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## **Defects in skein theory and HOMFLY parabolic restriction**

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*A mis padres.*



«*Non est ad astra mollis e terris via.*»

«*No hay un camino fácil desde la tierra hasta las estrellas.*»

— Séneca, *Hercules Furens* (Hércules furioso)

«Κι αν πτωχική την βρεις, η Ιθάκη δεν σε γέλασε.  
Έτσι σοφός που έγινες, με τόση πείρα,  
ήδη θα το κατάλαβες η Ιθάκης τι σημαίνουν.»

«*Y si la encuentras pobre, Ítaca no te ha engañado.  
Así sabio como te has vuelto, con tanta experiencia,  
ya habrás comprendido qué significan las Ítacas.*»

— Καβάφης (Cavafis), *Ιθάκη* (Ítaca)



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# Abstract

This thesis deals with parabolic restriction and defects in skein theory. We develop the necessary formalism to produce local relations around codimension 1 defects in a stratified 3-manifold, using the algebraic data of centered bimodule categories. We construct an interpolating version of the parabolic restriction formalism based on the diagrammatic calculus for pivotal categories. By employing both defect skein theory and parabolic restriction, we construct a categorified version of the Turaev coproduct on the HOMFLY algebra of a framed surface. The thesis presents three new contributions:

The first contribution consists in developing the formalism of line defects in skein theory. The algebraic data required to construct these theories are provided by the objects and morphisms of the 4-category  $\text{BrTens}$  of braided monoidal categories. After reviewing these algebraic structures, we construct a local model for line defects in stratified 3-manifolds and define an evaluation functor to a centered bimodule category, allowing us to produce local relations around codimension 2 defects.

The second contribution consists in constructing an interpolating version of the parabolic restriction formalism. We recall the definition of the HOMFLY category, a universal version of the category of quantum representations of  $\text{GL}_N$ , and use it to describe an analogous construction for its Levi subgroup of two-block diagonal matrices. We define a planar diagrammatic category  $\text{Rep}_q P_t$  that interpolates the category of quantum representations of the parabolic subgroup of block-upper triangular matrices of  $\text{GL}_{m+n}$ , and we construct four restriction functors interpolating the standard functors between these categories. Finally, we show that the category  $\text{Rep}_q P_t$  carries the structure of a central algebra, providing the algebraic data necessary to implement a defect skein theory.

The final contribution consists in explicitly describing the theory associated with the universal parabolic restriction formalism. By restricting this formalism, we recover a bimodule over the framed surface skein algebras of  $\text{GL}_{m+n}$  and  $\text{GL}_m \times \text{GL}_n$ . We demonstrate that the vector space underlying this bimodule is isomorphic to the surface skein algebra of  $\text{GL}_m \times \text{GL}_n$ , and that the action of  $\text{GL}_{m+n}$  on it induces the Turaev coproduct.

**Keywords** skein theory, defects, parabolic restriction, HOMFLY, TQFTs, quantum groups.



# Résumé en français

Cette thèse porte sur la restriction parabolique et les défauts en théorie d'écheveaux. Nous développons le formalisme nécessaire pour produire des relations locales autour de défauts de codimension 1 dans une 3-variété stratifiée, en utilisant les données algébriques des catégories de bimodules centrées. Nous construisons une version interpolante du formalisme de restriction parabolique basée sur le calcul diagrammatique pour les catégories pivotales. En employant à la fois la théorie d'écheveaux avec défauts et la restriction parabolique, nous construisons une version catégorifiée du coproduit de Turaev sur l'algèbre HOMFLY d'une surface parallélisée. La thèse présente trois nouvelles contributions :

La première contribution consiste à développer le formalisme des défauts de ligne en théorie d'écheveaux. Les données algébriques requises pour construire ces théories sont fournies par les objets et les morphismes de la 4-catégorie  $\text{BrTens}$  des catégories monoïdales tressées. Après avoir revisité ces structures algébriques, nous construisons un modèle local pour les défauts de ligne dans les 3-variétés stratifiées et définissons un foncteur d'évaluation vers une catégorie de bimodule centrée, nous permettant de produire des relations locales autour des défauts de codimension 2.

La deuxième contribution consiste à construire une version interpolante du formalisme de restriction parabolique. Nous rappelons la définition de la catégorie HOMFLY, une version universelle de la catégorie des représentations quantiques de  $\text{GL}_N$ , et l'utilisons pour décrire une construction analogue pour son sous-groupe de Levi de matrices diagonales à deux blocs. Nous définissons une catégorie diagrammatique plane  $\text{Rep}_q\text{P}_t$  qui interpole la catégorie des représentations quantiques du sous-groupe parabolique des matrices triangulaires supérieures par blocs de  $\text{GL}_{m+n}$ , et nous construisons quatre foncteurs de restriction interpolant les foncteurs standards entre ces catégories. Enfin, nous montrons que la catégorie  $\text{Rep}_q\text{P}_t$  porte la structure d'une algèbre centrale, fournissant les données algébriques nécessaires pour implémenter une théorie d'écheveaux avec défauts.

La dernière contribution consiste à décrire explicitement la théorie associée au formalisme de restriction parabolique universelle. En restreignant ce formalisme, nous récupérons un bimodule sur les algèbres d'écheveaux de surfaces parallélisées de  $\text{GL}_{m+n}$  et  $\text{GL}_m \times \text{GL}_n$ . Nous démontrons que l'espace vectoriel sous-jacent à ce bimodule est isomorphe à l'algèbre d'écheveaux de la surface de  $\text{GL}_m \times \text{GL}_n$ , et que l'action de  $\text{GL}_{m+n}$  sur celui-ci induit le coproduit de Turaev.

**Mots-clés** théorie d'écheveaux, défauts, restriction parabolique, HOMFLY, TQFTs, quantum groups.

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# Introduction

## 1 Context

This thesis lies at the intersection of low-dimensional topology and representation theory. It deals with a family of topological field theories arising from the graphical calculus for monoidal and braided monoidal categories, and the formalism of parabolic restriction.

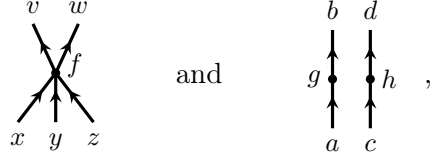
### Monoidal categories and their graphical calculus

A *monoidal category* is a category  $\mathcal{C}$  endowed with a *tensor product* functor, which is associative and unital up to prescribed isomorphisms. First introduced by MacLane [Mac63; Mac78], they can be seen as categorical versions of the notions of ring and monoid, and are now ubiquitous in mathematics, physics and computer science. As was pointed out by Penrose [Pen71], one special feature of monoidal categories is the fact that they admit a very rich graphical calculus, which was formalised by Joyal and Street in [JSa; JS91; JSb; JS93]. Indeed, a morphism in any category can be represented using a *linear* diagram of the form

$$\begin{array}{c} y \\ \uparrow \\ \bullet \\ \uparrow \\ x \end{array} \quad f \quad ,$$

where the edges represent the source and target objects  $x$  and  $y$ , respectively, and the vertex is the morphism itself. This is only a fancy way of exploiting the usual custom in all fields of mathematics of denoting maps between sets with any structure using arrows. The single direction represents the only available operation of composing morphisms, which is given by just concatenating graphs. In the presence of a monoidal structure, we can make sense of a more general class of graphs, now lying in the plane, the new dimension encoding the additional operation provided by the tensor product. For instance, the

diagrams

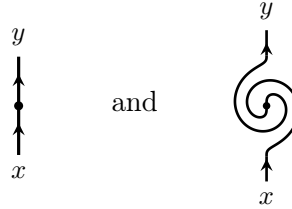


represent, respectively, a morphism  $f: x \otimes y \otimes z \rightarrow v \otimes w$  and the tensor product  $g \otimes h$  of two morphisms  $g: a \rightarrow b$  and  $h: c \rightarrow d$ .

The more structure and properties the underlying category has, the richer and more interesting this diagrammatic calculus becomes. A *rigid* category is a monoidal category  $\mathcal{C}$  whose objects have *duals*. The operation of taking duals can be reflected diagrammatically by switching orientations or allowing edges to turn around, which results in sophisticated diagrams. For example,



represent now a morphism  $f: x^* \otimes y \otimes z^* \rightarrow v^* \otimes w$  and the dual  $g^*: y^* \rightarrow x^*$  of a morphism  $g: x \rightarrow y$ , respectively. These graphs are *rigid* in the sense that they are not invariant by the deformations: for instance,

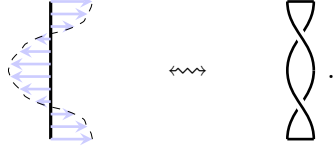


can be obtained from one another by rotating by  $2\pi$  the vertex. However, the first diagram represents a morphism  $x \rightarrow y$ , whereas the second one represents a morphism  $x^{**} \rightarrow y^{**}$  (since edges turn around twice).

A *pivotal* category is a category  $\mathcal{C}$  equipped with a natural family of isomorphisms  $x \cong x^{**}$  identifying objects with their double duals. Under this identification, both diagrams above can actually be seen as morphisms  $x \rightarrow y$  and, consequently, the resulting graphical calculus for pivotal categories is compatible with the topology of  $\mathbb{R}^2$ .

When the categories are braided, the graphical calculus extends to three dimensions, where the braiding is represented by two strands crossing in 3-dimensional space. This becomes particularly interesting when  $\mathcal{C}$  is a *ribbon* category. A ribbon category is a braided rigid category endowed with a compatible notion of *twist*, a special natural automorphism of the identity functor. A fundamental result by Reshetikhin–Turaev [RT90] and Shum [Shu94] shows that the algebraic properties of these categories are

governed by the topology of *ribbon graphs*. These are graphs whose edges are endowed with a non-vanishing normal vector field, allowing them to be regarded as actual ribbons:



The topology of ribbon graphs reflects diagrammatically the algebraic structure of ribbon categories: given a ribbon category  $\mathcal{C}$ , the construction of Reshetikhin–Turaev yields a full (but not faithful!) monoidal evaluation functor

$$\mathrm{RT}_{\mathcal{C}}: \mathrm{Rib}(\mathcal{C}) \rightarrow \mathcal{C},$$

where  $\mathrm{Rib}(\mathcal{C})$  is a category whose morphisms are isotopy classes of ribbon graphs coloured with the objects and morphisms of  $\mathcal{C}$ . This functor provides a unified framework for the polynomial quantum invariants introduced after the discovery of the Jones polynomial [Jon85], and lies at the heart of the fruitful interaction between low-dimensional topology and representation theory.

## String-nets and skein theory

The graphical calculus for monoidal categories is *local*, in the sense that it takes place inside rectangles and cubes in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ . A generalisation to arbitrary oriented surfaces and 3-manifolds is provided by *string-nets* and *skein theory*, respectively.

String-net spaces are vector spaces whose generators are decorated graphs (the string-nets) embedded in a surface and considered up to relations induced locally by the algebraic data of a rigid or pivotal category  $\mathcal{C}$ . They have their origin in physics, where they provide models for topological phases of matter [LW05; LW06]. They were formulated mathematically by Kirillov in [Kir11] using a spherical fusion category as input, and extended later to non-spherical fusion categories in [Run20]. As in the local case, a richer structure from the algebraic side reflects in a less rigid structure on the geometric one: rigid categories give rise to theories only for *framed* surfaces [KST24], which lift to oriented surfaces in the presence of a pivotal structure [HSS26].

The 3-dimensional version of the story is known as *skein theory*. Skein theory is concerned with the study of (skein) modules [Prz06; Tur90], algebras [PS97] and categories [Yet92; Wal06; Joh19; Coo23] generated by ribbon graphs in a 3-manifold, modulo the relations induced locally by the Reshetikhin–Turaev construction, using a ribbon category as input data. Over the past decade, skein theory has undergone a significant development, with skein modules, algebras and categories becoming central objects in quantum topology. The introduction of stated skein algebras [BW11] endowed skein theory with powerful excision properties, studied in [CL22; CL25]. On the other hand, admissible skein modules [CGP23; RST24; BH24] have extended this framework to non-semisimple ribbon categories, opening the door to new TQFTs beyond the semisimple setting [Cos+26; Hai25]. At the categorical level, skein categories have emerged as a natural

framework for studying factorization homology [BBJ18a; BBJ18b; Coo23; BH24]. Recently, derived versions of skein modules and categories have been introduced in [Bai26]. These developments have been accompanied by major theoretical breakthroughs, most notably the proof of Witten’s finiteness conjecture for skein modules [GJS23] (see also [JR25; BD25]).

## TQFTs and defects

Both string-nets and skein theories are examples of *Topological Quantum Field Theories* (TQFTs) defined by generators and relations [Wal06]. An  $n$ -dimensional TQFT is a symmetric monoidal functor

$$\mathcal{Z}: \text{Cob}_n^{\sqcup} \rightarrow \mathcal{C}^{\otimes},$$

where  $\text{Cob}_n^{\sqcup}$  is a category whose objects are closed  $(n - 1)$ -dimensional manifolds and whose morphisms are equivalence classes of framed or oriented cobordisms between them, and  $\mathcal{C}$  is a monoidal category. These theories provide simplified models for quantum field theories in physics and, since their axiomatization by Atiyah [Ati88], have been proven to be a rich source of mathematics, as well as a powerful tool to organize structures in geometry and topology.

Richer algebraic structures can be encountered by enriching the topology of cobordism through the introduction of *defects*. In physics, defects are localized structures in a theory where the bulk dynamics is modified, often occurring at interfaces between different phases or as a result of localized perturbations. Mathematically, they are described by embedded submanifolds of positive codimension, called *strata*, each of which carries its own algebraic data and mediates the interaction between the theories assigned to the adjacent bulk regions. An axiomatic definition for  $n$ -dimensional TQFTs with defects was given in [CMS20; CRS18a; CRS19; CRS18b] as symmetric monoidal functors

$$\mathcal{Z}: \text{Cob}_n^{\text{dec}, \sqcup} \rightarrow \mathcal{C}^{\otimes}$$

from a category  $\text{Cob}_n^{\text{dec}}$  of stratified  $n$ -dimensional cobordism labelled with appropriate algebraic data to a monoidal category  $\mathcal{C}$ .

The study of defects in skein theory was initiated in [BJ25], where the notions of skein modules, skein algebras, and skein categories were extended to 3-manifolds endowed with oriented codimension-one defects, in the context of the study of the quantum  $A$ -polynomial. The algebraic data modelling such defects is given by a *pivotal central algebra*  $\mathcal{C}$  over the ribbon categories  $\mathcal{A}$  and  $\mathcal{B}$  decorating the adjacent bulk regions. Concretely, a central algebra consists of a pivotal braided monoidal functor

$$F: \mathcal{A} \boxtimes \mathcal{B}^{\text{bop}} \longrightarrow Z_{\text{Dr}}(\mathcal{C}),$$

where  $\mathcal{B}^{\text{bop}}$  denotes the braided opposite of  $\mathcal{B}$  and  $Z_{\text{Dr}}(\mathcal{C})$  is the Drinfeld center of  $\mathcal{C}$ .

Given a stratified 3-manifold  $M$ , its *defect skein module* is spanned by isotopy classes of graphs embedded in  $M$  such that, in each bulk region, they form ribbon graphs coloured

by either  $\mathcal{A}$  or  $\mathcal{B}$ , while their intersection with the defect wall is a  $\mathcal{C}$ -coloured string-net. The central structure provides the local algebraic data needed to project ribbon graphs onto the defect and evaluate them in  $\mathcal{C}$  via the graphical calculus for pivotal categories, thereby producing the skein relations in a neighbourhood of the defect.

On the other hand, defect skein modules for manifolds with line and point defects on their boundary have been recently introduced in [KR26], with the aim of giving a skein-theoretic description of the state spaces for the Reshetikhin–Turaev defect TQFT from [CRS18a]. In the same spirit, line defects are coloured by modules over the bulk ribbon category, point defects are coloured by bimodule functors and an evaluation functor producing skein relations near the boundary is defined using the graphical calculus for the underlying structures.

## HOMFLY categories

An important source for the algebraic data inducing all these diagrammatic and topological constructions is representation theory. Prototypical examples of ribbon categories are categories of representations of quantum groups. Fix a commutative ground ring  $\mathbb{k}$ . Given a reductive affine algebraic group  $G$  and a sufficiently generic parameter  $q \in \mathbb{k}^\times$ , the category  $\text{Rep}_q G$  is equivalent as abelian category to the category  $\text{Rep } G$  of algebraic representations of  $G$ , but it has a non-trivial ribbon structure which is a deformation of the (trivial) one of  $\text{Rep } G$ . When  $G = \text{GL}_N$ , the relations generating the kernel of  $\text{RT}_{\mathcal{C}}$  depend in a very simple way on the integer parameter  $N \in \mathbb{N}$ . Replacing  $N$  with a formal parameter  $t$ , one obtains the *HOMFLY relations*

$$\begin{array}{c} \begin{array}{ccc} \begin{array}{c} \nearrow \quad \nwarrow \\ \nwarrow \quad \nearrow \end{array} & - & \begin{array}{c} \nwarrow \quad \nearrow \\ \nearrow \quad \nwarrow \end{array} & = & (q - q^{-1}) \begin{array}{c} \uparrow \\ \uparrow \end{array} \\ \begin{array}{c} \uparrow \\ \text{p} \end{array} & = & q^t \begin{array}{c} \uparrow \end{array} & \text{and} & \begin{array}{c} \circlearrowright \end{array} & = & \frac{q^t - q^{-t}}{q - q^{-1}} \mathbf{1}_\emptyset. \end{array}$$

The *HOMFLY category* [Bru17] is defined as the quotient of the category of *framed tangles* (a subcategory of  $\text{Rib}(\mathcal{C})$ ) by the HOMFLY skein relations. This category is a quantum version of the *Deligne category* [Del07] and can be understood as a universal category for the quantum representations of  $\text{GL}_N$ : for every  $N \in \mathbb{N}$ , there exists a monoidal specialising functor

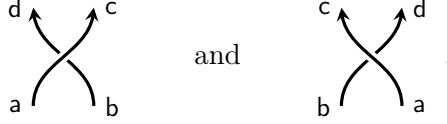
$$\text{sp}_N: \text{Rep}_q \text{GL}_t \rightarrow \text{Rep}_q \text{GL}_N$$

allowing one to recover  $\text{Rep}_q \text{GL}_N$  as a quotient of the specialisation of  $\text{Rep}_q \text{GL}_t$  to  $t = N$ .

## The Turaev coproduct

The polynomial link invariant obtained for  $\mathcal{C} = \text{Rep}_q \text{GL}_t$  via the Reshetikhin–Turaev construction is the *HOMFLY polynomial* [Fre+85; PT16]. Write  $H_{q,t}(D)$  for the HOM-

FLY polynomial of a link diagram  $D$ . In [Jae89], Jaeger gave a composition formula allowing one to compute  $H_{q,t_1+t_2}(D)$  as a state sum over  $H_{q,t_1}(D_1)$  and  $H_{q,t_2}(D_2)$ , where  $D_1, D_2$  are some subdiagrams of  $D$ . Concretely, a link diagram  $D$  is a 4-valent graph with two distinguished classes of vertices: positive and negative crossings. Given a vertex  $v$ , we denote by  $a$  (resp.  $b$ ) the upper (resp. lower) incoming edge, and by  $c$  (resp.  $d$ ) the upper (resp. lower) outgoing edge, as shown in the following pictures:



Let  $E_D$  be the set of edges of  $D$ . An *admissible labelling* is a map  $f : E_D \rightarrow \{1, 2\}$  such that, at any vertex  $v$ , either  $f(a) = f(c)$  and  $f(b) = f(d)$ , or  $f(a) = f(d) > f(b) = f(c)$ . When the second condition holds, we say that  $v$  is a *cutting vertex*. Given an admissible labelling  $f$  and a vertex  $v$ , we set

$$\langle v \mid f \rangle := \begin{cases} \operatorname{sgn}(v)(q - q^{-1}), & \text{if } v \text{ is a cutting vertex,} \\ 1, & \text{otherwise,} \end{cases}$$

where  $\operatorname{sgn}(v)$  is 1 if  $v$  is a positive crossing and  $-1$  otherwise. We define the *interaction of  $D$  with  $f$*  as

$$\langle D \mid f \rangle := \prod_v \langle v \mid f \rangle,$$

where the product runs over the set of vertices of  $D$ . Jaeger showed that

$$H_{q,t_1+t_2}(D) = \sum_{f \text{ admissible}} \langle D \mid f \rangle (q^{-t_1})^{r(D_{f,1})} (q^{t_2})^{r(D_{f,2})} H_{q,t_1}(D_{f,1}) H_{q,t_2}(D_{f,2}),$$

where  $D_{f,i} := f^{-1}(i)$  and  $r(D_{f,i})$  is the rotation number of  $D_{f,i}$ , for  $i \in \{1, 2\}$ .

