Soergel bimodules

Aim: understand Soergel's categorification theorem

1. Categorification

1.1. The idea of categorification

Categorification [Crane - Frenkel]: replacing set-theoretic notions by their corresponding category-theoretic analogues

- we get a richer structure and deeper relations hidden in the initial object

- all the initial information keep accessible via an inverse process of "decategorification"

$$\text{set-theoretic} \quad \underset{\text{decategorification}}{\overset{\text{categorification}}{\rightleftarrows}} \quad \text{category-theoretic}$$

Baby examples: (1) $(\mathbb{N}, +, \cdot)$ is categorified by $\mathrm{Vect}_{\mathbb{K}}^{<\infty}$

$$\mathbb{N} \underset{\dim}{\rightleftarrows} \mathrm{Vect}_{\mathbb{K}}^{<\infty} \qquad \begin{aligned} \dim(V \oplus W) &= \dim(V) + \dim(W) \\ \dim(V \otimes W) &= \dim(V) \cdot \dim(W) \end{aligned}$$

(2) Laurent polynomials $\mathbb{Z}[v, v^{-1}]$ are categorified by (bounded) complexes of graded spaces:

$$V^\bullet : \dots \xrightarrow{d} V^i \xrightarrow{d} V^{i+1} \longrightarrow \dots \rightsquigarrow \chi(V^\bullet) = \sum_{i \in \mathbb{Z}} (-1)^i \, g\dim V^i$$

where $\operatorname{gdim}\left(\bigoplus_{n\in\mathbb{Z}} V_n\right) = \sum_{n\in\mathbb{Z}} q^n \dim V$

(3) The homology of a CW-complex X obtained by gluing k_i cells of dimension $i \leq n$ is a categorification of its Euler characteristic:

$$H^\bullet(X) \rightsquigarrow \chi(X) = \sum_{i=0}^{\infty} \operatorname{rk} H_i(X)$$

⊛ To categorify an algebraic structure we need:

- a suitable category
- a way to decategorify the information in a category $\boxed{\text{Grothendieck group}}$

1.2. Grothendieck group

$\mathcal{A}$ essentially small additive category

Def $K^{\oplus}(\mathcal{A})$ is the free abelian group generated by isomorphism classes of objects of $\mathcal{A}$ modulo the relations

$$[X \oplus Y] = [X] + [Y], \quad \forall X, Y \in \operatorname{Ob}(\mathcal{A})$$

- Additive functors $F: \mathcal{A} \to \mathcal{B}$ induce group morphisms

$$[F]: K^{\oplus}(\mathcal{A}) \longrightarrow K^{\oplus}(\mathcal{B})$$
$$[X] \longmapsto [F(X)]$$

- If $\mathcal{A}$ is monoidal, $K^{\oplus}(\mathcal{A})$ is a ring with multiplication given on generators by

$$[X] \cdot [Y] := [X \otimes Y]$$

and extending by additivity.

Examples (1) $K^{\oplus}(\mathrm{Vect}_{\mathbb{K}}^{<\infty}) \cong \mathbb{Z}$, via $[X] \longmapsto \dim X$.

(2) $K^{\oplus}(\mathrm{Vect}_{\mathbb{K}}) = \{0\}$. Indeed, for $V \in \mathrm{Vect}_{\mathbb{K}}$, we have

$$[V] + \left[\bigoplus_{i=1}^{\infty} V\right] = \left[V \oplus \bigoplus_{i=1}^{\infty} V\right] = \left[\bigoplus_{i=1}^{\infty} V\right] \Rightarrow [V] = 0$$

(3) R commutative ring, $R\text{-fmod}_{\mathbb{Z}}$ category of finitely generated graded R-modules. Then, $K^{\oplus}(R\text{-fmod}) \cong \mathbb{Z}[v, v^{-1}]$ via

grading shift

$$[R] \longmapsto 1, \quad [R(d)] \longmapsto v^d$$

$$\left[M = \bigoplus_{i=1}^{m} R\, m_i\right] \longmapsto \sum_{i=1}^{m} v^{\deg m_i}$$

homogeneous basis

* If the category $\mathcal{A}$ "behaves well" wrt decompositions, $K^{\oplus}(\mathcal{A})$ has a nice basis

Def (1) An object X is indecomposable if

$$X \cong \bigoplus_{i \in I} X_i \Rightarrow \exists!\, i \in I \text{ st. } X \cong X_i \text{ and } X_j \cong 0 \text{ for } i \neq j$$

(2) $\mathcal{A}$ has the Krull-Schmidt property if every object has a decomposition into direct sum of indecomposable objects which is unique up to isomorphism and permutation of the factors.

