Spectral sequences

Plan of talk :

1. Idea and definitions
2. Exact couples and double complexes
3. Homology of a group with coefficients in a chain complex
4. Equivariant homology

References

[Brown] Group cohomology

[Rotman] Homological algebra

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1 Idea and definitions

We begin with a family $\{E^0_{p,q}\}_{p,q\geq 0}$ of R-modules equipped with maps

$$d^0_{p,q} : E^0_{p,q} \longrightarrow E^0_{p,q-1} \qquad \text{st.} \qquad d^0_{p,q-1} \circ d^0_{p,q} = 0$$

Since columns are chain complexes, we can construct a new family $\{E^1_{p,q}\}_{p,q\geq 0}$ by taking homologies, i.e.,

$$E^1_{p,q} = \operatorname{Ker}(d^0_{p,q}) / \operatorname{im}(d^0_{p,q+1})$$

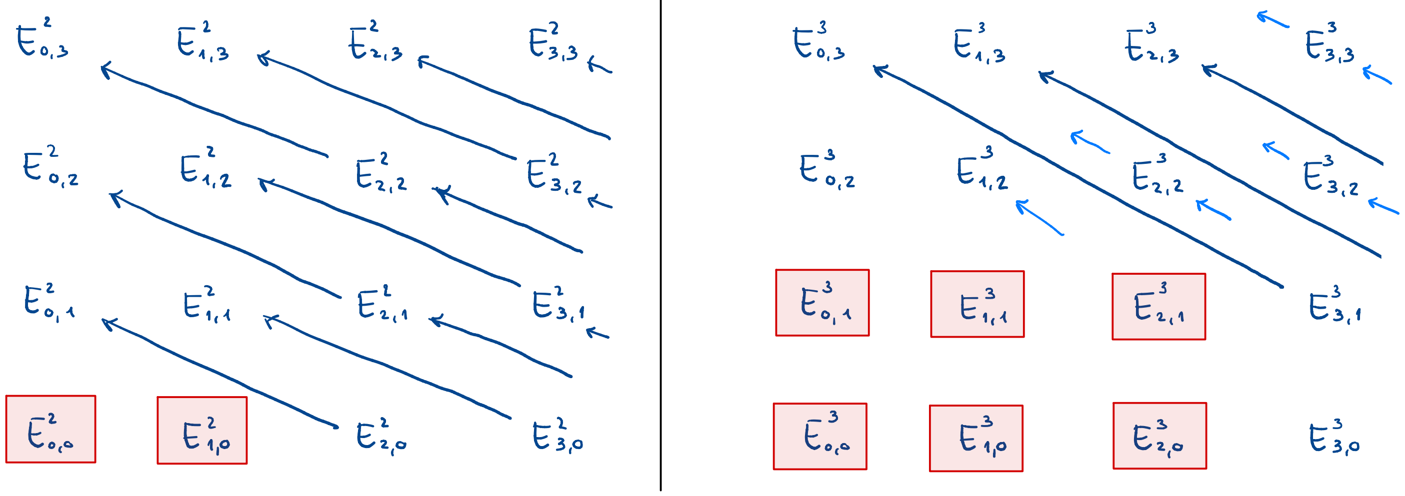
and we define (somehow) new differentials

$$d^1_{p,q} : E^1_{p,q} \longrightarrow E^1_{p-1,q}$$

that make rows into chain complexes. Repeating this process, we will have a family $\{E^r_{p,q}\}_{p,q\geq 0}$ at each step, that we endow with differentials

$$d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,\, q+r-1}$$

<u>Example</u> $r = 2$ ("2nd page") | $r = 3$ ("3rd page")



The maps leaving and entering $E^2_{0,0}$ and $E^2_{1,0}$ are both 0, so

$$E^3_{0,0} = \operatorname{Ker}(d^2_{0,0}) / \operatorname{im}(d^2_{2,-1}) = E^2_{0,0} =: E^\infty_{0,0}$$

$$E^3_{1,0} = \operatorname{Ker}(d^2_{1,0}) / \operatorname{im}(d^2_{3,-1}) = E^2_{1,0} =: E^\infty_{0,1}$$

As r increases, more and more entries stabilises ($E^r_{p,q}$ stabilises at least at $r = p+q+2$, and often earlier), so we can regard each $E^r_{p,q}$ as an approximation of $E^\infty_{p,q}$.

