}_2$ and $(\boldsymbol{f}_1, \boldsymbol{f}_2) \mapsto \boldsymbol{f}_1\boldsymbol{f}_2$. $\square$

Corollary 2.9. *For any oriented surface S, we have an equivalence of $(\Bbbk \otimes_{\mathbb{C}(q)} \Bbbk)$-linear categories*

$$\mathit{SkCat}_{\mathrm{L}_{t,t}}(S) \cong \mathit{SkCat}_{\mathrm{GL}_t}(S) \boxtimes \mathit{SkCat}_{\mathrm{GL}_t}(S).$$

In particular, this induces an algebra isomorphism

$$\mathit{Sk}_{\mathrm{L}_{t,t}}(S) \cong \mathit{Sk}_{\mathrm{GL}_t}(S) \otimes_{\mathbb{C}(q)} \mathit{Sk}_{\mathrm{GL}_t}(S).$$

Proof. Again, we define a functor $\mathrm{Rib}_{\mathrm{Rep}_q\mathrm{L}_t}(S) \to \mathrm{SkCat}_{\mathrm{GL}_t}(S) \boxtimes \mathrm{SkCat}_{\mathrm{GL}_t}(S)$ by separating the green and the red colour. It follows from the previous theorem that it is compatible with skein relations, so it factors through the skein category, yielding the desired equivalence of categories. $\square$

2.3 The restriction functor.

The restriction functor

$$\mathrm{res}_{m,n} \colon \mathrm{Rep}_q\mathrm{GL}_{m+n} \to \mathrm{Rep}_q\mathrm{L}$$

admits also a HOMFLY version that we construct diagrammatically in this section. We first define a coproduct $\Delta : \Bbbk \to \Bbbk \otimes_{\mathbb{C}(q)} \Bbbk$ by setting

$$\Delta\left(q^t\right) = q^{t_1}q^{t_2}, \quad \Delta(\delta) = \delta_1\, q^{t_2} + q^{-t_1}\, \delta_2 \tag{2.19}$$

on generators and extending $\mathbb{C}(q)$-linearly. Here, $q^{t_1} := q^t \otimes 1$, $q^{t_2} := 1 \otimes q^t$, $\delta_1 := \delta \otimes 1$ and $\delta_2 := 1 \otimes \delta$.

Lemma 2.10. *The map $\Delta : \Bbbk \to \Bbbk \otimes_{\mathbb{C}(q)} \Bbbk$ is well-defined and coassociative.*

Proof. This is an easy computation. Firstly,

$$\begin{aligned}\Delta(q^t - q^{-t}) &= q^{t_1}q^{t_2} - q^{-t_1}q^{-t_2} \\ &= (q^{t_1} - q^{-t_1})q^{t_2} + q^{-t_1}(q^{t_2} - q^{-t_2}) \\ &= (q - q^{-1})(\delta_1 q^{t_2} + q^{-t_1}\delta_2) = (q - q^{-1})\Delta(\delta),\end{aligned}$$

so the map is well-defined. For the coassociativity, we have

$$(1 \otimes \Delta) \circ \Delta(q^t) = q^{t_1}q^{t_2}q^{t_3} = (\Delta \otimes 1) \circ \Delta(q^t),$$

and

$$\begin{aligned}(1 \otimes \Delta) \circ \Delta(\delta) &= \delta_1(q^{t_2}q^{t_3}) + q^{-t_1}(\delta_2 q^{t_3} + q^{-t_2}\delta_3) \\ &= (\delta_1 q^{t_2} + q^{-t_1}\delta_2)q^{t_3} + (q^{-t_1}q^{-t_2})\delta_3 = (\Delta \otimes 1) \circ \Delta(\delta).\end{aligned}$$

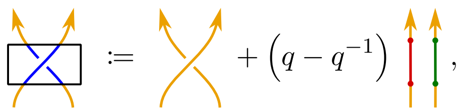
□

Remark 2.11. Specialising formulas (2.19) at $t = m + n$, $t_1 = m$ and $t_2 = n$, with $m, n \in \mathbb{N}$, we recover the identities

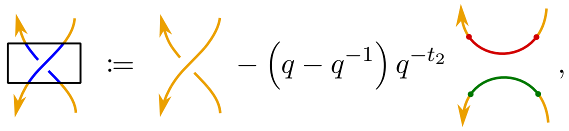
$$q^{m+n} = q^m q^n, \qquad [m + n]_q = q^n[m]_q + q^{-m}[n]_q,$$

expressing the quantum dimension of the fundamental representation of $U_q(\mathfrak{gl}_{m+n})$ in terms of the quantum dimensions of the fundamental representations of $U_q(\mathfrak{gl}_m)$ and $U_q(\mathfrak{gl}_n)$. Again, we may think of Δ as interpolating these formulas. ♢

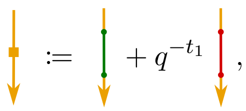
We introduce the following notations:



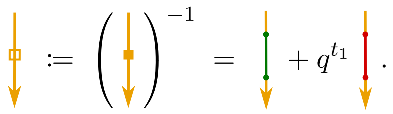
(2.20)



(2.21)



(2.22)

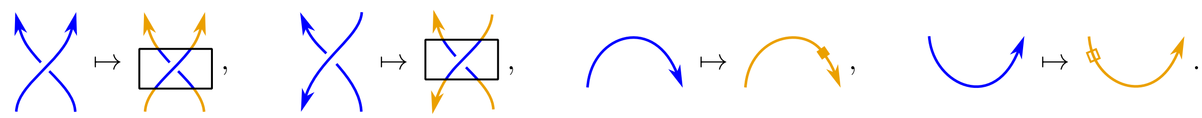


(2.23)

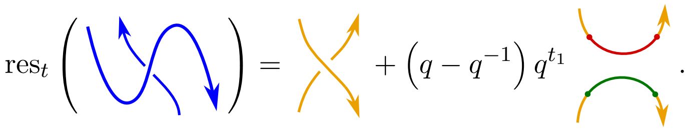
Theorem 2.12. *There exists a unique Δ-linear functor*

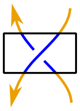
$$res_t : Rep_q GL_t \to Rep_q L_t \tag{2.24}$$

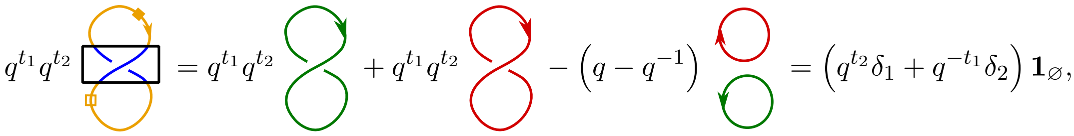
preserving the tensor product and such that



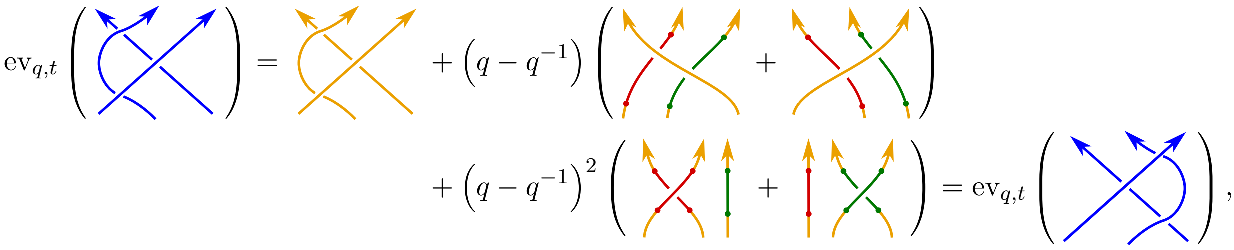
Proof. Proposition 2.2 provides a presentation of $\mathrm{Rep}_q\mathrm{GL}_t$ by generators and relations. Since res_t is defined on generators, it will be unique if it is well defined. We check that it preserves relations (2.4) - (2.8). For the skein relation (2.4) and the zigzags (2.5), this is an easy computation using the corresponding green and red relations and (2.14). Applying the functor to the right-hand side of (2.6), we have



Composing with  and applying the one-colour twist relations to the terms involving the (co)evaluations appearing, we get the identity, so (2.6) is preserved. On the other hand,



so relation (2.7) is preserved. Finally, using the relations in Remark 2.14, one gets



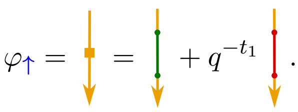
so the Reidemeister relation (2.8) is also preserved. □

Along the same lines as Remarks 2.3 and 2.7, the functor 2.24 interpolates the restriction functors $\mathrm{res}_{m,n} : \mathrm{Rep}_q\mathrm{GL}_{m+n} \to \mathrm{Rep}_q\mathrm{L}$. That is, the diagram

$$\begin{tikzcd}
\mathrm{Rep}_q\mathrm{GL}_t \arrow[r, "\mathrm{res}_t"] \arrow[d, "\mathrm{ev}^{\mathrm{G}}_{m+n}"'] & \mathrm{Rep}_q\mathrm{L}_t \arrow[d, "\mathrm{ev}^{\mathrm{L}}_{m,n}"] \\
\mathrm{Rep}_q\mathrm{GL}_{m+n} \arrow[r, "\mathrm{res}_{m,n}"] & \mathrm{Rep}_q\mathrm{L}
\end{tikzcd}$$

commutes for any $m, n \in \mathbb{N}$.

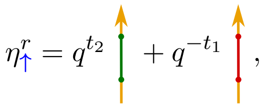
Remark 2.13. The restriction functor res_t is strictly monoidal, but it does not preserve the distinguished dualities. The natural isomorphism from (1.3) is determined by



Moreover, res_t is not pivotal: the categories $\mathrm{Rep}_q\mathrm{GL}_t$ and $\mathrm{Rep}_q\mathrm{L}_t$ are endowed with trivial pivotal structures, but the isomorphism

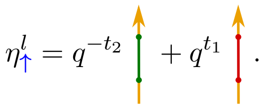
$$\eta^r_{\uparrow} : \uparrow = \mathrm{res}_t(\uparrow)^{**} \xrightarrow{(\varphi^r_{\uparrow})^*} \mathrm{res}_t(\uparrow^*)^* \xrightarrow{(\varphi^r_{\downarrow})^{-1}} \mathrm{res}_t(\uparrow^{**}) \xrightarrow{\mathrm{res}_t(\mathrm{id}_{\downarrow})} \mathrm{res}_t(\uparrow) = \uparrow$$

from (1.4) is represented by



(2.25)

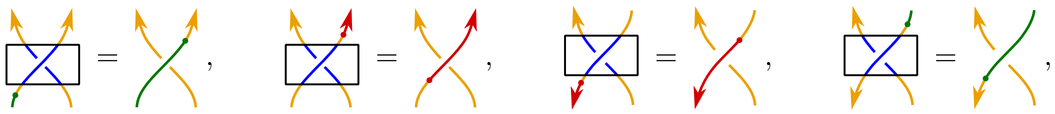
which is not the identity of ↑ in $\mathrm{Rep}_q\mathrm{L}_t$. Similarly, for the left dualities induced by the ribbon structures of $\mathrm{Rep}_q\mathrm{GL}_t$ and $\mathrm{Rep}_q\mathrm{L}_t$, we get an isomorphism



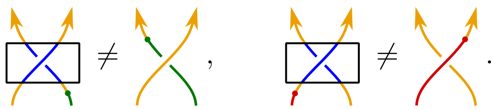
(2.26)

◊

Remark 2.14. Using (2.18) and the definition of the boxes, one easily checks that



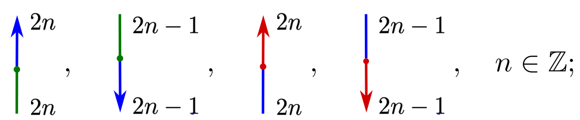
but, for instance,



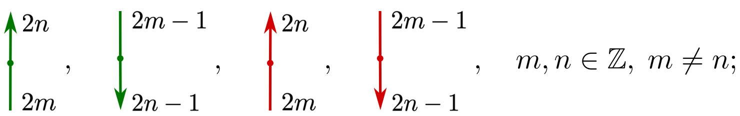
This is a diagrammatic counterpart of the fact that the R-matrix of $U_q(\mathfrak{gl}_{m+n})$ belong to (a completion of) $U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^-)$, where $\mathfrak{b}^\pm$ are the corresponding positive and negative. Consequently, the restriction of the braiding of $\mathrm{Rep}_q\mathrm{GL}_{m+n}$ to $\mathrm{Rep}_q\mathrm{L}$ is natural in the first factor (resp. in the second) for morphisms of $U_q(\mathfrak{l})$-modules commuting with the action of $U_q(\mathfrak{p})$ (resp. $U_q(\mathfrak{p}^-)$). ◊

2.4 The planar category $\mathrm{Rep}_q\mathrm{P}_t$. Finally, we define a category $\mathrm{Rep}_q\mathrm{P}_t$ based on the graphical calculus for rigid categories introduced in Section 1.3. The category $\mathrm{Rep}_q\mathrm{P}$ of locally finite $U_q(\mathfrak{p})$-modules does not admit a pivotal structure compatible with those of $\mathrm{Rep}_q\mathrm{GL}_{m+n}$ and $\mathrm{Rep}_q\mathrm{L}$. This is because the restriction of the pivotal structure of $\mathrm{Rep}_q\mathrm{GL}_{m+n}$ is not natural for morphisms from $\mathrm{Rep}_q\mathrm{L}$. Hence, we define a universal version of $\mathrm{Rep}_q\mathrm{P}$ using the graphical calculus.

