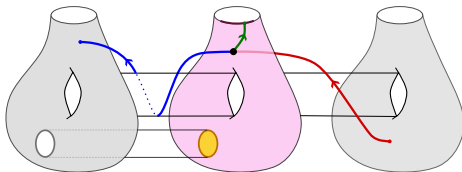


Defects in skein theory and HOMFLY parabolic restriction



Juan Ramón Gómez García

under the supervision of
Adrien Brochier and Emmanuel Wagner

IMJ-PRG, Université Paris Cité

September 25, 2026

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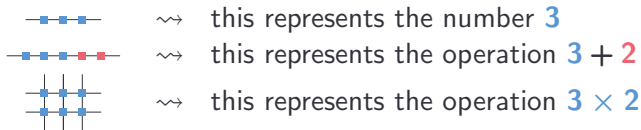
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 $\rightsquigarrow$ this represents the number **3**

 $\rightsquigarrow$ this represents the operation **3 + 2**

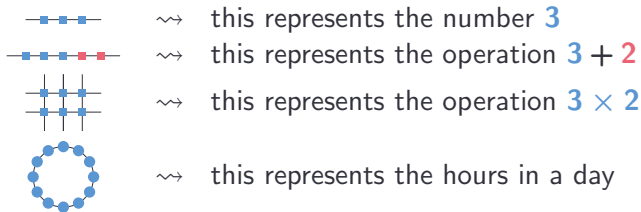
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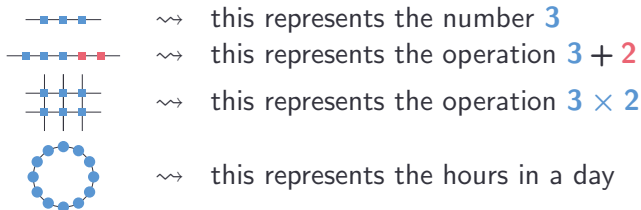
-  $\rightsquigarrow$ this represents the number 3
-  $\rightsquigarrow$ this represents the operation $3 + 2$
-  $\rightsquigarrow$ this represents the operation 3×2
-  $\rightsquigarrow$ this represents the hours in a day

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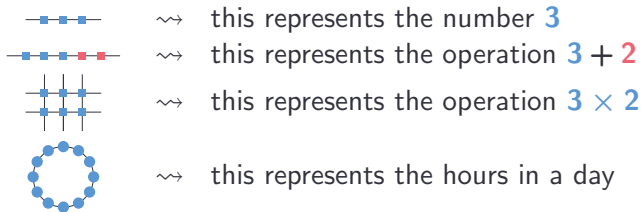
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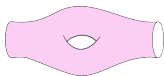
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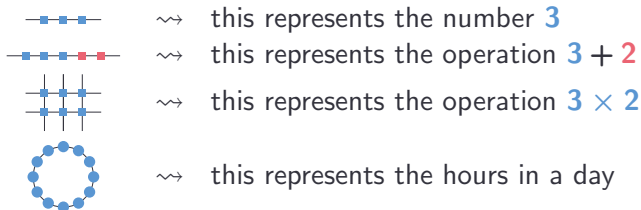
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Topology



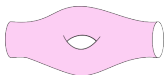
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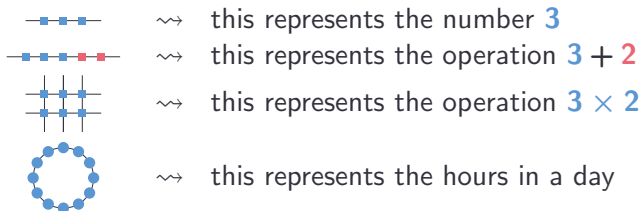


Algebra

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$p(x) = x^2 + 1$$

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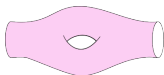


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Quantum Topology

Topology



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Plan of talk:

1. Mathematical context
2. Motivation: HOMFLY formalism and the Turaev coproduct
3. Line defects in skein theory
4. HOMFLY parabolic restriction
5. The HOMFLY skein bialgebra

Mathematical context

We use the language of **Topological Quantum Field Theories**.

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Definition (Atiyah, Segal '88)