Thm If $\mathcal{A}$ has the KS property, then

$$\{[S] : S \in \mathrm{Ob}(\mathcal{A}) \text{ indecomposable}\}$$

is a basis of $K^{\oplus}(\mathcal{A})$

Example In R-fmod$_{\mathbb{Z}}$, we have

$$M = \bigoplus_{i=1}^{m} R m_i \cong \bigoplus_{i=1}^{m} R(-\deg m_i)$$

Hence, $\{[R(d)]\}_{d\in\mathbb{Z}}$ is a basis of $K^{\oplus}(R\text{-fmod})$, which is identified with $\{v^d\}_{d\in\mathbb{Z}}$ via $K^{\oplus}(A) \cong \mathbb{Z}[v,v^{-1}]$.

1.3. Towards a categorification of Hecke algebras

Reminder: H_{n+1} is the $\mathbb{Z}[v,v^{-1}]$-algebra given by generators $T_1,\dots,T_n$ and relations

$$T_i^2 = (v^{-2}-1)T_i + v^{-2},$$
$$T_iT_j = T_jT_i \quad \text{for } |i-j|\geq 2,$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}.$$

We set $T_w := T_{i_1,\dots,i_m}$ for a reduced expression $s_{i_1}\cdots s_{i_m}$ of $w \in S_{n+1}$, which is well-defined by Matsumoto's and the braid relation. This gives rise to the standard basis $\{T_w\}_{w\in S_{n+1}}$ of H_{n+1}.

Remark We have $T_i\cdot(v^2T_i + v^2 - 1) = 1$ so all basis elements are invertible.

Def. The ring homomorphism

$$d: H_{n+1} \longrightarrow H_{n+1}, \quad v \longmapsto v^{-1}, \quad T_w \longmapsto (T_{w^{-1}})^{-1}$$

is called the Kazhdan-Lusztig involution.

Note that

$$d(vT_i + v) = v^{-1}T_i^{-1} + v^{-1} = vT_i + v - v + v^{-1} = \overbrace{vT_i + v^{-1}}^{=:\,b_i}$$

Thm For every $w \in S_n$, there exists a unique self-dual element $b_w \in H_{n+1}$ st.

$$b_w = v^{\ell(w)} \cdot T_w + \sum_{w' < w} h_{w'w} T_{w'}$$

The set $\{b_w : w \in S_n\}$ is a $\mathbb{Z}[v, v^{-1}]$-basis of H_{n+1} called the Kazhdan-Lusztig basis.

Using this basis, the relations in the definition of the Hecke algebra transform into

$$b_i^2 = (v + v^{-1})\, b_i,$$

$$b_i b_j = b_j b_i, \quad \text{for } |i-j| \geq 2,$$

$$b_i b_{i+1} b_i + b_{i+1} = b_{i+1} b_i b_{i+1} + b_i$$

This presentation is known as the Kazhdan-Lusztig presentation.

Let $\varepsilon : H_{n+1} \to \mathbb{Z}[v, v^{-1}]$ be the $\mathbb{Z}[v, v^{-1}]$-linear map given by

$$\varepsilon(T_e) = 1; \quad \varepsilon(T_w) = 0, \quad w \neq 0$$

Def The standard form of H_{n+1} is the pairing of $\mathbb{Z}$-modules

$$(\,,\,) : H_{n+1} \times H_{n+1} \longrightarrow \mathbb{Z}[v, v^{-1}], \quad (x, y) = \varepsilon(d(x) y)$$

Proposition It is the unique paring satisfying:

(1) semi-linearity: $(ax, y) = \bar{a}(x, y)$ and $(x, ay) = a(x, y)$ for $a \in \mathbb{Z}[v, v^{-1}]$;

(2) b_i is self-adjoint i.e. $(x b_i, y) = (x, y b_i)$ and $(b_i x, y) = (x, b_i y)$;

(3) for any increasing sequence $i_1 < \cdots < i_d$, $(1, b_{\underline{i}}) = v^d$, where $\underline{i}$ is the corresponding element of S_{n+1}.