Turaev showed in [Tur91] that Jaeger's formula extends to the HOMFLY skein algebra of framed surfaces. More precisely:

**Theorem 1.1** ([Tur91]). *Let  $(S, f)$  be a framed surface. The linear map*

$$\Delta_f : \operatorname{SkAlg}_{\operatorname{GL}_t}(S) \rightarrow \operatorname{SkAlg}_{\operatorname{GL}_t}(S) \otimes \operatorname{SkAlg}_{\operatorname{GL}_t}(S)$$

*defined by extending Jaeger's formula is a morphism of algebras. It is coassociative, so  $\operatorname{SkAlg}_{\operatorname{GL}_t}(S)$  is a bialgebra.* ■

## 2 Results

In this thesis, we develop a framework for line defects in skein theory, provide a HOMFLY version of parabolic restriction, and combine both formalisms to give a new construction of the Turaev coproduct that extends to the HOMFLY skein category.

## HOMFLY parabolic restriction

Jaeger's formula has been traditionally interpreted as a universal version of the restriction of representations from  $\mathrm{GL}_{m+n}$  to  $\mathrm{GL}_m \times \mathrm{GL}_n$ , in the sense that specialising to  $t_1 = m$  and  $t_2 = n$ , the diagram

$$\begin{array}{ccc} D & \xrightarrow{\text{Jaeger}} & \sum_f D_{f,1} \sqcup D_{f,2} \\ \text{RT}_{\mathrm{Rep}_q \mathrm{GL}_{m+n}} \downarrow & & \downarrow \text{RT}_{\mathrm{Rep}_q(\mathrm{GL}_m \times \mathrm{GL}_n)} \\ H_{q,m+n}(D) & \xrightarrow{\text{res}_{m,n}} & \sum_f \alpha(f) H_{q,m}(D_{f,1}) H_{q,n}(D_{f,2}) \end{array}$$

is commutative for every link diagram  $D$ , where

$$\text{res}_{m,n}: \mathrm{Rep}_q \mathrm{GL}_{m+n} \rightarrow \mathrm{Rep}_q(\mathrm{GL}_m \times \mathrm{GL}_n)$$

is the restriction functor induced by the block-diagonal embedding  $\mathrm{GL}_m \times \mathrm{GL}_n \hookrightarrow \mathrm{GL}_{m+n}$ , and  $\alpha(f)$  is the coefficient appearing in Jaeger's formula.

From this point of view, the existence of Turaev's coproduct is surprising. The restriction functor is monoidal, so it induces a morphism between the planar theories associated with  $\mathrm{Rep}_q \mathrm{GL}_{m+n}$  and  $\mathrm{Rep}_q(\mathrm{GL}_m \times \mathrm{GL}_n)$ . However, it is not braided, so there is a priori no reason for this morphism to lift to three dimensions nor for it to be multiplicative (the product of  $\mathrm{SkAlg}_{\mathrm{GL}_t}(S)$  being a purely 3-dimensional construction).

A different approach can be carried out using parabolic restriction and skein theory with defects. Let  $P$  be the parabolic subgroup of block-upper triangular matrices of  $\mathrm{GL}_{m+n}$  and set  $L := \mathrm{GL}_m \times \mathrm{GL}_n$  for its Levi subgroup of block-diagonal matrices. Then,  $L$  is both a quotient and a subgroup of  $P$  and we have morphisms of groups

$$\iota: P \hookrightarrow \mathrm{GL}_{m+n}, \quad j: L \hookrightarrow P \quad \text{and} \quad \pi: P \twoheadrightarrow L.$$

Restricting representations along these morphisms induce the following family of functors:

$$\begin{array}{ccc} & \mathrm{Rep}_q P & \\ \iota^* \nearrow & & \searrow j^* \\ \mathrm{Rep}_q \mathrm{GL}_{m+n} & \xrightarrow{\text{res}_{m,n}} & \mathrm{Rep}_q L. \end{array} \quad (2.1)$$

The functor

$$\iota^* \otimes \pi^*: \mathrm{Rep}_q \mathrm{GL}_{m+n} \boxtimes \mathrm{Rep}_q L \rightarrow \mathrm{Rep}_q P$$

lifts to the Drinfeld center of  $\mathrm{Rep}_q P$ , endowing this category with the structure of a  $(\mathrm{Rep}_q \mathrm{GL}_{m+n}, \mathrm{Rep}_q L)$ -central algebra. The associated *parabolic defect* has been studied in detail in [BJ25] (for the case of  $\mathrm{SL}_2$  instead) and it associates, to a framed surface  $S$ , a bimodule functor

$$\mathrm{Sk}^{\mathrm{def}}(S): \mathrm{SkCat}_{\mathrm{GL}_{m+n}}(S) \boxtimes \mathrm{SkCat}_L(S)^{\mathrm{op}} \rightarrow \mathrm{VECT}_{\mathbb{k}}$$

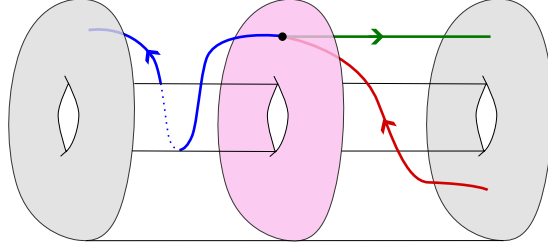


Figure 1: Defect bimodule on  $\mathbb{T}^2 \times [-1, 1]$ . The skein categories  $\text{SkCat}_{\text{GL}_{m+n}}(\mathbb{T}^2)$  and  $\text{SkCat}_L(\mathbb{T}^2)$  act by gluing skeins on the left and right bases, respectively. The defect wall  $\mathbb{T}^2 \times \{0\}$  is decorated by  $\text{Rep}_q P$  and interfaces both 3-dimensional theories.

whose components are defect skein modules as in Figure 3, lying at the boundary between the  $\text{Rep}_q \text{GL}_{m+n}$  and the  $\text{Rep}_q L$  theories.

Our first contribution is the construction of an interpolating version of this parabolic defect, extending the defect bimodule above to a bimodule between the HOMFLY theories associated with  $\text{Rep}_q \text{GL}_t$  and  $\text{Rep}_q L_t := \text{Rep}_q \text{GL}_t \boxtimes \text{Rep}_q \text{GL}_t$ . To do so, first note that, the restriction functor

$$j^*: \text{Rep}_q P \rightarrow \text{Rep}_q L$$

being faithful, we can describe the hom-spaces of  $\text{Rep}_q P$  as subspaces of the hom-spaces of  $\text{Rep}_q L$ . Identifying these subspaces and interpolating them *à la Deligne*, we define a category  $\text{Rep}_q P_t$  which is universal among the categories of quantum representations of  $P$ . Together with this category, we obtain a family of functors

$$\begin{array}{ccc} & \text{Rep}_q P_t & \\ \iota_t^* \nearrow & & \nwarrow j_t^* \\ \text{Rep}_q \text{GL}_t & \xrightarrow{\text{res}_t} & \text{Rep}_q L_t \\ & \nwarrow \pi_t^* & \end{array}$$

defined diagrammatically and interpolating the restriction functors in (2.2). We obtain the following structures:

**Theorem 2.1** ([Góm26]). *The functor*

$$\iota_t^* \otimes \pi_t^*: \text{Rep}_q \text{GL}_t \boxtimes \text{Rep}_q L_t \rightarrow \text{Rep}_q P_t$$

*lifts to the Drinfeld center of  $\text{Rep}_q P_t$ , turning  $\text{Rep}_q P_t$  into a  $(\text{Rep}_q \text{GL}_t, \text{Rep}_q L_t)$ -central algebra. Moreover,  $\text{Rep}_q L_t$  is a  $\text{Rep}_q P_t$ -centred  $(\text{Rep}_q \text{GL}_t, \text{Rep}_q L_t)$ -bimodule. ■*

Restricting the defect bimodule functor obtained by applying this formalism to closed components induces a bimodule over the HOMFLY skein algebras  $\text{SkAlg}_{\text{GL}_t}(S)$  and  $\text{SkAlg}_{L_t}(S) \cong \text{SkAlg}_{\text{GL}_t}(S) \otimes \text{SkAlg}_{\text{GL}_t}(S)$ . In order to upgrade this bimodule to an actual morphism of algebras recovering the Turaev coproduct, we modify the construction by incorporating line defects.

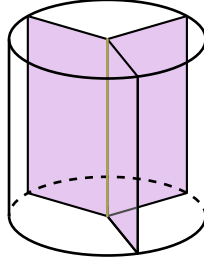


Figure 2: Local model for a line defect lying at the intersection of three defect walls.

## Line defects in skein theory

Our second contribution is joint work in progress with Jennifer Brown, David Jordan and Matthias Vancraeynest [Bro+26], which consists of the formalization of line defects in skein theory. In the light of extended TQFTs, the central algebras modelling surface defects in TQFTs corresponds to 1-morphisms between the adjacent ribbon categories in the Morita 4-category BRTENS studied in [BJS21]. Objects of this category are braided tensor categories, 1-morphisms are central algebras, 2-morphisms are centred bimodules, and 3 and 4-morphisms are bimodule functors and natural transformations, respectively. In this context, the algebraic data attached to line defects lying at the intersection of several surface defects correspond to centred bimodules between the underlying central algebras.

In [Bro+26] we develop a graphical calculus for centred bimodules and construct an evaluation functor extending the Reshetikhin–Turaev and the Brown–Jordan evaluation functors to accommodate defect lines:

**Theorem 2.2** ([Bro+26]). *Let  $\mathcal{A}_1, \dots, \mathcal{A}_n$  be ribbon categories and, for each  $i$  (taken modulo  $n$ ), let  $\mathcal{C}_i$  be an  $(\mathcal{A}_{i+1}, \mathcal{A}_i)$ -central algebra. Let  $\mathcal{L}$  be a left-module over*

$$\mathrm{tr}_{\mathcal{A}_1} (\mathcal{C}_1 \boxtimes_{\mathcal{A}_2} \mathcal{C}_2 \boxtimes_{\mathcal{A}_3} \dots \boxtimes_{\mathcal{A}_n} \mathcal{C}_n).$$

*There exists a unique full functor of centred bimodules*

$$\mathrm{ev}_{\mathcal{L}}^{\mathrm{CENTRED}}: \mathrm{CENTRED}(\mathcal{L}) \rightarrow \mathcal{L},$$

*where  $\mathrm{CENTRED}(\mathcal{L})$  is the free centred bimodule category on  $\mathcal{L}$ , described by coloured graphs in the stratified cylinder of Figure 4.* ■

The notions of skein module, skein algebra and skein category extend in a natural way to surfaces and 3-manifolds with both line and surface defects. By modifying the stratification that we consider on cylinders to include properly chosen defect lines coloured with  $\mathrm{Rep}_q L_t$  (cf. Theorem 2.1), we obtain a *HOMFLY defect bimodule functor*

$$\mathrm{Sk}_H^{\mathrm{def}}(S): \mathrm{SkCat}_{\mathrm{GL}_t}(S) \boxtimes \mathrm{SkCat}_{L_t}(S)^{\mathrm{op}} \rightarrow \mathrm{VECT}_{\mathbb{k}}$$

for any framed surface  $S$ .

## The Turaev coproduct

As our last contribution, we show that this construction is compatible with cutting and gluing surfaces, and we describe its components explicitly. In particular, we prove that it recovers the Turaev coproduct:

**Theorem 2.3** ([G6m26]). *Let  $(S, f)$  be a framed surface endowed with a suitably chosen stratification. We have an isomorphism of vector spaces*

$$\mathrm{Sk}_H^{\mathrm{def}}(S)(\emptyset) \cong \mathrm{SkAlg}_{\mathrm{GL}_t}(S) \otimes \mathrm{SkAlg}_{\mathrm{GL}_t}(S).$$

*Under this identification, the action of  $\mathrm{SkAlg}_{\mathrm{GL}_t}(S)$  on the empty diagram induces the Turaev coproduct*

$$\Delta_f: \mathrm{SkAlg}_{\mathrm{GL}_t}(S) \rightarrow \mathrm{SkAlg}_{\mathrm{GL}_t}(S) \otimes \mathrm{SkAlg}_{\mathrm{GL}_t}(S)$$

*on the HOMFLY skein algebra.* ■

The HOMFLY defect bimodule functor can thus be seen as an extension of the Turaev coproduct to the entire HOMFLY skein category. The multiplicativity is now a consequence of the functoriality of the construction, which becomes moreover compatible with cutting and gluing surfaces in a non-trivial way.

## 3 Outlook

Turaev showed in [Tur91] that the HOMFLY skein algebra provides a canonical quantization of the so-called *Goldman–Turaev Lie bialgebra*. Consider the vector space  $\mathcal{G}(S)$  generated by homotopy classes of free loops in a surface  $S$ . In [Gol86], Goldman constructed a Lie bracket  $[-, -]_{\mathrm{Goldman}}$  on  $\mathcal{G}(S)$  in terms of intersections of curves on  $S$ , providing a description of the Atiyah–Bott symplectic structure on the moduli space of flat connections on  $S$  [AB83]. In the presence of a framing  $f$ , the cobracket  $\delta_{\mathrm{Turaev}}$  defined by Turaev in [Tur91] lifts to a framing-dependent cobracket  $\delta_{\mathrm{Turaev}}^f$  on  $\mathcal{G}(S)$ , turning  $(\mathcal{G}(S), [-, -]_{\mathrm{Goldman}}, \delta_{\mathrm{Turaev}}^f)$  into a Lie bialgebra.

The Goldman–Turaev Lie bialgebra comes with a natural filtration, induced by the weight filtration on the group algebra of the fundamental group of  $S$ . The *formality problem* for the Goldman–Turaev bialgebra consists of determining the existence of a filtration-preserving isomorphism

$$\theta: \widehat{\mathcal{G}(S)} \xrightarrow{\sim} \mathrm{gr} \widehat{\mathcal{G}(S)}$$

between the completion of  $\mathcal{G}(S)$  with respect to the filtration and its associated graded space, such that  $\mathrm{gr} \theta = \mathrm{id}$ . This problem was given a positive answer in [Ale+18b; Ale+18c; Ale+18a]. In the case where  $S$  is a pair of pants, all such maps  $\theta$  can be written as  $\theta = F^{-1} \circ \theta_{\mathrm{exp}}$ , where  $F$  is an automorphism of a free Lie algebra on two

generators and  $\theta_{\text{exp}}$  is a distinguished map called the *exponential expansion*. The authors show that, in this case,  $\theta$  is a solution to the formality problem if and only if  $F$  is a solution to the Kashiwara–Vergne (KV) problem [KV78], a famous problem in Lie theory concerning the properties of the Baker–Campbell–Hausdorff series. The existence of solutions to the KV problem was established in [AT12] using the theory of Drinfeld associators, and an explicit solution to the formality problem was constructed in [Mas18] using the Kontsevich integral.

These results lay bare a surprising connection between low-dimensional topology on one hand, and Lie theory, the theory of Drinfeld associators, and the Kontsevich integral on the other. This connection should have a clearer interpretation in terms of factorization homology by extending the formalism developed in this thesis to a quantum version of Goldman–Turaev formality. A first step in this direction is to produce a construction similar to the one provided by Theorem 2.3 for the coproduct on the associated graded of the HOMFLY skein bialgebra. This coproduct has been described explicitly in [Sch05] and should have a reformulation and generalization in terms of chord diagrams, Drinfeld associators, and factorization homology, in light of the Kohno–Drinfeld theorem for the HOMFLY category [Eti+26].

On the other hand, in [BJ25], Brown–Jordan study a quantization of the  $A$ -polynomial using the framework of parabolic defect skein modules. Their construction relies on a module over the boundary skein algebra, defined via parabolic restriction for  $\text{SL}_2$ , rather than a direct map between algebras. Specializing our construction to  $t = 2$  in the absence of line defects (and modulo the reduction from  $\text{GL}_2$  to  $\text{SL}_2$ ), we recover the structure of the parabolic central algebra appearing in their work. The introduction of line defects allows us to upgrade the bimodule obtained by Brown–Jordan to an actual morphism of algebras. Namely, we can repeatedly apply Turaev’s coproduct to construct an algebra map

$$\Delta_f^N : \text{SkAlg}_{\text{GL}_t}(S) \rightarrow \text{SkAlg}_{\text{GL}_t}(S) \otimes \cdots \otimes \text{SkAlg}_{\text{GL}_t}(S).$$

Specializing to  $t = N$  on the left and to  $t_1 = \cdots = t_N = 1$  on the right, we obtain a morphism (rather than a bimodule)

$$\tilde{\Delta}_f^N : \text{Sk}_{\text{GL}_N}(S) \rightarrow \text{Sk}_{\mathbb{C}^N}(S)$$

between the  $\text{GL}_N$ - and the  $\mathbb{C}^N$ -skein algebras of the surface. This should provide a more natural setting for the quantization of the  $A$ -polynomial.

Finally, the formalism for line defects that we develop here allows us to consider, as particular cases, line defects embedded in a 3-dimensional region, viewing them as lying at the interface of two trivial surface defects. The algebraic data attached to a line is equivalent to the data of a *braided module category*  $\mathcal{A}$  [Bro12; Bro13]. In [BBJ18b], it was shown that providing a braided module structure is equivalent to specifying a *quantum moment map*. For the case  $\mathcal{A} = \text{Rep}_q G$ , examples of such moment maps were constructed using Verma modules in [DDM02; DM04] and were used to construct equivariant quantizations of conjugacy classes. We expect that a new construction, as

well as a skein-theoretic interpretation of these maps, can be given using the formalism of parabolic restriction and line defects.

## Outline of the thesis

The thesis is divided into four chapters:

- Chapter 1 contains mainly background material. In Section 1.1, we provide an overview of the formalism of monoidal categories and fix the conventions for the rest of the document. In Section 1.2, we review the graphical calculus for monoidal, rigid, pivotal and ribbon categories. In Section 1.3, we introduce string-nets and skein theory.
- Chapter 2 is devoted to defect theories. In Section 2.1, we introduce the algebraic machinery needed to extend string-nets and skein theory to stratified surfaces and 3-manifolds. In Section 2.2, we provide local models for planar line defects in string-net theories and for both surface and line defects in skein theory, based on the graphical calculus for bimodule categories, central algebras and centred bimodules. In Section 2.3, we introduce defects in string-nets and skein theory and explain the construction in the language of factorisation homology.
- In Chapter 3, we construct a HOMFLY analogue of the formalism of parabolic restriction. In Section 3.1, we give a quick overview of quantum groups and parabolic restriction. In Section 3.2 we recall the definition of the HOMFLY category  $\text{Rep}_q \text{GL}_t$  and construct an interpolating version of the restriction functor for representations from  $\text{GL}_{m+n}$  to  $\text{GL}_m \times \text{GL}_n$ . In Section 3.3, we construct a HOMFLY version of parabolic restriction and describe a central algebra and a centred bimodule arising from this formalism.
- In Chapter 4 we study the defect string-net and skein theories associated with the HOMFLY formalism and use it to extend the Turaev coproduct to the HOMFLY skein category. In Section 4.1, we describe explicitly the state spaces provided by the HOMFLY defect theories. In Section 4.2, we recover the Turaev coproduct as a particular instance of this formalism. In Section 4.3, we extend the counit map of the HOMFLY bialgebra to the HOMFLY skein category.

# Introduction en français

## 1 Contexte

Cette thèse se situe à l’intersection de la topologie de basse dimension et de la théorie des représentations. Elle traite d’une famille de théories quantiques des champs topologiques issues du calcul graphique pour les catégories monoïdales et monoïdales tressées, ainsi que du formalisme de la restriction parabolique.

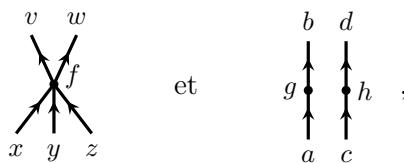
### Catégories monoïdales et leur calcul graphique

Une *catégorie monoïdale* est une catégorie  $\mathcal{C}$  munie d’un foncteur *produit tensoriel*, qui est associatif et unitaire à des isomorphismes prescrits près. Introduites pour la première fois par MacLane [Mac63; Mac78], elles peuvent être vues comme des versions catégoriques des notions d’anneau et de monoïde, et sont aujourd’hui omniprésentes en mathématiques, en physique et en informatique. Comme l’a souligné Penrose [Pen71], une caractéristique remarquable des catégories monoïdales est le fait qu’elles admettent un calcul graphique très riche, formalisé par Joyal et Street dans [JSa; JS91; JSb; JS93]. En effet, un morphisme dans n’importe quelle catégorie peut être représenté par un diagramme *linéaire* de la forme

$$\begin{array}{c} y \\ \uparrow \\ \bullet \\ \uparrow \\ x \end{array} \quad f \quad ,$$

où les arêtes représentent respectivement les objets source et cible  $x$  et  $y$ , et le sommet est le morphisme lui-même. Il ne s’agit là que d’une manière élégante d’exploiter l’habitude usuelle, dans tous les domaines des mathématiques, de dénoter les applications entre ensembles munis d’une structure quelconque à l’aide de flèches. L’unique direction représente la seule opération disponible, à savoir la composition de morphismes, qui est donnée par la simple concaténation de graphes. En présence d’une structure monoïdale, nous pouvons donner un sens à une classe plus générale de graphes, se trouvant désormais dans le plan, la nouvelle dimension encodant l’opération supplémentaire fournie par le

produit tensoriel. Par exemple, les diagrammes

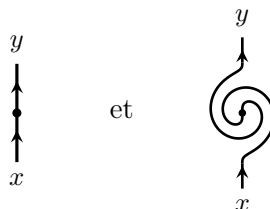


représentent, respectivement, un morphisme  $f: x \otimes y \otimes z \rightarrow v \otimes w$  et le produit tensoriel  $g \otimes h$  de deux morphismes  $g: a \rightarrow b$  et  $h: c \rightarrow d$ .