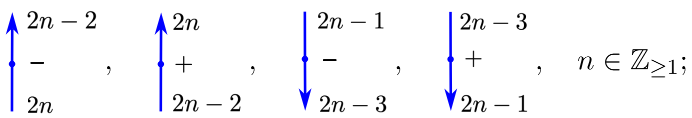
As in Section 1.3, we consider rectilinear isotopy classes of vertical planar graphs embedded in $\mathbb{R} \times [0,1]$. Edges are coloured with either ■, ■ or ■, with endpoints labelled by integers following the same rules defined in that section. Vertices are coupons admitting the following decorations:



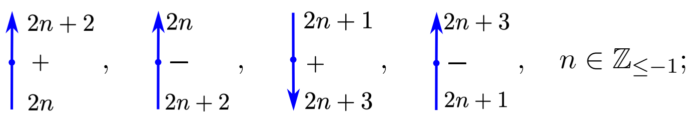
(2.27)



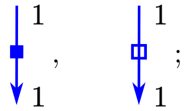
(2.28)



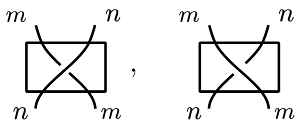
(2.29)



(2.30)

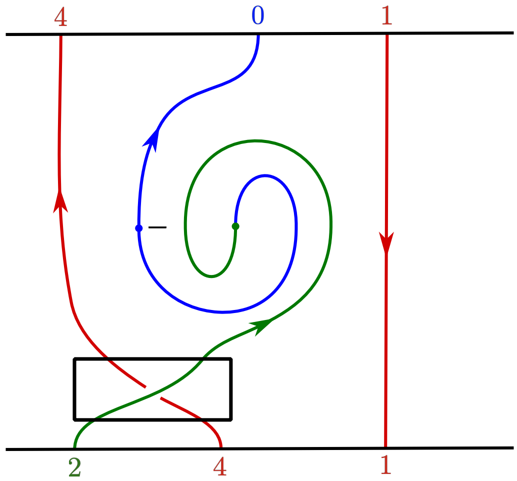


(2.31)



with $m, n \in \mathbb{Z}$ and all possible colourings and orientations. (2.32)

Let Rig(P) be the $(\Bbbk \otimes_{\mathbb{C}(q)} \Bbbk)$-linear category whose objects are sequences of oriented points with integer values (compatible with orientations, cf. section 1.3) and coloured with ■, ■ and ■. The hom-spaces are spanned by rectilinear isotopy classes of planar graphs decorated as above. For instance, the diagram



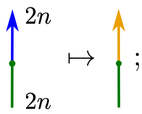
represents a morphism in Rig(P). Note that the decoration of the components of the graph is uniquely determined by the source and the target.

We define a functor

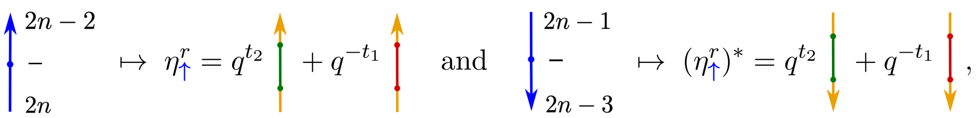
$$\mathrm{ev}^{\mathrm{P}} : \mathrm{Rig}(\mathrm{P}) \to \mathrm{Rep}_q \mathrm{L}_t$$

as follows. On objects, it just forgets the integer labels and switch colours ■ $\rightsquigarrow$ ■. On morphisms, we interpret generators as follows:

- the morphisms in (2.27) represents the inclusion ■ $\equiv V_m \to V \equiv$ ■ and the projection ■ $\equiv V \to V_n \equiv$ ■, so the evaluation functor just forgets the integer labels and switches colours ■ $\rightsquigarrow$ ■; for instance,

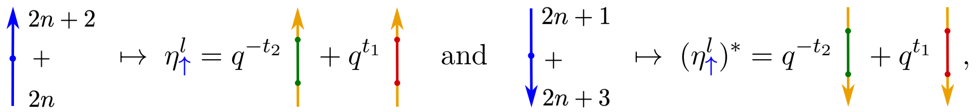


- the morphisms in (2.28) represent the restriction of the pivotal structure of $\mathrm{Rep}_q \mathrm{L}_t$, which is trivial, so they are all sent to the corresponding identity;
- the morphisms in (2.29) represent the image of the pivotal structure of $\mathrm{Rep}_q \mathrm{GL}_t$ by the restriction functor (see (1.4)), hence their images by the evaluation functor are

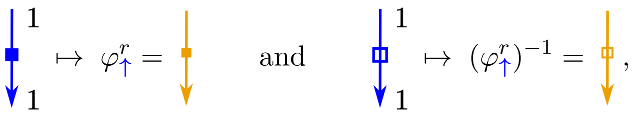


  and the two remaining morphisms are the inverses of these ones;

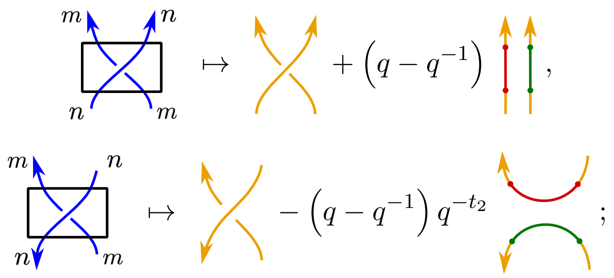
- similarly, for $n \in \mathbb{Z}_{\leq -1}$, we set



  and the two remaining morphisms in (2.30) are the inverses of these ones;
- the morphisms in (2.31) represent the natural isomorphism from (1.3) identifying the duality induced by the restriction functor with the distinguished duality of $\mathrm{Rep}_q \mathrm{L}_t$, so



  where φ^r is the isomorphism in (2.3);
- for crossings whose strands are both blue, we apply the same rules as for computing the restriction functor (2.24), i.e.,



- for the rest of morphisms in (2.32), the evaluation functor just replaces the coupon by the underlying crossing.

Lemma 2.15. *The evaluation functor $\mathrm{ev}^P : Rig(P) \to Rep_q L_t$ is well-defined.*

Proof. This is straightforward, since rectilinear isotopy is a relation holding in $\mathrm{Rep}_q \mathrm{L}_t$. □

Definition 2.16. We define the category $\mathrm{Rep}_q \mathrm{P}_t$ as the quotient of $\mathrm{Rig(P)}$ by the kernel of ev^{P}.

By construction, the evaluation functor induces a faithful functor

$$j_t^* : \mathrm{Rep}_q \mathrm{P}_t \to \mathrm{Rep}_q \mathrm{L}_t \cong \mathrm{Rep}_q \mathrm{GL}_t \boxtimes \mathrm{Rep}_q \mathrm{GL}_t \tag{2.33}$$

analogue the restriction functor $j^* \colon \mathrm{Rep}_q \mathrm{P} \to \mathrm{Rep}_q \mathrm{L}$.

Proposition 2.17. *Let $n, m \in \mathbb{N}$ and $\varphi_{m,n} \colon \Bbbk \otimes_{\mathbb{C}(q)} \Bbbk \to \mathbb{C}(q)$ the evaluation morphism defined in Remark 2.7. Then, there exists a monoidal $\varphi_{m,n}$-linear functor*

$$\mathrm{ev}^P_{m,n} \colon Rep_q P_t \to Rep_q P$$

mapping

$$(\textcolor{blue}{\uparrow}, 2n) \mapsto \iota^*(V_{m+n}), \quad (\textcolor{green}{\uparrow}, 2n) \mapsto \pi^*(V_m), \quad (\textcolor{red}{\uparrow}, 2n) \mapsto \pi^*(V_n),$$

where ι^* and π^* are the restriction functors defined in section 1.5.

Proof. The restriction functor $j^*\colon \mathrm{Rep}_q\mathrm{P} \to \mathrm{Rep}_q\mathrm{L}$ is faithful so we may describe $\mathrm{Rep}_q\mathrm{P}$ as a subcategory of $\mathrm{Rep}_q\mathrm{L}$. By construction, the image of

$$\mathrm{Rep}_q\mathrm{P}_t \xrightarrow{j_t^*} \mathrm{Rep}_q\mathrm{L}_t \xrightarrow{\mathrm{ev}^{\mathrm{L}}_{m,n}} \mathrm{Rep}_q\mathrm{L}$$

lies in this subcategory, so it lifts to a monoidal functor $\mathrm{ev}^{\mathrm{P}}_{m,n}$ as in the statement. □

We get a commutative diagram of functors

$$\begin{tikzcd} \mathrm{Rep}_q\mathrm{P}_t \arrow[r, "j_t^*"] \arrow[d, "\mathrm{ev}^{\mathrm{P}}_{m,n}"'] & \mathrm{Rep}_q\mathrm{L}_t \arrow[d, "\mathrm{ev}^{\mathrm{L}}_{m,n}"] \\ \mathrm{Rep}_q\mathrm{P} \arrow[r, "j^*"] & \mathrm{Rep}_q\mathrm{L}, \end{tikzcd}$$

hence we may interpret again the functor j_t^* as interpolating the functors j^*, for $m, n \in \mathbb{N}$.

3 Universal parabolic restriction

In this section, we complete the picture by defining HOMFLY analogues of all the restriction functors appearing in (1.6). These functors induce a $(\mathrm{Rep}_q\mathrm{GL}_t, \mathrm{Rep}_q\mathrm{GL}_t \boxtimes \mathrm{Rep}_q\mathrm{GL}_t)$-central algebra structure on $\mathrm{Rep}_q\mathrm{P}_t$, interpolating the one described in section 1.5.

3.1 Central algebra structure. We define functors

$$\iota_t^* : \mathrm{Rep}_q\mathrm{GL}_t \to \mathrm{Rep}_q\mathrm{P}_t \qquad \text{and} \qquad \pi_t^* : \mathrm{Rep}_q\mathrm{GL}_t \boxtimes \mathrm{Rep}_q\mathrm{GL}_t \to \mathrm{Rep}_q\mathrm{P}_t$$

analogue to the restriction functors $\mathrm{Rep}_q\mathrm{GL}_{m+n} \xrightarrow{\iota^*} \mathrm{Rep}_q\mathrm{P}$ and $\mathrm{Rep}_q\mathrm{L} \xrightarrow{\pi^*} \mathrm{Rep}_q\mathrm{P}$ from section 1.5. Let us introduce first some terminology, that will be also useful later to extend the graphical calculus for rigid categories to framed surfaces.

Let S be a framed surface, i.e., a surface S endowed with a trivialization $f = (v_p, w_p)_{p\in S}$ of its tangent bundle. Let $\alpha : I \to S$ be an immersed curve such that $\dot\alpha(0)$ and $\dot\alpha(1)$ are in the direction of $w_{\alpha(0)}$ and $w_{\alpha(1)}$, respectively. Consider the map

$$u_\alpha^f :\ I \xrightarrow{\dot\alpha} \mathrm{T}S \setminus \{0\} \xrightarrow{f} S \times (\mathbb{R}^2 \setminus \{0\}) \xrightarrow{\text{proj.}} \mathbb{R}^2 \setminus \{0\} \xrightarrow{\frac{x}{||x||}} \mathbb{S}^1.$$

Hence, $u_\alpha^f(0) = \pm i$ and we can set

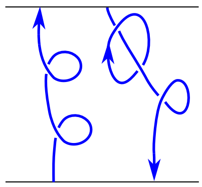
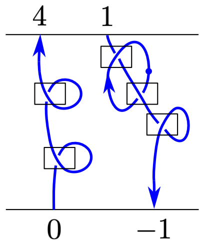
$$u_\alpha^f(t) = \exp\left(-i\pi\left(\theta_\alpha^f(t) \mp \frac{1}{2}\right)\right),$$

with $\theta_\alpha^f(0) = 0$. The conditions imposed to α imply that $\theta_\alpha^f(1) \in \mathbb{Z}$.

Definition 3.1. The *rotation number* of α with respect to f is the integer

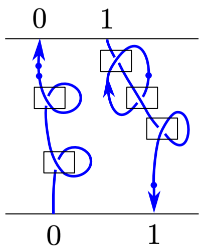
$$\mathrm{rot}^f(\alpha) := \theta_\alpha^f(1) \in \mathbb{Z}.$$

For the rest of this subsection, we fix $S = \mathbb{R} \times [0,1]$ with its canonical framing. From a given ribbon diagram Γ in $\mathbb{R} \times [0,1]$, we construct a decorated planar diagram $\widehat\Gamma$ as follows. Any open strand of Γ defines an immersed curve $\alpha : I \to \mathbb{R} \times [0,1]$. If α is oriented upwards at $\alpha(0)$, we decorate this endpoint with $n_{\alpha(1)} = 0$; otherwise, we set $n_{\alpha(0)} = 1$. A decoration for $\alpha(1)$ is then uniquely determined by setting $n_{\alpha(1)} = n_{\alpha(0)} + \mathrm{rot}^f(\alpha)$. Next, if α is a closed strand (i.e. a link component) with rotation number $r_\alpha = \mathrm{rot}^f(\alpha)$, we add $|\frac{r_\alpha}{2}|$ dots at $\alpha(0)$. Finally, replace every crossing by the corresponding coupon. For example:

$\Gamma =$  $\rightsquigarrow$ 

.

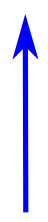
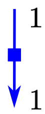
Precomposing and postcomposing again with the morphisms in (2.29)-(2.30), we can modify the source and target of the planar graph obtained so that points oriented upwards (resp. downwards) are all decorated by 0 (resp. -1). Let $\hat{\Gamma}$ be the decorated planar graph thus obtained. For instance, for the ribbon diagram depicted above, we get

$\hat{\Gamma} =$ 

.

Roughly speaking, this procedure comes to forget the braided and the pivotal structure of $\mathrm{Rep}_q\mathrm{GL}_t$, so that the planar graph $\widehat{\Gamma}$ represents the same morphism as Γ but in the rigid category underlying the ribbon category $\mathrm{Rep}_q\mathrm{GL}_t$.

Recall that we have a restriction functor $\iota^*\colon \mathrm{Rep}_q\mathrm{GL}_{m+n} \to \mathrm{Rep}_q\mathrm{P}$. This functor is faithful, but it does not preserve the distinguished dualities. We reflect this fact diagrammatically by modifying the distinguished duality for $\uparrow$ as follows. Set

$\varphi_0 :=$  and $\varphi_1 :=$ 

.

Given a sequence $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$ with $\varepsilon_i \in \{0,1\}$, we define

$$\varphi_\varepsilon := \varphi_{\varepsilon_1} \otimes \cdots \otimes \varphi_{\varepsilon_k}. \tag{3.1}$$

Finally, if $D : \varepsilon_1 \to \varepsilon_2$ is a planar diagram between two such sequences, we define

$$\iota_t^*(D) := \varphi_{\varepsilon_2} \circ D \circ \varphi_{\varepsilon_1}^{-1}.$$

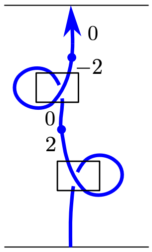
This planar graph

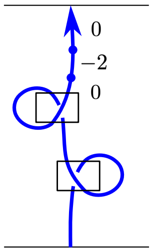
Proposition 3.2. *The assignment* $\uparrow \mapsto 0$, $\downarrow \mapsto 1$, $\Gamma \mapsto \iota_t^*(\widehat{\Gamma})$ *yields a well-defined strict monoidal functor*

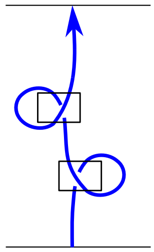
$$\iota_t^* : \mathit{Rep}_q GL_t \to \mathit{Rep}_q P_t. \tag{3.2}$$

Moreover, $\mathit{res}_t = j_t^* \circ \iota_t^*$, *where* j_t^* *is the functor in* 2.33.