Motivation: we are going to construct a category $\mathcal{SC}$ st:

→ $\mathcal{SC}$ has the KS property;

→ indecomposables induce the KL basis;

→ the standard form describes the graded rank of hom-sets.

2. The category of Soergel bimodules

2.1. Gradings and shifts

Set $R = \mathbb{K}[x_1, \dots, x_{n+1}]$, $\deg(x_i) = 2$. $(\mathrm{char}\,\mathbb{K} \neq 2)$.

We work in the category $R\text{-mod}_{\mathbb{Z}}\text{-}R$ of graded R-bimodules that are finitely generated as both left and right modules

→ Objects: $M = \underbrace{\bigoplus_{i \in \mathbb{Z}} M^i}_{\text{decomposition as a group}}$ st. $R^i M^j \subseteq M^{i+j}$ and $M^i R^j \subseteq M^{i+j}$

→ Morphisms: $f : M \to N$ st. $f(M^i) \subseteq N^i$

- Grading shift: automorphism

$$(1) : R\text{-mod}_{\mathbb{Z}}\text{-}R \longrightarrow R\text{-mod}_{\mathbb{Z}}\text{-}R$$

$$M = \bigoplus_{j \in \mathbb{Z}} M^j \longmapsto M(1) = \bigoplus_{j \in \mathbb{Z}} M^{j+1}$$

$$\begin{array}{ccc} M & & M(1) \\ \vdots & & \vdots \\ M^1 & \text{—— } \deg 1 \text{ ——} & M^2 \\ M^0 & \text{—— } \deg 0 \text{ ——} & M^1 \\ M^{-1} & \text{—— } \deg -1 \text{ ——} & M^0 \end{array}$$

↳ applying it $n \in \mathbb{Z}$ times: $M(n)^j = M^{n+j}$

- Category with shift $\mathcal{C}$ ⇝ $\underbrace{\text{graded category}}_{\text{enriched over graded } \mathbb{Z}\text{-modules}}$ $\mathcal{C}^{gr}$

st. $\mathrm{Hom}^i(Y,Z) \circ \mathrm{Hom}^j(X,Y) \subseteq \mathrm{Hom}^{i+j}(X,Z)$

→ Objects: the same as $\mathcal{C}$

→ Morphisms: $\mathrm{Hom}_{\mathcal{C}^{gr}}(X, Y) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X, Y(n))$

⇝ Composition $X \xrightarrow{f} Y(n)$, $Y \xrightarrow{g} Z(k)$

$$\Rightarrow X \xrightarrow{f} Y(n) \xrightarrow{(n)(g)} Z(k+n)$$

⇝ Identity: $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$

- Conversely,

Graded category $\mathscr{C}$ $\rightsquigarrow$ category with shift $\mathscr{C}^{sh}$

→ Objects: pairs $(X, n) =: X(n)$, $X \in Ob(\mathscr{C})$, $n \in \mathbb{Z}$.

→ Morphisms: $Hom_{\mathscr{C}^{sh}}(X(n), Y(m)) := Hom_{\mathscr{C}}^{m-n}(X, Y)$

* Grothendieck group of a category with shift:

$\mathscr{C}$ additive category with shift functor $\rightsquigarrow$ Natural action $\mathbb{Z}[v, v^{-1}] \curvearrowright K^{\oplus}(\mathscr{C})$

$$v \cdot [X] = [X(1)]$$

$$v^{-1}[X] = [X(-1)]$$

Remark (n) is additive, so if $X = \bigoplus_{i \in I} X_i$, then $X(n) = \bigoplus_{i \in I} X_i(n)$. In particular, X indecomposable $\Rightarrow$ $X(n)$ indecomposable

Lemma If the degrees of $End_{\mathscr{C}}^{\bullet}(X) := \bigoplus_{n \in \mathbb{Z}} Hom_{\mathscr{C}}(X, X(n))$ are bounded below, then $X \cong X(n)$ implies $n = 0$ or $X \cong 0$.