Plus la catégorie sous-jacente possède de structure et de propriétés, plus ce calcul diagrammatique devient riche et intéressant. Une catégorie *rigide* est une catégorie monoïdale  $\mathcal{C}$  dont les objets admettent des *duaux*. L'opération de passage au dual peut se traduire diagrammatiquement par l'inversion des orientations ou en permettant aux arêtes de faire demi-tour, ce qui donne lieu à des diagrammes plus sophistiqués. Par exemple,



représentent désormais un morphisme  $f: x^* \otimes y \otimes z^* \rightarrow v^* \otimes w$  et le dual  $g^*: y^* \rightarrow x^*$  d'un morphisme  $g: x \rightarrow y$ , respectivement. Ces graphes sont *rigides* au sens où ils ne sont pas invariants par déformations : par exemple,

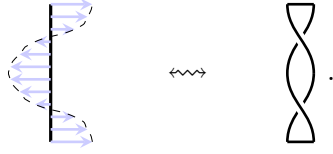


peuvent être obtenus l'un à partir de l'autre en effectuant une rotation de  $2\pi$  autour du sommet. Cependant, le premier diagramme représente un morphisme  $x \rightarrow y$ , tandis que le second représente un morphisme  $x^{**} \rightarrow y^{**}$  (puisque les arêtes font demi-tour deux fois).

Une catégorie *pivotale* est une catégorie  $\mathcal{C}$  munie d'une famille naturelle d'isomorphismes  $x \cong x^{**}$  identifiant les objets à leurs doubles duaux. Sous cette identification, les deux diagrammes ci-dessus peuvent en fait être vus comme des morphismes  $x \rightarrow y$  et, par conséquent, le calcul graphique résultant pour les catégories pivotales est compatible avec la topologie de  $\mathbb{R}^2$ .

Lorsque les catégories sont tressées, le calcul graphique s'étend en trois dimensions, où le tressage est représenté par deux brins se croisant dans l'espace tridimensionnel. Cela devient particulièrement intéressant lorsque  $\mathcal{C}$  est une catégorie *enrubannée*. Une

catégorie enrubannée est une catégorie rigide tressée munie d’une notion compatible de *twist*, un automorphisme naturel spécial du foncteur identité. Un résultat fondamental de Reshetikhin–Turaev [RT90] et Shum [Shu94] montre que les propriétés algébriques de ces catégories sont régies par la topologie des *graphes en rubans*. Ce sont des graphes dont les arêtes sont munies d’un champ de vecteurs normaux non nul, ce qui permet de les considérer comme de véritables rubans :



La topologie des graphes en rubans reflète diagrammatiquement la structure algébrique des catégories enrubannées : étant donné une catégorie enrubannée  $\mathcal{C}$ , la construction de Reshetikhin–Turaev fournit un foncteur d’évaluation monoïdal plein (mais non fidèle !)

$$\mathrm{RT}_{\mathcal{C}}: \mathrm{Rib}(\mathcal{C}) \rightarrow \mathcal{C},$$

où  $\mathrm{Rib}(\mathcal{C})$  est une catégorie dont les morphismes sont des classes d’isotopie de graphes en rubans coloriés par les objets et les morphismes de  $\mathcal{C}$ . Ce foncteur offre un cadre unifié pour les invariants quantiques polynomiaux introduits après la découverte du polynôme de Jones [Jon85], et se trouve au cœur de l’interaction fructueuse entre la topologie de basse dimension et la théorie des représentations.

## Réseaux de cordes et théorie des écheveaux

Le calcul graphique pour les catégories monoïdales est *local*, au sens où il a lieu à l’intérieur de rectangles et de cubes dans  $\mathbb{R}^2$  ou  $\mathbb{R}^3$ . Une généralisation aux surfaces orientées et aux variétés de dimension 3 arbitraires est fournie respectivement par les *réseaux de cordes* (string-nets) et la *théorie des écheveaux* (skein theory).

Les espaces de réseaux de cordes sont des espaces vectoriels dont les générateurs sont des graphes décorés (les réseaux de cordes) plongés dans une surface et considérés à des relations près, induites localement par les données algébriques d’une catégorie rigide ou pivotale  $\mathcal{C}$ . Ils trouvent leur origine en physique, où ils fournissent des modèles pour les phases topologiques de la matière [LW05; LW06]. Ils ont été formulés mathématiquement par Kirillov dans [Kir11] en utilisant une catégorie de fusion sphérique comme donnée initiale, et ont ensuite été étendus aux catégories de fusion non sphériques dans [Run20]. Comme dans le cas local, une structure plus riche du côté algébrique se traduit par une structure moins rigide du côté géométrique : les catégories rigides donnent lieu à des théories uniquement pour les surfaces *parallélisées* [KST24], qui se relèvent aux surfaces orientées en présence d’une structure pivotale [HSS26].

La version tridimensionnelle de l’histoire est connue sous le nom de *théorie des écheveaux*. La théorie des écheveaux s’intéresse à l’étude des modules (dits d’écheveaux) [Prz06; Tur90], algèbres [PS97] et catégories [Yet92; Wal06; Joh19; Coo23] engendrés

par des graphes en rubans dans une variété de dimension 3, modulo les relations induites localement par la construction de Reshetikhin–Turaev, en utilisant une catégorie enrubannée comme donnée initiale. Au cours de la dernière décennie, la théorie des écheveaux a connu un développement important, les modules, algèbres et catégories d’écheveaux devenant des objets centraux en topologie quantique. L’introduction des algèbres d’écheveaux à états (*stated skein algebras*) [BW11] a doté la théorie des écheveaux de puissantes propriétés d’excision, étudiées dans [CL22; CL25]. D’autre part, les modules d’écheveaux *admissibles* [CGP23; RST24; BH24] ont étendu ce cadre aux catégories en rubans non semi-simples, ouvrant la voie à de nouvelles TQFTs au-delà du cadre semi-simple [Cos+26; Hai25]. Au niveau catégorique, les catégories d’écheveaux sont apparues comme un cadre naturel pour l’étude de l’homologie de factorisation [BBJ18a; BBJ18b; Coo23; BH24]. Récemment, des versions dérivées de modules et de catégories d’écheveaux ont été introduites dans [Bai26]. Ces développements se sont accompagnés d’avancées théoriques majeures, notamment la preuve de la conjecture de finitude de Witten pour les modules d’écheveaux [GJS23] (voir aussi [JR25; BD25]).

## TQFTs et défauts

Les réseaux de cordes ainsi que les théories des écheveaux sont des exemples de *théories quantiques des champs topologiques* (TQFTs) définies par générateurs et relations [Wal06]. Une TQFT de dimension  $n$  est un foncteur monoïdal symétrique

$$\mathcal{Z}: \text{Cob}_n^{\sqcup} \rightarrow \mathcal{C}^{\otimes},$$

où  $\text{Cob}_n^{\sqcup}$  est une catégorie dont les objets sont des variétés fermées de dimension  $n - 1$  et dont les morphismes sont des classes d’équivalence de cobordismes parallélisés ou orientés entre elles, et où  $\mathcal{C}$  est une catégorie monoïdale. Ces théories fournissent des modèles simplifiés pour les théories quantiques des champs en physique et, depuis leur axiomatisation par Atiyah [Ati88], se sont révélées être une riche source de mathématiques, ainsi qu’un outil puissant pour organiser les structures en géométrie et topologie.

Des structures algébriques plus complexes peuvent être rencontrées en enrichissant la topologie des cobordismes via l’introduction de *défauts*. En physique, les défauts sont des structures dans une théorie où la dynamique ambiante est modifiée, apparaissant souvent aux interfaces entre différentes phases ou résultant de perturbations abruptes. Mathématiquement, ils sont décrits par des sous-variétés plongées de codimension strictement positive, appelées *strates*, dont chacune porte ses propres données algébriques et régit l’interaction entre les théories attribuées aux régions adjacentes. Une définition axiomatique pour les TQFTs de dimension  $n$  avec défauts a été donnée dans [CMS20; CRS18a; CRS19; CRS18b] en tant que foncteurs monoïdaux symétriques

$$\mathcal{Z}: \text{Cob}_n^{\text{dec}, \sqcup} \rightarrow \mathcal{C}^{\otimes}$$

allant d’une catégorie  $\text{Cob}_n^{\text{dec}}$  de cobordismes stratifiés de dimension  $n$  étiquetés avec des données algébriques appropriées vers une catégorie monoïdale  $\mathcal{C}$ .

L'étude des défauts dans la théorie des écheveaux a été initiée dans [BJ25], où les notions de modules, d'algèbres et de catégories d'écheveaux ont été étendues aux variétés de dimension 3 munies de défauts orientés de codimension 1, dans le cadre de l'étude du  $A$ -polynôme quantique. Les données algébriques modélisant de tels défauts sont données par une *algèbre centrale pivotale*  $\mathcal{C}$  sur les catégories enrubannées  $\mathcal{A}$  et  $\mathcal{B}$  décorant les régions tridimensionnelles adjacentes. Concrètement, une algèbre centrale consiste en un foncteur monoïdal tressé pivot

$$F: \mathcal{A} \boxtimes \mathcal{B}^{\text{bop}} \longrightarrow Z_{\text{Dr}}(\mathcal{C}),$$

où  $\mathcal{B}^{\text{bop}}$  désigne l'opposée tressée de  $\mathcal{B}$  et  $Z_{\text{Dr}}(\mathcal{C})$  est le centre de Drinfeld de  $\mathcal{C}$ .

Étant donné une variété stratifiée  $M$  de dimension 3, son *module d'écheveaux à défauts* est engendré par des classes d'isotopie de graphes plongés dans  $M$  tels que, dans chaque région tridimensionnelle, ils forment des graphes en rubans coloriés par soit  $\mathcal{A}$  soit  $\mathcal{B}$ , tandis que leur intersection avec la paroi du défaut est un réseau de cordes colorié par  $\mathcal{C}$ . La structure centrale fournit les données algébriques locales nécessaires pour projeter les graphes en rubans sur le défaut et les évaluer dans  $\mathcal{C}$  via le calcul graphique des catégories pivotales, produisant ainsi les relations d'écheveaux dans des voisinages du défaut.

D'autre part, les modules d'écheveaux à défauts pour des variétés avec des défauts de ligne et de point sur leur bord ont été récemment introduits dans [KR26], dans le but de donner une description en termes de théorie des écheveaux des espaces d'états pour la TQFT à défauts de Reshetikhin–Turaev issue de [CRS18a]. Dans le même esprit, les défauts de ligne sont coloriés par des modules sur la catégorie enrubannée décorant la région tridimensionnelle, les défauts de point sont coloriés par des foncteurs bimodules, et un foncteur d'évaluation produisant des relations d'écheveaux près du bord est défini en utilisant le calcul graphique des structures sous-jacentes.

## Catégories HOMFLY

Une source importante pour les données algébriques induisant toutes ces constructions diagrammatiques et topologiques est la théorie des représentations. Des exemples prototypiques de catégories enrubannées sont les catégories de représentations de *groupes quantiques*. Fixons un anneau commutatif  $\mathbb{k}$ . Étant donné un groupe algébrique affine réductif  $G$  et un paramètre suffisamment générique  $q \in \mathbb{k}^\times$ , la catégorie  $\text{Rep}_q G$  est équivalente en tant que catégorie abélienne à la catégorie  $\text{Rep } G$  des représentations algébriques de  $G$ , mais elle possède une structure enrubannée non triviale qui est une déformation de celle (triviale) de  $\text{Rep } G$ . Lorsque  $G = \text{GL}_N$ , les relations engendrant le noyau de  $\text{RT}_{\mathcal{C}}$  dépendent de façon très simple du paramètre entier  $N \in \mathbb{N}$ . En remplaçant  $N$  par un paramètre formel  $t$ , on obtient les *relations HOMFLY*

$$\begin{array}{c} \text{X} \\ \text{X} \end{array} - \begin{array}{c} \text{X} \\ \text{X} \end{array} = (q - q^{-1}) \begin{array}{c} \uparrow \\ \uparrow \end{array},$$

$$\text{blue loop} = q^t \text{blue vertical line} \quad \text{et} \quad \text{blue circle} = \frac{q^t - q^{-t}}{q - q^{-1}} \mathbf{1}_\emptyset.$$

La catégorie *HOMFLY* [Bru17] est définie comme le quotient de la catégorie des *entrelacs en rubans* (une sous-catégorie de  $\text{Rib}(\mathcal{C})$ ) par les relations d'écheveaux HOMFLY. Cette catégorie est une version quantique de la *catégorie de Deligne* [Del07] et peut être comprise comme une catégorie universelle pour les représentations quantiques de  $\text{GL}_N$  : pour tout  $N \in \mathbb{N}$ , il existe un foncteur de spécialisation monoïdal

$$\text{sp}_N : \text{Rep}_q \text{GL}_t \rightarrow \text{Rep}_q \text{GL}_N$$

permettant de retrouver  $\text{Rep}_q \text{GL}_N$  comme un quotient de la spécialisation de  $\text{Rep}_q \text{GL}_t$  en  $t = N$ .

## Le coproduit de Turaev

L'invariant polynomial d'entrelacs obtenu pour  $\mathcal{C} = \text{Rep}_q \text{GL}_t$  via la construction de Reshetikhin–Turaev est le *polynôme HOMFLY* [Fre+85; PT16]. Notons  $H_{q,t}(D)$  le polynôme HOMFLY d'un diagramme d'entrelacs  $D$ . Dans [Jae89], Jaeger a donné une formule de composition permettant de calculer  $H_{q,t_1+t_2}(D)$  comme une somme d'états sur  $H_{q,t_1}(D_1)$  et  $H_{q,t_2}(D_2)$ , où  $D_1, D_2$  sont des sous-diagrammes de  $D$ . Concrètement, un diagramme d'entrelacs  $D$  est un graphe 4-valent avec deux classes distinguées de sommets : les croisements positifs et négatifs. Étant donné un sommet  $v$ , on note  $a$  (resp.  $b$ ) l'arête entrante supérieure (resp. inférieure), et  $c$  (resp.  $d$ ) l'arête sortante supérieure (resp. inférieure), comme illustré sur les figures suivantes :



Soit  $E_D$  l'ensemble des arêtes de  $D$ . Un *étiquetage admissible* est une application  $f : E_D \rightarrow \{1, 2\}$  telle que, à chaque sommet  $v$ , on ait soit  $f(a) = f(c)$  et  $f(b) = f(d)$ , soit  $f(a) = f(d) > f(b) = f(c)$ . Lorsque la seconde condition est vérifiée, on dit que  $v$  est un *sommet de coupure*. Étant donné un étiquetage admissible  $f$  et un sommet  $v$ , on pose

$$\langle v \mid f \rangle := \begin{cases} \text{sgn}(v)(q - q^{-1}), & \text{si } v \text{ est un sommet de coupure,} \\ 1, & \text{sinon,} \end{cases}$$

où  $\text{sgn}(v)$  vaut 1 si  $v$  est un croisement positif et  $-1$  sinon. On définit *l'interaction de  $D$  avec  $f$*  par

$$\langle D \mid f \rangle := \prod_v \langle v \mid f \rangle,$$

où le produit parcourt l'ensemble des sommets de  $D$ . Jaeger a montré que

$$H_{q,t_1+t_2}(D) = \sum_{f \text{ admissible}} \langle D \mid f \rangle (q^{-t_1})^{r(D_{f,1})} (q^{t_2})^{r(D_{f,2})} H_{q,t_1}(D_{f,1}) H_{q,t_2}(D_{f,2}),$$

où  $D_{f,i} := f^{-1}(i)$  et  $r(D_{f,i})$  est le nombre de rotation de  $D_{f,i}$ , pour  $i \in \{1, 2\}$ .

Turaev a montré dans [Tur91] que la formule de Jaeger s'étend à l'algèbre d'écheveaux HOMFLY des surfaces parallélisées. Plus précisément :

**Theorem 1.1** ([Tur91]). *Soit  $(S, f)$  une surface parallélisée. L'application linéaire*

$$\Delta_f : \text{SkAlg}_{\text{GL}_t}(S) \rightarrow \text{SkAlg}_{\text{GL}_t}(S) \otimes \text{SkAlg}_{\text{GL}_t}(S)$$

*définie en prolongeant la formule de Jaeger est un morphisme d'algèbres. Elle est coassociative, donc  $\text{SkAlg}_{\text{GL}_t}(S)$  est une bigèbre.* ■

## 2 Résultats

Dans cette thèse, nous développons un cadre pour les défauts de ligne en théorie des écheveaux, proposons une version HOMFLY de la restriction parabolique, et combinons les deux formalismes pour donner une nouvelle construction du coproduit de Turaev qui s'étend à la catégorie d'écheveaux HOMFLY.

### Restriction parabolique HOMFLY

La formule de Jaeger a traditionnellement été interprétée comme une version universelle de la restriction des représentations de  $\text{GL}_{m+n}$  à  $\text{GL}_m \times \text{GL}_n$ , au sens où en spécialisant en  $t_1 = m$  et  $t_2 = n$ , le diagramme

$$\begin{array}{ccc} D & \xrightarrow{\text{Jaeger}} & \sum_f D_{f,1} \sqcup D_{f,2} \\ \text{RT}_{\text{Rep}_q \text{GL}_{m+n}} \downarrow & & \downarrow \text{RT}_{\text{Rep}_q(\text{GL}_m \times \text{GL}_n)} \\ H_{q,m+n}(D) & \xrightarrow{\text{res}_{m,n}} & \sum_f \alpha(f) H_{q,m}(D_{f,1}) H_{q,n}(D_{f,2}) \end{array}$$

est commutatif pour tout diagramme d'entrelacs  $D$ , où

$$\text{res}_{m,n} : \text{Rep}_q \text{GL}_{m+n} \rightarrow \text{Rep}_q(\text{GL}_m \times \text{GL}_n)$$

est le foncteur de restriction induit par le plongement bloc-diagonal  $\text{GL}_m \times \text{GL}_n \hookrightarrow \text{GL}_{m+n}$ , et  $\alpha(f)$  est le coefficient apparaissant dans la formule de Jaeger.

De ce point de vue, l'existence du coproduit de Turaev est surprenante. Le foncteur de restriction étant monoïdal, il induit un morphisme entre les théories planaires associées à  $\text{Rep}_q \text{GL}_{m+n}$  et  $\text{Rep}_q(\text{GL}_m \times \text{GL}_n)$ . Cependant, il n'est pas tressé, il n'y a donc a priori aucune raison pour que ce morphisme se relève en trois dimensions ni pour qu'il soit multiplicatif (le produit de  $\text{SkAlg}_{\text{GL}_t}(S)$  étant une construction purement tridimensionnelle).

Une approche différente peut être menée en utilisant la restriction parabolique et la théorie des écheveaux avec défauts. Soit  $P$  le sous-groupe parabolique des matrices

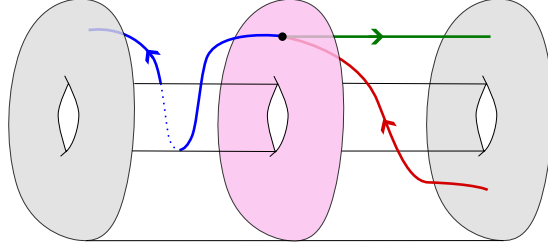


Figure 3: Bimodule de défaut sur  $\mathbb{T}^2 \times [-1, 1]$ . Les catégories d'écheveaux  $\text{SkCat}_{\text{GL}_{m+n}}(\mathbb{T}^2)$  et  $\text{SkCat}_L(\mathbb{T}^2)$  agissent par recollement d'écheveaux sur les bases gauche et droite, respectivement. La paroi du défaut  $\mathbb{T}^2 \times \{0\}$  est décorée par  $\text{Rep}_q P$  et assure l'interface entre les deux théories tridimensionnelles.

triangulaires supérieures par blocs de  $\text{GL}_{m+n}$  et notons  $L := \text{GL}_m \times \text{GL}_n$  son sous-groupe de Levi de matrices diagonales par blocs. Alors,  $L$  est à la fois un quotient et un sous-groupe de  $P$  et nous avons des morphismes de groupes

$$\iota: P \hookrightarrow \text{GL}_{m+n}, \quad j: L \hookrightarrow P \quad \text{et} \quad \pi: P \twoheadrightarrow L.$$

La restriction des représentations le long de ces morphismes induit la famille de foncteurs suivante :

$$\begin{array}{ccc} & \text{Rep}_q P & \\ \iota^* \nearrow & & \nwarrow j^* \\ \text{Rep}_q \text{GL}_{m+n} & \xrightarrow{\text{res}_{m,n}} & \text{Rep}_q L \\ & \nwarrow \pi^* & \end{array} \quad (2.2)$$

Le foncteur

$$\iota^* \otimes \pi^*: \text{Rep}_q \text{GL}_{m+n} \boxtimes \text{Rep}_q L \rightarrow \text{Rep}_q P$$

se relève au centre de Drinfeld de  $\text{Rep}_q P$ , dotant cette catégorie de la structure d'une algèbre centrale sur  $(\text{Rep}_q \text{GL}_{m+n}, \text{Rep}_q L)$ . Le *défaut parabolique* associé a été étudié en détail dans [BJ25] (pour le cas de  $\text{SL}_2$ ) et il associe, à une surface parallélisée  $S$ , un foncteur bimodule

$$\text{Sk}^{\text{def}}(S): \text{SkCat}_{\text{GL}_{m+n}}(S) \boxtimes \text{SkCat}_L(S)^{\text{op}} \rightarrow \text{Vect}_{\mathbb{k}}$$

dont les composantes sont des modules d'écheveaux à défauts comme dans la Figure 3, situés à la frontière entre les théories de  $\text{Rep}_q \text{GL}_{m+n}$  et de  $\text{Rep}_q L$ .