Proof. First note that the isomorphisms $\eta_\uparrow^r$, $\eta_\uparrow^l$ and $\varphi_\uparrow^r$ commute with every endomorphism of $\uparrow$ in $\mathrm{Rep}_q\mathrm{L}_t$. This implies that $\iota_t^*(\widehat{\Gamma_1 \circ \Gamma_2}) = \iota_t^*(\widehat{\Gamma}_1) \circ \iota_t^*(\widehat{\Gamma}_2)$ for every pair of composable diagrams Γ_1, Γ_2. Indeed, we can slide the dots appearing between Γ_1 and Γ_2 in $\iota_t^*(\widehat{\Gamma}_1) \circ \iota_t^*(\widehat{\Gamma}_2)$ so that they lie all at the end of the corresponding strand. By replacing each pair dot - inverse dot appearing this way by the identity, we get exactly $\iota_t^*(\widehat{\Gamma_1 \circ \Gamma_2})$. For instance,

$\iota_t^*(\widehat{\Gamma}_1) \circ \iota_t^*(\widehat{\Gamma}_2) =$ 

$=$ 

$=$  $= \iota_t^*(\widehat{\Gamma_1 \circ \Gamma_2})$.

Moreover, we trivially have that $\iota_t^*(\widehat{\Gamma_1 \otimes \Gamma_2}) = \iota_t^*(\widehat{\Gamma}_1) \otimes \iota_t^*(\widehat{\Gamma}_2)$, hence the functor is strictly monoidal.

On the other hand, it is straightforward to check that, if Γ is one of the generators of $\mathrm{Rep}_q\mathrm{GL}_t$ (cf. Proposition 2.2), then $j_t^*(\Gamma) \circ \iota_t^*(\widehat{\Gamma}) = \mathrm{res}_t(\Gamma)$. The compatibility with the composition and the tensor product implies then that the same is true for any diagram Γ. In particular, if Γ_1 and Γ_2 are two diagrams representing the same morphism in $\mathrm{Rep}_q\mathrm{GL}_t$, then

$$j_t^* \circ \iota_t^*(\widehat{\Gamma}_1) = \mathrm{res}_t(\Gamma_1) = \mathrm{res}_t(\Gamma_2) = j_t^* \circ \iota_t^*(\widehat{\Gamma}_2).$$

Since j_t^* is faithful, we get that $\iota_t^*(\widehat{\Gamma}_1) = \iota_t^*(\widehat{\Gamma}_2)$, which proves that the functor is well-defined. $\square$

Proposition 3.3. *The assignment*

$$(\uparrow, \varnothing) \mapsto 0, \quad (\downarrow, \varnothing) \mapsto 1, \quad \Gamma \mapsto \widehat{\Gamma},$$

$$(\varnothing, \uparrow) \mapsto 0, \quad (\varnothing, \downarrow) \mapsto 1, \quad \Gamma \mapsto \widehat{\Gamma},$$

yields a well defined strict monoidal functor

$$\pi_t^* : \; Rep_q GL_t \boxtimes Rep_q GL_t \;\to\; Rep_q P_t. \tag{3.3}$$

Proof. The assignment is functorial and strictly monoidal by the same arguments as in the previous proof. The well-definiteness follows again from the fact that j_t^* is faithful. Indeed, identifying $\mathrm{Rep}_q\mathrm{L}_t$ and $\mathrm{Rep}_q\mathrm{GL}_t \boxtimes \mathrm{Rep}_q\mathrm{GL}_t$ via the equivalence from Proposition 2.8, we have that $j_t^* \circ \pi_t^* = \mathrm{id}_{\mathrm{Rep}_q\mathrm{GL}_t \boxtimes \mathrm{Rep}_q\mathrm{GL}_t}$. Therefore, if Γ_1 represent the same morphism Γ_2 in $\mathrm{Rep}_q\mathrm{GL}_t \boxtimes \mathrm{Rep}_q\mathrm{GL}_t$, then the equality

$$j_t^* \circ \pi_t^*(\Gamma_1) = \Gamma_1 = \Gamma_2 = j_t^* \circ \pi_t^*(\Gamma_2)$$

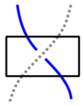
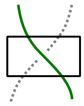
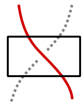
implies that $\pi_t^*(\Gamma_1) = \pi_t^*(\Gamma_2)$. $\square$

To sum up, we have four functors

$$\begin{tikzcd}
 & \mathrm{Rep}_q\mathrm{P}_t \arrow[dr, "j_t^*"] & \\
\mathrm{Rep}_q\mathrm{GL}_t \arrow[ur, "\iota_t^*"] \arrow[rr, "\mathrm{res}_t"'] & & \mathrm{Rep}_q\mathrm{GL}_t \boxtimes \mathrm{Rep}_q\mathrm{GL}_t \arrow[ul, bend right, "\pi_t^*"']
\end{tikzcd} , \tag{3.4}$$

such that the inner triangle is commutative, but not the outer one. This is a diagrammatic version of the situation described in subsection 1.5. We will see now that the functors ι_t^* and π_t^*

induce a $(\mathrm{Rep}_q\mathrm{GL}_t, \mathrm{Rep}_q\mathrm{GL}_t \boxtimes \mathrm{Rep}_q\mathrm{GL}_t)$-central algebra structure on $\mathrm{Rep}_q\mathrm{P}_t$. By remark 2.14 and the relations defining $\mathrm{Rep}_q\mathrm{L}_t$, the families of morphisms given by

 ,  and  ,

where the dotted strands can be replaced by any colour, are natural isomorphisms in $\mathrm{Rep}_q\mathrm{P}_t$. Hence, they define a canonical monoidal functor

$$Z: \ \mathrm{Rep}_q\mathrm{P}_t \ \to \ Z(\mathrm{Rep}_q\mathrm{P}_t).$$

Let

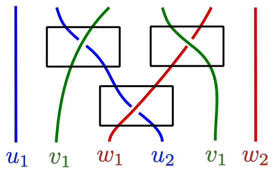
$$\begin{array}{rlcl} F \coloneqq \iota_t^* \boxtimes \pi_t^* : & \mathrm{Rep}_q\mathrm{GL}_t \boxtimes (\mathrm{Rep}_q\mathrm{GL}_t \boxtimes \mathrm{Rep}_q\mathrm{GL}_t) & \to & \mathrm{Rep}_q\mathrm{P}_t, \\ & (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) & \mapsto & \boldsymbol{uvw} \\ & (\Gamma_1, \Gamma_2, \Gamma_3) & \mapsto & \iota_t^*(\widehat{\Gamma}_1)\widehat{\Gamma}_2\widehat{\Gamma}_3. \end{array} \tag{3.5}$$

Theorem 3.4. *F lifts to a braided monoidal functor*

$$(Z \circ F, J): \ \mathrm{Rep}_q GL_t \boxtimes (\overline{\mathrm{Rep}_q GL_t \boxtimes \mathrm{Rep}_q GL_t}) \ \to \ Z(\mathrm{Rep}_q P_t),$$

where $\overline{(-)}$ stands for the opposite braided category. In particular, $\mathrm{Rep}_q P_t$ is a $(\mathrm{Rep}_q GL_t, \mathrm{Rep}_q L_t)$-central algebra.

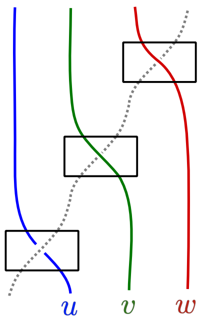
Proof. The family of isomorphisms



$$: \ F(\boldsymbol{u_1}, \boldsymbol{v_1}, \boldsymbol{w_1}) \otimes F(\boldsymbol{u_2}, \boldsymbol{v_2}, \boldsymbol{w_2}) \ \to \ F(\boldsymbol{u_1 u_2}, \boldsymbol{v_1 v_2}, \boldsymbol{w_1 w_2})$$

defines a monoidal structure on $Z \circ F$. Checking that $(Z \circ F, J)$ is braided is an easy computation. □

Note that, for an object $\boldsymbol{uvw}$, the half braiding is given by



$$: \ - \otimes \boldsymbol{uvw} \ \to \ \boldsymbol{uvw} \otimes -.$$

3.2 The centred bimodule $\mathrm{Rep}_q\mathrm{L}_t$. We will now briefly describe additional structure induced by the functors constructed in the previous section, which will serve as motivation for the topological construction we will give later in this paper. In addition to the $(\mathrm{Rep}_q\mathrm{GL}_t, \mathrm{Rep}_q\mathrm{L}_t)$-central structure on $\mathrm{Rep}_q\mathrm{P}_t$ defined in the previous section, we will also consider the following three central algebras:

- $\mathrm{Rep}_q\mathrm{GL}_t$ is a $(\mathrm{V{\small ECT}}_\Bbbk, \mathrm{Rep}_q\mathrm{GL}_t)$-central algebra via the central functor

$$\mathrm{V{\small ECT}}_\Bbbk \boxtimes \mathrm{Rep}_q\mathrm{GL}_t \simeq \mathrm{Rep}_q\mathrm{GL}_t \xrightarrow{\mathrm{id}} \mathrm{Rep}_q\mathrm{GL}_t,$$

- $\mathrm{Rep}_q\mathrm{L}_t$ is a $(\mathrm{V\small{ECT}}_{\Bbbk}, \mathrm{Rep}_q\mathrm{L}_t)$-central algebra with central functor

$$\mathrm{V\small{ECT}}_{\Bbbk} \boxtimes \mathrm{Rep}_q\mathrm{L}_t \simeq \mathrm{Rep}_q\mathrm{L}_t \xrightarrow{\mathrm{id}} \mathrm{Rep}_q\mathrm{L}_t,$$

- and finally $\mathrm{Rep}_q\mathrm{L}_t$ is also a $(\mathrm{Rep}_q\mathrm{L}_t, \mathrm{Rep}_q\mathrm{L}_t)$-central algebra with central structure

$$\mathrm{Rep}_q\mathrm{L}_t \boxtimes \mathrm{Rep}_q\mathrm{L}_t \xrightarrow{\mathrm{id}\otimes\mathrm{id}} \mathrm{Rep}_q\mathrm{L}_t.$$

These four algebras define 1-morphisms in the Morita 4-category $\mathrm{B\small{R}T\small{ENS}}$ studied in [BJS21]. Composing $\mathrm{Rep}_q\mathrm{P}_t$ with $\mathrm{Rep}_q\mathrm{GL}_t$, we get a central functor

$$\mathrm{V\small{ECT}}_{\Bbbk} \boxtimes \mathrm{Rep}_q\mathrm{L}_t \to \mathrm{Rep}_q\mathrm{GL}_t \underset{\mathrm{Rep}_q\mathrm{GL}_t}{\boxtimes} \mathrm{Rep}_q\mathrm{P}_t \simeq \mathrm{Rep}_q\mathrm{P}_t,$$

where the action of $\mathrm{Rep}_q\mathrm{P}_t$ is the one induced by the functor j_t^*. Similarly, composing the two central structures on $\mathrm{Rep}_q\mathrm{L}_t$, we get another central functor

$$\mathrm{V\small{ECT}}_{\Bbbk} \boxtimes \mathrm{Rep}_q\mathrm{L}_t \to \mathrm{Rep}_q\mathrm{L}_t \underset{\mathrm{Rep}_q\mathrm{L}_t}{\boxtimes} \mathrm{Rep}_q\mathrm{L}_t \simeq \mathrm{Rep}_q\mathrm{L}_t,$$

where $\mathrm{Rep}_q\mathrm{L}_t$ acts on itself via the identity functor. On the other hand, note that $\mathrm{Rep}_q\mathrm{L}_t$ is a $(\mathrm{Rep}_q\mathrm{P}_t, \mathrm{Rep}_q\mathrm{L}_t)$-bimodule via the functor

$$\mathrm{Rep}_q\mathrm{P}_t \boxtimes \mathrm{Rep}_q\mathrm{L}_t \xrightarrow{j^*\otimes\mathrm{id}} \mathrm{Rep}_q\mathrm{L}_t$$

and it is straightforward to check that the braiding induces a $(\mathrm{Rep}_q\mathrm{P}_t, \mathrm{Rep}_q\mathrm{L}_t)$-centered structure (see [BJS21, section 3] for the definitions). Centred structures are the 2-morphisms in $\mathrm{B\small{R}T\small{ENS}}$, so we have the following diagram in $\mathrm{B\small{R}T\small{ENS}}$:

$$\begin{tikzcd}
\mathrm{V\small{ECT}}_{\Bbbk} \arrow[r, "\mathrm{Rep}_q\mathrm{GL}_t"] \arrow[d, "\mathrm{Rep}_q\mathrm{L}_t"'] & \mathrm{Rep}_q\mathrm{GL}_t \arrow[d, "\mathrm{Rep}_q\mathrm{P}_t"] \arrow[dl, Rightarrow, "\mathrm{Rep}_q\mathrm{L}_t"'] \\
\mathrm{Rep}_q\mathrm{L}_t \arrow[r, "\mathrm{Rep}_q\mathrm{L}_t"] & \mathrm{Rep}_q\mathrm{L}_t.
\end{tikzcd}$$

In the language of factorization algebras (see [BJS21, section 3] and the figures therein), this structures are governed by embedding disks in the following stratified space:

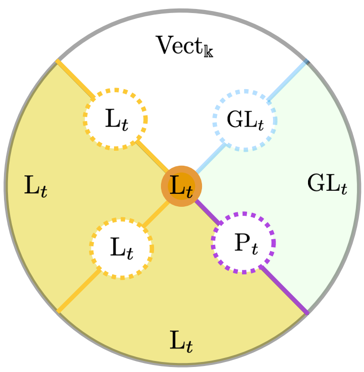


FIGURE 3: Centered $(\mathrm{Rep}_q\mathrm{P}_t, \mathrm{Rep}_q\mathrm{L}_t)$-bimodule structure of $\mathrm{Rep}_q\mathrm{L}_t$ as a factorization algebra.