A n -dimensional TQFT is a symmetric monoidal functor

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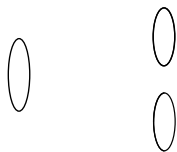
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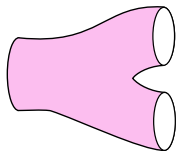
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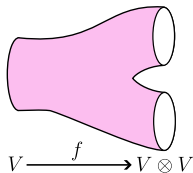
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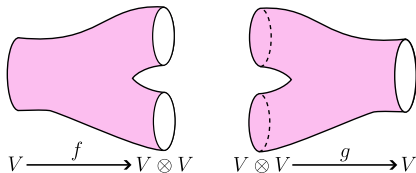
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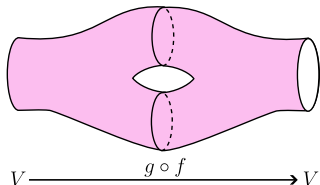
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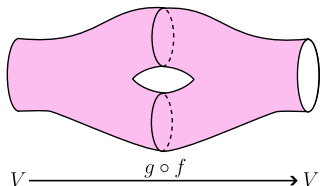
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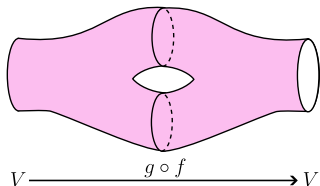
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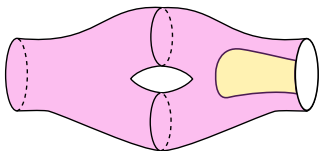
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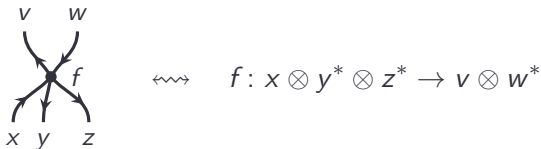
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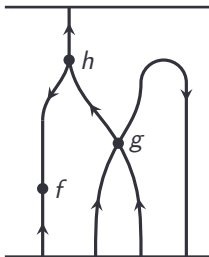
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Graphical calculus based on decorated graphs:

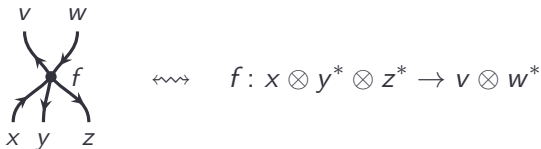
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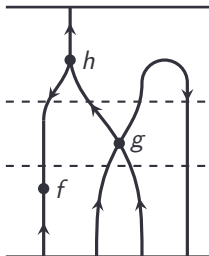
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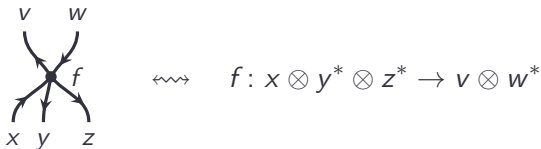
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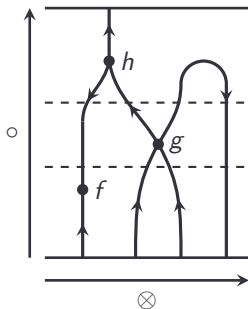
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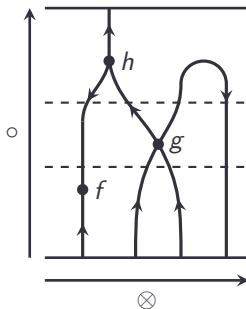


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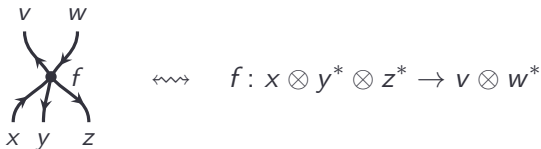
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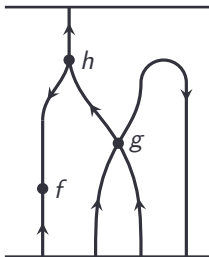
$$\begin{aligned} &\mapsto h \otimes \text{ev} \\ &\mapsto \text{id} \otimes g \otimes \text{id} \\ &\mapsto f \otimes \text{id} \otimes \text{id} \otimes \text{id} \end{aligned}$$

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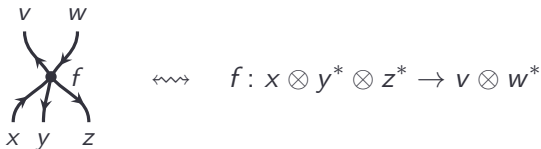