Notre première contribution est la construction d'une version interpolante de ce défaut parabolique, qui étend le bimodule avec défaut ci-dessus à un bimodule entre les théories HOMFLY associées à  $\text{Rep}_q \text{GL}_t$  et  $\text{Rep}_q L_t := \text{Rep}_q \text{GL}_t \boxtimes \text{Rep}_q \text{GL}_t$ . Pour ce faire, remarquons tout d'abord que, le foncteur de restriction

$$j^*: \text{Rep}_q P \rightarrow \text{Rep}_q L$$

étant fidèle, nous pouvons décrire les espaces de morphismes de  $\text{Rep}_q P$  comme des sous-espaces des espaces de morphismes de  $\text{Rep}_q L$ . En identifiant ces sous-espaces et en les interpolant à la Deligne, nous définissons une catégorie  $\text{Rep}_q P_t$  qui est universelle parmi les catégories de représentations quantiques de  $P$ . Parallèlement, nous construisons une famille de foncteurs

$$\begin{array}{ccc} & \text{Rep}_q P_t & \\ \iota_t^* \nearrow & & \nwarrow j_t^* \\ \text{Rep}_q GL_t & \xrightarrow{\text{res}_t} & \text{Rep}_q L_t \\ & \nwarrow \pi_t^* & \end{array}$$

définis diagrammatiquement et interpolant les foncteurs de restriction de (2.2). Nous obtenons les structures suivantes :

**Theorem 2.1** ([Góm26]). *Le foncteur*

$$\iota_t^* \otimes \pi_t^* : \text{Rep}_q GL_t \boxtimes \text{Rep}_q L_t \rightarrow \text{Rep}_q P_t$$

se relève au centre de Drinfeld de  $\text{Rep}_q P_t$ , faisant de  $\text{Rep}_q P_t$  une algèbre centrale sur  $(\text{Rep}_q GL_t, \text{Rep}_q L_t)$ . De plus,  $\text{Rep}_q L_t$  est un  $(\text{Rep}_q GL_t, \text{Rep}_q L_t)$ -bimodule  $\text{Rep}_q P_t$ -centré. ■

La restriction du foncteur bimodule avec défaut obtenu par application de ce formalisme à des composantes fermées induit un bimodule sur les algèbres d'écheveaux HOMFLY  $\text{SkAlg}_{GL_t}(S)$  et  $\text{SkAlg}_{L_t}(S) \cong \text{SkAlg}_{GL_t}(S) \otimes \text{SkAlg}_{GL_t}(S)$ . Afin de promouvoir ce bimodule au rang de véritable morphisme d'algèbres permettant de retrouver le coproduit de Turaev, nous modifions la construction en incorporant des défauts de ligne.

## Défauts de ligne en théorie des écheveaux

Notre deuxième contribution est un travail en cours en collaboration avec Jennifer Brown, David Jordan et Matthias Vancraeynest [Bro+26], qui consiste en la formalisation des défauts de ligne en théorie des écheveaux. À la lumière des TQFTs étendues, les algèbres centrales modélisant les défauts de surface dans les TQFTs correspondent aux 1-morphismes entre les catégories enrubannées adjacentes dans la 4-catégorie de Morita BRTENS étudiée dans [BJS21]. Les objets de cette catégorie sont les catégories monoïdales tressées, les 1-morphismes sont les algèbres centrales, les 2-morphismes sont les bimodules centrés, et les 3- et 4-morphismes sont respectivement les foncteurs bimodules et les transformations naturelles. Dans ce contexte, les données algébriques attachées aux défauts de ligne se trouvant à l'intersection de plusieurs défauts de surface correspondent à des bimodules centrés entre les algèbres centrales sous-jacentes.

Dans [Bro+26], nous développons un calcul graphique pour les bimodules centrés et construisons un foncteur d'évaluation étendant les foncteurs d'évaluation de Reshetikhin–Turaev et de Brown–Jordan pour accommoder les lignes de défaut :

**Theorem 2.2** ([Bro+26]). *Soient  $\mathcal{A}_1, \dots, \mathcal{A}_n$  des catégories enrubannées et, pour chaque  $i$  (modulo  $n$ ), soit  $\mathcal{C}_i$  une algèbre centrale sur  $(\mathcal{A}_{i+1}, \mathcal{A}_i)$ . Soit  $\mathcal{L}$  un module à gauche sur*

$$\text{tr}_{\mathcal{A}_1} (\mathcal{C}_1 \boxtimes_{\mathcal{A}_2} \mathcal{C}_2 \boxtimes_{\mathcal{A}_3} \dots \boxtimes_{\mathcal{A}_n} \mathcal{C}_n).$$

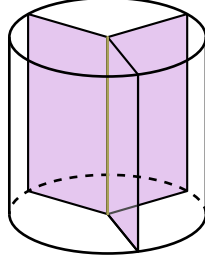


Figure 4: Modèle local pour un défaut de ligne situé à l'intersection de trois parois de défaut.

Il existe un unique foncteur plein de bimodules centrés

$$\mathrm{ev}_{\mathcal{L}}^{\mathrm{CENTRED}}: \mathrm{CENTRED}(\mathcal{L}) \rightarrow \mathcal{L},$$

où  $\mathrm{CENTRED}(\mathcal{L})$  est la catégorie de bimodules centrés libre sur  $\mathcal{L}$ , décrite par des graphes coloriés dans le cylindre stratifié de la Figure 4. ■

Les notions de module, d'algèbre et de catégorie d'écheveaux s'étendent de manière naturelle aux surfaces et aux variétés de dimension 3 présentant à la fois des défauts de ligne et de surface. En modifiant la stratification que nous considérons sur les cylindres pour inclure des lignes de défaut bien choisies et coloriées par  $\mathrm{Rep}_q L_t$  (cf. Théorème 2.1), nous obtenons un *foncteur bimodule de défaut HOMFLY*

$$\mathrm{Sk}_H^{\mathrm{def}}(S): \mathrm{SkCat}_{\mathrm{GL}_t}(S) \boxtimes \mathrm{SkCat}_{L_t}(S)^{\mathrm{op}} \rightarrow \mathrm{Vect}_{\mathbb{k}}$$

pour toute surface parallélisée  $S$ .

## Le coproduit de Turaev

Comme dernière contribution, nous montrons que cette construction est compatible avec le découpage et le recollement des surfaces, et nous décrivons ses composantes de manière explicite. En particulier, nous prouvons qu'elle permet de retrouver le coproduit de Turaev :

**Theorem 2.3** ([Góm26]). *Soit  $(S, f)$  une surface parallélisée munie d'une stratification bien choisie. Nous avons un isomorphisme d'espaces vectoriels*

$$\mathrm{Sk}_H^{\mathrm{def}}(S)(\varnothing) \cong \mathrm{SkAlg}_{\mathrm{GL}_t}(S) \otimes \mathrm{SkAlg}_{\mathrm{GL}_t}(S).$$

*Sous cette identification, l'action de  $\mathrm{SkAlg}_{\mathrm{GL}_t}(S)$  sur le diagramme vide induit le coproduit de Turaev*

$$\Delta_f: \mathrm{SkAlg}_{\mathrm{GL}_t}(S) \rightarrow \mathrm{SkAlg}_{\mathrm{GL}_t}(S) \otimes \mathrm{SkAlg}_{\mathrm{GL}_t}(S)$$

*sur l'algèbre d'écheveaux HOMFLY.* ■

Le foncteur bimodule avec défaut HOMFLY peut ainsi être vu comme une extension du coproduit de Turaev à l'ensemble de la catégorie d'écheveaux HOMFLY. La multiplicativité est maintenant une conséquence de la fonctorialité de la construction, qui devient de plus compatible de façon non triviale avec le découpage et le recollement des surfaces.

### 3 Perspectives

Turaev a montré dans [Tur91] que l'algèbre d'écheveaux HOMFLY fournit une quantification canonique de ce qu'on appelle la *bigèbre de Lie de Goldman–Turaev*. Considérons l'espace vectoriel  $\mathcal{G}(S)$  engendré par les classes d'homotopie des lacets libres dans une surface  $S$ . Dans [Gol86], Goldman a construit un crochet de Lie  $[-, -]_{\text{Goldman}}$  sur  $\mathcal{G}(S)$  en termes d'intersections de courbes sur  $S$ , fournissant une description de la structure symplectique d'Atiyah–Bott sur l'espace des modules des connexions plates sur  $S$  [AB83]. En présence d'un framing  $f$ , le cocrochet  $\delta_{\text{Turaev}}$  défini par Turaev dans [Tur91] se relève en un cocrochet dépendant du framing  $\delta_{\text{Turaev}}^f$  sur  $\mathcal{G}(S)$ , faisant de  $(\mathcal{G}(S), [-, -]_{\text{Goldman}}, \delta_{\text{Turaev}}^f)$  une bigèbre de Lie.

La bigèbre de Lie de Goldman–Turaev est munie d'une filtration naturelle, induite par la filtration par les poids sur l'algèbre du groupe du groupe fondamental de  $S$ . Le *problème de la formalité* pour la bigèbre de Goldman–Turaev consiste à déterminer l'existence d'un isomorphisme préservant la filtration

$$\theta: \widehat{\mathcal{G}(S)} \xrightarrow{\sim} \text{gr } \widehat{\mathcal{G}(S)}$$

entre la complétion de  $\mathcal{G}(S)$  par rapport à la filtration et son espace gradué associé, tel que  $\text{gr } \theta = \text{id}$ . Ce problème a reçu une réponse positive dans [Ale+18b; Ale+18c; Ale+18a]. Dans le cas où  $S$  est un *pantalon*, toutes ces applications  $\theta$  peuvent s'écrire sous la forme  $\theta = F^{-1} \circ \theta_{\text{exp}}$ , où  $F$  est un automorphisme d'une algèbre de Lie libre à deux générateurs et  $\theta_{\text{exp}}$  est une application distinguée appelée *l'expansion exponentielle*. Les auteurs montrent que, dans ce cas,  $\theta$  est une solution au problème de la formalité si et seulement si  $F$  est une solution au problème de Kashiwara–Vergne (KV) [KV78], un problème célèbre en théorie de Lie concernant les propriétés de la série de Baker–Campbell–Hausdorff. L'existence de solutions au problème KV a été établie dans [AT12] en utilisant la théorie des associateurs de Drinfeld, et une solution explicite au problème de la formalité a été construite dans [Mas18] en utilisant l'intégrale de Kontsevich.

Ces résultats mettent en évidence un lien surprenant entre la topologie de basse dimension d'une part, et la théorie de Lie, la théorie des associateurs de Drinfeld et l'intégrale de Kontsevich d'autre part. Cette connexion devrait trouver une interprétation plus claire en termes d'homologie de factorisation en étendant le formalisme développé dans cette thèse à une version quantique de la formalité de Goldman–Turaev. Une première étape dans cette direction consiste à produire une construction similaire à celle fournie par le Théorème 2.3 pour le coproduit sur le gradué associé de la bigèbre

d'écheveaux HOMFLY. Ce coproduit a été décrit explicitement dans [Sch05] et devrait avoir une reformulation et une généralisation en termes de diagrammes de cordes, d'associateurs de Drinfeld et d'homologie de factorisation, à la lumière du théorème de Kohno–Drinfeld pour la catégorie HOMFLY [Eti+26].

D'autre part, dans [BJ25], Brown–Jordan étudient une quantification du  $A$ -polynôme en utilisant le cadre des modules d'écheveaux à défauts paraboliques. Leur construction repose sur un bimodule sur les algèbres d'écheveaux, défini via la restriction parabolique pour  $SL_2$ , plutôt que sur une application directe entre algèbres. En spécialisant notre construction en  $t = 2$  en l'absence de défauts de ligne (et modulo la réduction de  $GL_2$  à  $SL_2$ ), nous retrouvons la structure de l'algèbre centrale parabolique apparaissant dans leurs travaux. L'introduction de défauts de ligne nous permet d'élever le bimodule obtenu par Brown–Jordan au rang de véritable morphisme d'algèbres. Concrètement, nous pouvons appliquer de manière répétée le coproduit de Turaev pour construire une application d'algèbres

$$\Delta_f^N : \text{SkAlg}_{GL_t}(S) \rightarrow \text{SkAlg}_{GL_t}(S) \otimes \cdots \otimes \text{SkAlg}_{GL_t}(S).$$

En spécialisant en  $t = N$  à gauche et en  $t_1 = \cdots = t_N = 1$  à droite, nous obtenons un morphisme (plutôt qu'un bimodule)

$$\tilde{\Delta}_f^N : \text{Sk}_{GL_N}(S) \rightarrow \text{Sk}_{\mathbb{C}^N}(S)$$

entre les algèbres d'écheveaux de  $GL_N$  et de  $\mathbb{C}^N$  de la surface. Cela devrait fournir un cadre plus naturel pour la quantification du  $A$ -polynôme.

Enfin, le formalisme des défauts de ligne que nous développons ici nous permet de considérer, comme cas particuliers, les défauts de ligne plongés dans une région tridimensionnelle, en les considérant comme étant à l'interface de deux défauts de surface triviaux. Les données algébriques attachées à telle ligne sont équivalentes aux données d'une *catégorie module tressée*  $\mathcal{A}$  [Bro12; Bro13]. Dans [BBJ18b], il a été montré que fournir une structure de module tressé est équivalent à la spécification d'une *application moment quantique*. Pour le cas  $\mathcal{A} = \text{Rep}_q G$ , des exemples de telles applications moment ont été construits en utilisant des modules de Verma dans [DDM02; DM04] et ont été utilisés pour construire des quantifications équivariantes de classes de conjugaison. Nous nous attendons à ce qu'une nouvelle construction, ainsi qu'une interprétation en théorie des écheveaux de ces applications, puissent être données en utilisant le formalisme de la restriction parabolique et des défauts de ligne.

## Organisation de la thèse

La thèse est divisée en quatre chapitres :

- Le Chapitre 1 contient principalement du matériel de base. Dans la Section 1.1, nous fournissons un aperçu du formalisme des catégories monoïdales et fixons les conventions pour le reste du document. Dans la Section 1.2, nous passons en revue

le calcul graphique pour les catégories monoïdales, rigides, pivotales et enrubannées. Dans la Section 1.3, nous introduisons les réseaux de cordes et la théorie des écheveaux.

- Le Chapitre 2 est consacré aux théories avec défauts. Dans la Section 2.1, nous introduisons la machinerie algébrique nécessaire pour étendre les réseaux de cordes et la théorie des écheveaux aux surfaces et aux variétés de dimension 3 stratifiées. Dans la Section 2.2, nous fournissons des modèles locaux pour les défauts de ligne planaires dans les théories de réseaux de cordes, et pour les défauts de surface et de ligne dans la théorie des écheveaux, en nous basant sur le calcul graphique pour les catégories bimodules, les algèbres centrales et les bimodules centrés. Dans la Section 2.3, nous introduisons les défauts dans les réseaux de cordes et la théorie des écheveaux et expliquons la construction dans le langage de l'homologie de factorisation.
- Dans le Chapitre 3, nous construisons un analogue HOMFLY du formalisme de la restriction parabolique. Dans la Section 3.1, nous donnons un bref aperçu des groupes quantiques et de la restriction parabolique. Dans la Section 3.2, nous rappelons la définition de la catégorie  $\text{HOMFLY Rep}_q \text{GL}_t$  et construisons une version interpolatrice du foncteur de restriction pour les représentations de  $\text{GL}_{m+n}$  vers  $\text{GL}_m \times \text{GL}_n$ . Dans la Section 3.3, nous construisons une version HOMFLY de la restriction parabolique et décrivons une algèbre centrale et un bimodule centré découlant de ce formalisme.
- Dans le Chapitre 4, nous étudions les théories de réseaux de cordes et d'écheveaux à défauts associées au formalisme HOMFLY et l'utilisons pour étendre le coproduit de Turaev à la catégorie d'écheveaux HOMFLY. Dans la Section 4.1, nous décrivons de manière explicite les espaces d'états fournis par les théories à défauts HOMFLY. Dans la Section 4.2, nous retrouvons le coproduit de Turaev comme une instance particulière de ce formalisme. Dans la Section 4.3, nous étendons l'application counité de la bigèbre HOMFLY à la catégorie d'écheveaux HOMFLY.



# Chapter 1

## Monoidal categories and associated theories

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### 1.1 The language of monoidal categories

We recall the basics of the language of monoidal categories and fix some of the notation that we use throughout the rest of the document. We follow closely [TV17] to which we refer for complete definitions. Other excellent references for this section are [Eti+15; BK01; Kas95].

### 1.1.1 Monoidal categories

Monoidal categories are a categorification of the notion of monoid:

**Definition 1.1.1.** (i) A *monoidal category* consists of the data  $(\mathcal{C}, \otimes, \mathbf{1}_{\mathcal{C}}, \alpha, l, r)$  given by:

- a category  $\mathcal{C}$  with a distinguished *unit object*  $\mathbf{1}_{\mathcal{C}} \in \mathcal{C}$ ;
- a *tensor product* functor  $- \otimes - : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ ;
- a natural isomorphism  $\alpha : (- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$  called the *associator* satisfying the usual pentagon axiom;
- natural isomorphisms  $l : \mathbf{1}_{\mathcal{C}} \otimes - \xrightarrow{\sim} \text{id}_{\mathcal{C}}$  and  $r : - \otimes \mathbf{1}_{\mathcal{C}} \xrightarrow{\sim} \text{id}_{\mathcal{C}}$  called *left* and *right unitor*, respectively, and satisfying the usual triangle axioms.

When the context is clear, we will suppress the extra data in the notation and denote a monoidal category simply by its underlying category. Moreover, if the category  $\mathcal{C}$  is  $\mathbb{k}$ -linear for some commutative ring  $\mathbb{k}$ , we will assume that  $\otimes$  is a  $\mathbb{k}$ -bilinear functor.

(ii) A monoidal category is *strict* if the associators and the unitors are identity natural transformations.  $\diamond$

We will write

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n := (\cdots (x_1 \otimes x_2) \otimes \cdots) \otimes x_n.$$

The pentagon and triangle axioms imply that all possible ways of removing the unit object and reparenthesizing a given expression involving the tensor product coincide. It turns out that this is equivalent to the fact that every monoidal category is equivalent to a strict monoidal one (see [Mac78, Section XI.3]). Consequently, whenever associators or unitors are needed to make a diagram well-typed, they will be understood implicitly and will not be displayed.

**Definition 1.1.2.** Given a monoidal category  $\mathcal{C}$ , we define its *opposite monoidal category*  $\mathcal{C}^{\text{mop}}$  as the monoidal category with the same underlying category  $\mathcal{C}$  and tensor product given by

$$x \otimes^{\text{mop}} y := y \otimes x.$$

The associator is  $\alpha_{x,y,z}^{\text{mop}} := \alpha_{z,y,x}^{-1}$  and the unitors are  $l_x^{\text{mop}} = r_x$  and  $r_x^{\text{mop}} = l_x$ .  $\diamond$

Monoidal functors are functors endowed with extra structure expressing the compatibility with the monoidal structures of the source and target category:

**Definition 1.1.3.** A (*strong*) *monoidal functor* between two monoidal categories  $\mathcal{C}$  and  $\mathcal{D}$  is the data  $(F, J, J_0)$  given by:

- a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  between the underlying categories;
- a natural isomorphism  $J: F(-) \otimes F(-) \xrightarrow{\sim} F(- \otimes -)$  compatible with the associators;
- an isomorphism  $J_0: \mathbf{1}_{\mathcal{D}} \xrightarrow{\sim} F(\mathbf{1}_{\mathcal{C}})$  compatible with the unitors.

We call  $J$  and  $J_0$  the *monoidal coherences* and we say that the functor is *strict* if the coherences are the identity maps. When the context is clear, we will omit the monoidal coherences from the notation and denote monoidal functors just by their underlying functors.  $\diamond$

The compatibility conditions imply again that every diagram constructed using the monoidal coherences, the associator and the unitors is commutative, so we write

$$J_{x_1, \dots, x_m}: F(x_1) \otimes \dots \otimes F(x_m) \rightarrow F(x_1 \otimes \dots \otimes x_m)$$

for the unique isomorphism induced by the monoidal structure.