Proof Suppose $n \neq 0$ and $X \cong X(n)$. Then,

$$X(-n) \cong (X(n))(-n) = X(0) = X,$$

$$X \cong X(\pm n) \cong (X(\pm n))(\pm n) = X(\pm 2n)$$

and by induction, $X \cong X(kn)$, $\forall k \in \mathbb{Z}$. Choose k st. kn is less than the minimal degree of $End_{\mathscr{C}}^{\bullet}(\mathscr{C})$, then the iso. $X \cong X(kn)$ lies in $Hom_{\mathscr{C}}(X, X(kn)) = \{0\}$. Hence $X = 0$. $\boxtimes$

Corollary If $\mathcal{C}$ has the KS property and the degrees of $\mathrm{End}^{\bullet}_{\mathcal{C}}(X)$ are bounded below for each $X \in \mathcal{C}$, then $K^{\oplus}(\mathcal{C})$ is a free $\mathbb{Z}[v, v^{-1}]$-module with basis the orbits of shifts of isomorphism classes of indecomposable objects of $\mathcal{C}$.

Proof By the remark, X indecomposable $\Rightarrow$ $X(n)$ indecomposable and, by the lemma, $[X] \neq [X(n)]$. $\boxtimes$

2.2. Invariant polynomials and Demazure operators

$S_{n+1} \curvearrowright R = \mathbb{R}[x_1, \dots, x_{n+1}]$ by permutation of the variables

$R^i \subseteq R$ set of s_i-invariants $\rightsquigarrow$ by restriction, each R-bimodule is also an R^i-bimodule

$\hookrightarrow$ Notation: $\cdot\ M \otimes N := M \otimes_R N$ (also $MN := M \otimes_R N$)

$\cdot\ M \otimes_i N := M \otimes_{R^i} N$

Def For each $i = 1, \dots, n$, we have a graded map

$$\partial_i : R \longrightarrow R^i(-2)$$
$$f \longmapsto \frac{f - s_i(f)}{x_i - x_{i+1}}$$

Equivalently, we can take $\tilde{P}_i(f) = f + x_{i+1} \partial_i(f)$

$\rightsquigarrow$ we will use this later

Lemma $R \cong R^i \oplus R^i(-2)$ as R^i-bimodules

Proof For $f \in R$, set $P_i(f) = f - x_i \partial_i(f)$. Then,

$$P_i(f) = \frac{\cancel{x_i f} - x_{i+1} f - \cancel{x_i f} + x_i s_i(f)}{x_i - x_{i+1}} = s_i(f) - x_{i+1} \cdot \partial_i(f) = s_i(P_i(f))$$

$\pm x_{i+1} \cdot s_i(f)$

so $P_i(f) \in R^i$ and $f = P_i(f) + x_i \partial_i(f)$. Hence,

$$f \longmapsto (P_i(f), \partial_i(f))$$

is an isomorphism of inverse $(g, h) \mapsto g + x_i h$. $\boxtimes$

<u>Remark</u> A basis of homogeneous elements of R is $\{1, x_i\}$

2.3. Bott-Samelson bimodules

For $i = 1, \dots, n$, set $B_i := R \otimes_i R(-1)$

<u>Def</u> The Bott-Samelson bimodule corresponding to a (not necessarily reduced) word $\underline{w} = s_{i_1} \cdots s_{i_k}$ is defined to be

$$BS(\underline{w}) = B_{i_1} \cdots B_{i_k}$$

<u>Remark</u> (1) If M, N are graded bimodules, then $M \otimes N$ is also graded, with $(M \otimes N)^i$ given by the image of

$$\bigoplus_{j+k=i} M^i \otimes_{\mathbb{Z}} N^j$$

in $M \otimes N$.

(2) $M(n) \otimes N = M \otimes N(n) = (M \otimes N)(n)$ (canonical identifications)
In particular,

$$BS(\underline{w}) = R \otimes_{i_1} \cdots \otimes_{i_k} R\,(\ell(\underline{w}))$$

<u>Example</u> $\boxed{n=2}$ $B(s_1 s_2) = R \otimes_1 R \otimes_2 R(2)$ and $xz^2 \underset{xy}{|} xyz \underset{yz}{|} x^3$ is an element of degree 16.