Definition A (first quadrant homological) spectral sequence is a collection of R-modules $\{E^r_{p,q}\}_{p,q,r\geq 0}$ together with maps

$$d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,\,q+r-1}$$

such that

(i) $d^r_{p-r,\,q+r-1} \circ d^r_{p,q} = 0$;

(ii) $E^{r+1}_{p,q} \cong \operatorname{Ker}(d^r_{p,q}) / \operatorname{im}(d^r_{p+r,\,q-r+1})$

In any spectral sequence, the (p,q) entry eventually stabilises. We denote the stable value of $E^r_{p,q}$ by $E^\infty_{p,q}$.

Definition A spectral sequence $E^r_{p,q}$ is said to converge to modules H_n (written $E^r_{p,q} \Rightarrow H_{p+q}$), if there exists a filtration

$$0 = H_n^{-1} \subseteq H_n^0 \subseteq H_n^1 \subseteq \cdots \subseteq H_n^{n-2} \subseteq H_n^{n-1} \subseteq H_n^n = H_n$$

such that

$$E^\infty_{p,n-p} \cong H_n^p / H_n^{p-1}$$

We have the following picture:

$$0 \underbrace{\hookrightarrow}_{E^\infty_{0,n}} H_n^0 \underbrace{\hookrightarrow}_{E^\infty_{1,n-1}} H_n^1 \hookrightarrow \cdots \hookrightarrow H_n^{p-1} \underbrace{\hookrightarrow}_{E^\infty_{p,n-p}} H_n^p \hookrightarrow \cdots \hookrightarrow H_n^{n-1} \underbrace{\hookrightarrow}_{E^\infty_{0,n}} H_n$$

so the stable modules $E^\infty_{p,n-p}$ are the graded pieces of the R-module H_n.

Definition A spectral sequence is said to collapse at E^r $(r \geq 2)$ if there is exactly one nonzero row or column in $\{E^r_{p,q}\}_{p,q\geq 0}$.

If $\{E^r_{p,q}\}$ collapses at $r \geq 2$, we can read the H_n off : it is the unique nonzero $E^r_{p,q}$ with $p+q=n$.

we require $r \geq 2$ so that all differentials have domain and codomain in different rows, hence they are 0 and the nonzero column/row gives the limit term.

2 Exact couples and double complexes

Definition An exact couple is a triangle

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

such that D, E are bi-graded modules and the triangle is exact at each vertex.

Recall that a filtration of a complex (C, d) is a sequence $(F^pC)_{p\in\mathbb{Z}}$ of graded sub-modules such that

$$\cdots \subseteq F^{p-1}C \subseteq F^pC \subseteq F^{p+1}C \subseteq \cdots$$

and

$$d : (F^pC)_n \longrightarrow (F^pC)_{n-1}.$$

Notation $C^p_n := (F^pC)_n$

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\[-10pt]
 & \cap\!\mid & \cap\!\mid & \cap\!\mid & \\[-10pt]
\cdots \arrow[r] & C^{p-1}_{n+1} \arrow[r, "d"] & C^{p-1}_n \arrow[r, "d"] & C^{p-1}_{n-1} \arrow[r] & \cdots \\[-10pt]
 & \cap\!\mid & \cap\!\mid & \cap\!\mid & \\[-10pt]
\cdots \arrow[r] & C^{p}_{n+1} \arrow[r, "d"] & C^{p}_n \arrow[r, "d"] & C^{p}_{n-1} \arrow[r] & \cdots \\[-10pt]
 & \cap\!\mid & \cap\!\mid & \cap\!\mid & \\[-10pt]
\cdots \arrow[r] & C^{p+1}_{n+1} \arrow[r, "d"] & C^{p+1}_n \arrow[r, "d"] & C^{p+1}_{n-1} \arrow[r] & \cdots \\[-10pt]
 & \cap\!\mid & \cap\!\mid & \cap\!\mid & \\[-10pt]
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Proposition 1 Every filtration $(F^pC)_{p\in\mathbb{Z}}$ of a complex induces an exact couple

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

with $\deg(\alpha) = (1,-1)$, $\deg(\beta) = (0,0)$ and $\deg(\gamma) = (-1, 0)$.