Finally, we consider the following class of natural transformations, compatible with the monoidal structure of the involved functors:

**Definition 1.1.4.** A *monoidal natural transformation* between two monoidal functors  $F, G: \mathcal{C} \rightarrow \mathcal{D}$  is a natural transformation  $\eta: F \Rightarrow G$  such that

$$\begin{array}{ccc} F(x) \otimes F(y) & \xrightarrow{\eta_x \otimes \eta_y} & G(x) \otimes G(y) \\ J_{x,y}^F \downarrow & & \downarrow J_{x,y}^G \\ F(x \otimes y) & \xrightarrow{\eta_{x \otimes y}} & G(x \otimes y) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{1}_{\mathcal{D}} & \xrightarrow{J_0^F} & F(\mathbf{1}_{\mathcal{C}}) \\ & \searrow J_0^G & \downarrow \eta_{\mathbf{1}_{\mathcal{C}}} \\ & & G(\mathbf{1}_{\mathcal{C}}) \end{array}$$

are commutative diagrams for every  $x, y \in \mathcal{C}$ .  $\diamond$

### 1.1.2 Rigid categories

Rigid categories are a special class of monoidal categories whose objects admit left and right dualities. Throughout this subsection, we fix a monoidal category  $\mathcal{C}$ .

**Definition 1.1.5.** A *right duality* for an object  $x \in \mathcal{C}$  consists of an object  $y \in \mathcal{C}$  and morphisms

$$\text{ev}_x: x \otimes y \rightarrow \mathbf{1}_{\mathcal{C}} \quad \text{and} \quad \text{coev}_x: \mathbf{1}_{\mathcal{C}} \rightarrow y \otimes x,$$

called *right evaluation* and *right coevaluation*, respectively, and satisfying the so-called *right zig-zag identities*:

$$\begin{aligned} \text{id}_x &= \left( x \xrightarrow{\text{id}_x \otimes \text{coev}_x} x \otimes y \otimes x \xrightarrow{\text{ev}_x \otimes \text{id}_x} x \right), \\ \text{id}_y &= \left( y \xrightarrow{\text{coev}_x \otimes \text{id}_y} y \otimes x \otimes y \xrightarrow{\text{id}_y \otimes \text{ev}_x} y \right). \end{aligned}$$

The object  $y$  is then a *right dual* of  $x$ . Similarly, a *left duality* for  $x \in \mathcal{C}$  consists of an object  $y' \in \mathcal{C}$  together with morphisms

$$\mathrm{ev}'_x : y' \otimes x \rightarrow \mathbf{1}_{\mathcal{C}} \quad \text{and} \quad \mathrm{coev}'_x : \mathbf{1}_{\mathcal{C}} \rightarrow x \otimes y',$$

called *left evaluation* and *left coevaluation*, respectively, and satisfying the *left zig-zag identities*:

$$\begin{aligned} \mathrm{id}_x &= \left( x \xrightarrow{\mathrm{coev}'_x \otimes \mathrm{id}_x} x \otimes y' \otimes x \xrightarrow{\mathrm{id}_x \otimes \mathrm{ev}'_x} x \right), \\ \mathrm{id}_{y'} &= \left( y' \xrightarrow{\mathrm{id}_{y'} \otimes \mathrm{coev}'_x} y' \otimes x \otimes y' \xrightarrow{\mathrm{ev}'_x \otimes \mathrm{id}_{y'}} y' \right). \end{aligned}$$

We say that  $y'$  is a *left dual* of  $x$ . ◇

Duals are unique up to canonical isomorphism. Indeed, if  $(y_1, \mathrm{ev}^1, \mathrm{coev}^1)$  and  $(y_2, \mathrm{ev}^2, \mathrm{coev}^2)$  are right dualities of  $x$ , then

$$\varphi := \left( y_1 \xrightarrow{\mathrm{coev}^2_x \otimes \mathrm{id}_{y_1}} y_2 \otimes x \otimes y_1 \xrightarrow{\mathrm{id}_{y_2} \otimes \mathrm{ev}^1_x} y_2 \right)$$

is the unique isomorphism such that

$$\mathrm{ev}^1_x = \mathrm{ev}^2_x \circ (\mathrm{id}_x \otimes \varphi) \quad \text{and} \quad \mathrm{coev}^1_x = (\varphi^{-1} \otimes \mathrm{id}_x) \circ \mathrm{coev}^2_x.$$

The same holds for left duals.

**Definition 1.1.6. (i)** The monoidal category  $\mathcal{C}$  is *right rigid* (resp. *left rigid*) if every object has a right (resp. left) dual.

**(ii)** We say that  $\mathcal{C}$  is *rigid* if it is both left and right rigid. ◇

We will always assume that rigid categories come endowed with *distinguished dualities*, that is, with the choice of both a right and a left duality for every object  $x \in \mathcal{C}$ . We will denote by  $x^*$  and  ${}^*x$  the distinguished right and left duals of  $x \in \mathcal{C}$ , respectively. The choice of distinguished dualities induces strong equivalences of monoidal categories

$${}^*(-), (-)^* : \mathcal{C}^{\mathrm{op}, \mathrm{mop}} \rightarrow \mathcal{C}$$

sending objects to their distinguished duals and defined on morphisms by

$$f^* := (\mathrm{id}_{y^*} \otimes \mathrm{ev}_y) \circ (\mathrm{id}_{x^*} \otimes f \otimes \mathrm{id}_{y^*}) \circ (\mathrm{coev}_x \otimes \mathrm{id}_{y^*}) : y^* \rightarrow x^*$$

and

$${}^*f := (\mathrm{ev}'_y \otimes \mathrm{id}_{x^*}) \circ (\mathrm{id}_{y^*} \otimes f \otimes \mathrm{id}_{x^*}) \circ (\mathrm{id}_{y^*} \otimes \mathrm{coev}'_x) : {}^*y \rightarrow {}^*x,$$

where  $f : x \rightarrow y$ . We call these functors the *duality functors*.

**Remark 1.1.7** (Coherence for rigid categories). In a rigid category with distinguished dualities, the uniqueness of the duals up to isomorphism implies that  $*(-)^* \simeq \text{id}_{\mathcal{C}}$ , i.e., the left and right dual functors are quasi-inverses of each other. If the category is skeletal, then they are actual inverses. By choosing representatives for every isomorphism class of objects, we can show that any category is equivalent to a skeletal one. In particular, every rigid category is equivalent to a rigid category such that  $*(-)^* = \text{id}_{\mathcal{C}}$ .  $\diamond$

Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a monoidal functor between rigid categories. For any object  $x \in \mathcal{C}$ , the morphisms

$$\begin{aligned} F(x) \otimes F(x^*) &\xrightarrow{J_{x,x^*}} F(x \otimes x^*) \xrightarrow{\text{ev}_x} F(\mathbf{1}_{\mathcal{C}}) \xrightarrow{J_0^{-1}} \mathbf{1}_{\mathcal{D}}, \\ \mathbf{1}_{\mathcal{D}} &\xrightarrow{J_0} F(\mathbf{1}_{\mathcal{C}}) \xrightarrow{\text{coev}_x} F(x^* \otimes x) \xrightarrow{J_{x^*,x}^{-1}} F(x^*) \otimes F(x), \end{aligned}$$

exhibit  $F(x^*)$  as a right dual of  $F(x)$ . By the uniqueness of the duals, we get a natural transformation

$$\varphi_x^r: F(x^*) \rightarrow F(x)^* \quad (1.1.3)$$

identifying it with the distinguished dual  $F(x)^*$ . When  $\varphi^r$  is the identity, we say that  $F$  *preserves right dualities*. Similarly,  $F(x^*)$  is a left dual of  $F(x)$  so we have natural isomorphisms

$$\varphi_x^l: F(x^*) \rightarrow {}^*F(x) \quad (1.1.4)$$

identifying it with the distinguished left dual  ${}^*F(x)$  and we say that  $F$  *preserves left dualities* when  $\varphi^l$  is the identity.

### 1.1.3 Pivotal categories

In general, the left and right duality functors of a rigid category do not coincide. We introduce in this subsection pivotal categories, a class of rigid categories with distinguished dualities whose left and right duality functors coincide.

**Definition 1.1.8.** A *pivotal category* is a right rigid category  $\mathcal{C}$  endowed with a monoidal natural isomorphism

$$\iota: \text{id}_{\mathcal{C}} \xrightarrow{\sim} (-)^{**}$$

that we call *pivotal structure*.  $\diamond$

Every pivotal category is rigid. Indeed, using the pivotal structure we can exhibit right duals as left duals: for every  $x \in \mathcal{C}$ , the maps

$$\begin{aligned} \text{ev}'_x &:= \left( x^* \otimes x \xrightarrow{\text{id}_{x^*} \otimes \iota_x} x \otimes x^{**} \xrightarrow{\text{ev}_{x^*}} \mathbf{1}_{\mathcal{C}} \right), \\ \text{coev}'_x &:= \left( \mathbf{1}_{\mathcal{C}} \xrightarrow{\text{coev}_{x^*}} x^{**} \otimes x^* \xrightarrow{\iota_x^{-1} \otimes \text{id}_{x^*}} x \otimes x^* \right) \end{aligned}$$

are left evaluation and coevaluation, so  $x^*$  is also a left dual of  $x$ . When  $\mathcal{C}$  is endowed with this distinguished left duality, we say that it carries a *compatible* or *pivotal* duality. In that case, the left and right duality functors coincide as monoidal functors.

**Remark 1.1.9.** Alternatively, we can define a pivotal category as a rigid category with distinguished dualities in which

$$(-)^* = {}^*(-)$$

as monoidal functors. The pivotal structure can thus be recovered as

$$\iota_x := \left( x \xrightarrow{\text{coev}_{x^*} \otimes \text{id}_x} x^{**} \otimes x^* \otimes x \xrightarrow{\text{id}_{x^{**}} \otimes \text{ev}'_x} x^{**} \right).$$

Note that in both cases, a pivotal category is a category with extra structure: either a pivotal structure or a compatible left duality.  $\diamond$

**Remark 1.1.10** (Coherence for pivotal categories). Every pivotal category is equivalent to a *strict* pivotal category, i.e., a pivotal category where  $x = x^{**}$  and whose pivotal structure is the identity (see [NS07, Theorem 2.2]). We can construct strictifications using the graphical calculus developed in Section 1.2.  $\diamond$

In a pivotal category  $\mathcal{C}$  with pivotal duality, we can compute traces of endomorphisms and dimensions of objects:

**Definition 1.1.11.** (i) The *left* and *right traces* of an endomorphism  $f: x \rightarrow x$  of  $\mathcal{C}$  are the endomorphisms of  $\mathbf{1}_{\mathcal{C}}$  defined by

$$\text{tr}_l(f) = \text{ev}_x \circ (\text{id}_{x^*} \otimes f) \circ \text{coev}'_x \quad \text{and} \quad \text{tr}_r(f) = \text{ev}'_x \circ (f \otimes \text{id}_{x^*}) \circ \text{coev}_x,$$

respectively. In general, left and right traces are different. When they agree for every endomorphism, we say that  $\mathcal{C}$  is a *spherical* category.

(ii) The *left* and *right dimensions* of an object are the left and right traces, respectively, of its identity morphism.  $\diamond$

Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a monoidal functor between right rigid categories. Using the isomorphism  $\varphi^r$  defined in (1.1.3), we obtain a natural isomorphism

$$\eta_x^r := \left( F(x^{**}) \xrightarrow{\varphi_{x^*}^r} F(x^*)^* \xrightarrow{[(\varphi_x^r)^*]^{-1}} F(x)^{**} \right). \quad (1.1.5)$$

**Definition 1.1.12.** A monoidal functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  between pivotal categories is *pivotal* if the diagram

$$\begin{array}{ccc} & & F(x^{**}) \\ & \nearrow F(\iota_x) & \downarrow \eta_x^r \\ F(x) & \xrightarrow[\iota_{F(x)}^{\mathcal{D}}]{} & F(x)^{**} \end{array}$$

commutes for every  $x \in \mathcal{C}$ .  $\diamond$

Pivotal functors preserve traces and dimensions in the sense that

$$\text{tr}_r(F(f)) = J_0^{-1} \circ F(\text{tr}_r(x)) \circ J_0 \quad \text{and} \quad \dim_r(F(x)) = J_0^{-1} \circ F(\dim_r(x)) \circ J_0,$$

and similarly with  $r$  replaced by  $l$  (see [TV17, Lemma 2.8]).

### 1.1.4 Braided, balanced and ribbon categories

Finally, we review the notions of braided, balanced and ribbon categories. Braided categories are monoidal categories with the extra structure of a constraint expressing the commutativity of the tensor product. A balanced category is a braided category endowed with a compatible natural automorphism of the identity functor. When this isomorphism satisfies certain conditions, the category is ribbon. As we will see in the next sections, these conditions reflect the compatibility of the algebraic structure of ribbon categories with the topology of a certain class of graphs in  $\mathbb{R}^3$ .

**Definition 1.1.13.** (i) A *braided category* is a monoidal category  $\mathcal{C}$  endowed with a natural isomorphism

$$c: - \otimes - \xrightarrow{\sim} - \otimes^{\text{op}} -,$$

called the *braiding*, such that the diagrams

$$\begin{array}{ccc} x \otimes y \otimes z & \xrightarrow{c_{x,y} \otimes \text{id}_z} & y \otimes x \otimes z \\ & \searrow c_{x,y \otimes z} \quad \swarrow \text{id}_y \otimes c_{x,z} & \\ & y \otimes z \otimes x & \end{array} \quad \text{and} \quad \begin{array}{ccc} x \otimes y \otimes z & \xrightarrow{\text{id}_x \otimes c_{y,z}} & x \otimes z \otimes y \\ & \searrow c_{x \otimes y, z} \quad \swarrow c_{x,z} \otimes \text{id}_y & \\ & z \otimes x \otimes y & \end{array}$$

are commutative for all  $x, y, z \in \mathcal{C}$ .

(ii) If the braiding satisfies

$$c_{y,x} \circ c_{x,y} = \text{id}_{x \otimes y}$$

for all  $x, y \in \mathcal{C}$ , then we say that it is a *symmetry* and that  $\mathcal{C}$  is a *symmetric category*.

(iii) Given a braided monoidal category  $\mathcal{C}$  with braiding  $c$ , we will write  $\mathcal{C}^{\text{bop}}$  for its *braided opposite*, that is, the braided category whose underlying monoidal category is still  $\mathcal{C}$  but with braiding

$$c_{x,y}^{\text{op}} := c_{y,x}^{-1}. \quad \diamond$$

Using the braiding in a right rigid category, we can flip the factors  $x$  and  $x^*$  in the evaluation and coevaluation map, so we have:

**Remark 1.1.14.** Any right rigid braided monoidal category  $\mathcal{C}$  is rigid: the right dual  $x^*$  of every object  $x$  is also a left dual with left evaluation and coevaluation given by

$$\text{ev}'_x = \left( x^* \otimes x \xrightarrow{c_{x^*,x}} x \otimes x^* \xrightarrow{\text{ev}_x} \mathbf{1}_{\mathcal{C}} \right)$$

and

$$\text{coev}'_x = \left( \mathbf{1}_{\mathcal{C}} \xrightarrow{\text{coev}_x} x^* \otimes x \xrightarrow{c_{x,x^*}^{-1}} x \otimes x^* \right).$$

In particular, both  $x$  and  $x^{**}$  are right duals of  $x^*$ , and we get a natural transformation

$$\lambda_x: x \xrightarrow{\text{coev}_{x^*} \otimes \text{id}_x} x^{**} \otimes x^* \otimes x \xrightarrow{\text{id}_{x^{**}} \otimes \text{ev}'_x} x^{**}. \quad (1.1.6)$$

Yet, this natural transformation is not monoidal in general, so it does not provide  $\mathcal{C}$  with a pivotal structure. Moreover, it is not canonical: for instance,

$$\lambda'_x: x \xrightarrow{\text{id}_x \otimes \text{coev}'_{x^*}} x \otimes x^* \otimes x^{**} \xrightarrow{\text{ev}_x \otimes \text{id}_{x^{**}}} x^{**} \quad (1.1.7)$$

is also a natural isomorphism  $(-) \cong (-)^{**}$ .  $\diamond$

Braided functors are monoidal functors that commute with braidings:

**Definition 1.1.15.** A monoidal functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  between braided categories is *braided* if the diagram

$$\begin{array}{ccc} F(x) \otimes F(y) & \xrightarrow{J_{x,y}} & F(x \otimes y) \\ c_{x,y}^{\mathcal{D}} \downarrow & & \downarrow F(c_{x,y}^{\mathcal{C}}) \\ F(y) \otimes F(x) & \xrightarrow{J_{y,x}} & F(y \otimes x) \end{array}$$

commutes for all  $x, y \in \mathcal{C}$ .  $\diamond$

The lack of monoidality of the isomorphism  $(-) \cong (-)^{**}$  from Remark 1.1.14 can be corrected by precomposing it with a natural automorphism of the identity functor satisfying appropriate compatibility conditions. This motivates the following definition:

**Definition 1.1.16. (i)** A *twist* or *balance* in a braided monoidal category  $\mathcal{C}$  is a natural isomorphism

$$\theta: \text{id}_{\mathcal{C}} \xrightarrow{\sim} \text{id}_{\mathcal{C}}$$

such that

$$\theta_{x \otimes y} = c_{y,x} \circ c_{x,y} \circ (\theta_x \otimes \theta_y),$$

for all  $x, y \in \mathcal{C}$ , where  $c$  is the braiding.

**(ii)** A *balanced category* is a braided monoidal category endowed with a twist.  $\diamond$

The axiom in the definition of twist implies that

$$\iota_x := \left( x \xrightarrow{\theta_x} x \xrightarrow{\lambda_x} x^{**} \right),$$

where  $\lambda$  is the natural transformation defined in (1.1.6), is a monoidal natural isomorphism hence a pivotal structure. Conversely, starting from a pivotal structure  $\iota$ , one recovers a twist by setting  $\theta := \iota \circ \lambda^{-1}$  (see [Sel10, Lemma 4.20]). Therefore, in any braided rigid category, twists are in bijection with pivotal structures. This bijection is however not unique. Indeed, starting from a braided pivotal category, another twist can be defined by setting  $\theta' := \lambda' \circ \iota^{-1}$ , where  $\lambda'$  is the isomorphism defined in (1.1.7) (and there are indeed countably many such choices). While  $\theta$  and  $\theta'$  are in general different, their pictorial representations in the graphical calculus that we develop in the next sections correspond to the diagrams

$$\text{d} \quad \text{and} \quad \text{p},$$

which are equivalent under a natural notion of isotopy, called *regular isotopy*. In order to make the algebraic structure of our categories compatible with topology (in the sense that will be made precise in Section 1.2), we need these two isomorphisms to coincide.

**Definition 1.1.17.** A *ribbon category* is a braided pivotal category such that the associated twists  $\theta$  and  $\theta'$  coincide.  $\diamond$

Under the equivalence between braided pivotal and balanced rigid categories sketched above, ribbon categories can be characterised in a different way:

**Remark 1.1.18.** Equivalently, ribbon categories can be defined as balanced rigid categories for which the pivotal structures induced using  $\lambda$  and  $\lambda'$  coincide or, as it is more usual, as balanced categories whose twist is compatible with the duality, meaning that

$$\theta_{x^*} = \theta_x^*$$

for every object  $x$  (see [HPT15, Proposition A4]).  $\diamond$

## 1.2 The graphical calculus for monoidal categories

Following [JSa; JS91; JSb], we introduce in this section the graphical calculus for monoidal categories. Other references are [TV17], for a detailed exposition, and [Sel10] and [HPT15], where nice overviews can be found.

Before diving into details, we fix some terminology that we will use throughout the rest of the document. A *graph* is the data of a finite 1-dimensional CW complex  $\Gamma$  and a finite closed subset  $\Gamma_0 \subset \Gamma$  such that  $\Gamma \setminus \Gamma_0$  is a smooth 1-manifold without boundary. The elements of  $\Gamma_0$  are called *vertices*. A connected component  $c$  of  $\Gamma \setminus \Gamma_0$  is called an *edge* if it is homeomorphic to an interval, and a *circle* otherwise.

Given a graph  $(\Gamma, \Gamma_0)$ , we distinguish two classes of vertices, depending on whether they lie at the interior or at an endpoint of  $\Gamma$ . More precisely, we say that  $v \in \Gamma_0$  is an *endpoint* if, for any connected neighbourhood  $U \subset \Gamma$  of  $v$ ,  $U \setminus \{v\}$  is still connected. Otherwise, we say that  $v$  is an *interior vertex*. The set of endpoints is denoted by  $\partial\Gamma_0$ . An *elementary graph* is a connected graph containing exactly one interior vertex.

An *oriented edge* is an edge endowed with an orientation. A *parametrization* of an oriented edge is an orientation-preserving homeomorphism  $\bar{e} \cong [0, 1]$ , where  $\bar{e}$  is the closed edge obtained by adjoining its endpoints to  $e$ , and  $[0, 1]$  is endowed with its positive orientation. We write  $e(0)$  and  $e(1)$  for the images of 0 and 1 by

$$[0, 1] \cong \bar{e} \rightarrow \Gamma,$$

where the last arrow is obtained by continuously extending the inclusion  $e \hookrightarrow \Gamma$ . We call these points the *source* and *target* of the edge, respectively. We write  $\partial_0(e)$  and  $\partial_1(e)$  for the velocity vectors at the source and target, respectively, for a given parametrization. When  $v$  is a vertex with  $v = e(0)$  or  $v = e(1)$ , we also write  $\partial_v(e)$  for the velocity vector at  $v$ .