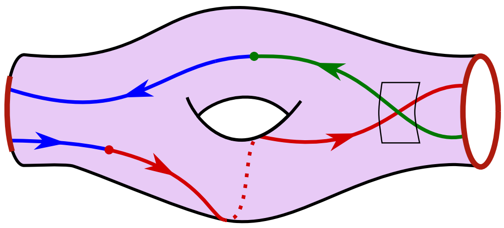


FIGURE 4: Decorated planar graph on a framed surface (we omit the integer labels for simplicity).

4 DEFECT PLANAR THEORIES

The graphical calculus for rigid categories naturally generalizes to surfaces with a chosen framing, producing a 2-dimensional framed TFT constructed from a rigid monoidal category's algebraic data. The $(\mathrm{Rep}_q\mathrm{L}_t, \mathrm{Rep}_q\mathrm{P}_t)$-module structure on $\mathrm{Rep}_q\mathrm{L}_t$ produces a morphism between the theories associated with $\mathrm{Rep}_q\mathrm{L}_t$ and $\mathrm{Rep}_q\mathrm{P}_t$, that we will describe through line defects on surfaces.

4.1 One-coloured theories. We describe here the planar theory induced by $\mathrm{Rep}_q\mathrm{P}_t$. A similar description hold for the rigid category underlying $\mathrm{Rep}_q\mathrm{L}_t$.

Definition 4.1. Let M be a d-dimensional oriented manifold, and let $n \geq d$. An *n-dimensional framing* of M is defined as a homotopy class of orientation-preserving isomorphisms $f : TM \oplus \mathbb{R}^{n-d} \to M \times \mathbb{R}^n$ between vector bundles over M. When $n = d$, we refer to this as a *framing*.

Note that if $M = I$ or $M = \mathbb{S}^1$, each choice of orientation admits exactly one 2-framing. We write I, $\mathbb{S}^1$ for the standard positive orientation and $\overline{I}$, $\overline{\mathbb{S}^1}$ for the negative one.

Definition 4.2. A *marking* on an oriented surface S is a (possibly empty) collection $\mathcal{B}$ of orientation preserving embeddings $C \hookrightarrow \partial S$, with C a 1-dimensional connected oriented manifold. Every marking has a splitting $\mathcal{B} = \mathcal{B}_+ \sqcup \mathcal{B}_-$ into positively and negatively oriented submanifolds.

Definition 4.3. A *rectilinear embedding* is an embedding $M \hookrightarrow N$ between framed manifolds which preserves the framing up to rescaling in each direction.

Let $(S, f, \mathcal{B})$ be a marked framed surface. We consider the following class of planar graphs Γ embedded in S (see Figure 4):

- edges are coloured with ■, ■ or ■;
- the endpoints $\alpha(0)$ and $\alpha(1)$ of every edge α are attached to either a coupon or one of the marked boundary components. Moreover, they are labelled with integers $n_{\alpha(0)}$ and $n_{\alpha(1)}$ such that $n_{\alpha(1)} = n_{\alpha(0)} + \mathrm{rot}^f(\alpha)$;
- coupons are rectilinearly embedded in the surface and decorated with morphisms of $\mathrm{Rep}_q\mathrm{P}_t$.

Set $\mathcal{B}_+ = \{C_i\}_{i=1,\dots,m}$ and $\mathcal{B}_- = \{C'_j\}_{j=1,\dots,n}$. The intersection of Γ with the marking determines a family of configurations of points $(a_{C_1}, \dots, a_{C_m}, b_{C'_1}, \dots, b_{C'_n})$ that we refer as *boundary conditions*. We denote by $\mathcal{Z}^{\mathrm{P}_t}(S, f, \mathcal{B})(a_{C_1}, \dots, a_{C_m}, b_{C'_1}, \dots, b_{C'_n})$ the $(\Bbbk \otimes_{\mathbb{C}(q)} \Bbbk)$-linear space generated by planar graphs with given boundary conditions, modulo the following relations:

- <u>rectilinear isotopies</u>: these are isotopies of the surface S fixing ∂S such that coupons remain parallel to the framing throughout the isotopy;

- framed skein relations: $\sum_i \lambda_i(\varphi, \Gamma_i) \sim 0$ if there exists a rectilinear embedding $\iota : [0,1]^2 \hookrightarrow S$ such that $\sum_i \lambda_i \iota^{-1}(\Gamma \cap \iota([0,1])) = 0$ in $\mathrm{Rep}_q \mathrm{P}_t$.

As a particular case, we have:

Definition 4.4. Let $(\mathbb{A}, f_{\mathrm{rad}})$ be the annulus endowed with its radial framing. The *cylinder category* $\mathrm{Cyl}(\mathrm{Rep}_q \mathrm{P}_t)$ is the category with:

- objects: finite configurations of coloured directed points on the circle $\mathbb{S}^1$ decorated by integers compatibly with the orientation;
- morphisms: $(\Bbbk \otimes_{\mathbb{C}(q)} \Bbbk)$-linear combinations of planar diagrams on $(\mathbb{A}, f_{\mathrm{rad}})$, modulo rectilinear isotopy and framed skein relations.

Composition is given by inserting one copy of $\mathbb{A}$ into another one.

We set

$$\mathcal{Z}^{\mathrm{P}_t}(I) = \mathrm{Rep}_q \mathrm{P}_t \quad \text{and} \quad \mathcal{Z}^{\mathrm{P}_t}(\mathbb{S}^1) = \mathrm{Cyl}(\mathrm{Rep}_q \mathrm{P}_t),$$

and negatively oriented 1-manifolds are assigned the opposite categories. If $C \in \mathcal{B}_+$, then $\mathcal{Z}^{\mathrm{P}_t}(C)$ acts on $\mathcal{Z}^{\mathrm{P}_t}(S, f, \mathcal{B})$ by gluing rectangles/cylinders. Namely, every configuration of points on C determines an object b of $\mathcal{Z}^{\mathrm{P}_t}(C)$. If $\Gamma\colon a \to b$ is a morphism in $\mathcal{Z}^{\mathrm{P}_t}(C)$, then gluing $C \times I$ along the marking induces a linear map

$$\mathcal{Z}^{\mathrm{P}_t}(S, f, \mathcal{B})(\Gamma)\colon \mathcal{Z}^{\mathrm{P}_t}(S, f, \mathcal{B})(-, b, -) \to \mathcal{Z}^{\mathrm{P}_t}(S, f, \mathcal{B})(-, a, -).$$

The analogue holds for negative orientations, so we get a functor

$$\mathcal{Z}^{\mathrm{P}_t}(S, f, \mathcal{B}) : \left(\boxtimes_{C \in \mathcal{B}_+} \mathcal{Z}^{\mathrm{P}_t}(C) \right) \boxtimes \left(\boxtimes_{\overline{C} \in \mathcal{B}_-} \mathcal{Z}^{\mathrm{P}_t}(C) \right)^{\mathrm{op}} \to \textsc{Vect}.$$

Lemma 4.5. *Let $(S, f, \mathcal{B})$ be a marked framed surface surface with $C, \overline{C} \in \mathcal{B}$ for some 1-dimensional manifold C. Let $(gl_C(S), gl_C(f), gl_C(\mathcal{B}))$ be the marked framed surface obtained by gluing S along C. Then,*

$$\mathcal{Z}^{P_t}\left(gl_C(S), gl_C(f), gl_C(\mathcal{B})\right)(-,-) = \int^{c \in \mathcal{Z}^{P_t}(C)} \mathcal{Z}^{P_t}(S, f, \mathcal{B})(-, c, c, -).$$

Proof. The proof is the same as in [Wal06, Theorem 4.4.2]. □

We have now all the ingredients to define a framed TFT with target category $\textsc{Bimod}$. Let $\mathrm{Bord}_1^{\mathrm{fr}}(2)$ the category of two-dimensional framed cobordisms, whose objects are 1-dimensional 2-framed manifolds and whose morphisms are marked framed cobordisms. It follows straightforward that:

Theorem 4.6. *The construction above defines a symmetric monoidal functor*

$$\mathcal{Z}^{P_t}\colon \mathit{Bord}_1^{\mathrm{fr}}(2) \to \textsc{Bimod},$$

hence a 2-dimensional framed TFT. ■

Considering the category $\mathrm{Rep}_q \mathrm{L}_t$ just as a rigid monoidal category, we can define an associated 2-dimensional framed TFT that can be described in the same lines as $\mathcal{Z}^{\mathrm{P}_t}$. We denote it by $\mathcal{Z}^{\mathrm{L}_t}$.

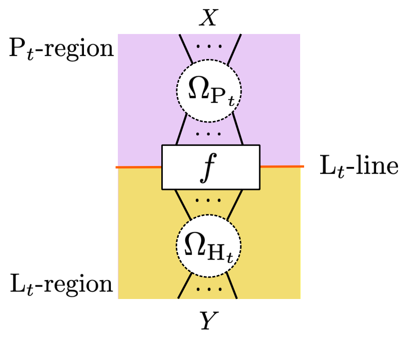


FIGURE 5: Planar graph on a stratified square.

4.2 Planar model around the defect line. The $(\mathrm{Rep}_q\mathrm{L}_t, \mathrm{Rep}_q\mathrm{P}_t)$-bimodule structure of $\mathrm{Rep}_q\mathrm{L}_t$ has a diagrammatic interpretation in terms of planar diagrams on a stratified square. We ignore for now its $(\mathrm{Rep}_q\mathrm{L}_t, \mathrm{Rep}_q\mathrm{P}_t)$-centred structure, that will be described later in section 5.2 when we address three-dimensional theories on surfaces. Recall the functor $j^*\colon \mathrm{Rep}_q\mathrm{P}_t \to \mathrm{Rep}_q\mathrm{L}_t$ defined in (2.33). It induces two morphisms in BIMOD, namely the bimodules

$$F_1 \coloneqq \mathrm{Hom}_{\mathrm{Rep}_q\mathrm{L}_t}(-, j^*(-))\colon \mathrm{Rep}_q\mathrm{P}_t \boxtimes \mathrm{Rep}_q\mathrm{L}_t^{\mathrm{op}} \to \mathrm{VECT}$$

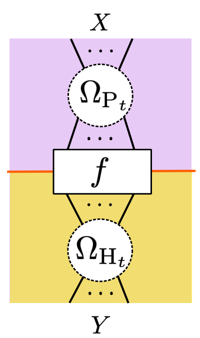
and

$$F_2 \coloneqq \mathrm{Hom}_{\mathrm{Rep}_q\mathrm{L}_t}(j^*(-), -)\colon \mathrm{Rep}_q\mathrm{L}_t \boxtimes \mathrm{Rep}_q\mathrm{P}_t^{\mathrm{op}} \to \mathrm{VECT}.$$

Let I^* be the unit interval with a point marked in the middle and $E = I \times I^*$ a square with a horizontal line defect $I \times \left\{\frac{1}{2}\right\}$. We will call *$\mathrm{L}_t$-region* (resp. *$\mathrm{P}_t$-region*) the half-square under (resp. above) the defect. The defect itself will be decorated by L_t as well and we will call it *L_t-line*. We consider planar graphs Ω in E of the following form (see Figure 5):

- endpoints are attached to the top and bottom bases of the square;
- Ω is coloured by $\mathrm{Rep}_q\mathrm{P}_t$ (resp. by $\mathrm{Rep}_q\mathrm{L}_t$) on the P_t-region (resp. on the L_t-region);
- edges can meet transversally the line defect at a coupon decorated with a morphism of $\mathrm{Rep}_q\mathrm{L}_t$ in a compatible way: if the edges coming from the L_t-region (resp. the P_t-region) are coloured with $X_1, \ldots, X_m$ (resp. $Y_1, \ldots, Y_k$), then the coupon is decorated with a morphism $f\colon X_1 \otimes \cdots \otimes X_m \to j^*(Y_1) \otimes \cdots \otimes j^*(Y_k)$.

The assignment

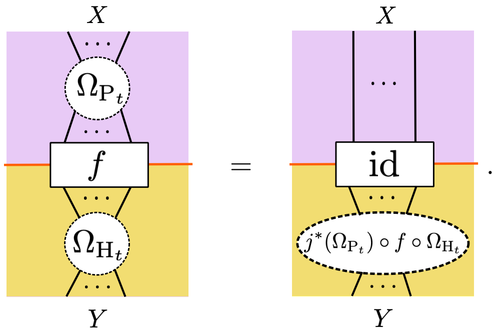


$$\mapsto \quad j^*(\Omega_{\mathrm{P}_t}) \circ f \circ \Omega_{\mathrm{L}_t} \in \mathrm{Hom}_{\mathrm{Rep}_q\mathrm{L}_t}(Y, j^*(X)),$$

extends to a linear map defined on the $(\Bbbk \otimes_{\mathbb{C}(q)} \Bbbk)$-linear space spanned by diagrams with Y and X fixed. Its kernel is given by generated by:

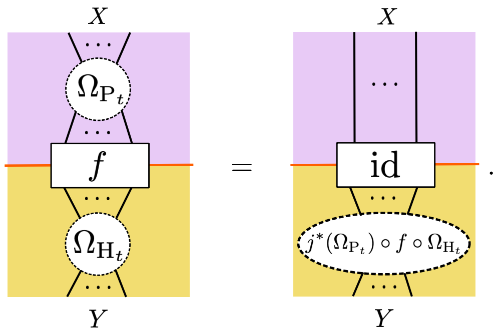
- planar skein relations induced by $\mathrm{Rep}_q\mathrm{L}_t$ and $\mathrm{Rep}_q\mathrm{P}_t$ on each side of the defect line;

- relations of the form



(4.1)

Moding out by all of them, we get a diagrammatic description of F_1, where the actions of $\mathrm{Rep}_q\mathrm{P}_t$ and $\mathrm{Rep}_q\mathrm{L}_t$ are just given by stacking morphisms above and below. Note that, since every stratified diagram Ω represents a morphism in $\mathrm{Rep}_q\mathrm{L}_t$, (4.2) implies that it is equivalent to another diagram $\widetilde{\Omega}$ where $\widetilde{\Omega}_{\mathrm{P}_t}$ and f are just identities, that is:



We will not represent the identity coupon in the pictures, so whenever a bunch of strands crosses the L_t-line (changing their colour), they must be thought as attached to an identity coupon. The bimodule F_2 can be described equally by just exchanging the P_t and the L_t region in the diagrams. Similarly, we have functors

$$G_1 \colon \mathrm{Cyl}(\mathrm{Rep}_q\mathrm{P}_t) \boxtimes \mathrm{Cyl}(\mathrm{Rep}_q\mathrm{L}_t)^{\mathrm{op}} \to \mathrm{V\small{ECT}},$$

$$G_2 \colon \mathrm{Cyl}(\mathrm{Rep}_q\mathrm{L}_t) \boxtimes \mathrm{Cyl}(\mathrm{Rep}_q\mathrm{P}_t)^{\mathrm{op}} \to \mathrm{V\small{ECT}},$$

mapping an object $X \boxtimes Y$ to the linear space spanned by planar diagrams with boundary conditions (X, Y) on a stratified cylinder $\mathbb{S}^1 \times I^*$, modulo skein relations on each side of the defect and the local relations depicted in (4.2) near the defect.