Lemma BS bimodules are graded free of finite rank as left (resp. right) R-modules.

only preserves the left R-mod structure because P_i and ∂_i are not R-linear

Proof $B_i = R \otimes_i R(1) \cong R \otimes_i (R^i \oplus R^i(-2))(1) \simeq R(1) \oplus R(-1)$

The tensor product of bimodules which are free of finite rank as left R-modules is also fofr as left R-module $\boxtimes$

Explicitly,

$$f \otimes_i g = f \otimes_i (P_i(g) + x_i \partial_i(g)) = \underbrace{P_i(g) f \otimes_i 1}_{\deg(g)+\deg(f)-1} + \underbrace{\partial_i(g) f \otimes_i x_i}_{\deg(g)+\deg(f)+2-1}$$

$$f \otimes_i g = (P_i(f) + x_i \partial_i(f)) \otimes_i g = 1 \otimes_i P_i(f) g + x_i \otimes_i \partial_i(f) g$$

- Polynomial forcing relations $\begin{cases} 1 \otimes_i g = P_i(g) \otimes_i 1 + \partial_i(g) \otimes_i x_i \\ f \otimes_i 1 = 1 \otimes_i P_i(f) + x_i \otimes_i \partial_i(f) \end{cases}$

↳ when we force a non-invariant polynomial across a tensor product we left a 1 and a x_i (or x_{i+1}) behind

Lemma $B_i B_i \cong B_i(1) \oplus B_i(-1)$ ← these are not BS bimodules !!

Proof $B_i B_i \simeq R \otimes_i R \otimes_i R(2)$ by construction. Using the polynomial forcing relations, we can write $f \otimes g \otimes h \in B_i \otimes B_i$ as

$$f \otimes g \otimes h = P_i(g) f \otimes 1 \otimes h + \partial_i(g) f \otimes x_i \otimes h$$ ← $\deg(f \otimes g \otimes h) = D(f) + D(g) + D(g) - 2$

Then, $f \otimes g \otimes h \longmapsto (P_i(g) f \otimes h, \partial_i(g) f \otimes h)$ is an R-bimodule iso. of inverse $(f_1 \otimes h_1, f_2 \otimes h_2) \longmapsto f_1 \otimes 1 \otimes h_1 + f_2 \otimes x_i \otimes h_2$ $\boxtimes$

in B_i $\deg = D(g) + D(f) + D(h) - 1$ $\deg = D(g) - 2 + D(f) + D(h) - 1$

Corollary A spanning set of $B_i B_i$ as R-bimodule is

$$\{1 \otimes 1 \otimes 1,\ 1 \otimes x_i \otimes 1\}$$

Lemma A spanning set of $B_i B_j$, $i \neq j$, is $\{1 \otimes 1 \otimes 1\}$.

Proof Same as above, but now the x_i (or x_{i+1}) left behind can be slided to the right. $\boxtimes$

Corollary (1) $B_i B_j$, $i \neq j$, is indecomposable.
(2) $B_i \not\cong B_j$, $i \neq j$ ($B_i B_i$ is not indecomposable but $B_i B_j$ is)

Similarly, for $f \otimes g \otimes h \otimes K \in B_i B_{i+1} B_i$, we have

$$f \otimes_i g \otimes_{i+1} h \otimes_i K = f \otimes_i g \otimes_{i+1} 1 \otimes_i P_i(h)K + f \otimes_i g \otimes_{i+1} x_i \otimes_i \partial_i(h)K$$

$$= P_i(g) f \otimes_i 1 \otimes_{i+1} 1 \otimes_i P_i(h)K + \partial_i(g) f \otimes_i x_i \otimes_{i+1} 1 \otimes_i P_i(h)K$$

$$+ P_i(x_i g) f \otimes_i 1 \otimes_{i+1} 1 \otimes_i \partial_i(h)K + \partial_i(g x_i) f \otimes_i x_i \otimes_{i+1} 1 \otimes_i \partial_i(h)K$$

so a generating set for $B_i \otimes B_{i+1} \otimes B_i$ is $\{1 \otimes 1 \otimes 1 \otimes 1,\ 1 \otimes x_i \otimes 1 \otimes 1\}$.