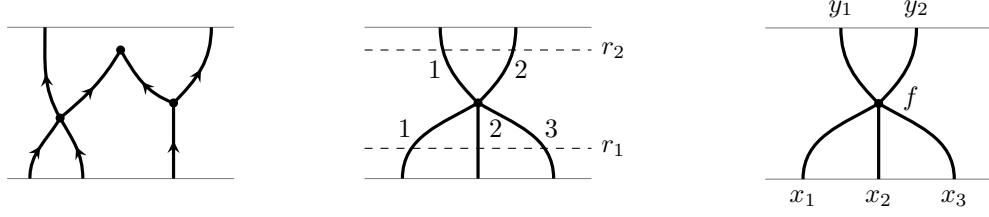


Figure 1.1: **Left:** a progressive planar graph with the orientation induced by the second coordinate function. **Middle:** order on the set of incoming and outgoing edges induced by the first coordinate function. **Right:** a  $\mathcal{C}$ -colouring of the previous vertex with  $f: x_1 \otimes x_2 \otimes x_3 \rightarrow y_1 \otimes y_2$ .

Similarly, an *oriented circle* is a circle  $c \in \pi_0(\Gamma \setminus \Gamma_0)$  endowed with an orientation. A *parametrization* of an oriented circle is an orientation-preserving homeomorphism  $c \cong \mathbb{S}^1$ , where  $\mathbb{S}^1$  is oriented clockwise. A graph where every edge and circle is endowed with an orientation is called an *oriented graph*.

We say that an edge  $e$  is *adjacent* to a vertex  $v$  if  $v \in \bar{e}$  and we denote by  $\Gamma_v \subset \Gamma_0$  the set of edges adjacent to  $v$ . An *ordered graph* is a graph  $\Gamma$  together with the choice, for every interior vertex  $v$ , of two ordered subsets  $\Gamma_v^i, \Gamma_v^o \subset \Gamma_v$  (possibly not disjoint) covering  $\Gamma_v$ . We call them *incoming* and *outgoing* edges for  $v$ , respectively. Interior vertices of an ordered graph will be also called *coupons*.

### 1.2.1 Free monoidal categories

We begin the exposition with the graphical calculus for monoidal categories. Broadly, we define a strict monoidal category whose morphisms are a certain class of decorated graphs embedded in  $\mathbb{R} \times [0, 1]$ . This category comes with an evaluation functor to the underlying monoidal category  $\mathcal{C}$ , allowing us to interpret diagrams as actual morphisms. Topological operations and transformations at the level of graphs will reflect the algebraic structure of  $\mathcal{C}$ , so that we can use diagrams to perform algebraic computations.

**Definition 1.2.1.** A *progressive planar graph* is a graph  $(\Gamma, \Gamma_0)$  together with a smooth embedding  $\iota: \Gamma \rightarrow \mathbb{R} \times [0, 1]$  such that

- the image of  $\Gamma$  intersects  $\mathbb{R} \times \{0, 1\}$  exactly at its endpoints, i.e.  $\iota(\partial\Gamma_0) = \iota(\Gamma) \cap (\mathbb{R} \times \{0, 1\})$ ;
- the restriction of the second coordinate function  $\text{pr}_2: \mathbb{R} \times [0, 1] \rightarrow [0, 1]$  to any connected component of  $\Gamma \setminus \Gamma_0$  is an injective function.  $\diamond$

In particular, a progressive planar graph contains no circles. We will abuse notation and denote progressive planar graphs simply by their underlying abstract graph  $\Gamma$ ,

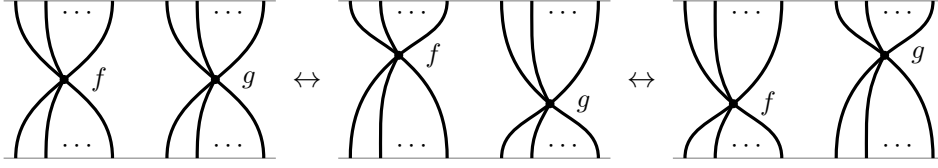


Figure 1.2: Three isotopically equivalent progressive planar graphs. Evaluating them in  $\mathcal{C}$  we obtain an identity that expresses exactly the functoriality of the tensor product.

omitting the embedding from the notation.

Let  $\Gamma$  be a progressive planar graph. It becomes an ordered graph as follows. First, using the injectivity on edges of the second coordinate, we endow every edge  $e$  with the orientation determined by  $\text{pr}_2 e(0) < \text{pr}_2 e(1)$  (i.e., the orientation is chosen so that any parametrization satisfying this property is orientation-preserving). Given an interior vertex  $v \in \Gamma_0 \setminus \partial\Gamma_0$ , we define

$$\Gamma_v^i := \{e \in \Gamma_v \mid v = e(1)\} \quad \text{and} \quad \Gamma_v^o := \{e \in \Gamma_v \mid v = e(0)\}.$$

Note that the injectivity condition implies that  $\Gamma_v^i \cap \Gamma_v^o = \emptyset$ . We define next an order for these two subsets. To do so, choose  $t_1 \in [0, \text{pr}_2(v))$  big enough so that  $r_1 := \mathbb{R} \times \{t_1\}$  intersects every  $e \in \Gamma_v^i$  exactly once. Then, we set

$$e < e' \quad \text{if} \quad \text{pr}_1(e \cap r_1) < \text{pr}_1(e' \cap r_1)$$

for every  $e, e' \in \Gamma_v^i$ . Similarly, choosing  $t_2 \in (\text{pr}_2(v), 1]$  small enough, we order the elements of  $\Gamma_v^o$  increasingly on the first coordinate of their intersection with  $r_2 := \mathbb{R} \times \{t_2\}$  (see Figure 1.1).

We will consider progressive planar graphs up to deformation:

**Definition 1.2.2.** Let  $\Gamma$  be a progressive planar graph. An *isotopy of progressive planar graphs* of  $\Gamma$  is an isotopy

$$H: (\mathbb{R} \times [0, 1]) \times [0, 1] \rightarrow \mathbb{R} \times [0, 1]$$

of  $\mathbb{R} \times [0, 1]$  preserving  $\mathbb{R} \times \{0, 1\}$  and such that  $H(\Gamma, t)$  is a progressive planar graph for every  $t \in [0, 1]$  (see Figure 1.2).  $\diamond$

For the rest of the section, we fix a monoidal category  $\mathcal{C}$ , that we use to decorate our graphs as follows:

**Definition 1.2.3. (i)** A  $\mathcal{C}$ -colouring of a progressive planar graph  $\Gamma$  is a pair of functions

$$\phi_0: \Gamma_0 \setminus \partial\Gamma_0 \rightarrow \text{mor}(\mathcal{C}) \quad \text{and} \quad \phi_1: \pi_0(\Gamma \setminus \Gamma_0) \rightarrow \text{ob}(\mathcal{C})$$

$$\phi_0(v) \in \text{Hom}_{\mathcal{C}} \left( \phi_1(e_1) \otimes \cdots \otimes \phi_1(e_m), \phi_1(e'_1) \otimes \cdots \otimes \phi_1(e'_n) \right).$$

(ii) Given an endpoint  $v$  with adjacent edge  $e_v$ , we set  $\phi_1(v) := \phi_1(e_v)$ . Let

$$\Gamma \cap (\mathbb{R} \times \{0\}) = \{v_1 < v_2 < \cdots < v_m\} \quad \text{and} \quad \Gamma \cap (\mathbb{R} \times \{1\}) = \{w_1 < w_2 < \cdots < w_n\}$$

$$\text{source}(\Gamma) = (\phi_1(v_1), \dots, \phi_1(v_m)) \quad \text{and} \quad \text{target}(\Gamma) = (\phi_1(w_1), \dots, \phi_1(w_n)),$$


**Definition 1.2.4.** The *free monoidal category on  $\mathcal{C}$*  is the category  $\text{Mon}(\mathcal{C})$  given by:

- objects are finite sequences of objects of  $\mathcal{C}$ ;
- a morphism between two such sequences  $X, Y$  is the isotopy class of a  $\mathcal{C}$ -coloured progressive planar graph  $\Gamma$  with  $\text{source}(\Gamma) = X$  and  $\text{target}(\Gamma) = Y$ .

Composition is given by vertically stacking graphs and rescaling the second coordinate. The identity of a sequence is represented by a graph with no coupons and whose edges are vertical and coloured with the objects of the sequence. A strict tensor product is defined by concatenating sequences and horizontally stacking progressive planar graphs. The unit object is the empty sequence.  $\diamond$

The category  $\text{Mon}(\mathcal{C})$  is generated as a strict monoidal category by  $\mathcal{C}$ -coloured elementary progressive planar graphs, as the right-most graph in Figure 1.1. Indeed, any planar graph can be written as a composition of tensor products of elementary graphs with identity morphisms, and two such decompositions are related by an isotopy of progressive planar graphs. Therefore, the assignments

$$(x_1, \dots, x_m) \mapsto x_1 \otimes \dots \otimes x_m \quad \text{and} \quad \begin{array}{c} y_1 \ y_2 \ \dots \ y_n \\ \hline \text{diagram of } f \\ \hline x_1 \ x_2 \ \dots \ x_m \end{array} \mapsto f$$

determine a well-defined strict monoidal evaluation functor

$$\mathrm{ev}_{\mathcal{C}}^{\mathrm{Mon}}: \mathrm{Mon}(\mathcal{C}) \rightarrow \mathcal{C}$$

to the underlying monoidal category  $\mathcal{C}$  (see Figure 1.3).

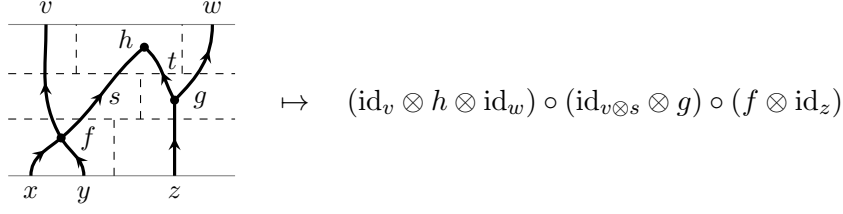


Figure 1.3: Monoidal decomposition of a  $\mathcal{C}$ -coloured progressive planar graph into elementary graphs, and evaluation in  $\mathcal{C}$ .

### 1.2.2 Free rigid categories

The injectivity condition in the definition of progressive planar graphs prevents the existence of cups, caps, and circles. This is a diagrammatic counterpart of the fact that in a monoidal category one does not have, in general, dualities or traces. In order to extend our graphical calculus to a more general class of graphs containing at least cups and caps, we need the underlying category to be rigid. Moreover, since left and right duals may be different and since the double dual of an object is different from the object itself, we need to endow our graphs with the extra data of an orientation and an integer label for every edge. The orientation will allow us to distinguish between left and right duals, while the integer label will indicate how many times the duality functors are applied. In this way, we obtain a graphical calculus for rigid categories that extends the one for monoidal categories.

**Definition 1.2.5.** A *polarised planar graph* is an oriented graph  $(\Gamma, \Gamma_0)$  without circles together with a smooth embedding  $\iota: \Gamma \rightarrow \mathbb{R} \times [0, 1]$  such that

- the image of  $\Gamma$  intersects  $\mathbb{R} \times \{0, 1\}$  exactly at its endpoints, i.e.  $\iota(\partial\Gamma_0) = \iota(\Gamma) \cap (\mathbb{R} \times \{0, 1\})$ ;
- the restriction of the second coordinate function  $\text{pr}_2: \mathbb{R} \times [0, 1] \rightarrow [0, 1]$  to any connected component of  $\Gamma \setminus \Gamma_0$  has finitely many critical points, none of them at a vertex.

Given an edge  $e$ , every local maximum or minimum of  $\text{pr}_2|_e$  is a *redirection point*. We call *cup* (resp. *cap*) a local neighbourhood in  $\Gamma$  of a local minimum (resp. maximum) containing no other critical points.  $\diamond$

Let  $\Gamma$  be a polarised planar graph. The second condition in the previous definition implies that, forgetting the orientation, every interior vertex  $v$  admits a neighbourhood  $U$  such that  $U \cap \Gamma$  is an elementary progressive planar graph. Applying the same procedure as in the previous section, we then obtain two ordered subsets  $\Gamma_v^i$  and  $\Gamma_v^o$  that may be now non-disjoint. We stress the fact that incoming and outgoing edges are determined using the orientation induced by the second coordinate function, which may be different from the orientation of  $\Gamma$  itself (see Figure 1.4).

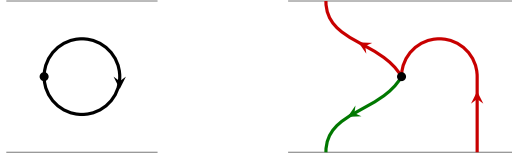


Figure 1.4: Examples of polarised planar graphs. The only edge of the graph on the left is both an incoming and an outgoing edge. For the graph on the right, the green edge is an incoming edge, while the red ones are outgoing.

**Definition 1.2.6.** (i) Given a redirection point  $p$ , we define its *turning number*  $\gamma(p)$  to be 1 if  $p$  lies in a clockwise oriented cup or cap, and  $-1$  otherwise.

(ii) The *rotation number* of an edge  $e$  is the integer

$$\text{rot}(e) = \sum_p \gamma(p),$$

where the sum is taken over all the redirection points lying in  $e$ . ◇

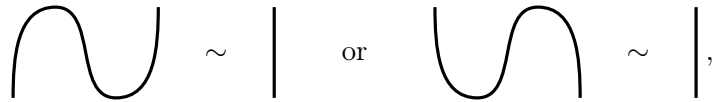
We will consider polarised planar graphs up to deformation:

**Definition 1.2.7.** Let  $\Gamma$  be a polarised planar graph. An *isotopy of polarised planar graphs* of  $\Gamma$  is an isotopy

$$H: (\mathbb{R} \times [0, 1]) \times [0, 1] \rightarrow \mathbb{R} \times [0, 1]$$

of  $\mathbb{R} \times [0, 1]$  preserving  $\mathbb{R} \times \{0, 1\}$  and such that  $H(\Gamma, t)$  is a polarised planar graph for every  $t \in [0, 1]$  (see Figure 1.2). ◇

Isotopies of polarised planar graphs are precisely the isotopies preserving the rotation number of each edge. Indeed, since we are preventing critical points from happening at a vertex, isotopies of polarised planar graphs cannot fully rotate vertices (otherwise, the vertical coordinate of  $\partial_v(e)$  would change of sign throughout an isotopy, resulting in a critical point at  $v$ ). On the other hand, isotopies of polarised planar graphs may perform a *zigzag transformation*



making appear a pair of new redirection points. Yet, they have opposite turning number, so the rotation number of the segment is preserved.

Let  $e$  be an edge of a polarised graph  $\Gamma$  with its orientation-preserving parametrization  $\bar{e} \cong [0, 1]$ . Let  $\partial_0(e) \in T_{e(0)}(\mathbb{R}^2 \times [0, 1])$  be the velocity vector at the source  $e(0)$ :

**Definition 1.2.8. (i)** We say that  $e$  is *upwards* if the second coordinate of  $\partial_0(e)$  in the standard basis of  $T_{e(0)}(\mathbb{R}^2 \times [0, 1])$  is positive, and we say that it is *downwards* otherwise.

**(ii)** An *integer labelling* of a polarised graph  $\Gamma$  is a function

$$N_0: \pi_0(\Gamma \setminus \Gamma_0) \rightarrow \mathbb{Z}$$

such that, on edges,  $N_0(e)$  is even if  $e$  is upwards, and odd otherwise.  $\diamond$

Given an integer labelling  $N_0$  of  $\Gamma$ , we define

$$N_1(e) := N_0(e) + \text{rot}(e).$$

Note that  $N_1$  is invariant under isotopy of polarised planar graphs. We will think of the integers  $N_0(e)$  and  $N_1(e)$  as decorating the endpoints  $e(0)$  and  $e(1)$  of  $e$ , respectively. Fix a rigid category  $\mathcal{C}$  with distinguished dualities and write

$$x^n := \begin{cases} x^{*\overset{\times n}{\dots} *} & \text{if } n \geq 0, \\ *^{\times(-n)} \overset{\times(-n)}{\dots} x & \text{if } n < 0, \end{cases} \quad (1.2.8)$$

for every integer  $n \in \mathbb{Z}$  and every object  $x \in \mathcal{C}$ .

**Definition 1.2.9. (i)** A  $\mathcal{C}$ -*colouring* of a polarised planar graph  $\Gamma$  consists of an integer labelling  $N_0$  and a pair of functions

$$\phi_0: \Gamma_0 \setminus \partial\Gamma_0 \rightarrow \text{mor}(\mathcal{C}) \quad \text{and} \quad \phi_1: \pi_0(\Gamma \setminus \Gamma_0) \rightarrow \text{ob}(\mathcal{C})$$

satisfying the following compatibility condition: for every coupon  $v$  with incoming and outgoing edges  $\Gamma_v^i = \{e_1 < e_2 < \dots < e_m\}$  and  $\Gamma_v^o = \{e'_1 < e'_2 < \dots < e'_n\}$ , respectively, we have

$$\phi_0(v) \in \text{Hom}_{\mathcal{C}} \left( \phi_1(e_1)^{k(e_1)} \otimes \dots \otimes \phi_1(e_m)^{k(e_m)}, \phi_1(e'_1)^{k(e'_1)} \otimes \dots \otimes \phi_1(e'_n)^{k(e'_n)} \right)$$

where  $k(e)$  is the integer label of  $e$  at  $v$ , i.e.,  $k(e) = N_0(e)$  if  $v = e(0)$  and  $k(e) = N_1(e)$  if  $v = e(1)$ .

**(ii)** Given an endpoint  $v$  with adjacent edge  $e_v$ , we set  $\phi_1(v) := \phi_1(e_v)$  and  $k(v) = k(e_v)$  and we let

$$\Gamma \cap (\mathbb{R} \times \{0\}) = \{v_1 < v_2 < \dots < v_m\} \quad \text{and} \quad \Gamma \cap (\mathbb{R} \times \{1\}) = \{w_1 < w_2 < \dots < w_n\}$$

be the set of bottom and top endpoints, respectively, ordered by their first coordinate. We define the *source* and *target* of  $\Gamma$  as the tuples of pairs

$$\text{source}(\Gamma) = ((\phi_1(v_1), k(v_1)), \dots, (\phi_1(v_m), k(v_m))),$$

$$\text{target}(\Gamma) = ((\phi_1(w_1), k(w_1)), \dots, (\phi_1(w_n), k(w_n))),$$

respectively.  $\diamond$



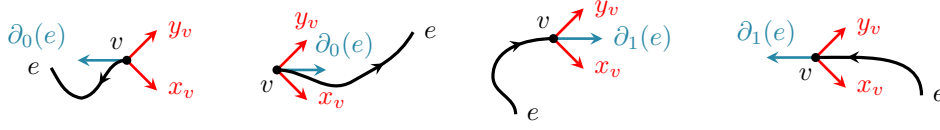


Figure 1.6: The edges in the first and third pictures are incoming, while the remaining two are outgoing.

In particular, cups and caps are mapped to the distinguished dualities. The compatibility with isotopy follows since the zigzag identities hold in  $\mathcal{C}$ .

### 1.2.3 Free pivotal categories

In a pivotal category, left and right duals as well as multiple duals are canonically identified via the pivotal structure and we have well-defined notions of traces and dimensions for endomorphisms and objects. A graphical calculus for pivotal categories can thus be obtained by dropping the integer labels from the graphical calculus for rigid categories and by allowing circles and a more general class of isotopies, performing rotations of vertices.

**Definition 1.2.11.** A *cyclic planar graph* is an oriented graph  $(\Gamma, \Gamma_0)$  together with the data of

- a smooth embedding  $\iota: \Gamma \rightarrow \mathbb{R} \times [0, 1]$  such that  $\iota(\partial\Gamma_0) = \iota(\Gamma) \cap (\mathbb{R} \times \{0, 1\})$ ;
- a *positive framing*  $(x_v, y_v)$  for every vertex  $v$ , that is, the choice of a positive basis  $(x_v, y_v)$  of  $T_v(\mathbb{R} \times [0, 1])$  for every interior vertex  $v$ , which is moreover supposed to be the canonical one for endpoints;

subject to the condition that, for every interior vertex  $v$ , none of the velocity vectors at  $v$  of its adjacent edges is parallel to  $x_v$ .  $\diamond$

We determine the sets of incoming and outgoing edges for a vertex as follows. Let  $v$  be a vertex and  $e \in \Gamma_v$  be an edge parameterized following its orientation and such that  $e(0) = v$  (i.e., the edge points away from the vertex). Let  $\partial_0(e)$  denote its velocity vector at  $v$ . The edge  $e$  is classified as incoming if  $(x_v, \partial_0(e))$  forms a negative basis for  $T_v(\mathbb{R} \times [0, 1])$ , and outgoing if it forms a positive basis. Conversely, if  $e(1) = v$  (i.e., pointing towards the vertex), let  $\partial_1(e)$  denote its velocity vector at  $v$ . In this case,  $e$  is incoming if  $(x_v, \partial_1(e))$  forms a positive basis, and outgoing otherwise. Roughly speaking, incoming edges attach to  $v$  from “below” and outgoing edges attach from “above,” where “below” and “above” are determined by the local framing (see Figure 1.6).