Lemma 4.7. *In* B*IMOD, we have*

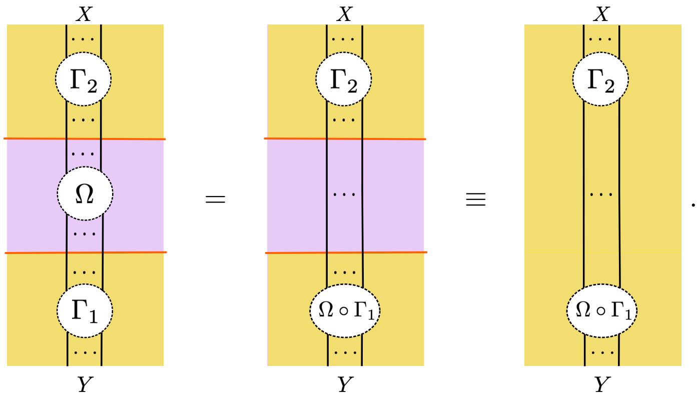
$$F_2 \circ F_1 = \mathrm{id}_{\mathrm{Rep}_q\mathrm{L}_t} \quad \text{and} \quad G_2 \circ G_1 = \mathrm{id}_{\mathrm{Cyl}(\mathrm{Rep}_q\mathrm{L}_t)}.$$

Proof. By definition,

$$(F_2 \circ F_1)(X \boxtimes Y) = \int^{Z \in \mathrm{Rep}_q\mathrm{P}_t} \mathrm{Hom}_{\mathrm{Rep}_q\mathrm{L}_t}(X, j^*(Z)) \otimes \mathrm{Hom}_{\mathrm{Rep}_q\mathrm{L}_t}(j^*(Z), Y),$$

and the canonical map $f \otimes g \to g \circ f$ yields an isomorphism between the coend on the right hand side and $\mathrm{Hom}_{\mathrm{Rep}_q\mathrm{L}_t}(X, Y)$. The same holds for $G_2 \circ G_1$. □

Remark 4.8. We can interpret the proof pictorially. The coend can be represented via diagrams in a squared with two horizontal L_t-defects. The region between them is decorated by P_t, while the regions above and below are decorated by L_t. Applying relations (4.2), we have



This suggests that it may be possible to modify stratifications making appear/disappear regions decorated by P_t in the planar theory associated with L_t. As we will see, the GL_t-skein algebra acts on the planar theory associated with P_t, we may use this to induce an action of the GL_t-skein algebra on the L_t-theory, by making appear P_t-regions and acting on them. These ideas will be made precise later on by describing topologically the central algebras and the centred bimodule from sections 3.1 and 3.2. ◊

4.3 Planar defect theory. The planar theories $\mathcal{Z}^{\mathrm{P}_t}$ and $\mathcal{Z}^{\mathrm{L}_t}$ induced by the rigid categories $\mathrm{Rep}_q\mathrm{P}_t$ and $\mathrm{Rep}_q\mathrm{L}_t$, respectively, glue together via the bimodule structure, yielding a two-dimensional theory on stratified surfaces. Recall that a bipartite surface is a surface S with a stratification

$$\phi\colon S \to (A \geq B \leq C)\,.$$

We label $\phi^{-1}(A)$ and $\phi^{-1}(C)$ with L_t and P_t, respectively, and we call them the *L_t-region* and the *P_t-region*. The 1-dimensional stratum $\phi^{-1}(B)$ is decorated with L_t and we refer to it as the *L_t-line*.

Definition 4.9. A *compatible marking* of (S, f, ϕ) is a marking $\mathcal{B}$ such that each of its components is contained either in the L_t-region or the P_t-region. Therefore, we have a splitting $\mathcal{B} = \mathcal{B}_{\mathrm{L}_t} \sqcup \mathcal{B}_{\mathrm{P}_t}$.

Let $(S, f, \phi, \mathcal{B})$ be a bipartite framed surface with compatible marking. For each $C \in \mathcal{B}$, we set

$$\mathcal{Z}(C) := \begin{cases} \mathcal{Z}^{\mathrm{L}_t}(C), & \text{if } C \in \mathcal{B}_{\mathrm{L}_t}, \\ \mathcal{Z}^{\mathrm{P}_t}(C), & \text{if } C \in \mathcal{B}_{\mathrm{P}_t}. \end{cases}$$

Take $U_C \in \mathcal{Z}(C)$ for each $C \in \mathcal{B}$ and consider the $(\Bbbk \otimes_{\mathbb{C}(q)} \Bbbk)$-linear space spanned by planar graphs as in section 4.1, with given boundary conditions, coloured with $\mathrm{Rep}_q\mathrm{L}_t$ (resp. $\mathrm{Rep}_q\mathrm{P}_t$) in the L_t-region (resp. the P_t-region) and meeting transversally the defect at coupons decorated with morphisms from $\mathrm{Rep}_q\mathrm{L}_t$ as explained in section 4.2. We set $\mathcal{Z}(S, f, \phi, \mathcal{B})\,((U_C)_{C\in\mathcal{B}})$ for its quotient by

- skein relations induced by $\mathrm{Rep}_q\mathrm{L}_t$ and $\mathrm{Rep}_q\mathrm{P}_t$ at the interior of the 2-dimensional regions;
- stratified skein relations around the L_t-line: these are the relations induced by the local model described in the previous section.

The assignment

$$\begin{aligned} \mathcal{Z}(S, f, \phi, \mathcal{B})\colon \quad \Big(\boxtimes_{C\in\mathcal{B}^+} \mathcal{Z}(C)\Big) \boxtimes \Big(\boxtimes_{\bar{C}\in\mathcal{B}_-} \mathcal{Z}(C)\Big)^{\mathrm{op}} &\to \qquad \textsc{Vect}, \\ (U_C)_{C\in\mathcal{B}} &\mapsto \mathcal{Z}(S, f, \phi, \mathcal{B})\,((U_C)_{C\in\mathcal{B}}) \end{aligned}$$

is well-defined and functorial, since all the relations from $\mathrm{Rep}_q\mathrm{L}_t$ and $\mathrm{Rep}_q\mathrm{P}_t$ hold. We shall abbreviate by $\mathcal{Z}(S)$ whenever there is no risk of confusion.

Lemma 4.10 *([Wal06, Theorem 4.4.2]).* *Let $(S, f, \phi, \mathcal{B})$ be a bipartite framed surface with compatible marking. Let $\mathrm{gl}_C(S)$ the surface obtained by gluing S along a boundary component C, with $C, \overline{C} \in \mathcal{B}_{\mathrm{P}_t}$ (alternatively, in $\mathcal{B}_{\mathrm{L}_t}$). Then,*

$$\mathcal{Z}(\mathrm{gl}_C(S))(-) = \int^{U \in \mathcal{Z}(C)} \mathcal{Z}(S)(-, U, U),$$

where the components decorated by U on the RHS are C and $\overline{C}$. ∎

Theorem 4.11. *The assignment*

$$\mathcal{Z} \colon \mathit{Bord}_1^{bip,fr}(2) \to \textsc{Bimod}$$

defines a symmetric monoidal functor from the category of 2-dimensional bipartite framed cobordisms, hence a 2-dimensional framed TFT.

Proof. By the previous lemma, the construction is compatible with gluing surfaces along boundary components of a single color. For the general case, since we are working with compatible markings, it suffices to observe that we may first glue along the subregion coloured by P_t, and then along that coloured by L_t. □

5 Defect three-dimensional theories

In this section, we extend the theories introduced in the above to three-dimensional theories. The local models around the surface and defect lines are given by the algebraic structures described in section 1.5.

5.1 Parabolic surface defects. Following [BJ25], we construct a local model for surface defects using the central algebra structure on $\mathrm{Rep}_q\mathrm{P}_t$ constructed in section 3.1. Let $S = [0, 1]^2$ endowed with its canonical framing and let I^* be the unit interval marked at $1/2$, so that $E^* := [0, 1]^2 \times I^*$ is a stratified cube with a *framed defect wall* $[0, 1]^2 \times \{1/2\}$. We consider the following class of stratified ribbon graphs in E^* (see Figure 6):

- endpoints are attached to one of the following regions:

$$E_0 := [0, 1] \times \{0\} \times \{0\}, \qquad E_{1/2} := [0, 1] \times \{1/2\} \times \{0, 1\} \quad \text{or} \quad E_1 := [0, 1] \times \{1\} \times \{1\};$$

- the defect wall cuts the ribbon graph Ω into three pieces:
  1. $\Omega^{\mathrm{L}} := \Omega \cap \left([0, 1]^2 \times \left[0, \frac{1}{2}\right)\right)$ represents a morphism of $\mathrm{SkCat}_{\mathrm{L}_t}([0, 1]^2)$;
  2. $\Omega^{\mathrm{G}} := \Omega \cap \left([0, 1]^2 \times \left(\frac{1}{2}, 1\right]\right)$ represents a morphism of $\mathrm{SkCat}_{\mathrm{GL}_t}([0, 1]^2)$;
  3. $\Omega^{\mathrm{P}} := \Omega \cap ([0, 1] \times \{1/2\})$ is a planar graph as defined in section 2.4, with the only difference that we allow endpoints to lie in the interior of S if they are attached to either Ω^{G} or Ω^{L};
- Ω^{G} and Ω^{L} meet transversally the defect at a coupon of Ω^{P} compatible with the colour of the strands.

If Ω is such a ribbon graph, we set

$$V \coloneqq \Omega \cap E_0, \quad W \coloneqq \Omega \cap E_1, \quad a \coloneqq \Omega \cap [0,1] \times \{1/2\} \cap \{0\}, \quad b \coloneqq \Omega \cap [0,1] \times \{1/2\} \cap \{1\}$$

for the objects of $\mathrm{Rep}_q\mathrm{GL}_t$, $\mathrm{Rep}_q\mathrm{L}_t$ and $\mathrm{Rep}_q\mathrm{P}_t$ determined by its endpoints. We say that Ω is a *stratified* (a,b,V,W)*-ribbon graph on the disk*.

Fix a,b,V,W and consider the $(\Bbbk \otimes_{\mathbb{C}(q)} \Bbbk)$-linear space spanned by stratified (a,b,V,W)-ribbon graphs modulo local relations happening on either one side of the defect wall or the defect wall itself. This means that $\sum_i \lambda_i \Omega_i \sim 0$ if either $\sum_i \lambda_i \Omega_i^{\mathrm{G}} = 0$ in $\mathrm{SkCat}_{\mathrm{GL}_t}([0,1]^2)$, $\sum_i \lambda_i \Omega_i^{\mathrm{L}} = 0$ in $\mathrm{SkCat}_{\mathrm{L}_t}([0,1]^2)$ or $\sum_i \lambda_i \Omega_i^{\mathrm{P}} = 0$ via the local relations introduced in section 2.4 applied to the defect. We also mod out by *framed stratified isotopy*: that is, by isotopies of E^* fixing the boundary, preserving the interior of the defect and such that coupons in the defect remain parallel to the framing throughout the isotopy. Let $\mathcal{L}_{a,b,V,W}$ be the resulting space. We define next an evaluation morphism for stratified ribbon graphs. To a stratified (a,b,V,W)-ribbon graph Ω as above, we associate a morphism in $\mathrm{Rep}_q\mathrm{P}_t$ in the following way:

- we modify Ω^{G} as in the definition of the restriction functor (3.2), that is:
  1. we label the initial point of any open strand α by $n_{\alpha(0)} = 0$ or $n_{\alpha(0)} = 1$ depending on the orientation;
  2. we label the endpoint by $n_\alpha + \mathrm{rot}^f \alpha$;
  3. we use the pivotal structure of $\mathrm{Rep}_q\mathrm{GL}_t$ (morphisms in (2.29) and (2.30)) to switch the label of every endpoint to 0 or 1;
  4. we replace the graph $\widetilde{\Omega}^{\mathrm{G}}$ thus obtained by $\varphi_{\varepsilon_2}^{-1} \circ \widetilde{\Omega}^{\mathrm{G}} \circ \varphi_{\varepsilon_1}$, where φ_ε is the isomorphism in (3.1);
- we apply the same procedure to Ω^{L} (here the pivotal structure is the trivial one);
- we switch the decoration of every endpoint of Ω^{P} to either 0 or 1 by concatenating with the pivotal structures of $\mathrm{Rep}_q\mathrm{GL}_t$ and $\mathrm{Rep}_q\mathrm{L}_t$ (morphisms in (2.28), (2.29) and (2.30));
- we project the resulting ribbon graph into the defect wall and replace all the crossings appearing by the corresponding half-twist from $Z(\mathrm{Rep}_q\mathrm{P}_t)$ (cf. section 4.1).

Let $\widehat{\Omega}$ be obtained by applying this procedure to Ω. Projecting E_0 and E_1 onto $E_{1/2}$, we get two configurations of labelled points V_a and W_b. The planar graph $\widehat{\Omega}$ represents thus a morphism $f_\Omega \in \mathrm{Hom}_{\mathrm{Rep}_q\mathrm{P}_t}(V_a, W_b)$ (see Figure 6 for an example). We define the *evaluation morphism* by

$$\begin{array}{rccc} \mathrm{ev}_{a,b,V,W}: & \mathcal{L}_{a,b,V,W} & \to & \mathrm{Hom}_{\mathrm{Rep}_q\mathrm{P}_t}(V_a, W_b), \\ & \Omega & \mapsto & f_\Omega, \end{array} \tag{5.1}$$

on generators and extending linearly.