Proof For each $p \in \mathbb{Z}$, we have a short exact sequence

$$0 \longrightarrow C^{p-1} \xrightarrow{\iota^{p-1}} C^p \xrightarrow{\pi^p} C^p/C^{p-1} \longrightarrow 0$$

which induces a long exact sequence in homology:

$$\cdots \longrightarrow H_{p+q}(C^{p-1}) \xrightarrow{\iota^{p-1}_*} H_{p+q}(C^p) \xrightarrow{\pi^*_p} H_{p+q}(C^p/C^{p-1}) \xrightarrow{\partial} H_{p+q-1}(C^{p-1}) \longrightarrow \cdots$$

We get an exact couple $(E, D, \alpha, \beta, \gamma)$ with:

$$D_{p,q} := H_{p+q}(C^p), \qquad E_{p,q} := H_{p+q}(C^p/C^{p-1})$$

$$\alpha := \iota_*, \qquad \beta := \pi_*, \qquad \gamma := \partial$$

$\boxtimes$

Proposition 2 If

$$\begin{tikzcd} D \arrow[rr, "\alpha\,(a,a')"] & & D \arrow[dl, "\beta(b,b')"] \\ & E \arrow[ul, "\gamma(c,c')"] & \end{tikzcd}$$

is an exact couple, then $d^1 = \beta\circ\gamma : E \to E$ is a differential and there exists an exact couple

$$\begin{tikzcd} D^2 \arrow[rr, "\alpha^2\,(a,a')"] & & D^2 \arrow[dl, "\beta\,(b-a,\,b'-a')"] \\ & E^2 \arrow[ul, "\gamma^2(c,c')"] & \end{tikzcd}$$

with $E^2 = H(E^1, d^1)$.

Proof (sketch) One sets $D^2 := \operatorname{im}\alpha \subseteq D$, $\alpha^2 := \alpha|_{D^2}$,

$$\beta^2 : D^2 \longrightarrow E^2, \qquad y \in D^2_{p,q} \longmapsto [\beta(\alpha^{-1}(y))] \in E^2_{p+b-a,\, q+b'-a'}$$

$$\gamma^2 : E^2 \longrightarrow D^2, \qquad [z] \in E^2_{p,q} \longmapsto \gamma(z) \in D_{p+c,\, p+c'}$$

and check that everything is well-defined and yields an exact triangle. $\boxtimes$

The exact couple $(E^2, D^2, \alpha^2, \beta^2, \gamma^2)$ is called the derived exact couple.

Iterating the process, we a family $\{E^r_{p,q}\}^{r\geq 0}_{p,q\geq 0}$ of R-modules together with differentials $d^r : E^r \to E^r$ of bidegree $(-r, r-1)$ such that $E^{r+1} = H(E^r, d^r)$.

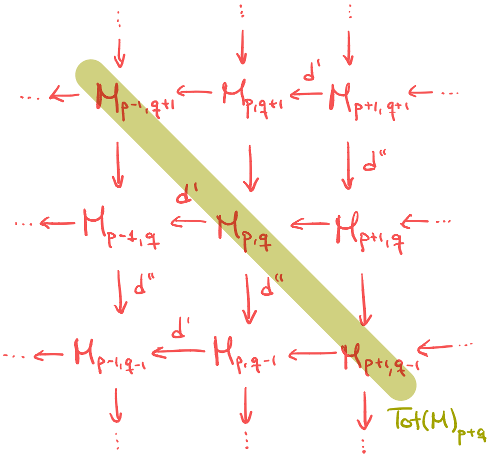
Theorem 3 Let (C, d) be a chain complex.

(1) every filtration $(F^pC)_{p\in\mathbb{Z}}$ yields a spectral sequence $\{E^r_{p,q}\}_{p,q,r\geq 0}$, constructed by taking iterated derived exact couples;

(2) if the filtration is bounded, then the associated spectral sequence converges to the homology of (C, d):

$$E^r_{p,q} \Longrightarrow H_{p+q}(C).$$

We now apply this theory to the particular case of double complexes. A double complex is a bigraded module $M = (M_{p,q})_{p,q\in\mathbb{Z}}$ together with differentials $d', d'' : M \to M$ of bidegree $(-1, 0)$ and $(0, -1)$, respectively, and such that



$$d'_{p,q-1} \circ d''_{p,q} + d''_{p-1,q} \circ d'_{p,q} = 0.$$

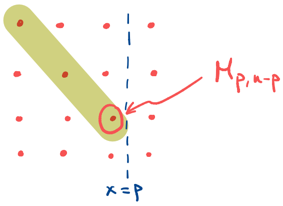
Definition The total complex $\mathrm{Tot}(M)$ of a double complex M is the complex

$$\mathrm{Tot}(M)_n = \bigoplus_{p+q=n} M_{p,q}$$

with differentials $D_n : \mathrm{Tot}(M)_n \to \mathrm{Tot}(M)_{n-1}$ given by

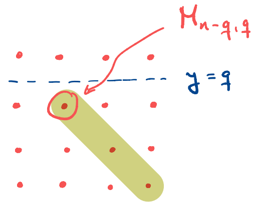
$$D_n = \sum_{p+q=n} (d'_{p,q} + d''_{p,q}).$$