Next we endow the sets  $\Gamma_v^i$  and  $\Gamma_v^o$  with an order. To do so, we consider now oriented lines  $r_1$  (below) and  $r_2$  (above) parallel to  $x_v$  and close enough to  $v$  so that every adja-

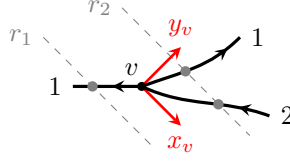


Figure 1.7: Order of the incoming and outgoing edges of a framed vertex.

cent edge intersects one of them. We order the edges following the linear order of the intersection points (see Figure 1.7).

**Definition 1.2.12.** An *isotopy of cyclic planar graphs* is just an isotopy of  $\mathbb{R} \times [0, 1]$  fixing  $\mathbb{R} \times \{0, 1\}$ .  $\diamond$

Let  $\Gamma$  be a cyclic planar graph and  $\alpha$  an isotopy of cyclic planar graphs. Then,  $\alpha(\Gamma, t)$  is also a cyclic planar graph, for all  $t \in [0, 1]$ . In particular, if a vertex  $v \in \Gamma_0$  has framing  $(x_v, y_v)$ , then  $\alpha(v, t) \in \alpha(\Gamma, t)_0$  is also framed with framing  $(\alpha^*(x_v, t), \alpha^*(y_v, t))$ .

Fix a pivotal category  $\mathcal{C}$  for the rest of the section. We introduce now colourings over  $\mathcal{C}$  for cyclic planar graphs. Given a framed vertex  $v$  in a cyclic planar graph  $\Gamma$  and  $e \in \Gamma_v$ , we set

$$\varepsilon(e) := \begin{cases} 0 & \text{if the velocity vector } \partial_e(v) \text{ has positive } y_v \text{ coordinate,} \\ 1 & \text{otherwise,} \end{cases}$$

and  $\varepsilon(e)^* := (\varepsilon(e) + 1) \bmod 2$ . For instance, for the first and last vertices in Figure 1.6 we have  $\varepsilon(e) = 1$ , while for the second and the third one  $\varepsilon(e) = 0$ .

**Definition 1.2.13. (i)** A  $\mathcal{C}$ -colouring for a cyclic planar graph  $\Gamma$  is a pair of functions

$$\phi_0: \Gamma_0 \setminus \partial\Gamma_0 \rightarrow \text{mor}(\mathcal{C}) \quad \text{and} \quad \phi_1: \pi_0(\Gamma \setminus \Gamma_0) \rightarrow \text{ob}(\mathcal{C})$$

satisfying the following compatibility condition: for every coupon  $v$  with incoming and outgoing edges  $\Gamma_v^i = \{e_1 < e_2 < \dots < e_m\}$  and  $\Gamma_v^o = \{e'_1 < e'_2 < \dots < e'_n\}$ , respectively, we have

$$\phi_0(v) \in \text{Hom}_{\mathcal{C}} \left( \phi_1(e_1)^{\varepsilon(e_1)} \otimes \dots \otimes \phi_1(e_m)^{\varepsilon(e_m)}, \phi_1(e'_1)^{\varepsilon(e'_1)} \otimes \dots \otimes \phi_1(e'_n)^{\varepsilon(e'_n)} \right),$$

where  $x^0 := x$  and  $x^1 := x^*$ , for every  $x \in \mathcal{C}$ .

**(ii)** Given an endpoint  $v$  with adjacent edge  $e_v$ , we set  $\phi_1(v) := \phi_1(e_v)$  and  $\varepsilon(v) = \varepsilon(e_v)$  and we let

$$\Gamma \cap (\mathbb{R} \times \{0\}) = \{v_1 < v_2 < \dots < v_m\} \quad \text{and} \quad \Gamma \cap (\mathbb{R} \times \{1\}) = \{w_1 < w_2 < \dots < w_n\}$$

be the set of bottom and top endpoints, respectively, ordered by their first coordinate. We define the *source* and *target* of  $\Gamma$  as the tuples of pairs

$$\text{source}(\Gamma) = ((\phi_1(v_1), \varepsilon(v_1)), \dots, (\phi_1(v_m), \varepsilon(v_m)),$$

$$\text{target}(\Gamma) = ((\phi_1(w_1), \varepsilon(w_1)), \dots, (\phi_1(w_n), \varepsilon(w_n))).$$

**Definition 1.2.14.** The *free pivotal category* on  $\mathcal{C}$  is the category  $\text{Piv}(\mathcal{C})$  whose

- objects are finite sequences of pairs  $(x, \varepsilon)$ , where  $x \in \mathcal{C}$  and  $\varepsilon \in \{0, 1\}$ ;
- morphisms are isotopy classes of  $\mathcal{C}$ -coloured cyclic planar graphs in  $\mathbb{R} \times [0, 1]$  with given source and target.

The composition and the monoidal structure are defined as in the previous sections.  $\diamond$

The category  $\text{Piv}(\mathcal{C})$  is rigid: the object  $((x_n, \varepsilon_n^*), \dots, (x_1, \varepsilon_1^*))$  is both a left and a right dual of  $((x_1, \varepsilon_1), \dots, (x_n, \varepsilon_n))$ . The evaluations and coevaluations are given by the cups and caps, so that  $\text{Piv}(\mathcal{C})$  admits a pivotal structure given by the identity morphisms. As a monoidal category, it is generated by  $\mathcal{C}$ -coloured cups, caps and elementary planar graphs, and we have again a strict monoidal evaluation functor

$$\mathrm{ev}_{\mathcal{C}}^{\mathrm{Piv}}: \mathrm{Piv}(\mathcal{C}) \rightarrow \mathcal{C}$$

to the underlying pivotal category determined by

$$\mathrm{ev}_{\mathcal{C}}^{\mathrm{Piv}}((x, 0)) = x, \quad \mathrm{ev}_{\mathcal{C}}^{\mathrm{Piv}}((x, 1)) = x^*, \quad \mathrm{ev}_{\mathcal{C}}^{\mathrm{Piv}} \left( \begin{array}{c} y_1 \ y_2 \ \cdots \ y_n \\ \hline \begin{array}{c} \text{diagram of a crossing with } n \text{ inputs and } m \text{ outputs} \end{array} \\ \hline x_1 \ x_2 \ \cdots \ x_m \end{array} \right) = f,$$

$$\mathrm{ev}_{\mathcal{C}}^{\mathrm{Piv}} \left( \text{curved arrow pointing down to } x \right) = \mathrm{ev}_x, \quad \mathrm{ev}_{\mathcal{C}}^{\mathrm{Piv}} \left( \text{curved arrow pointing up to } x \right) = \mathrm{coev}_x,$$

$$\mathrm{ev}_{\mathcal{C}}^{\mathrm{Piv}} \left( x \curvearrowright \right) = \mathrm{ev}'_x, \quad \mathrm{ev}_{\mathcal{C}}^{\mathrm{Piv}} \left( x \curvearrowleft \right) = \mathrm{coev}'_x.$$

**Remark 1.2.15.** The evaluation functor for pivotal categories is pivotal, but it does not preserve the distinguished dualities. Indeed, the right dual of  $(x, 1)$  in  $\text{Rig}(\mathcal{C})$  is  $(x, 0)$ , with evaluation and coevaluation given by

 and .

This duality is sent to  $(x, \text{ev}'_x, \text{coev}'_x)$ , which is the pivotal left duality of  $x$  in  $\mathcal{C}$ . Since this duality is induced from the right one, the isomorphism

$$\varphi_{(x,1)}^r: x = \mathrm{ev}_{\mathcal{C}}^{\mathrm{Piv}}((x,1)^*) \rightarrow \mathrm{ev}_{\mathcal{C}}^{\mathrm{Piv}}((x,1))^* = x^{**}$$

is exactly the pivotal structure, so  $\text{ev}_C^{\text{Piv}}$  is pivotal.

**Remark 1.2.16.** The evaluation functor is full but not faithful. Taking the quotient by its kernel we have an equivalence

$$\mathcal{C} \simeq \text{Piv}(\mathcal{C}) / \ker \text{ev}_{\mathcal{C}}^{\text{Piv}},$$

providing a strictification of  $\mathcal{C}$  as a pivotal category (see Remark 1.1.10). The same happens for monoidal and rigid categories.  $\diamond$

**Remark 1.2.17.** The left and right traces of a morphism  $f$  in a pivotal category  $\mathcal{C}$  are represented graphically by the diagrams

$$f \bullet \text{ (circle with arrow) } \text{ and } \text{ (circle with arrow) } \bullet f.$$

These diagrams are not isotopic in  $\mathbb{R} \times [0, 1]$ , but they are in  $\mathbb{S}^2$ , so the graphical calculus for pivotal categories extends to a graphical calculus for spherical categories (cf. Definition 1.1.11) by considering the same class of diagrams modulo isotopies of the sphere.  $\diamond$

#### 1.2.4 Free ribbon categories

The extra structure provided by a braiding allows one to lift the graphical calculus developed in the previous sections to a graphical calculus in three dimensions. On the other hand, the structure of balanced categories can be reflected diagrammatically by replacing edges with “thickened ribbons”. Under this picture, the algebraic twist then corresponds to actually twisting a ribbon by  $360^\circ$ . When the underlying category is a ribbon category, the construction becomes compatible with isotopies, so that the algebraic structure of ribbon categories is governed by the topology of so-called *ribbon graphs*.

**Definition 1.2.18.** A *ribbon graph* is an oriented graph  $(\Gamma, \Gamma_0)$  together with the data of

- a smooth embedding  $\iota: \Gamma \rightarrow \mathbb{R}^2 \times [0, 1]$ ;
- a framing  $(x_v, y_v, z_v)$  for every vertex  $v \in \Gamma_0$ , which is supposed to be the canonical one for endpoints;
- a framing for every connected component  $c$  of  $\Gamma \setminus \Gamma_0$ , meaning a non-vanishing section of the normal bundle  $N(c) \subset T(\mathbb{R}^2 \times [0, 1])$ ;

satisfying the following conditions:

- the image of  $\Gamma$  intersects  $\partial(\mathbb{R}^2 \times [0, 1])$  exactly at its endpoints, which are aligned in the top and bottom boundary planes, i.e.,  $\iota(\partial\Gamma_0) = \iota(\Gamma) \cap (\mathbb{R} \times \{0\} \times \{0, 1\})$ ;
- for every vertex  $v$  and every oriented edge  $e \in \Gamma_v$ , the velocity vector  $\partial_v(e)$  at  $v$  is contained in the  $(x_v, y_v)$ -plane and is not parallel to  $x_v$ ;

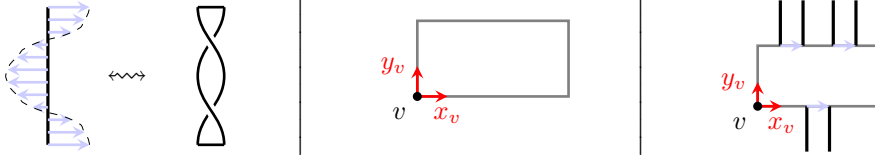


Figure 1.8: The framing on points and edges allows us to think of them as thickened ribbons and boxes. The boundary condition on the framing for edges prescribes a preferred side for ribbons, which is visible from the preferred side of the box to which it is attached. This is how ribbon graphs are presented, for instance, in [Tur16; RT90].

- for every vertex  $v$  and every oriented edge  $e \in \Gamma_v$ , the normal vector  $n_v(e)$  of  $e$  provided by the framing is contained in the direction of  $x_v$  and  $(n_v(e), \partial_v(e))$  is form a positive basis of  $T_v(\mathbb{R}^2 \times [0, 1])$ .  $\diamond$

The framing on edges can be thought of as a thickening of edges, so that they can be considered as actual ribbons. The framing on vertices allows us to think of them as boxes with sides parallel to the  $(x_v, y_v)$ -plane to which the ribbons are attached. Finally, the boundary condition on the framing can be understood as choosing a preferred side for the ribbons, which is always visible near the vertices (see Figure 1.8).

Let  $\Gamma$  be a ribbon graph. The ordered set  $\Gamma_v^i$  and  $\Gamma_v^o$  of incoming and outgoing edges of a given vertex  $v$  can be determined exactly as for pivotal graphs, using the fact that the velocity vector of an edge lies in the  $(x_v, y_v)$ -plane near every vertex.

By pushing the different framings of a ribbon graph  $\Gamma$  through an isotopy  $H(-, t)$  of the cube  $\mathbb{R}^2 \times [0, 1]$ , we obtain an image of  $\Gamma$  that is a ribbon graph for every  $t \in [0, 1]$ , so we define:

**Definition 1.2.19.** An *isotopy of ribbon graphs* is an isotopy of  $\mathbb{R}^2 \times [0, 1]$  fixing the top and bottom boundary planes  $\mathbb{R}^2 \times \{0, 1\}$  pointwise.  $\diamond$

As usual, we will represent ribbon graphs through planar diagrams, obtained by projecting a generic ribbon graph into the plane and keeping track of the relative position of the strands at intersection points. More precisely:

**Definition 1.2.20. (i)** Let  $\pi: \mathbb{R}^2 \times [0, 1] \rightarrow \mathbb{R} \times [0, 1]$  be a projection onto the planar strip that preserves the boundary. A ribbon graph  $\Gamma$  is in *generic position* with respect to  $\pi$  if

- the restriction  $\pi|_\Gamma$  has a finite number of self-intersections, all of which are transverse double points not involving vertices;
- for every point  $p$  on an edge, the normal vector determining the framing does not lie in the kernel of the differential  $T_p\pi$ ;

- the differential  $T_v\pi$  is injective on the  $(x_v, y_v)$ -plane, for every interior vertex  $v$ .

(ii) A *ribbon diagram* is a cyclic planar graph (cf. Definition 1.2.11) with two distinguished classes of interior vertices, called *positive crossings* and *negative crossings*, that we represent as



respectively.  $\diamond$

Viewing the coupons of a ribbon graph as boxes and its edges as actual ribbons, the genericity condition guarantees that these are always projected parallel to the plane. Consequently, all the information about the ribbon graph can be recovered from its diagram by thickening the edges in the direction of the plane (i.e., by endowing the diagram with a *blackboard framing*) and lifting it to three dimensions using the crossing data. In particular, twists of ribbons must be deformed as

$$\text{twist} \sim \text{p} \sim \text{d}. \quad (1.2.9)$$

Deformations of ribbon graphs have the following diagrammatic counterpart:

**Definition 1.2.21.** A *regular isotopy* of a ribbon diagram  $D$  consists of a planar isotopy of  $\mathbb{R} \times [0, 1]$  preserving  $\mathbb{R} \times \{0, 1\}$  and a finite number of the following *Reidemeister moves*:

$$\text{d} \longleftrightarrow \text{p}, \quad (\text{RI}')$$

$$\text{crossing} \longleftrightarrow \text{parallel} \longleftrightarrow \text{crossing}, \quad (\text{RII})$$

$$\text{crossing} \longleftrightarrow \text{crossing}, \quad (\text{RIII})$$

for all the orientations of the strands and all the variants obtained by appropriately replacing the crossing signs.  $\diamond$

It is well-known that two ribbon diagrams represent the same isotopy class of ribbon graphs if and only if they are related by regular isotopy. Therefore, the set of isotopy classes of ribbon graphs up to isotopy is in bijection with the set of regular isotopy classes of ribbon diagrams, and we do not distinguish between them anymore.

Given a ribbon category  $\mathcal{C}$ , we define  $\mathcal{C}$ -colourings for ribbon graphs exactly in the same way as for cyclic planar graphs. This can be done since we are assuming that the velocity vector of every edge  $e$  adjacent to a vertex  $v$  is contained in the  $(x_v, y_v)$ -plane provided by the framing, so that it can be treated as a planar framed vertex and the same rules apply. The source and target of the edge are determined as in the previous cases. We omit the details.

**Definition 1.2.22.** The *free ribbon category on  $\mathcal{C}$*  is the category  $\text{Rib}(\mathcal{C})$  given by:

- objects are finite sequences of pairs  $(x, \varepsilon)$ , where  $x \in \mathcal{C}$  and  $\varepsilon \in \{0, 1\}$ ;
- a morphism between two such configurations  $X$  and  $Y$  is the isotopy class of a ribbon graph  $\Gamma$  with  $\text{source}(\Gamma) = X$  and  $\text{target}(\Gamma) = Y$ .

Composition is given by vertically stacking morphisms and a strictly monoidal structure is given by concatenation of objects and horizontally stacking diagrams. The unit of an object consists of a collection of non-twisted vertical ribbons and the monoidal unit is the empty sequence.  $\diamond$

Note that the boundary condition on the framing of edges at the top and bottom boundary surfaces guarantees that the framing of two composable ribbon graphs glues properly, so that the composition is a well-defined operation. The monoidal category  $\text{Rib}(\mathcal{C})$  is again strictly pivotal, with dualities given as with  $\text{Piv}(\mathcal{C})$ . Moreover, it is braided, with braiding defined by  $\mathcal{C}$ -coloured crossings, and ribbon. The twist is represented by the (equivalent) diagrams in (1.2.9). A set of monoidal generators is given by  $\mathcal{C}$ -colored cups, caps, crossings and elementary ribbon graphs (i.e., connected ribbon graphs containing exactly one coupon) and there is a strictly monoidal evaluation functor ([Shu94; Yet01; RT90])

$$\text{ev}_{\mathcal{C}}^{\text{Rib}} : \text{Rib}(\mathcal{C}) \rightarrow \mathcal{C}$$

to the underlying ribbon category  $\mathcal{C}$  determined by

$$(x, 0) \mapsto x, \quad (x, 1) \mapsto x^*,$$

$$\begin{array}{c} \nearrow \\ \searrow \\ x \quad y \end{array} \mapsto c_{x,y}, \quad \text{and} \quad \begin{array}{c} \text{---} \\ \text{---} \end{array} \mapsto \text{ev}_x \quad \text{and} \quad \begin{array}{c} \text{---} \\ \text{---} \end{array} \mapsto \text{coev}_x.$$

This functor is often called the *Reshetikhin–Turaev evaluation functor* and all the details about its construction can be found in [Tur16, Chapter I]. Note that it is a ribbon functor (the justification is analogous to the one given in the previous section for pivotal categories).

**Remark 1.2.23.** The diagrams

$$\begin{array}{c} \uparrow \\ \text{---} \end{array} \quad \text{and} \quad \begin{array}{c} \uparrow \\ \text{---} \end{array}$$

correspond to the twist  $\theta, \theta'$  in  $\text{Rib}(\mathcal{C})$  obtained from its (trivial) pivotal structure. Since both diagrams are regularly isotopic, their images in  $\mathcal{C}$  must coincide, so we need the underlying category to be ribbon. A graphical calculus for balanced (but not necessarily ribbon) categories could be obtained by considering ribbon diagrams only up to the (RII) and (RIII) moves.  $\diamond$

### 1.3 Associated theories

In this section, we extend the graphical calculus for monoidal categories introduced in Section 1.2 to arbitrary surfaces, yielding 2- and 3-dimensional TQFTs assigning vector spaces to closed manifolds and bimodules to cobordisms. When the input algebraic data is a monoidal or rigid category  $\mathcal{C}$ , we obtain framed theories whose state spaces are described through vector spaces generated by a class of graphs on surfaces, locally looking like progressive or polarised graphs [KST24]. When the category  $\mathcal{C}$  comes with a pivotal structure, the construction extends to oriented surfaces [string-nets-haioun].

On the other hand, using a ribbon category  $\mathcal{C}$  as algebraic data, skein theory yields a 3-dimensional TQFT [Hai25] assigning skein categories to surfaces and skein module functors to cobordisms [Wal06; Joh19; Coo23]. We review the basics of this theory in the last subsection.

#### 1.3.1 Framed and marked surfaces

We collect here some notions about framed surfaces that we will use throughout the rest of the section.

**Definition 1.3.1.** Let  $M$  be a  $d$ -dimensional manifold and  $n \geq d$ . An  $n$ -framing of  $M$  is an isomorphism  $f: TM \oplus \mathbb{R}^{n-d} \xrightarrow{\sim} M \times \mathbb{R}^n$  of vector bundles over  $M$ . When  $n = d$  we refer to this simply as a *framing*.  $\diamond$

Hence, an  $n$ -th framing on  $M$  consists of a family

$$f_p = (x_p^1, \dots, x_p^d, v_p^{d+1}, \dots, v_p^n) \in T_p M \oplus \mathbb{R}^{n-d}$$

varying smoothly with  $p$ . We define the *opposite  $n$ -framing* as the framing  $\bar{f}$  obtained by changing the sign of the last vector, i.e.,

$$\bar{f}_p = (x_p^1, \dots, x_p^d, v_p^{d+1}, \dots, -v_p^n).$$

If the manifold  $M$  comes with an orientation, the framing  $f$  may be compatible with it or not. When it is, we say that  $f$  is an *orientation framing*. When  $M$  is not oriented, we will endow it with the orientation prescribed by the framing, i.e., we declare that the framing yields a positive basis of every tangent space.

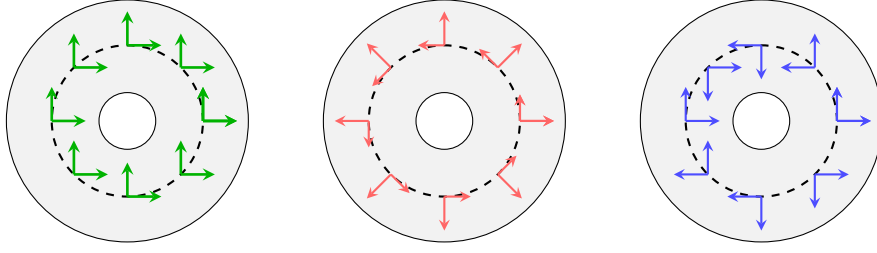


Figure 1.9: Three positive 2-framings of  $\mathbb{S}^1$  with winding numbers 0 (in green), 1 (in red) and 3 (in blue).