Lemma 5.1. *The evaluation morphism $\mathrm{ev}_{a,b,V,W}$ is well-defined, i.e., f_Ω does not depend on the equivalence class of Ω in $\mathcal{L}_{a,b,V,W}$.*

Proof. We have to check that f_Ω is invariant under skein and isotopy relations applied to Ω. For skein relations, this is just the fact that the restriction functors ι_t^* and π_t^* (cf. diagram (3.4)) are well-defined. For the topological relations, first note that stratified isotopy does not fix the defect wall, so the position a strand intersect the defect may change. In particular, we may move a crossing between two strands intersecting transversally the defect from one side to the other by applying such an isotopy. For instance, the stratified ribbon graphs representd (as seen

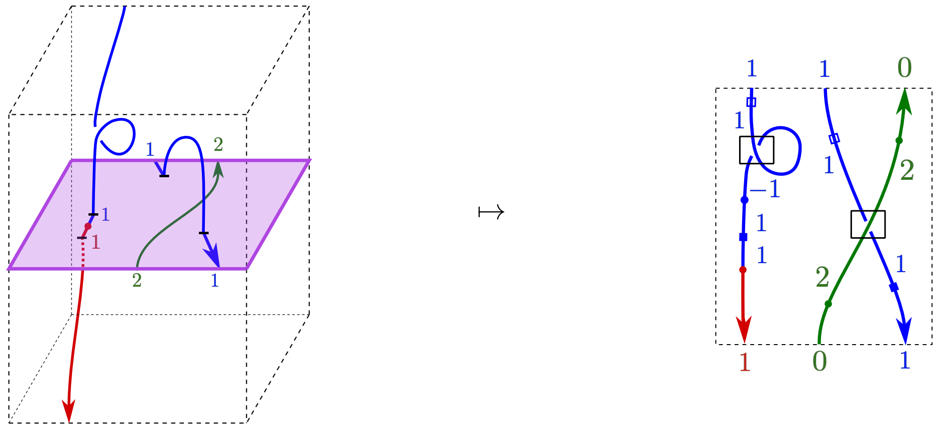
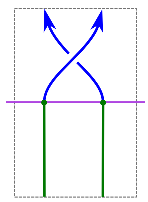
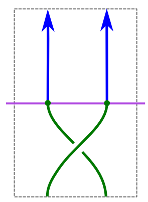
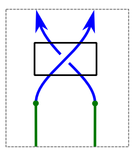
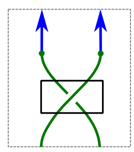


FIGURE 6: On the left, a stratified ribbon graph on the disk; on the right, its evaluation in $\mathrm{Rep}_q\mathrm{P}_t$.

from the front) by

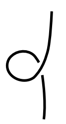
 and  (5.2)

are isotopic, however their projections are

 and  ,

which are not related by isotopy in $\mathrm{Rep}_q\mathrm{P}_t$. The naturality of the blue crossing (cf. Remark 2.14) guarantees that they have the same evaluation.

The previous observation implies that crossings can pass through the defect, hence we have to prove that the evaluation is invariant under framed Reidemeister moves happening either in one side of the defect or in the defect itself. For the Reidemeister moves II and III, it suffices to note that crossings are sent to half-braidings in $Z(\mathrm{Rep}_q\mathrm{P}_t)$, hence they are invertible and satisfy the corresponding Yang-Baxter equation. For the framed Reidemeister move I

 $\leftrightarrow$  ,

this is a consequence of the restriction 2.24 being well-defined and the fact that all relations holding in $\mathrm{Rep}_q\mathrm{P}_t$ come from relations in $\mathrm{Rep}_q\mathrm{L}_t$. □

5.2 Defect lines. We introduce finally defect lines decorated by $\mathrm{Rep}_q\mathrm{L}_t$ and we use the $(\mathrm{Rep}_q\mathrm{L}_t, \mathrm{Rep}_q\mathrm{P}_t)$-centre structure to define a local model around them. Let $E = I^3$ be the cube, that we endow with the stratification defined by the following figure:

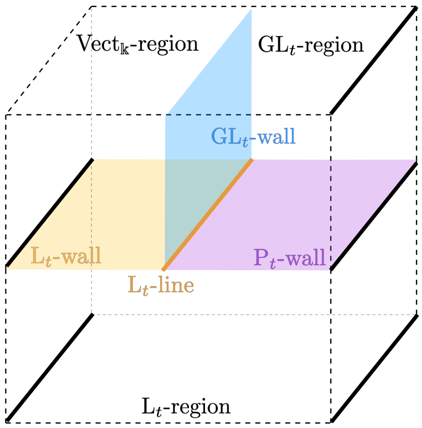


FIGURE 7: We consider two horizontal defect walls decorated with $\mathrm{Rep}_q\mathrm{L}_t$ and $\mathrm{Rep}_q\mathrm{P}_t$, and a vertical wall decorated with $\mathrm{Rep}_q\mathrm{GL}_t$. These walls meet at a 1-dimensional stratum decorated with $\mathrm{Rep}_q\mathrm{L}_t$. The region below the horizontal wall is the L_t-region, the one over the L_t-wall is the $\mathrm{VECT}_{\Bbbk}$-region and, finally, the region over the P_t-wall is the GL_t-region.

We consider the following class of ribbon graphs Ω, that generalizes the ones introduced in the previous section:

- the endpoints of Ω lie on the bold intervals and we suppose that they are all at different depths, so that projecting them into the horizontal wall produces no intersections;
- each part of the graph is decorated accordingly to its region. In particular, the $\mathrm{VECT}_{\Bbbk}$-region is empty;
- every edge lies entirely in one of the strata and can be attached to a coupon lying in a stratum whose dimension differ at most by one (that is, they can not go through the defect line directly from one of the three-dimensional regions).

As usual, we consider these graphs up to framed stratified isotopies of the cube.

Fix a configuration of coloured oriented points $\mathcal{P}$ on the bold intervals and let $E(\mathcal{P})$ be the $(\Bbbk \otimes_{\mathbb{C}(q)} \Bbbk)$-linear space spanned by isotopy classes of ribbon graphs whose endpoints match $\mathcal{P}$. Projecting $\mathcal{P}$ into the horizontal walls, we get two configurations X and Y representing objects of $\mathrm{Rep}_q\mathrm{L}_t$ and $\mathrm{Rep}_q\mathrm{P}_t$, respectively. Let $\Omega \in E(\mathcal{P})$ be a ribbon graph, following the definition of the evaluation morphism 5.1, we can project Ω into the horizontal defects, replace the crossings appearing by the corresponding half braiding and introduce the pivotal structures, so that we get a planar graph $\widetilde{\Omega}$ on a stratified square as in Figure 5, representing a morphism $f_\Omega \in \mathrm{Hom}_{\mathrm{Rep}_q\mathrm{L}_t}(Y, j^*(X))$.

Lemma 5.2. *The linear map defined by the assignment*

$$\begin{array}{ccc} E(\mathcal{P}) & \to & Hom_{\mathrm{Rep}_q\mathrm{L}_t}(Y, j^*(X)), \\ \Omega & \mapsto & f_\Omega, \end{array}$$

is well-defined, i.e., it does not depend on the isotopy class of Ω.

Proof. Extending the vertical wall up to the bottom of the cube, it cuts the ribbon graph Ω into two subgraphs: Ω_1, lying under the L_t-wall, and Ω_2, lying under and over the P_t-wall. The planar diagram $\widetilde{\Omega}$ can then be obtained by projecting Ω_1 and Ω_2 separately and gluing them along the L_t-defect line. The projections $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are obtained as in the definition of the evaluation morphism (5.1), using the trivial $(\mathrm{VECT}, \mathrm{Rep}_q\mathrm{L}_t)$-central structure of $\mathrm{Rep}_q\mathrm{L}_t$ for $\widetilde{\Omega}_1$ and the $(\mathrm{Rep}_q\mathrm{GL}_t, \mathrm{Rep}_q\mathrm{L}_t)$-central structure of $\mathrm{Rep}_q\mathrm{L}_t$ for $\widetilde{\Omega}_2$. Hence, by the same arguments

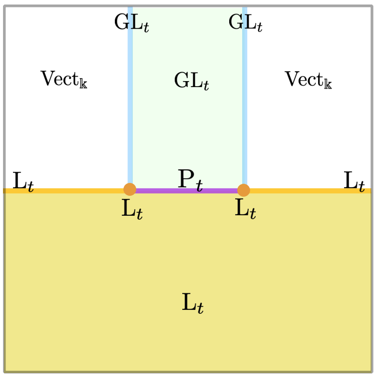


FIGURE 8: Schematic view of the section of a decorated cylinder. The horizontal line represents the defect wall. The orange points are sections of the defect lines. The blue vertical lines represent the GL_t-wall.

as in Lemma 5.1, they do not depend on the isotopy class of Ω_1 and Ω_2. This shows that f_Ω is invariant under isotopies of the stratified cube happening entirely at one of the sides of the vertical wall.

It remains to check that f_Ω is also invariant under isotopies making a part of the diagram move through the vertical wall. This is the diagrammatic counterpart of the fact that the bimodule $\mathrm{Rep}_q\mathrm{L}_t$ is centered: the L_t-region can act on the defect line both through the P_t-wall and through the L_t-wall, and we want to show that both actions in fact coincide. This follows from relation (4.2) and the definition of the central structures. Indeed, when we project Ω into the defect, we replace all the crossing appearing by the corresponding half-braidings. In particular, when one of the strands involved comes from the L_t-region (this is the only case of interest), the half-braiding is the braiding β of $\mathrm{Rep}_q\mathrm{L}_t$ (cf. section 3.1). Therefore, projecting into the P_t-wall makes appear a coupon decorated with $\pi^*(\beta)$, while applying an isotopy and projecting into the L_t-wall makes appear a coupon decorated with β itself. Relation (4.2) and the fact that $j^* \circ \pi^* = \mathrm{id}_{\mathrm{Rep}_q\mathrm{L}_t}$ imply that both diagrams represent the same morphism in $\mathrm{Hom}_{\mathrm{Rep}_q\mathrm{L}_t}(Y, j^*(X))$. □

5.3 Three-dimensional defect theory. We introduce finally the three-dimensional theory with defects. Let (S, f, ϕ) be a bipartite framed surface. We label $\phi^{-1}(A)$ and $\phi^{-1}(B)$ by L_t and $\phi^{-1}(C)$ by P_t. The stratification ϕ induces an stratification of the cylinder $S \times I$ decorated as follows (see Figure 8):

- the horizontal wall $S \times \left\{\frac{1}{2}\right\}$ is decorated as (S, ϕ);
- the cylinder $\phi^{-1}(C) \times \left(\frac{1}{2}, 1\right]$ and the vertical walls $\phi^{-1}(B) \times \left(\frac{1}{2}, 1\right]$ are labelled with GL_t;
- the cylinder $\phi^{-1}(A) \times \left(\frac{1}{2}, 1\right]$ is labelled with $\mathrm{VECT}_{\Bbbk}$;
- the region $S \times \left[0, \frac{1}{2}\right)$ under the horizontal wall is entirely decorated with L_t.

Let $\mathcal{B}$ be a compatible marking of the surface. We consider isotopy classes of ribbon graphs Ω on S (see Figure 9) such that:

- Ω does not intersect the $\mathrm{VECT}_{\Bbbk}$-region and it is coloured accordingly with the labelling on the other regions;
- the endpoints of Ω are attached to either $S \times \{\pm 1\}$ or one of the components marked on the boundary of the horizontal wall;
- every edge lies entirely in one of the strata and can be attached to a coupon rectilinearly

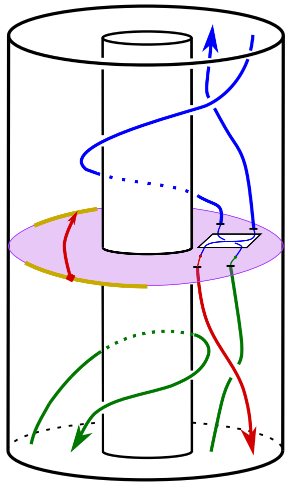


FIGURE 9: Stratified ribbon graph on an annulus entirely labelled by P_t.

embedded in a stratum whose dimension differ at most by one (this means, again, that cannot go directly from the GL_t-region to the L_t-line).

Call S^{P_t} the region of S decorated with P_t and fix boundary conditions $X \in \mathrm{SkCat}_{\mathrm{GL}_t}(S^{\mathrm{P}_t})$, $Y \in \mathrm{SkCat}_{\mathrm{L}_t}(S)$ and $U_C \in \mathcal{Z}(C)$ for every $C \in \mathcal{B}$. We set $E^S(X, Y, (U_C)_{C\in\mathcal{B}})$ for the linear space spanned by ribbon graphs as above, modulo local relations induced by $\mathrm{Rep}_q\mathrm{GL}_t$ and $\mathrm{Rep}_q\mathrm{L}_t$ on the three-dimensional regions, and by the local models described in sections 5.1 and 5.2 near the defects. The assignment

$$\begin{array}{rcll} \mathcal{A}(S,f,\phi,\mathcal{B})\colon & \left(\boxtimes_{C\in\mathcal{B}} \mathcal{Z}(C)\right) \boxtimes \mathrm{SkCat}_{\mathrm{GL}_t}(S^{\mathrm{P}_t}) \boxtimes \mathrm{SkCat}_{\mathrm{L}_t}(S)^{\mathrm{op}} & \to & \mathrm{V}\textsc{ect}, \\ & ((U_C)_{C\in\mathcal{B}}, X, Y) & \mapsto & E^S(X, Y, (U_C)_{C\in\mathcal{B}}), \end{array}$$

is functorial. The action on morphism is given by horizontally gluing cylinders along the components marked on the defect wall, and by vertically stacking diagrams representing morphisms in the skein categories.

Lemma 5.3. *Let $(S, f, \phi, \mathcal{B})$ be a bipartite framed surface with compatible marking. Let $\mathrm{gl}_C(S)$ the surface obtained by gluing S along a boundary component C, with $C, \overline{C} \in \mathcal{B}_{\mathrm{P}_t}$ (alternatively, in $\mathcal{B}_{\mathrm{L}_t}$). Then,*

$$\mathcal{A}(\mathrm{gl}_C(S))(-) = \int^{U\in\mathcal{Z}(C)} \mathcal{A}(S)(-, U, U),$$

where the components decorated with U on the RHS are C and $\overline{C}$.