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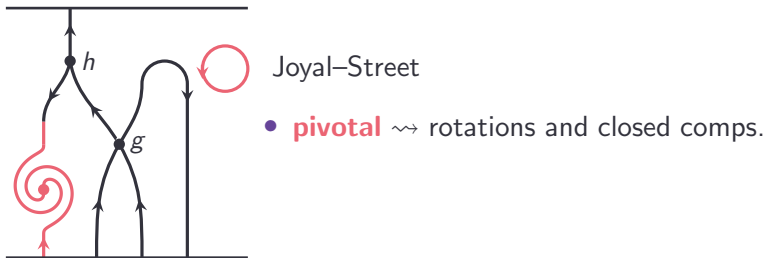
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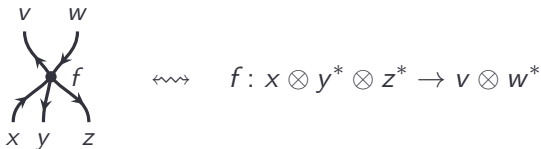
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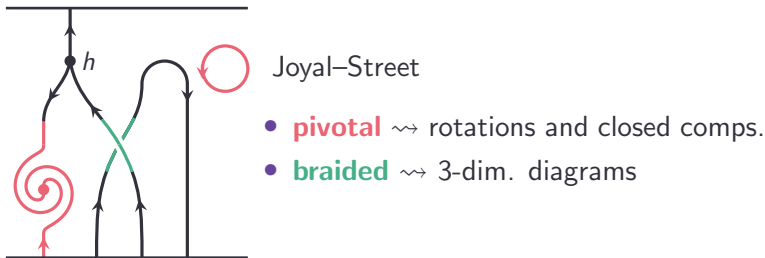
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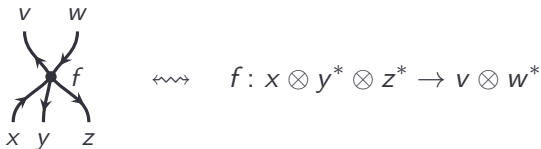
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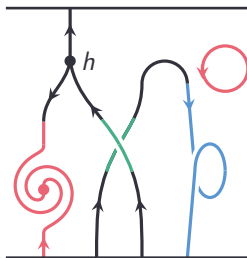
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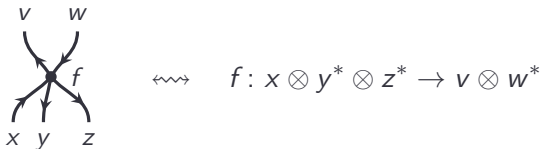
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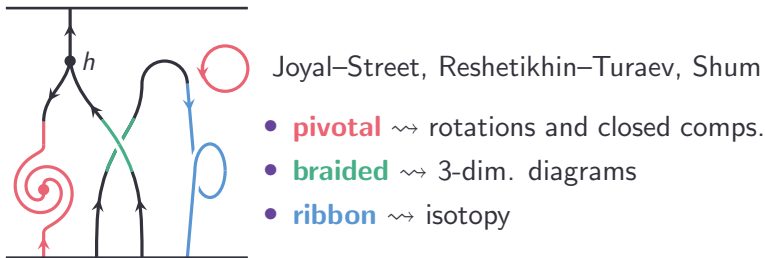
Joyal–Street, Reshetikhin–Turaev, Shum

- **pivotal** $\rightsquigarrow$ rotations and closed comps.
- **braided** $\rightsquigarrow$ 3-dim. diagrams
- **ribbon** $\rightsquigarrow$ isotopy

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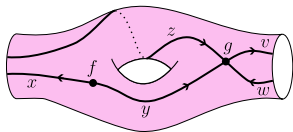
In any case, we obtain an evaluation functor $\text{ev}_{\mathcal{C}}: \mathbf{Free}(\mathcal{C}) \rightarrow \mathcal{C}$.

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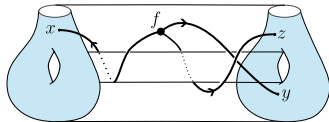
S framed/oriented surf., $\mathcal{C}$ rigid/pivotal



(Levin–Wen, Kirillov, Runkel)

Skeins

M oriented 3-manifold, $\mathcal{C}$ ribbon

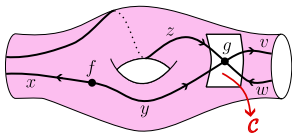


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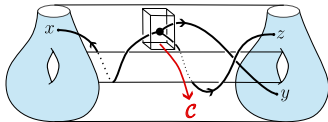
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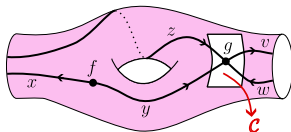


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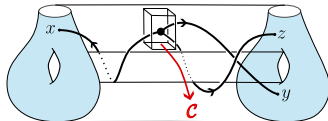
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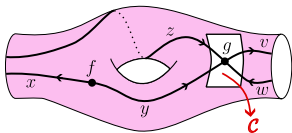
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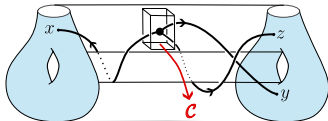
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Theorem (Walker)

String nets and skeins are compatible with cutting/gluing, so they yield 2- and 3-dim. TQFTs, respectively.

Motivation

HOMFLY formalism and the Turaev coproduct

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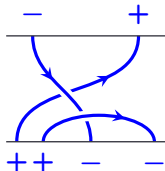
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HOMFLY
relations

[Deligne, Brundan] Replace N with a **variable** $t \rightsquigarrow \text{Rep}_q \text{GL}_t$



- objects: confs. of oriented points;
- morphisms: iso. classes of framed oriented tangles, modulo HOMFLY.

The prototypical example of **ribbon categories** are categories of representations of quantum groups.