Def Category $\mathcal{SC}_1$ of Bott-Samelson bimodules

→ Objects: BS bimodules

→ Morphisms: $\mathrm{Hom}_{\mathcal{SC}_1}(B, B') = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{R\text{-mof}_{\mathbb{Z}}\text{-}R}(B, B'(n))$

morphisms are homogeneous of all degrees

Remarks (1) $\mathcal{SC}_1$ is a graded category.

(2) $\mathcal{SC}_1$ lies in R-mof-R and not in R-mof$_{\mathbb{Z}}$-R

(3) All the lemmas and corollaries above do not hold in $\mathcal{SC}_1$, since we do not have direct sums and grading shifts

↳ we need to enlarge $\mathcal{SC}_1$ to include:

→ grading shifts ⇝ $\boxed{\text{category with shift}}$

→ direct sums ⇝ $\boxed{\text{additive envelope}}$

→ direct summands ⇝ $\boxed{\text{Karoubi envelope}}$

2.4. Additive and Karoubi envelopes

Additive envelope

preadditive category $\mathcal{C}$ ⇝ additive envelope $\mathcal{C}^{\oplus}$

→ Objects : formal finite direct sums $(X_i)_{i \in I}$, $X_i \in \mathrm{Ob}(\mathcal{C})$

→ Morphisms : $\mathrm{Hom}_{\mathcal{C}^{\oplus}}\big((X_i)_i, (Y_j)_j\big) = \underbrace{\bigoplus_{i,j} \mathrm{Hom}_{\mathcal{C}}(X, Y)}_{\text{matrices of morphisms}}$

↳ composition is given by the usual product of matrix

Clearly $\mathcal{C}^{\oplus}$ is additive and $\mathcal{C} \hookrightarrow \mathcal{C}^{\oplus}$

<u>Universal property</u> If $\mathcal{D}$ is an additive category and $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a pre-additive functor, then $\exists!$ additive functor $F^{\oplus} : \mathcal{C}^{\oplus} \longrightarrow \mathcal{D}$ st. the following diagram is commutative

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, hook] & F \\
\mathcal{C}^{\oplus} \arrow[ur, dashed, "F^{\oplus}"'] &
\end{tikzcd}$$

Lemma If $\mathcal{C}$ is a full subcategory of an additive category $\mathcal{D}$, then $\mathcal{C}^{\oplus}$ is equivalent to the full subcategory $\mathcal{C}'$ of $\mathcal{D}$ consisting of finite direct sums of objects of $\mathcal{D}$.

Proof $F : \mathcal{C}^{\oplus} \longrightarrow \mathcal{C}'$, $(X_i)_{i\in I} \longmapsto \bigoplus_i X_i$ is essentially surjective and

$$\mathrm{Hom}_{\mathcal{C}^{\oplus}}((X_i)_i, (Y_j)_j) \overset{\text{by definition}}{=} \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{D}}(X_i, Y_j) \overset{\text{by the up of } \sqcup \text{ and } \sqcap}{=} \mathrm{Hom}_{\mathcal{C}'}(\bigoplus_i X_i, \bigoplus_j Y_j)$$

so F is an equivalence of categories. $\boxtimes$

Remark If $\mathcal{C}$ is monoidal with shift, then so is $\mathcal{C}^{\oplus}$.

Karoubi envelope

Def-lemma An additive category $\mathcal{C}$ is Karoubian if the following equivalent conditions hold:

(i) every idempotent endomorphism has a Kernel;

(ii) every idempotent endomorphism $e : X \to X$ splits, i.e., there is a factorization $X \xrightarrow{\pi} Y \xrightarrow{i} X$ of e st. $\pi \circ i = \mathrm{id}_Y$;

(iii) every idempotent $e : X \to X$ is the projection onto Z of some decomposition $X \cong Y \oplus Z$.

Idea: idempotents are projectors $\rightsquigarrow$ a category is Karoubian if it contains all direct summand $\rightsquigarrow$ we add them to construct the Karoubi envelope.

Def Karoubi envelope $\mathrm{Kar}(\mathcal{C})$ of an additive category $\mathcal{C}$:

→ Objects: (X, e) with $X \in \mathrm{Ob}(\mathcal{C})$ and $e: X \to X$ idempotent

→ Morphisms $(X, e) \to (Y, \tilde{e})$: $X \xrightarrow{f} Y$ st $f \circ e = f = \tilde{e} \circ f$
i.e., f sends the image of e to the image of f and all the other factors to 0

↳ in particular, for $(X, e) \xrightarrow{f} (X, e)$, $f \circ e = f = e \circ f$
so $\mathrm{id}_{(X,e)} = e$.