Proof. Again, this is essentially the proof of [Wal06, Theorem 4.4.2]. Gluing along C induces morphisms

$$\mathrm{gl}_U :\ \mathcal{A}(S)(-, U, U) \ \to\ \mathcal{A}(\mathrm{gl}_C(S))(-)$$

for every $U \in \mathcal{Z}(C)$. Moreover, if $f : U \to V$ is a morphism in $\mathcal{Z}(C)$, Ω is a stratified ribbon graph representing an element of $\mathcal{A}(S)(-, U, V)$ and $f \star \Omega$ (resp. $\Omega \star f$) are the graphs obtained by gluing f to Ω along C (resp. $\overline{C}$), then the graphs $\mathrm{gl}_U(f \star \Omega)$ and $\mathrm{gl}_V(\Omega \star f)$ are related by an isotopy supported on a tubular neighbourhood of ∂S. Therefore, they define the same element

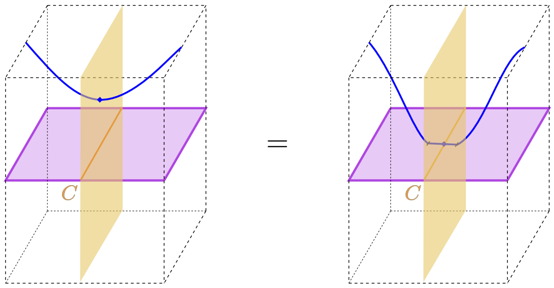


FIGURE 10: Applying isotopy and skein relations, we may always cut a graph along a planar component.

of $\mathcal{A}(\mathrm{gl}_C(S))(-)$ and the diagram

$$\begin{tikzcd} & \mathcal{A}(S)(-,U,U) \arrow[dr, "\mathrm{gl}_a"] & \\ \mathcal{A}(S)(-,U,V) \arrow[ur, "-\star f"] \arrow[dr, "f\star -"'] & & \mathcal{A}(\mathrm{gl}_C(S))(-) \\ & \mathcal{A}(S)(-,V,V) \arrow[ur, "\mathrm{gl}_b"'] & \end{tikzcd} \tag{5.3}$$

commutes. Let us show that $\mathcal{A}(\mathrm{gl}_C(S))(-)$ is universal for this property.

Let E be a vector space together with linear maps

$$\varphi_U : \mathcal{A}(S)(-,U,U) \to E$$

making the equivalent diagrams commute. We have to show that there is a unique linear map

$$\varphi : \mathcal{A}(\mathrm{gl}_C(S))(-) \to E$$

such that $\varphi_a = \varphi \circ \mathrm{gl}_a$ for every $U \in \mathcal{Z}(C)$. Let Ω be a stratified ribbon graph in $\mathrm{gl}_C(S) \times I$ representing an element of $\mathcal{Z}(\mathrm{gl}_C(S))(-)$. Applying isotopy and stratified skein relations, we may locally project Ω onto the defect, so that we may suppose that there is a neighborhood of $C \times I$ where Ω is represented by a planar graph contained in the defect (see Figure 10). Let U be the intersection of Ω with $C \times \left\{\frac{1}{2}\right\}$, which determines an object of $\mathcal{Z}(C)$. Cutting along $C \times I$, we get a ribbon graph Ω_U in $S \times I$ defining an element of $\mathcal{A}(S)(-,U,U)$. If φ exists, then $\varphi(\Omega) = \varphi_U(\Omega_U)$ so it is uniquely defined.

To prove the existence, suppose first that $\widetilde{\Omega}$ is another stratified ribbon graph representing the same element of $\mathcal{A}(\mathrm{gl}_C(S))(-)$ and related to Ω by an isotopy shifting a collar neighbourhood through $C \times I$. Then, cutting along $C \times I$ yields an ribbon graph $\widetilde{\Omega}_V$ in $S \times I$ and, since we may suppose that it is planar around $C \times I$, the commutativity of (5.3) implies that $\mathrm{gl}_U(\Omega_U) = \mathrm{gl}_V(\widetilde{\Omega}_V)$. On the other hand, if Ω and $\widetilde{\Omega}$ are related by an isotopy of $\mathrm{gl}_C(S)$ supported in a cube not intersecting $C \times I$, then Ω_U and $\widetilde{\Omega}_U$ are related by the same isotopy of S, so that they represent the same element of $\mathcal{A}(S)(-,U,U)$ and $\varphi_U(\Omega_U) = \varphi_U(\widetilde{\Omega}_U)$. Therefore, φ is compatible with isotopies.

Finally, if Ω and $\widetilde{\Omega}$ are related by a skein relation on a cube not intersecting $C \times I$, then Ω_U and $\widetilde{\Omega}_U$ are related by the same relation and they represent the same element. If the skein relation occurs in a cube intersecting S, we can apply an isotopy moving it off of $C \times I$. We get stratified ribbon graphs Ω' and $\widetilde{\Omega}'$ related by a skein relation on a cube not intersecting $C \times I$, so $\mathrm{gl}_{U'}(\Omega'_{U'}) = \mathrm{gl}_{U'}(\widetilde{\Omega}'_{U'})$. Moreover, the isotopy invariance of φ implies that $\mathrm{gl}_U(\Omega_U) = \mathrm{gl}_{U'}(\Omega'_{U'})$ and $\mathrm{gl}_U(\widetilde{\Omega}) = \mathrm{gl}_{U'}(\widetilde{\Omega}_{U'})$. This proves that φ is compatible with skein relations. $\square$

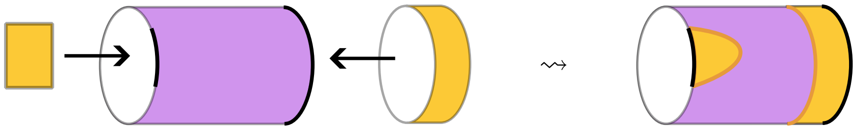


FIGURE 11: Gluing stratified cylinders along the boundary modifies the stratification.

Theorem 5.4. *The assignment*

$$\mathcal{A}\colon \mathrm{Bord}_1^{bip,fr}(2) \to \textsc{Bimod}$$

defines a symmetric monoidal functor from the category of 2*-dimensional bipartite framed cobordisms, hence a* 2*-dimensional framed TFT. It coincides with the planar theory* $\mathcal{Z}$ *on one-dimensional manifolds.*

Proof. As in Theorem 4.11, the compatibility with gluing/cutting surfaces follows from the previous lemma and the fact that we are considering compatible markings. □

6 The HOMFLY skein bialgebra

6.1 The coproduct map. Consider a bipartite framed surface $(S, f, \phi, \mathcal{B})$ with compatible marking. As above, denote by S^{P_t} the region of S decorated with P_t. Applying the three-dimensional TFT $\mathcal{A}$, we get a functor

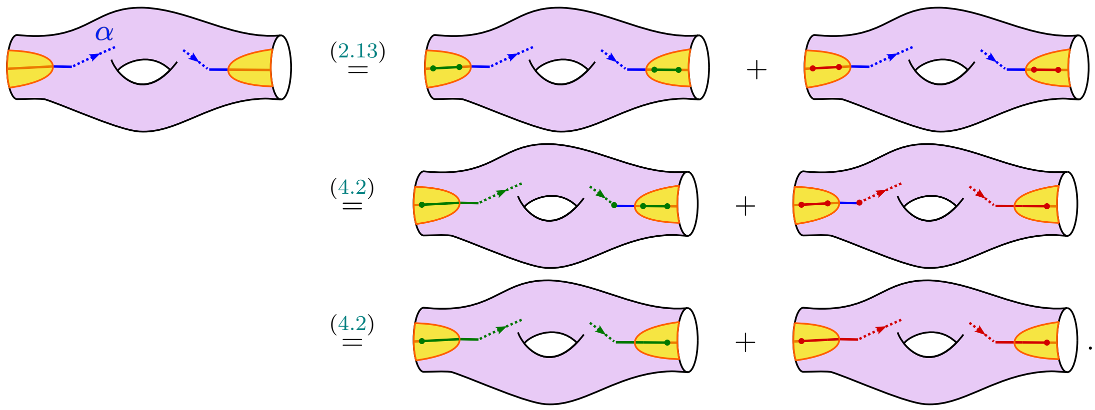
$$\mathcal{A}(S)\colon \left(\boxtimes_{C\in\mathcal{B}} \mathcal{Z}(C)\right) \boxtimes \mathrm{SkCat}_{\mathrm{GL}_t}(S_t^{\mathrm{P}}) \boxtimes \mathrm{SkCat}_{\mathrm{L}_t}(S) \to \textsc{Vect}.$$

Choose boundary conditions $U_C \in \mathcal{Z}(S)$ for every component $C \in \mathcal{B}$ and fix the empty configurations $\varnothing \in \mathrm{SkCat}_{\mathrm{GL}_t}(S^{\mathrm{P}_t})$ and $\varnothing \in \mathrm{SkCat}_{\mathrm{L}_t}(S)$ on the top and bottom bases of $S \times I$. By construction, the vector space $E^S((U_C)_{C\in\mathcal{B}})$ obtained from $\mathcal{A}(S)$ comes with a left and a right action of the skein algebras $\mathrm{Sk}_{\mathrm{GL}_t}(S^{\mathrm{P}_t})$ and $\mathrm{Sk}_{\mathrm{L}_t}(S)$, respectively. By choosing an appropriate stratification ϕ, this vector space is indeed isomorphic to $\mathrm{Sk}_{\mathrm{L}_t}(S)$:

Proposition 6.1. *Suppose that* $\mathcal{B} \neq \varnothing$ *and that* ϕ *consists of a family of cylinders inserted along the marked components as in Figure 11. Then, there is a natural family of isomorphisms of vector spaces*

$$\mathcal{A}(S)(-, \varnothing, \varnothing) \cong \mathcal{Z}^{\mathrm{L}_t}(-).$$

Proof. Fix boundary conditions U and let Ω be a stratified ribbon graph representing an element of $\mathcal{A}(S)(U, \varnothing, \varnothing)$. Since there is no strand attached to $S \times \{\pm 1\}$, the 3-dimensional stratified skein relations allow to project Ω into the defect, so that it can be represented by a planar graph on $S \times \left\{\frac{1}{2}\right\}$, coloured by $\mathrm{Rep}_q \mathrm{P}_t$ in the P_t-region and by $\mathrm{Rep}_q \mathrm{L}_t$ in the L_t-region. We first proof that this graph can be written as a linear combination of diagrams where none of the strands is coloured in ■. Indeed, suppose that α is a component coloured in blue. If α is an open strand, then it is attached to one of the intervals/circles marked on the boundary, hence it traverses the 1-dimensional L_t-defect. Crossing these defect lines, switches its colour to ■ near its endpoints. We can then apply relation (2.13) inside the L_t-disks/cylinders and make one of the inclusion/projections appearing on each term cross the defect (relation (4.2)) and slide along α (by naturality, cf. Remark 2.14) to reach the opposite endpoint (see figure below). The part of α lying in the P_t region is now entirely coloured with $\textcolor{green}{\blacksquare}$ and $\textcolor{red}{\blacksquare}$:



Similarly, if α is a closed strand, we can push it close to a marked component by applying and isotopy. Then, relation (4.2) allows to project it partially into the L_t-region where we can apply the same trick to write it as a combination of a green and a red component.

Let us call $\widetilde{\Omega}$ a diagram obtained by Ω by applying the previous transformations. Since it is entirely coloured by $\textcolor{orange}{\blacksquare}$, $\textcolor{green}{\blacksquare}$ and $\textcolor{red}{\blacksquare}$, it represents an element of $\mathcal{Z}^{\mathrm{L}_t}(S)(U)$. This elements is well-defined, since all the relations applied to switch $\textcolor{blue}{\blacksquare}$ to $\textcolor{orange}{\blacksquare}$ are local relations from $\mathrm{Rep}_q\mathrm{L}_t$, hence they hold in $\mathcal{Z}^{\mathrm{L}_t}(S)(U)$. We get a linear map

$$\mathcal{A}(S)(U,\varnothing,\varnothing) \to \mathcal{Z}^{\mathrm{L}_t}(S)(U), \quad \Omega \mapsto \widetilde{\Omega}.$$

An inverse of this map is constructed as follows. Given a planar diagram on Γ on S coloured by $\mathrm{Rep}_q\mathrm{L}_t$, we can apply isotopies and naturality to make all the "forbidden" morphisms slide along open strands and lie arbitrarily closed to the marked components of the boundary, so that they are all inside one of the embedded disks/cylinders defining an L_t-region. Note that this can be done in a unique way so that the intersection of the strand with the P_t-region is coloured in either $\textcolor{green}{\blacksquare}$ or $\textcolor{red}{\blacksquare}$. We get this way a well-defined graph $\widehat{\Gamma}$ on the stratified defect wall and it is straightforward to see that $\Gamma \mapsto \widehat{\Gamma}$ defines an inverse for of the linear map below. Naturality follows from the fact that these maps do not modify diagrams in a tubular neighborhood of the marked boundary components. $\square$

Therefore, the presence of regions coloured by $\mathrm{Rep}_q\mathrm{L}_t$ near the boundary of the defect wall allows to apply relation (2.13) to components coloured with $\textcolor{blue}{\blacksquare}$, yielding an identification $\textcolor{blue}{\blacksquare} = \textcolor{orange}{\blacksquare}$. In the case where there are strands crossing transversally the defect, this identification still holds, but only on closed components:

Proposition 6.2. *Let $\mathcal{B}$ and ϕ be as in the previous proposition and fix empty boundary conditions on each of the components marked on the defect. Then,*

$$\mathcal{A}(S)(\varnothing,-,-) \cong pRes(S)(\varnothing,-,-) \Big/ \langle \textcolor{blue}{\blacksquare} = \textcolor{orange}{\blacksquare} \text{ on closed components on the defect wall}\rangle \cdot$$

Proof. The idea of the proof is the same as in the previous preposition: we can construct linear maps between both spaces by pushing a part of the diagram near one the L_t-regions on the boundary, then apply relation (4.2) to project into the L_t-region and make the "forbidden" morphisms slide along closed components. This allows to decompose closed components coloured with $\textcolor{blue}{\blacksquare}$ into linear combinations of green and red diagrams, yielding the identification $\textcolor{blue}{\blacksquare} = \textcolor{orange}{\blacksquare}$. On open strands, this induces no new relations: since they are all attached to $S \times \{\pm 1\}$, "forbidden" morphisms cannot slide along then, so making them appear just produce a different way of writing the same morphism, but they induce no relation. $\square$

Combining both propositions, we see that $\mathcal{A}(S)(\varnothing)$ is isomorphic as a linear space to $\mathrm{Sk}_{\mathrm{L}_t}(S)$, and by functoriality of $\mathcal{A}(S)$ we have now actions of both $\mathrm{Sk}_{\mathrm{GL}_t}(S)$ and $\mathrm{Sk}_{\mathrm{L}_t}(S)$:

Corollary 6.3. *Let $(S, f, \mathcal{B})$ be a marked framed surface with $\mathcal{B} \neq \varnothing$. Then, the $Sk_{GL_t}(S) - Sk_{L_t}(S)$-bimodule $\mathcal{A}(S)(\varnothing)$ is isomorphic to $Sk_{L_t}(S)$ as a linear space. In particular, $Sk_{L_t}(S)$ becomes a left $Sk_{GL_t}(S)(S)$-module under this identification.* ■

We treat now the case of the torus $S = \mathbb{T}^2$. Let $(\mathbb{T}^2, f, \phi)$ be a framed torus endowed with a stratification ϕ whose H_t-region consist of only a disk embedded in $\mathbb{T}^2$. By the same argument as in the proof of Propositions 6.1 and 6.2,

$$\mathcal{A}(\mathbb{T}^2, f, \phi)(\varnothing) \cong \mathrm{Sk}_{\mathrm{L}_t}(\mathbb{T}^2)$$

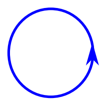
Moreover, the P_t-region is homeomorphic to a punctured torus, so we have an action

$$\mathrm{Sk}_{\mathrm{GL}_t}(\mathbb{T}^2 \setminus \mathbb{D}^2) \otimes \mathrm{Sk}_{\mathrm{L}_t}(\mathbb{T}^2) \to \mathrm{Sk}_{\mathrm{L}_t}(\mathbb{T}^2)$$

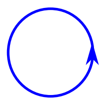
of the GL_t-skein algebra of this puncture torus.