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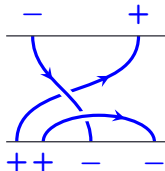
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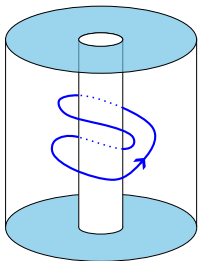


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Universal version of $\text{Rep}_q \text{GL}_N \rightsquigarrow \text{sp}_N: \text{Rep}_q \text{GL}_t \rightarrow \text{Rep}_q \text{GL}_N$

S oriented surface $\rightsquigarrow$ HOMFLY skein algebra $\mathbf{SkAlg}_{\mathrm{GL}_t}(S)$

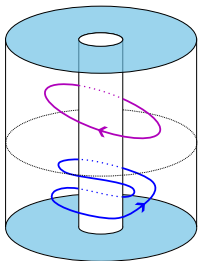
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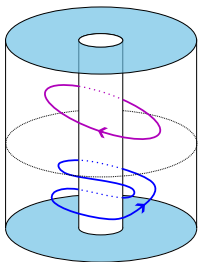


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Product: stacking cylinders and rescaling

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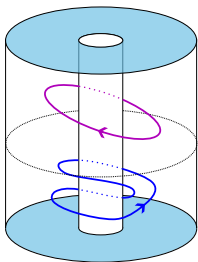
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Product: stacking cylinders and rescaling

Specialise at $t = 0$ $\rightsquigarrow$ $\mathcal{H}_q(S)$

Theorem (Turaev '91)

For any oriented surface S , there exists a coproduct

$$\Delta: \mathcal{H}_q(S) \rightarrow \mathcal{H}_q(S) \otimes \mathcal{H}_q(S)$$

turning $\mathcal{H}_q(S)$ into a bialgebra.

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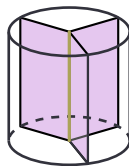
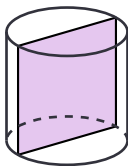
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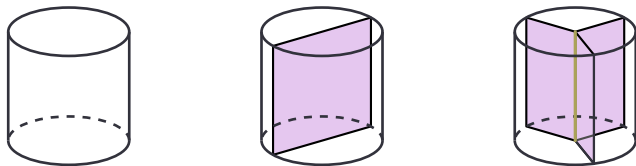
✓ **Approach:** defect skein theory and parabolic restriction

Line defects in skein theory

We extend skein theory to 3-manifolds modelled locally by

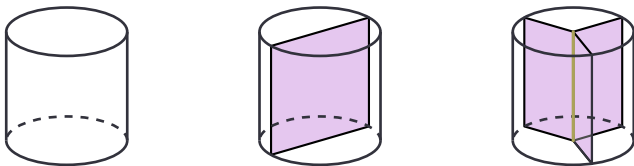


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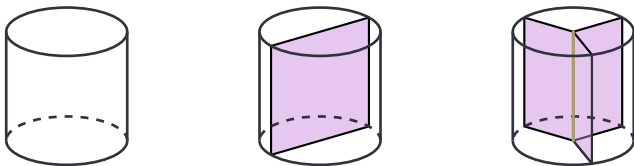
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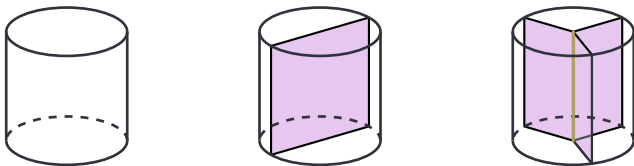
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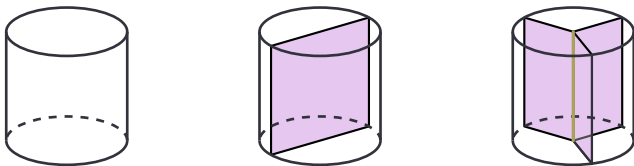
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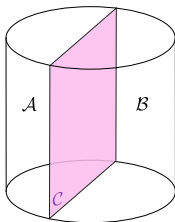
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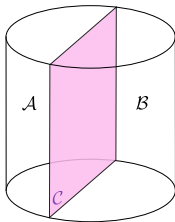
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- ...

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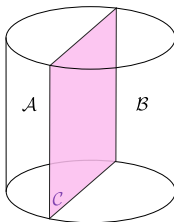
Codimension-1: a **central algebra** $\mathcal{A} \xrightarrow{\mathcal{C}} \mathcal{B}$ is a monoidal category $\mathcal{C}$ together with

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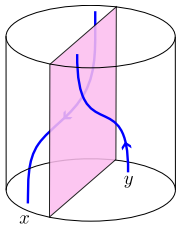
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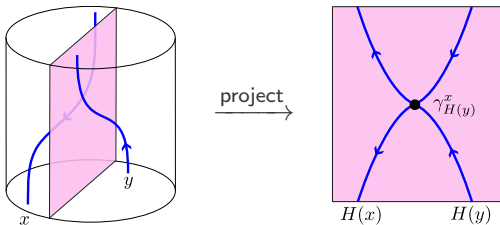
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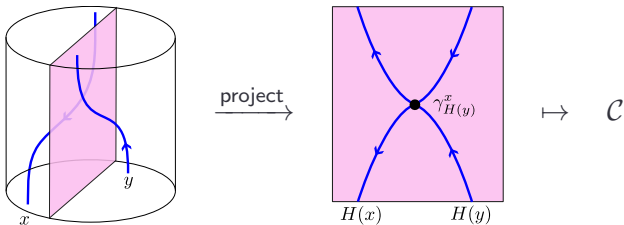
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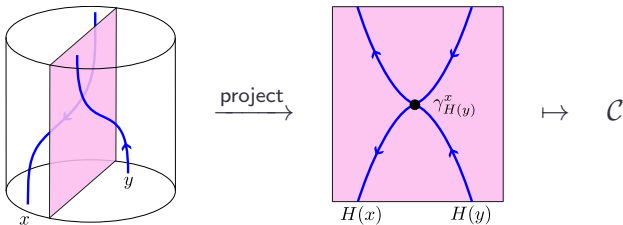
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Theorem (Brown–Jordan '25, GG '26)

This is invariant under stratified isotopies and yields a well-defined evaluation functor for stratified graphs locally around a surface defect.