Lemma $\mathrm{Kar}(\mathcal{C})$ is Karoubian.

think of this as $f: \mathrm{im}\, e \to \mathrm{im}\, e$

Proof Let $(X, e) \xrightarrow{f} (X, e)$ be an idempotent in $\mathrm{Kar}(\mathcal{C})$, i.e., $f^2 = f$ and $f \circ e = f = e \circ f$. Then,

$$(e - f)^2 = e^2 - ef - fe + f^2 = e - f$$

so $(X, e - f) \in \mathrm{Kar}(\mathcal{C})$. Moreover, we have an isomorphism

$$\begin{tikzcd} (X, e) \arrow[r, shift left, "{(e-f,\, f)}"] & (X, e-f) \oplus (X, f) \arrow[l, shift left, "(e-f) \oplus f"] \end{tikzcd}$$

so f splits (it is the projection onto (X, f)). $\boxtimes$

The initial category embeds into $\mathrm{Kar}(\mathcal{C})$ via $X \longmapsto (X, \mathrm{id}_X)$

Universal property If $\mathcal{D}$ is Karoubian additive and $F: \mathcal{C} \to \mathcal{D}$ is additive, then $\exists!$ $\mathrm{Kar}(F): \mathrm{Kar}(\mathcal{C}) \to \mathcal{D}$ st.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D} \\ \mathrm{Kar}(\mathcal{C}) \arrow[ur, dashed, "\mathrm{Kar}(F)"'] & \end{tikzcd}$$

Lemma If $\mathcal{C}$ is a full subcategory of a Karoubian category $\mathcal{D}$, then $\mathrm{Kar}(\mathcal{C})$ is equivalent to the full subcategory $\dot{\mathcal{C}}$ of $\mathcal{D}$ consisting of all direct summands of objects of $\mathcal{C}$.

Proof If $e: X \to X$ is an idempotent, then it is the projection onto a direct factor Z of X. We define $F: \mathrm{Kar}\,\mathcal{C} \to \dot{\mathcal{C}}$ by $(X, e) \mapsto Z$ and the identity on morphisms. Then,

$$\mathrm{Hom}_{\mathrm{Kar}(\mathcal{C})}((X,e),(Y,e'))$$

$$= \{ f \in \mathrm{Hom}_{\mathcal{D}}(X,Y) : f \text{ maps } Z \text{ into } Z' \text{ and the other factors to } 0 \}$$

$$\cong \mathrm{Hom}_{\mathcal{D}}(Z, Z') = \mathrm{Hom}_{\dot{\mathcal{C}}}(Z, Z')$$

$\boxtimes$

Example Given a ring R, $\mathrm{Kar}(R\text{-}\mathrm{fmod}) \simeq R\text{-}\mathrm{proj}$.

finitely generated free

finitely generated projective.

Thm Let $\mathcal{C}$ be a graded category. Under extra assumptions that hold in $R\text{-}\mathrm{mof}_{\mathbb{Z}}\text{-}R$, $\mathrm{Kar}(\mathcal{C}^{sh,\oplus})$ is Krull-Schmidt.

2.5. Soergel bimodules and categorification

Recall that $\mathcal{SC}_1$ is a graded category with

$$\mathrm{Hom}_{\mathcal{SC}_1}(B, B') = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{R\text{-}\mathrm{mof}_{\mathbb{Z}}\text{-}R}(B, B'(n))$$

Def Soergel category $\mathcal{SC}$: full-subcategory of $R\text{-}\mathrm{mof}_{\mathbb{Z}}\text{-}R$ obtained as:

$$\mathcal{SC}_1 \xrightarrow[\text{Additive envelope}]{\text{Grading}} \mathcal{SC}_2 \xrightarrow{\text{Karoubi envelope}} \mathcal{SC}$$

Remarks (1) By construction, objects are direct summands of finite direct sums of grading shifts of BS bimodules; morphisms are homogeneous of degree 0.