Proposition 6.4. *The action above descends to an action of the GL_t-skein algebra of the torus.*

Proof. A loop α around the puncture acts by multiplication by

$$\pi_t^* \circ \mathrm{res}_t \Big(\quad \Big)$$ 

on the L_t-disk hence the action descends to the quotient

$$\mathrm{Sk}_{\mathrm{GL}_t}(\mathbb{T}^2 \setminus \mathbb{D}^2) \Big/ \Big\langle \alpha = \quad \Big\rangle \cong \mathrm{Sk}_{\mathrm{GL}_{\approx}}(\mathbb{T}^2).$$ 

□

Let (S, f) be any framed surface and choose $\mathcal{B} \neq \varnothing$ if $\partial S \neq \varnothing$. Let ϕ be a stratification of S as in Propositions 6.1 and 6.4, so that $\mathrm{Sk}_{\mathrm{GL}_t}(S)$ acts on $\mathrm{Sk}_{\mathrm{L}_t}(S)$:

Proposition 6.5. *The morphism $Sk_{GL_t}(S) \to \mathcal{A}(S)(\varnothing, \varnothing)$ defined by acting the empty stratified diagram induces a $\mathbb{C}(q)$-algebra homomorphim*

$$\widetilde{\Delta}_f : \ Sk_{GL_t}(S) \to Sk_{L_t}(S).$$

Proof. The fact that $\widetilde{\Delta}_f$ is an algebra homomorphism follows from the functoriality of $\mathcal{A}(S)$. □

Composing $\widetilde{\Delta}_f$ with the isomorphism in Corollary 2.9, we obtain a $\mathbb{C}(q)$-algebra morphism

$$\Delta_f : \ \mathrm{Sk}_{\mathrm{GL}_t}(S) \ \to \ \mathrm{Sk}_{\mathrm{GL}_t}(S) \otimes_{\mathbb{C}(q)} \mathrm{Sk}_{\mathrm{GL}_t}(S) \tag{6.1}$$

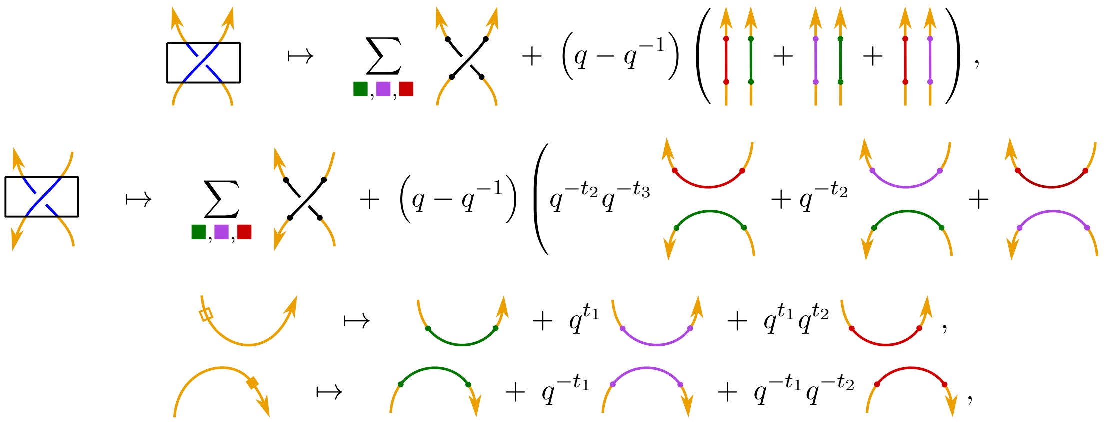
whose restriction to $\Bbbk \cong \Bbbk \cdot \mathbf{1}_{\varnothing}$ is (2.19). By construction, it can be computed on links by taking a diagram on the surface where every crossing, cup and cap has been replaced by their image by the restriction functor (2.24), and applying relations (2.20)-(2.3) and (2.13) to split the diagram into two coloured diagrams, that we consider as lying in two different copies of $\mathrm{Sk}_{\mathrm{GL}_t}(S)$. Here by cup and cap we mean a portion of the diagram where the rotation number increases or decreases by 1.

Proposition 6.6. *The coproduct in* (6.1) *is coassociative.*

Proof. The map

$$(\Delta_f \otimes \mathrm{id}) \circ \Delta_f : \ \mathrm{Sk}_{\mathrm{GL}_t}(S) \ \to \ \mathrm{Sk}_{\mathrm{GL}_t}(S) \otimes_{\mathbb{C}(q)} \mathrm{Sk}_{\mathrm{GL}_t}(S) \otimes_{\mathbb{C}(q)} \mathrm{Sk}_{\mathrm{GL}_t}(S)$$

can be computed as follows. Given a link diagram Γ, we first compute $\widetilde{\Delta}_f(\Gamma) \in \mathrm{Sk}_{\mathrm{L}_t}(S)$ as in the previous paragraph. Then, we split the green part into two colours (green again for the first copy of $\mathrm{Sk}_{\mathrm{GL}_t}(S)$ and violet for the second one) using the same rules: replacing all green crossings, cups and caps by their images by the restriction functor (2.24) and applying relations (2.20)-(2.3) and (2.13) with a proper choice of colours. Since applying $\widetilde{\Delta}$ does not produce new crossings, we can do all this at once from the initial diagram using the following relations:



(6.2)

(6.3)

(6.4)

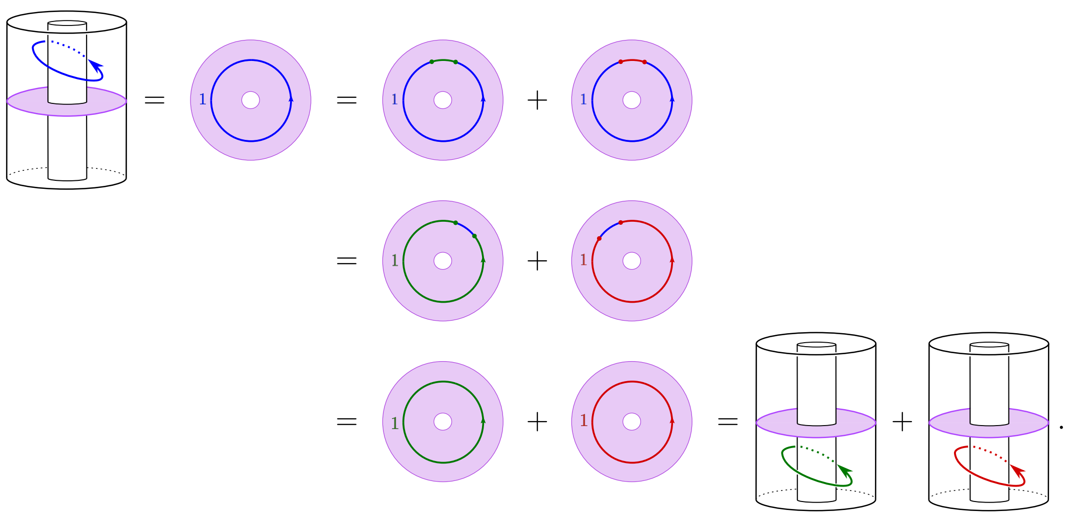
(6.5)

where the sums are taken over all the possible combinations of colours for the involved strands and

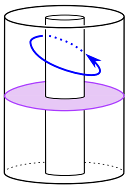
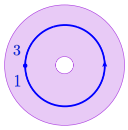
$$q^{t_1} = q^t \otimes 1 \otimes 1, \qquad q^{t_2} = 1 \otimes q^t \otimes 1, \qquad q^{t_3} = 1 \otimes 1 \otimes q^t,$$

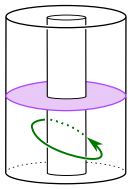
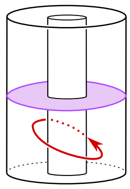
in $\Bbbk \otimes_{\mathbb{C}(q)} \Bbbk \otimes_{\mathbb{C}(q)} \Bbbk$. The same formulas compute $(\mathrm{id} \otimes \Delta_f) \circ \Delta_f$, hence the coproduct is coassociative. $\square$

Remark 6.7. The coproduct Δ_f depends on the choice of the framing f. For instance, let $S = \mathbb{A}$ be the annulus, which can be equipped with either a radial framing or the framing induced by $\mathbb{R}^2$. Consider first the case where f is the radial framing. We have



On the other hand, if f is the framing induced by $\mathbb{R}^2$, the local relations projecting the link into the defect make use of the pivotal structure of $\mathrm{Rep}_q\mathrm{GL}_t$:

 $=$ 

$= q^{t_2}$  $+\ q^{-t_1}$  ,

where the last equality follows from the definition of the pivotal structure of $\mathrm{Rep}_q\mathrm{GL}_t$ by doing the same manipulations as in the previous paragraph. $\lozenge$

6.2 The counit. Let $\overline{\mathrm{Rep}_q}\mathrm{GL}_n$ be the $\mathbb{C}$-linear category obtained by specialising

$$q^t \rightsquigarrow q^n, \qquad \delta \rightsquigarrow \frac{q^n - q^{-n}}{q - q^{-1}}$$

in Definition 2.1, with $q \in \mathbb{C}^\times$ and $q^2 \neq 1$. This is the category $\dot{\mathcal{OS}}(q - q^{-1}, q^n)$ from [Bru17]. Let $\varphi_n : \Bbbk \to \mathbb{C}(q)$ be the evaluation morphism in Remark 2.3. Then, the evaluation functor $\mathrm{ev}^{\mathrm{G}}_{q,n}$ factors as

$$\mathrm{Rep}_q\mathrm{GL}_t \ \to\ \overline{\mathrm{Rep}_q}\mathrm{GL}_n \ \to\ \mathrm{Rep}_q\mathrm{GL}_n,$$

where the first arrow is a φ_n-linear functor and the second one is $\mathbb{C}$-linear.

Definition 6.8. ([De07]) Let $\mathcal{C}$ be a ribbon category. A morphism $f : X \to Y$ is *negligible* if $\mathrm{Tr}_{\mathcal{C}}(f \circ g) = 0$ for all $g : Y \to X$ in $\mathcal{C}$.

Theorem 6.9. ([De07, Bru17]) *The functor $\overline{Rep_q}GL_n \to Rep_qGL_n$ above induces a monoidal equivalence*

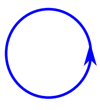
$$\Phi : \overline{Rep_q}GL_n \,/\mathcal{N} \ \xrightarrow{\approx}\ Rep_qGL_n,$$

where $\mathcal{N}$ is the tensor ideal of $\overline{Rep_q}GL_n$ consisting of negligible morphisms.

In particular, GL_0 is the trivial group, so that representations are just vector spaces and we have an embedding

$$\mathrm{Rep}_q\mathrm{GL}_0 \ \hookrightarrow\ \textsc{Vect}.$$

Moreover, if $n = 0$, then $\delta = 0$ and the dimension relation

 $= 0$

in $\overline{\mathrm{Rep}_q}\mathrm{GL}_0$ implies that the identity is a negligible morphism. Therefore, all nonempty diagrams are zero in $\overline{\mathrm{Rep}_q}\mathrm{GL}_n\,/\mathcal{N}$ and we get a $\mathbb{C}(q)$-linear functor

$$\mathcal{E} : \mathrm{Rep}_q\mathrm{GL}_t \ \to\ \overline{\mathrm{Rep}_q}\mathrm{GL}_0\,/\mathcal{N} \ \hookrightarrow\ \textsc{Vect}$$

annihilating all non-empty diagrams. If S is any oriented surface, this clearly extends to a functor

$$\mathcal{E}^S : \mathrm{SkCat}_{\mathrm{GL}_t}(S) \ \to\ \textsc{Vect}.$$

Proposition 6.10. *The linear map*

$$\varepsilon: \; Sk_{GL_t}(S) \; \to \; \mathbb{C}(q)$$

obtained by restricting $\mathcal{E}^S$ to $Sk_{GL_t}(S)$ provides a counit for the Turaev coproduct (6.1).

Proof. Recall the local relations (2.20)-(2.3). Since ε annihilates non-empty diagrams and maps q^t to 1, the map $(1 \otimes \varepsilon) \circ \Delta_f$ just switch ■ with ■. The same happens when ε is applied to the second factor, so it is a counit for Δ_f. □

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