Codimension-2:

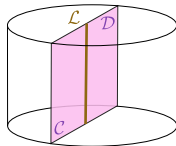
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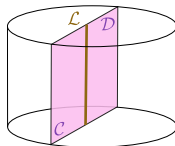
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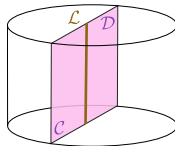
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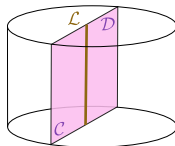
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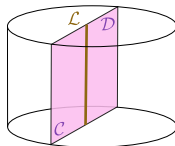
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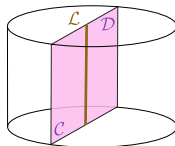
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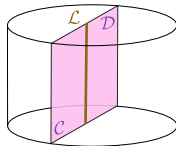
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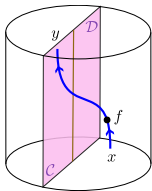
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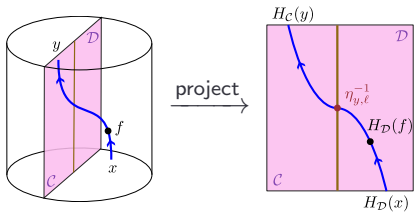
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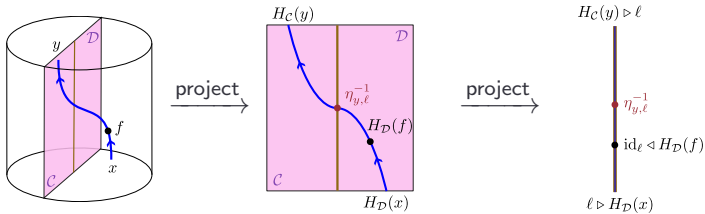
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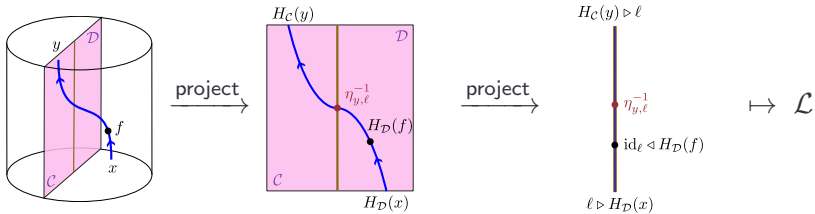
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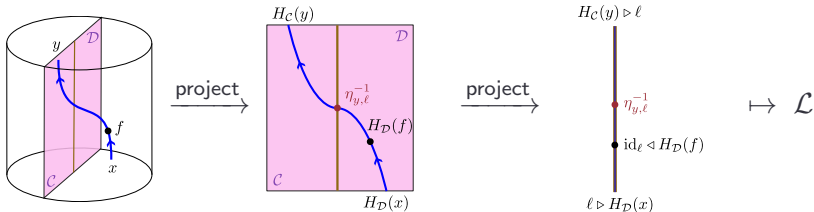
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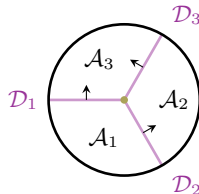


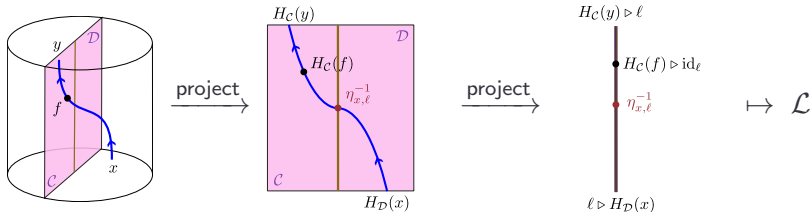
Theorem (Brown–GG–Jordan–Vancraeynest '26)

There exists a well-defined full evaluation functor

$$\text{ev}_{\mathcal{L}}: \text{Rib}^{\text{def}}(\mathcal{L}) \rightarrow \mathcal{L}$$

for graphs in a stratified cylinder.



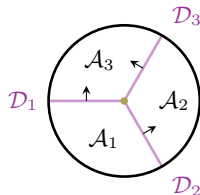


Theorem (Brown–GG–Jordan–Vancraeynest '26)

There exists a well-defined full evaluation functor

$$\text{ev}_{\mathcal{L}}: \text{Rib}^{\text{def}}(\mathcal{L}) \rightarrow \mathcal{L}$$

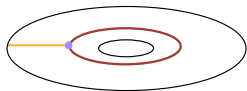
for graphs in a stratified cylinder.



Proof: invariance under the choice of the stratified isotopy representative.