(also right)

(2) Soergel bimodules are graded free of finite rank as left R-mod.

(3) $\mathcal{SC}$ is an additive, Karoubian, monoidal category with shift so $K^{\oplus}(\mathcal{SC})$ is a $\mathbb{Z}[v,v^{-1}]$-algebra free as $\mathbb{Z}[v,v^{-1}]$-module

- $\mathcal{SC}_2$ is generated as an additive monoidal category by grading shifts of the B_i's, $i=1,\dots,n$, which satisfy:

$$B_i \otimes B_i \cong B_i(1) \oplus B_i(-1),$$

$$B_i \otimes B_j \cong B_j \otimes B_i, \quad |i-j| \geq 2,$$

$$(B_i \otimes B_{i+1} \otimes B_i) \oplus B_{i+1} \cong (B_{i+1} \otimes B_i \otimes B_{i+1}) \oplus B_i$$

- These relations categorifies the KL presentation of H_{n+1} so

$$\begin{aligned} H_n &\longrightarrow K^{\oplus}(\mathcal{SC}_2) \\ b_i &\longmapsto [B_i] \end{aligned}$$

is a well defined surjective homomorphism of $\mathbb{Z}[v,v^{-1}]$-algebras.

- Taking the Karoubi envelope may add new indecomposables and enlarge the Grothendieck group

Thm [Soergel, 2006] Given any reduced expression $\underline{w}$ of $w \in S_{n+1}$, the Bott-Samelson bimodule $BS(\underline{w})$ contains up to iso. a unique indecomposable summand B_w which does not occur in

$BS(\underline{w}')$, for any expression $\underline{w}'$ of $w' \in S_{n+1}$ with $\ell(\underline{w}') < \ell(\underline{w})$. In addition, B_w does (up to iso.) not depend on the reduced expression $\underline{w}$ and the map

$$\begin{aligned} S_{n+1} \times \mathbb{Z} &\longrightarrow \left\{ \begin{matrix} \text{indecomposable} \\ \text{Soergel bimodules} \end{matrix} \right\} /_{\cong} \\ (w, n) &\longmapsto B_w(n) \end{aligned}$$

is a bijection.

Using this classification, the basis elements of $K^{\oplus}(SC)$ can be written in symbols of $K^{\oplus}(SC_2)$ by a triangular matrix. Using this, Soergel proved:

<u>Thm</u>. [Soergel's categorification thm, 2006]

$$\begin{aligned} \mathcal{E}: H_{n+1} &\longrightarrow K^{\oplus}(SC) \\ b_i &\longmapsto [B_i] \end{aligned}$$

is an isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras.

The inverse of $\mathcal{E}$ is called the character map. It was not explicit in Soergel's proof.

<u>Thm</u> [Soergel's conjecture, Elias-Williamson, 2014]

$$ch([B_w]) = b_w$$

finitely generated graded free R-mod

grK

Finally, note that $K^{\oplus}(R\text{-fmod}_{\mathbb{Z}}) \cong \mathbb{Z}$. Since Sorergel bimodules are fggf as left R-modules, so are the Hom-spaces.

Particularly,

since finitely generated graded free, only a finite number of hom. comp. B^i are $\neq 0$

$$\mathrm{Hom}^\bullet_{SC}(B, B') := \bigoplus_{n \in \mathbb{N}} \mathrm{Hom}_{SC}(B, B'(n))$$

lies in $R\text{-fmod}_{\mathbb{Z}}$ so the functor

$$\mathrm{Hom}^\bullet_{SC}(-,-) : SC^{op} \times SC \longrightarrow R\text{-fmod}_{\mathbb{Z}}$$

categorifies a semilinear form

$$\begin{aligned} H_{n+1} \times H_{n+1} &\longrightarrow \mathbb{Z}[v, v^{-1}] \\ ([B], [B']) &\longmapsto \mathrm{grK}\left(\mathrm{Hom}^\bullet_{SC}(B, B')\right) \end{aligned}$$

This semilinear form verifies all the conditions characterizing (uniquely) the standard form $(-,-)$ of H_{n+1} so:

Thm (Soergel, 2006)

$$\mathrm{grK}\left(\mathrm{Hom}^\bullet_{SC}(B, B')\right) = (ch(B), ch(B'))$$