It is straightforward to check that $(\mathrm{Tot}(M), D)$ is a chain complex. Associated to it, we have two filtrations:



(i) the vertical filtration is given by

$$({}^{I}F^{p}\,\mathrm{Tot}(M))_n = \bigoplus_{i\leq p} M_{i,n-i};$$



(ii) the horizontal filtration is given by:

$$({}^{II}F^{q}\,\mathrm{Tot}(M))_n = \bigoplus_{j\leq q} M_{n-j,j}$$

If (M,d) is a first quadrant double complex, both filtrations are bounded, so we have

$${}^{I}E^{r}_{p,q} \Rightarrow H_{p+q}(\mathrm{Tot}(M)) \qquad \text{and} \qquad {}^{II}E^{r}_{p,q} \Rightarrow H_{p+q}(\mathrm{Tot}(M)).$$

3 Homology of a group with coefficients in a chain complex

Reminder. Let G be a group and F the standard (or any other projective) resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. We define the homology groups of G to be

$$H_i(G) := H_i(F_G), \qquad i \geq 0.$$

coinvariants : $F/\langle gx - x\rangle$

A basis of F_n is given by the elements of the form $[g_1 | g_2 | \cdots | g_n]$ with $g_i \in G$, and the differential is $\partial = \sum_{i=0}^{n} (-1)^i d_i : F_n \to F_{n-1}$, with

$$d_i [g_1 | \cdots | g_n] = \begin{cases} g_1 [g_2 | \cdots | g_n], & i = 0 \\ [g_1 | \cdots | g_{i-1} | g_i g_{i+1} | g_{i+2} | \cdots | g_n], & 0 < i < n, \\ [g_1 | \cdots | g_{n-1}], & i = n. \end{cases}$$

In low dimensions, F_G looks like $(F_G)_2 \xrightarrow{\partial_2} (F_G)_1 \xrightarrow{0} \mathbb{Z}$, with $\partial_2 [g|h] = [g] - [gh] + [h]$

$$H_0(G) \cong \mathbb{Z} \qquad \text{and} \qquad H_1(G) \cong G/[G,G].$$

Definition Let $(C_n)_{n\geq 0}$ be a chain complex of $\mathbb{Z}G$-modules. The homology of G with coefficients in C as

$$H_*(G,C) := H_*(F \otimes_G C),$$

where F is the standard (or any projective) resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

Note that $F \otimes_G C = \mathrm{Tot}(F_p \otimes_G C_q)$, so we have two spectral sequences ${}^{I}E^r_{p,q}$ and ${}^{II}E^r_{p,q}$ converging to $H_*(G,C)$.

complex obtained from C applying $H_q(G,-)$ dimension-wise

Proposition 4 (i) ${}^{I}E^1_{p,q} = H_q(F_p \otimes_G C_*) = F_p \otimes_G H_q(C_*)$ and ${}^{I}E^2_{p,q} = H_p(G, H_q C)$
(ii) ${}^{II}E^1_{p,q} = H_q(F_* \otimes_G C_p) = H_q(G, C_p)$ and ${}^{II}E^2_{p,q} = H_p(H_q(G, C_*))$.

Proof Construct the exact couple as in Prop. 1 and compute its derived couples as in Prop. 2. ☒

chain complex concentrated at 0

Example Suppose that each C_p is acyclic, i.e., $H_i(G,C_p) = 0$ for $i>0$ (e.g. free or projective $\mathbb{Z}G$-modules). Then, by (ii), E^1 is concentrated in $q=0$, i.e., the spectral sequence collapses and we have

$$H_p(G,C) \cong {}^{II}E^2_{p,0} = H_p(H_0(G,C_*)) = H_p(C_G).$$

Theorem 5 (Hochschild-Serre spectral sequence) For any group extension $1 \to H \to G \to Q \to 1$ and any G-module M, there is a spectral sequence of the form

$$E^2_{pq} = H_p(Q, H_q(H,M)) \Rightarrow H_{p+q}(G,M).$$

Proof For such group extension, the Q-modules $C_p = (F_p \otimes M)_H$ are acyclic so we can apply prop. 4 (i), the previous example and the fact that

$$H_*(G,M) = H_*(C_Q) \qquad \text{and} \qquad H_*(H,M) \approx H_*(C).$$

☒

Corollary 6 Under the hypothesis of thm. 5, there is an exact sequence
$$H_2(G,M) \to H_2(Q,M_H) \to H_1(H,M)_Q \to H_1(G,M) \to H_1(Q,M_H) \to 0.$$