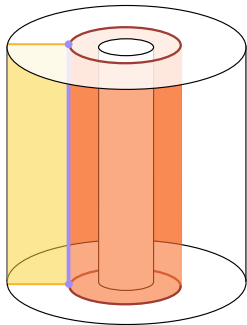
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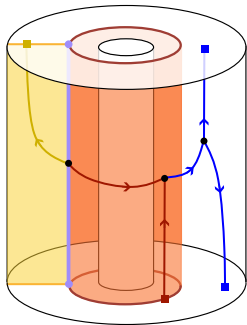
S decorated surface

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$$\mathbb{S} \text{ decorated surface} \rightsquigarrow \text{SkCat}_{\mathbb{S}}(\mathbb{S})$$

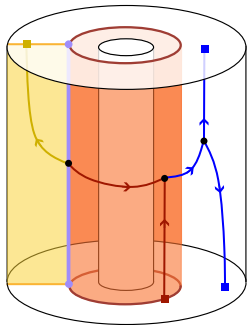
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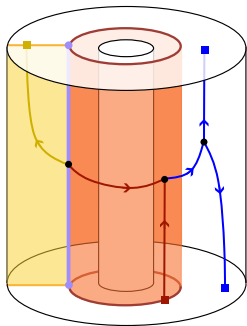


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Algebraic input: parabolic restriction

HOMFLY parabolic restriction

$$\text{GL}_{m+n}, \quad \text{P} = \left(\frac{\text{GL}_m}{0} \middle| \frac{*}{\text{GL}_n} \right), \quad \text{L} = \left(\frac{\text{GL}_m}{0} \middle| \frac{0}{\text{GL}_n} \right)$$

$$\mathrm{GL}_{m+n}, \quad \mathrm{P} = \left(\begin{array}{c|c} \mathrm{GL}_m & * \\ \hline 0 & \mathrm{GL}_n \end{array} \right), \quad \mathrm{L} = \left(\begin{array}{c|c} \mathrm{GL}_m & 0 \\ \hline 0 & \mathrm{GL}_n \end{array} \right)$$

Restriction functors:

$$\begin{array}{ccc}
 & \mathrm{Rep}_q \mathrm{P} & \\
 \iota^* \nearrow & & \nwarrow j^* \\
 \mathrm{Rep}_q \mathrm{GL}_{m+n} & \xrightarrow{\mathrm{res}_{m,n}} & \mathrm{Rep}_q \mathrm{L}
 \end{array}
 \quad \begin{array}{c} \leftarrow \pi^* \end{array}$$

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 \end{array}
 \quad \text{with a curved arrow } \pi^* \text{ from } \mathrm{Rep}_q \mathrm{L} \text{ to } \mathrm{Rep}_q \mathrm{P}$$

HOMFLY category $\mathrm{Rep}_q \mathrm{GL}_t \rightsquigarrow$ interpolating version of $\mathrm{Rep}_q \mathrm{GL}_N$

$$\mathrm{GL}_{m+n}, \quad P = \left(\begin{array}{c|c} \mathrm{GL}_m & * \\ \hline 0 & \mathrm{GL}_n \end{array} \right), \quad L = \left(\begin{array}{c|c} \mathrm{GL}_m & 0 \\ \hline 0 & \mathrm{GL}_n \end{array} \right)$$

Restriction functors:

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Interpolating version of parabolic restriction

There is an equivalence of monoidal categories

There is an equivalence of monoidal categories

$$\mathrm{Rep}_q L \simeq \mathrm{Rep}_q \mathrm{GL}_m \boxtimes \mathrm{Rep}_q \mathrm{GL}_n$$

There is an equivalence of monoidal categories $\rightsquigarrow$ interpolate

$$\mathrm{Rep}_q \mathbf{L}_t \simeq \mathrm{Rep}_q \mathbf{GL}_t \boxtimes \mathrm{Rep}_q \mathbf{GL}_t$$

Diagrammatic category:

There is an equivalence of monoidal categories $\rightsquigarrow$ interpolate

$$\mathrm{Rep}_q L_t \simeq \mathrm{Rep}_q \mathrm{GL}_t \boxtimes \mathrm{Rep}_q \mathrm{GL}_t$$

Diagrammatic category:

$$\text{yellow square} = \text{green square} \oplus \text{red square}$$

There is an equivalence of monoidal categories $\rightsquigarrow$ interpolate

$$\mathrm{Rep}_q L_{\mathbf{t}} \simeq \mathrm{Rep}_q \mathrm{GL}_{\mathbf{t}} \boxtimes \mathrm{Rep}_q \mathrm{GL}_{\mathbf{t}}$$

Diagrammatic category:

$$\blacksquare = \blacksquare \oplus \blacksquare$$

$$\begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array} = \begin{array}{c} \uparrow \\ \bullet \\ \uparrow \\ \bullet \\ \uparrow \end{array} + \begin{array}{c} \uparrow \\ \bullet \\ \uparrow \\ \bullet \\ \uparrow \end{array},$$

$$\begin{array}{c} \uparrow \\ \bullet \\ \uparrow \\ \bullet \\ \uparrow \end{array} = \begin{array}{c} \uparrow \\ \bullet \\ \uparrow \\ \bullet \\ \uparrow \end{array}, \quad \begin{array}{c} \uparrow \\ \bullet \\ \uparrow \\ \bullet \\ \uparrow \end{array} = 0 \quad \text{and} \quad \begin{array}{c} \uparrow \\ \bullet \\ \uparrow \\ \bullet \\ \uparrow \end{array} = \begin{array}{c} \uparrow \\ \bullet \\ \uparrow \\ \bullet \\ \uparrow \end{array}.$$

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Example. Dimension relation:

There is an equivalence of monoidal categories $\rightsquigarrow$ interpolate

$$\mathrm{Rep}_q L_{\textcolor{red}{t}} \simeq \mathrm{Rep}_q \mathrm{GL}_{\textcolor{red}{t}} \boxtimes \mathrm{Rep}_q \mathrm{GL}_{\textcolor{red}{t}}$$

Diagrammatic category:

$$\textcolor{brown}{\blacksquare} = \textcolor{green}{\blacksquare} \oplus \textcolor{red}{\blacksquare}$$

$$\begin{array}{c} \textcolor{brown}{\uparrow} = \textcolor{brown}{\uparrow} \text{ with two green dots} + \textcolor{brown}{\uparrow} \text{ with two red dots}, \\ \textcolor{green}{\uparrow} \text{ with one green dot} = \textcolor{green}{\uparrow}, \quad \textcolor{red}{\uparrow} \text{ with one green dot} = 0, \quad \textcolor{green}{\uparrow} \text{ with one red dot} = 0 \quad \text{and} \quad \textcolor{red}{\uparrow} \text{ with one red dot} = \textcolor{red}{\uparrow}. \end{array}$$

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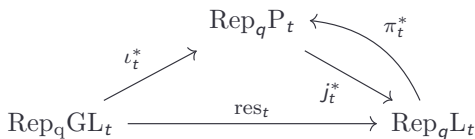


$$\text{ev}: \text{Rig}(P) \rightarrow \text{Rep}_q L \quad \rightsquigarrow \quad \boxed{\text{Rep}_q P_t := \text{Rig}(P) / \ker \text{ev}}$$

- not braided
- not pivotal

$$\text{twist} = q^{t_1} \neq \text{id}$$

$$\begin{array}{ccccc}
 & & \text{Rep}_q P_t & \xleftarrow{\pi_t^*} & \\
 \nearrow \iota_t^* & & & \searrow j_t^* & \\
 \text{Rep}_q \text{GL}_t & \xrightarrow{\text{res}_t} & & & \text{Rep}_q L_t
 \end{array}$$



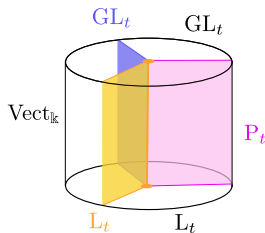
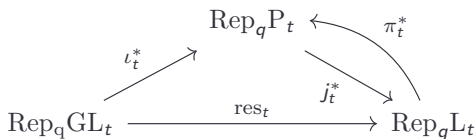
Theorem (GG' 26)

1. *There is a lift*

$$\iota_t^* \otimes \pi_t^*: \text{Rep}_q \text{GL}_t \boxtimes \text{Rep}_q \text{L}_t^{\text{bop}} \rightarrow \text{Z}_{\text{Dr}}(\text{Rep}_q \text{P}_t)$$

turning $\text{Rep}_q \text{P}_t$ into a $(\text{Rep}_q \text{GL}_t, \text{Rep}_q \text{L}_t)$ -central algebra.

2. *The $(\text{Rep}_q \text{L}_t, \text{Rep}_q \text{P}_t)$ - and the $(\text{Rep}_q \text{P}_t, \text{Rep}_q \text{GL}_t)$ -bimod. structures on $\text{Rep}_q \text{L}_t$ are centred by the braiding of $\text{Rep}_q \text{L}_t$.*



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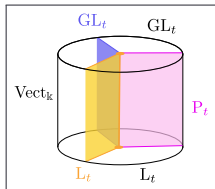
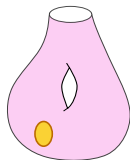
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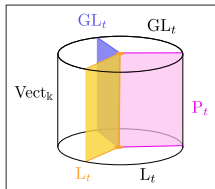
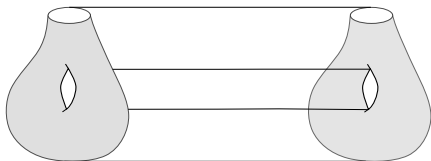
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The HOMFLY skein bialgebra

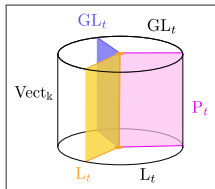
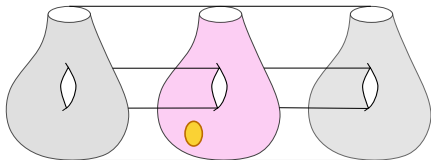
$\mathbb{S}$ decorated framed surface