Proof By the previous thm and the very first example of these notes, we have

② $E^2_{0,1} \twoheadrightarrow E^\infty_{0,1} = H_1^0(G,M) \hookrightarrow H_1(G,M)$, ③ $E^\infty_{1,0} = E^2_{1,0} = H_1(G,M)/E^\infty_{0,1}$,

stabilises at page 3

① $E^\infty_{2,0} = E^3_{2,0} = \mathrm{Ker}(d^2) \hookrightarrow E^2_{2,0}$,

where $d^2: E^2_{2,0} \to E^2_{0,1}$. Putting all these together we get an exact sequence
$$0 \to E^\infty_{2,0} \xhookrightarrow{①} E^2_{2,0} \xrightarrow{d^2} E^2_{0,1} \xrightarrow{②} H_1(G,M) \xrightarrow{③} E^\infty_{1,0} \to 0.$$

But now we have that
$$E^2_{2,0} = H_2(Q, H_0(H,M)) = H_2(Q,M_H), \qquad E^2_{0,1} = H_0(Q, H_1(H,M)) = H_1(H,M)_Q$$
$$E^\infty_{1,0} = E^2_{1,0} = H_1(Q, H_0(H,M)) = H_1(Q,M_H), \qquad E^\infty_{2,0} = E^3_{2,0} \cong H_2(G,M)/E^2_{2,0}$$

and the claim follows. ⊠

Example Let G be a finitely presented group, then $G \cong \langle F\rangle/\overline{\langle R\rangle}$, with $R \subseteq w(F)$ finite sets, $|F| = n$, $|R| = m$. We have a group extension
$$0 \to \overline{\langle R\rangle} \to \langle F\rangle \to G \to 0$$

Applying cor. 6 with $M = \mathbb{Q}$ (trivial action of G), we get
$$H_2(\langle F\rangle, \mathbb{Q}) \to H_2(G,\mathbb{Q}) \to H_1(\overline{\langle R\rangle}, \mathbb{Q})_G \to H_1(\langle F\rangle, \mathbb{Q}) \to H_1(G,\mathbb{Q}) \to 0.$$

Note that:

* $H_2(\langle F\rangle, \mathbb{Q}) = 0$, since $\langle F\rangle$ is free (hence $H_i(\langle F\rangle, \mathbb{Q}) = 0$, $i > 2$);
* $H_1(\langle F\rangle, \mathbb{Q}) \overset{(*)}{=} \langle F\rangle^{ab} \otimes \mathbb{Q} = \mathbb{Z}^n \otimes \mathbb{Q} = \mathbb{Q}^n$ (*) $\mathbb{Q}$ trivial module
* $H_1(\overline{\langle R\rangle}, \mathbb{Q})_G = H_0(G, H_1(\overline{\langle R\rangle}, \mathbb{Q})) \overset{(*)}{=} H_0(G, \overline{\langle R\rangle}^{ab} \otimes \mathbb{Q}) = (\overline{\langle R\rangle}^{ab} \otimes \mathbb{Q})_G$. But the action of G on $\overline{\langle R\rangle}^{ab} \otimes \mathbb{Q}$ is trivial (in the bar resolution $g([x]\otimes q) = [gxg^{-1}] \otimes q = [x] \otimes q$), so $H_1(\overline{\langle R\rangle}, \mathbb{Q})_G = \mathbb{Q}^k$, $k \leq m$

The sequence above thus yields:

$$0 \longrightarrow H_2(G,\mathbb{Q}) \longrightarrow \mathbb{Q}^k \longrightarrow \mathbb{Q}^n \longrightarrow H_1(G,\mathbb{Q}) \longrightarrow 0.$$

This proves that $\dim H_2(G,\mathbb{Q}) \leq m$.

4 Equivariant group homology

A G-complex consists of a CW complex X together with an action of G that permutes the cells. We will say that X is acyclic if $H_*(X) = H_*(pt)$.