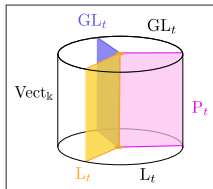
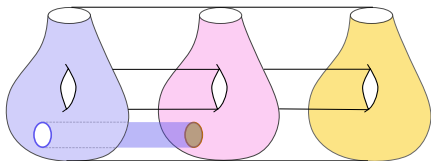
$\mathbb{S}$ decorated framed surface $\rightsquigarrow$ extend the decoration to $S \times [-1, 1]$



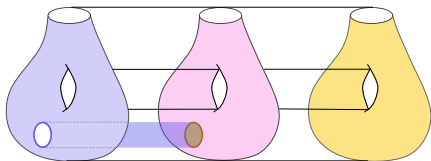
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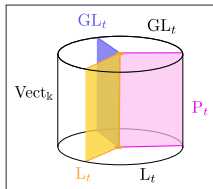
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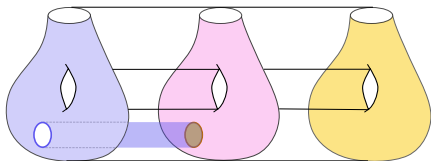
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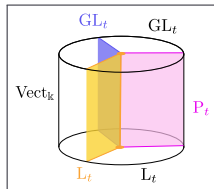


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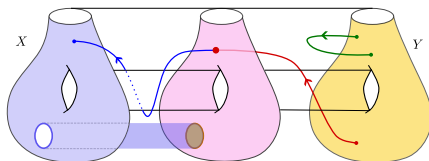


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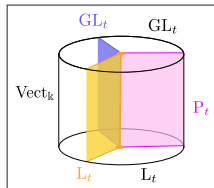


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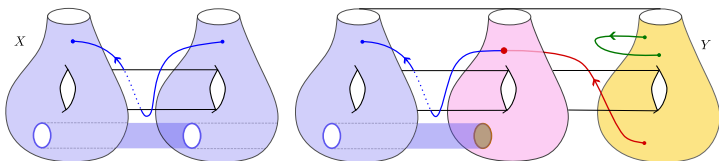


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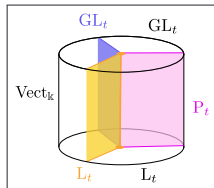


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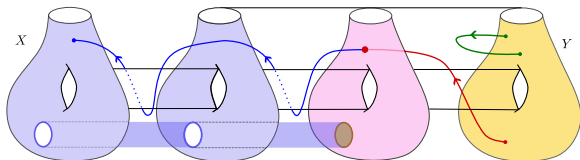


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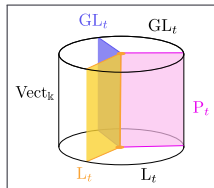


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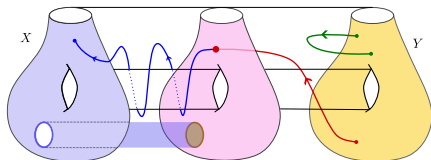


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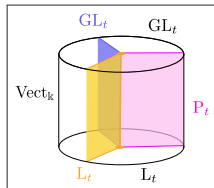


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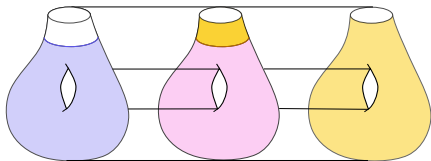
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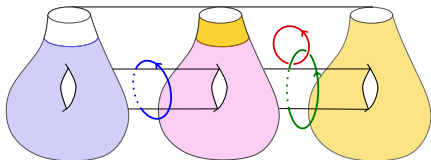


S framed surface with $\partial S \neq 0$

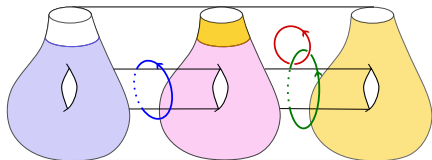
S framed surface with $\partial S \neq 0 \rightsquigarrow$ decorate it so that $S_p \cong S$



S framed surface with $\partial S \neq 0 \rightsquigarrow$ decorate it so that $S_p \cong S$



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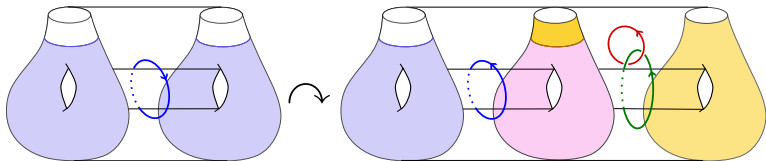


Theorem (GG '26)

We have an isomorphism of vector spaces

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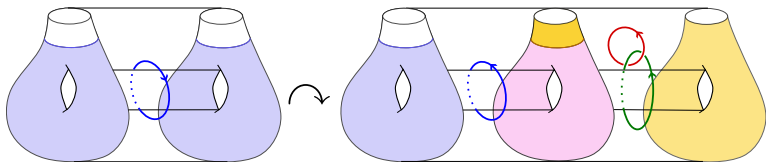


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Acting on the empty diagram induces a coassociative coproduct

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Outlook

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