Definition Let X be a G-complex and $C(X)$ its cellular chain complex. We define the equivariant homology of (G,X) as

$$H_*^G(X) := H_*(G, C(X)).$$

Moreover, if M is a G-module, we set

$$H_*^G(X,M) := H_*(G, C(X)\otimes M).$$

Example i) If $X = pt.$ then

$$H_*^G(X,M) = H_*(G, C(pt)\otimes M) = H_*(G,M).$$

ii) If G is the trivial group, the standard resolution is trivial hence

$$H_*^G(X,M) = H_*(G, C(X)\otimes M) = H_*(C(X)\otimes M) = H_*(X,M).$$

Proposition 7 If X is acyclic, then there is a canonical isomorphism

$$H_*^G(X,M) \xrightarrow{\sim} H_*(G,M).$$

Proof For any G-complex X, the unique map $X \to pt.$ induces a canonical map

$$H_*^G(X,M) \longrightarrow H_*(G,M).$$

By Thm. 3 and Prop. 4 (i), we have a spectral sequence

$${}^{I}E^2_{pq} = H_p(G, H_q(X,M)) \Rightarrow H^G_{p+q}(X,M).$$

But, since X is acyclic, $H_q(X,M) \cong H_q(pt.,M)$, and we have

$$
\begin{array}{c}
H_p(G, H_q(X,M)) \Rightarrow H^G_{p+q}(X,M) \\
\cong \\
H_p(G, H_q(pt, M)) \Rightarrow H^G_{p+q}(pt. M) \cong H_{p+q}(G,M).
\end{array}
$$

An isomorphism on the second page induces an isomorphism on the limits, so the canonical map above is an isomorphism. ⊠

Let X be a G-complex and σ a p-cell of X. We set

$$G_\sigma := \{ g \in G \mid g\sigma = \sigma \}.$$

Let $\mathbb{Z}_\sigma$ be the orientation module associated to σ, i.e., $\mathbb{Z}_\sigma$ it is an infinite cyclic group whose two generators correspond to the two orientations of σ with a G_σ-action given by the orientation character

$$\chi_\sigma(g) = \begin{cases} +1, & \text{if } g \text{ preserves the orientation of } \sigma, \\ -1, & \text{otherwise.} \end{cases}$$

Set $M_\sigma := \mathbb{Z}_\sigma \otimes M$. This is just M with G_σ-action twisted by χ_σ. We have

$$C_p(X) \otimes M = \bigoplus_{\sigma \in X_p} \mathbb{Z}_\sigma \otimes M = \bigoplus_{\sigma \in \Sigma_p} \bigoplus_{g \in G} \mathbb{Z}_{g \cdot \sigma} \otimes M \cong \bigoplus_{\sigma \in \Sigma_p} \mathrm{Ind}^G_{G_\sigma} M_\sigma.$$

$$g \cdot \sigma \otimes m \longmapsto g \otimes (\sigma \otimes m)$$

Lemma 8 (Shapiro's lemma) If $H \leq G$ and M is an H-module, then

$$H_*(H,M) \cong H_*(G, \mathrm{Ind}^G_H M)$$

Proof Let F be a resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Restricting to $\mathbb{Z}H$, F is also a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$, so

$$
\begin{aligned}
H_*(H,M) &= H_*(F \otimes_{\mathbb{Z}H} M) \\
&\cong H_*(F \otimes_{\mathbb{Z}G} (\mathbb{Z}G \otimes_{\mathbb{Z}H} M)) \\
&= H_*(F \otimes_{\mathbb{Z}G} \mathrm{Ind}^G_H M) \cong H_*(G, \mathrm{Ind}^G_H M)
\end{aligned}
$$

⊠

By Shapiro's lemma and the decomposition of $C(X,M)$ above, we get

$$H_q(G, C_p(X,M)) \approx \bigoplus_{\sigma \in \Sigma_p} H_q(G_\sigma, M_\sigma)$$

so the spectral sequence from Prop. 4 (ii) has the form

$${}^{II}E^1_{p,q} = \bigoplus_{\sigma \in \Sigma_p} H_p(G_\sigma, M_\sigma) \Rightarrow H^G_{p+q}(X,M).$$

If the complex is acyclic, we can now apply Prop. 7. to get

$$E^1_{p,q} = \bigoplus_{\sigma \in X_p/G} H_q(G_\sigma, M_\sigma) \Rightarrow H_{p+q}(G,M).$$

This provides a computational tool for the homology of groups, using acyclic spaces on which G acts.