**Example 1.3.2.** (i) The data of a 2-framing of the interval  $I$  is equivalent to a smooth map

$$f: I \rightarrow \mathrm{GL}_2(\mathbb{R}).$$

Considering them up to homotopy, we have

$$[I, \mathrm{GL}_2(\mathbb{R})] \cong \pi_0(\mathrm{GL}_2(\mathbb{R})) \cong \{0, 1\},$$

so there are exactly two 2-framings of the interval up to homotopy, corresponding to its two possible orientations.

(ii) For the circle  $\mathbb{S}^1$ , a 2-framing is equivalent to the data of a map

$$f: \mathbb{S}^1 \rightarrow \mathrm{GL}_2(\mathbb{R}).$$

Up to homotopy, we have

$$[\mathbb{S}^1, \mathrm{GL}_2(\mathbb{R})] \cong [\mathbb{S}^1, \mathrm{O}_2(\mathbb{R})] \cong \mathbb{Z} \times \mathbb{Z}_2.$$

The  $\mathbb{Z}_2$  factor represents the two possible orientations for a framing while the  $\mathbb{Z}$  factor is the winding number: it represents the number of complete rotations of the framing as we move once around the entire circumference of  $\mathbb{S}^1$  (see Figure 1.9).

(iii) The Poincaré–Hopf theorem implies that the only closed surface admitting a framing is the torus  $\mathbb{T}^2$ .  $\diamond$

Given a framed surface  $(S, f)$ , the framing  $f$  induces a framing of the boundary  $\partial S$  via

$$T(\partial S) \oplus \mathbb{R} \cong TS|_{\partial S} \xrightarrow{f|_{\partial S}} \mathbb{R}^2,$$

where the first isomorphism is induced by identifying  $\mathbb{R}$  with an outward-normal direction of  $TS|_{\partial S}$ .

**Definition 1.3.3.** A *marked framed surface* is a framed surface  $(S, f)$  endowed with a marking  $\mathcal{B}$ , i.e., a finite (possibly empty) collection of compact, connected, 2-framed 1-dimensional manifolds  $(C_1, f_1), \dots, (C_m, f_m)$  together with a frame preserving embedding

$$\bigsqcup_{i=1}^m C_i \hookrightarrow \partial S.$$

We will not distinguish between the  $C_i$ 's and their images in  $\partial S$ .  $\diamond$

**Definition 1.3.4. (i)** The *category of marked framed 2-cobordisms* is the category  $\text{Cob}_{2,1}^{\text{fr,mkd}}$  given by:

- objects are compact 2-framed 1-dimensional manifolds, i.e., disjoint unions of 2-framed intervals and circles;
- a morphism between two such manifolds  $X$  and  $Y$  is the equivalence class of a marked framed compact surface  $(S, f, \mathcal{B})$  with

$$\bigcup_{C \in \mathcal{B}} C \cong \overline{X} \sqcup Y \quad (1.3.10)$$

as framed manifolds, where  $\overline{X}$  is endowed with the opposite framing. Here, two cobordisms are considered to be equivalent if there exists a diffeomorphism between them, preserving the framing up to homotopy and the decomposition (1.3.10) of the boundary.

The composition is given by gluing cobordisms along collars of the marked components on the boundary and the disjoint union endows  $\text{Cob}_{2,1}^{\text{fr,mkd}}$  with a symmetric monoidal structure.

(ii) A *categorified framed (2,1)-TQFT* is a symmetric monoidal functor

$$\mathcal{Z}: \text{Cob}_{2,1}^{\text{fr,mkd}} \rightarrow \mathcal{C}$$

to a symmetric monoidal category  $\mathcal{C}$ .  $\diamond$

### 1.3.2 The framed string-net TQFT

In this subsection, we follow closely [KST24]. In the construction of the graphical calculus for monoidal and rigid categories, we used crucially the existence of two privileged directions in  $\mathbb{R}^2 \times [0, 1]$  provided by the two coordinate axes. Indeed, we defined progressive planar graphs as planar graphs whose edges never run parallel to the  $x$ -axis, so that they are injective in the  $y$ -coordinate. Similarly, a polarised planar graph is a planar graph whose edges may be tangent to the  $x$ -direction only in a finite number of points (the cups and caps). In order to generalize this graphical calculus to other surfaces, we need an analogue of the  $x$ - and  $y$ -directions allowing us to extend the definition of progressive and polarised graphs. This is provided by the extra data of a framing (cf. Definition 1.3.1).

A framing on a surface  $S$  is determined by two nowhere-vanishing vector fields  $(X, Y)$  on  $S$  such that  $(X_p, Y_p)$  is a positive basis of  $T_p S$ , for every point  $p \in S$ . Recall that the maximal flow of a vector field  $V$  is a collection of parametrized curves

$$\{\theta_p: I_p \rightarrow S\}_{p \in S}$$

defined on an open interval  $I_p$  containing 0, everywhere tangent to  $V$  and such that  $\theta_p(0) = p$ , for every  $p \in S$  (see [Lee13, Chapters 8-10] for precise definitions and a very complete discussion). The flow lines of  $X$  and  $Y$  are the analogues of the  $x$ - and  $y$ -axes for framed surfaces. Generalizations of progressive and polarised planar graphs can be obtained by asking edges to run always transversal to the  $X$ -flow line in the first case, and to be tangent to these lines in a finite number of points in the second. We formulate this more precisely as follows. Fix a framed surface  $(S, f)$  with framing  $f = (X, Y)$ :

**Definition 1.3.5.** (i) A *locally progressive graph* on  $S$  is a graph  $(\Gamma, \Gamma_0)$  equipped with a smooth embedding  $\iota: \Gamma \rightarrow S$  such that:

- the image of  $\Gamma$  intersects  $\partial S$  exactly at its endpoints, i.e.,  $\iota(\partial\Gamma_0) = \iota(\Gamma) \cap \partial S$ ;
- for every edge  $e \in \Gamma \setminus \Gamma_0$  and any smooth parametrization  $\bar{e} \cong [0, 1]$ , the velocity vector of the embedded edge is nowhere parallel to the  $X$ -field.

(ii) A *locally polarised graph* on  $S$  is an oriented graph  $(\Gamma, \Gamma_0)$  equipped with an embedding  $\iota: \Gamma \rightarrow S$  such that:

- the image of  $\Gamma$  intersects  $\partial S$  exactly at its endpoints, i.e.,  $\iota(\partial\Gamma_0) = \iota(\Gamma) \cap \partial S$ ;
- for every edge  $e \in \Gamma \setminus \Gamma_0$  and any smooth parametrization  $\bar{e} \cong [0, 1]$ , the velocity vector of the embedded edge is parallel to the  $X$ -field at only finitely many points, all of which lie in the interior of the edge.

As in the planar case, we will omit the embedding from the notation and denote embedded graphs by their underlying abstract graph.  $\diamond$

These are precisely the class of graphs on  $S$  that locally look like progressive and polarised graphs. Indeed, we can always embed  $\mathbb{R} \times [0, 1]$  in  $S$  in such a way that the  $x$ - and  $y$ - axes and their parallel lines are sent to the flow lines of the framing:

**Definition 1.3.6.** Given a progressive or polarised graph  $\Gamma \subset S$ , a *generic rectangle*  $R$  on  $S$  is the image of an embedding  $\Phi: \mathbb{R} \times [0, 1] \rightarrow S$  such that:

- $\Phi^{-1}(\Gamma \cap R)$  is contained in some compact subset of  $\mathbb{R} \times [0, 1]$  and intersects  $\mathbb{R} \times \{0, 1\}$  transversally away from vertices;
- the image of the canonical framing of  $\mathbb{R} \times [0, 1]$  under  $T\Phi$  is precisely  $(X, Y)$ .  $\diamond$

The straightening theorem for vector fields ([LPV13, Theorem B.2]) implies that every point  $p \in S$  is contained in a generic rectangle. The conditions in Definition 1.3.5 ensure that pulling back the restriction of a locally progressive (or polarised) graph to a generic rectangle yields a standard progressive (or polarised) planar graph.

We decorate locally progressive and locally polarised graphs with objects and morphisms of a suitable monoidal category by applying locally the same rules as with progressive planar graphs. Note first that we have well-defined notions of a cup and a cap:

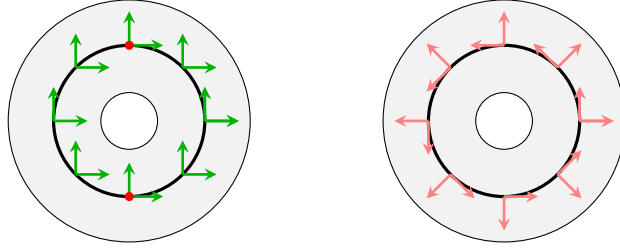


Figure 1.10: Two different framings on the annulus. The one on the left is inherited from  $\mathbb{R}^2$  and the other one is its *radial* framing. The circle embedded in the annulus has two redirection points for the first framing (drawn in red) and no redirection point for the radial framing, so its rotation number is 2 in the first picture and 0 in the second one.

these are points on an edge where the velocity vector is parallel to the  $X$ -field and such that the pullback of their restriction to any generic rectangle forms a planar cup or cap. As in the planar case, we call these points *redirection points*, and we define their *turning number* as the turning number of the planar redirection point obtained by restriction to any generic rectangle. The *rotation number* of an edge is again defined as the sum of the turning numbers of all its redirection points (see Figure 1.10).

**Definition 1.3.7.** Given a monoidal (resp. rigid) category  $\mathcal{C}$ , a  $\mathcal{C}$ -coloured locally progressive graph (resp.  $\mathcal{C}$ -coloured locally polarised graph) on  $S$  is a locally progressive (resp. polarised) graph  $\Gamma$  on  $S$  equipped with functions

$$\phi_0: \Gamma_0 \setminus \partial\Gamma_0 \rightarrow \text{mor}(\mathcal{C}) \quad \text{and} \quad \phi_1: \Gamma \setminus \Gamma_0 \rightarrow \text{ob}(\mathcal{C})$$

such that the pullback of the restriction of  $\Gamma$  to any generic rectangle is a  $\mathcal{C}$ -coloured progressive (resp. polarised) planar graph.  $\diamond$

We will consider graphs in marked surfaces, with endpoints lying in one of the marked components and satisfying some *boundary conditions*.

**Definition 1.3.8.** A *boundary condition* for locally progressive graphs on a marked framed surface  $(S, f, \mathcal{B})$  is a pair  $(B, \nu_B)$ , where  $B$  is a finite (possibly empty) collection of points

$$B = \{p_1, \dots, p_m\} \subset \bigcup_{C \in \mathcal{B}} C$$

and  $\nu_B: B \rightarrow \text{ob}(\mathcal{C})$ . Similarly, a boundary condition for locally polarised graphs is a pair  $(B, \nu_B)$ , where  $B$  is as above and

$$\nu_B: B \rightarrow \text{ob}(\mathcal{C}) \times \mathbb{N}.$$

We will refer to the data of  $(\mathcal{B}, (B, \nu_B))$  just by  $\nu_B$ .  $\diamond$

Let  $(S, f, \mathcal{B})$  be a marked framed surface. Let  $\mathbb{k}$  be a commutative ring and  $\mathcal{C}$  a  $\mathbb{k}$ -linear monoidal or rigid category. We write  $\text{Graph}_{\mathcal{C}}(S, f, \nu_B)$  for the  $\mathbb{k}$ -linear space spanned, depending on the context, by either  $\mathcal{C}$ -coloured locally progressive or locally polarised graphs on  $S$ , with boundary condition  $\nu_B$ : this means that the set of endpoints of  $\Gamma$  coincide with  $B$  and that their colours are precisely the ones given by  $\nu_B$ .

**Definition 1.3.9. (i)** We say that an element  $\Gamma := \sum_i \lambda_i \Gamma_i$  of  $\text{Graph}_{\mathcal{C}}(S, f, \nu_B)$  is *null* if there exists an embedding  $\Phi: \mathbb{R} \times [0, 1] \rightarrow S$  such that

- the image  $R$  of  $\Phi$  is a generic rectangle for all the  $\Gamma_i$ 's;
- all the  $\Gamma_i$  coincide outside  $R$  and

$$\sum_i \lambda_i \text{ev}_{\mathcal{C}}(\Phi^{-1}(\Gamma \cap R)) = 0$$

in  $\mathcal{C}$ , where  $\text{ev}_{\mathcal{C}}$  is  $\text{ev}_{\mathcal{C}}^{\text{Mon}}$  for locally progressive and  $\text{ev}_{\mathcal{C}}^{\text{Rig}}$  for locally polarised graphs.

We write  $\text{nGraph}_{\mathcal{C}}(S, f, \nu_B)$  for the  $\mathbb{k}$ -linear subspace of  $\text{Graph}_{\mathcal{C}}(S, f, \nu_B)$  spanned by null graphs.

(ii) The *framed  $\mathcal{C}$ -string-net space* with boundary condition  $\nu_B$  is the quotient

$$\text{SN}_{\mathcal{C}}^{\text{fr}}(S, f, \nu_B) := \text{Graph}_{\mathcal{C}}(S, f, \nu_B) / \text{nGraph}_{\mathcal{C}}(S, f, \nu_B). \quad \diamond$$

We explain next how string-net spaces yield a categorified  $(2, 1)$ -TQFT with values in  $\text{BIMOD}$ . Recall that the graphical calculus yields an equivalence

$$\mathcal{C} \simeq \text{Mon}(\mathcal{C}) / \ker \text{ev}_{\mathcal{C}}^{\text{Mon}}$$

for every monoidal category and an equivalence

$$\mathcal{C} \simeq \text{Rig}(\mathcal{C}) / \ker \text{ev}_{\mathcal{C}}^{\text{Rig}}$$

when  $\mathcal{C}$  is a rigid category. Under these equivalences, the hom-spaces of  $\mathcal{C}$  correspond precisely to framed string-net spaces on a square. On the other hand, let  $n \in \mathbb{Z}$  and  $(\mathbb{A}, f_n)$  be the annulus (seen as a cylinder) with the framing corresponding to  $n$ :

**Definition 1.3.10.** For a monoidal or rigid category  $\mathcal{C}$ , the  *$n$ -th framed cylinder category*  $\text{Cyl}_n(\mathcal{C})$  is the category with

- objects: finite configurations of points  $B = \{p_1, \dots, p_k\} \subset \mathbb{S}^1$  together with a labelling  $\nu: B \rightarrow \text{ob}(\mathcal{C})$  if  $\mathcal{C}$  is monoidal and  $\nu: B \rightarrow \text{ob}(\mathcal{C}) \times \mathbb{N}$  if  $\mathcal{C}$  is rigid;
- morphisms: given two objects  $(B_0, \nu_0)$  and  $(B_1, \nu_1)$ , we set

$$\text{Hom}_{\text{Cyl}_n(\mathcal{C})}((B_0, \nu_0), (B_1, \nu_1)) := \text{SN}_{\mathcal{C}}^{\text{fr}}(\mathbb{A}, f_n, \nu_{01})$$

where the boundary condition is supported on the set of points  $B_{01} := (B_0 \times \{0\}) \cup (B_1 \times \{1\})$ , with value map  $\nu_{01}(-, i) := \nu_i$ .

The composition is given by stacking cylinders and rescaling in the second coordinate and the identity of an object is represented by a collection of vertical edges on the cylinder.  $\diamond$

By gluing squares and cylinders along marked boundary components, we get actions of  $\mathcal{C}$  and their cylinder categories on the framed string-net spaces of a given surface. Let us make this precise. Given a marked framed surface  $(S, f, \mathcal{B})$ , the framing of the surface determines a 2-framing  $f$  on each component  $C \in \mathcal{B}$ . Then, we write

$$\mathrm{SN}_{\mathcal{C}}^{\mathrm{fr}}(C, f) := \begin{cases} \mathcal{C} & \text{if } (C, f) \simeq I; \\ \mathrm{Cyl}_n(\mathcal{C}) & \text{if } (C, f) \simeq (\mathbb{A}, f_n). \end{cases} \quad \text{and} \quad \mathrm{NS}_{\mathcal{C}}(C, \bar{f}) = \mathrm{SN}_{\mathcal{C}}(C, f)^{\mathrm{op}}.$$

Let  $\mathcal{B} = \{C_1, \dots, C_m\}$  and set  $f_i$  for the 2-framing induced by  $f$  on  $C_i$ .

**Definition 1.3.11.** The *framed string-net bimodule* on  $S$  is the functor

$$\mathrm{SN}_{\mathcal{C}}^{\mathrm{fr}}(S, f, \mathcal{B}) : \bigotimes_{i=1}^m \mathrm{SN}_{\mathcal{C}}^{\mathrm{fr}}(C_i, f_i) \rightarrow \mathrm{Vect}_{\mathbb{k}}$$

defined on objects by

$$((B_1, \nu_1), \dots, (B_m, \nu_m)) \mapsto \mathrm{SN}_{\mathcal{C}}^{\mathrm{fr}}(S, f, \nu_{1, \dots, m}),$$

where  $\nu_{1, \dots, m} : \bigcup B_i \times \{i\} \rightarrow \mathrm{ob}(\mathcal{C})$  is defined by  $\nu_{1, \dots, m}(-, i) = \nu_i$ . The action of morphisms is defined by gluing rectangles and cylinders along tubular neighbourhoods of the marked components.  $\diamond$

The proof of [Wal06, Theorem 4.4.2] generalizes to framed surfaces, yielding:

**Theorem 1.3.12.** For every monoidal or rigid category  $\mathcal{C}$ , there is a framed  $(2, 1)$ -TQFT

$$\mathrm{SN}_{\mathcal{C}}^{\mathrm{fr}} : \mathrm{Cob}_{2,1}^{\mathrm{fr}, \mathrm{mkd}} \rightarrow \mathrm{BIMOD}$$

assigning  $\mathcal{C}$  and  $\mathrm{Cyl}_n(\mathcal{C})$  to the interval and the  $n$ -th framed circle, respectively, and the framed string-net bimodule to a marked framed surface.  $\blacksquare$

### 1.3.3 The oriented string-net TQFT

The graphical calculus for pivotal categories is obtained from the graphical calculus for rigid categories by dropping the integer labels counting the rotation number of edges, and by replacing the rigid notion of progressive isotopy by a more flexible one, namely any planar isotopy of the band  $\mathbb{R} \times [0, 1]$ . The rotation number of the edges can be defined only in the presence of a framing (provided by the  $x$ - and  $y$ - axes of  $\mathbb{R} \times [0, 1]$ ), and progressive isotopies are the isotopies preserving this framing (up to homotopy). Dropping the labels and allowing general isotopies amounts to forgetting the framing, which results in a graphical calculus that can be extended now to any oriented surface, not necessarily framed, yielding an oriented version of the string-net construction defined in the previous section.

**Definition 1.3.13.** A *locally cyclic graph* in an oriented surface  $S$  is an oriented graph  $(\Gamma, \Gamma_0)$  together with the data of

- an embedding  $\iota: \Gamma \rightarrow S$  whose image intersects the boundary  $\partial S$  exactly at its endpoints, i.e.,  $\iota(\partial\Gamma_0) = \iota(\Gamma) \cap \partial S$ ;
- a positive framing for every interior vertex  $v$ , i.e., a basis  $(x_v, y_v)$  of the tangent space  $T_v S$  at  $v$ ;

subject to the condition that, for every interior vertex  $v$ , none of the velocity vectors at  $v$  of its adjacent edges is parallel to  $x_v$ . We will moreover assume that endpoints are endowed with a framing whose first vector is contained in  $T_v(\partial S)$ .  $\diamond$

Again, these are graphs that locally look like cyclic planar graphs. The set of incoming and outgoing edges for every vertex are defined exactly as in the planar case and, given a pivotal category  $\mathcal{C}$ , the notion of  $\mathcal{C}$ -colouring for planar cyclic graphs extends to locally cyclic graphs. We will use the following class of rectangles to evaluate  $\mathcal{C}$ -coloured graphs:

**Definition 1.3.14.** A *generic rectangle* on  $S$  for a locally cyclic graph  $\Gamma$  is the image of an embedding  $\Phi: \mathbb{R} \times [0, 1] \rightarrow S$  such that  $\Phi^{-1}(\Gamma \cap R)$  is a planar cyclic graph.  $\diamond$

As in the framed case, we will consider the spaces spanned by graphs satisfying some boundary condition:

**Definition 1.3.15. (i)** A *marking* on a oriented surface  $S$  consists of a finite (possibly empty) collection  $\mathcal{B} = \{C_1, \dots, C_m\}$  of compact, connected, 1-dimensional oriented manifolds together with an orientation-preserving embedding

$$\bigsqcup_{i=1}^m C_i \hookrightarrow \partial S.$$

As in the framed case, we will not distinguish between the  $C_i$ 's and their images in  $\partial S$ .

**(ii)** A *boundary condition* for locally cyclic graphs on a marked oriented surface  $(S, \mathcal{B})$  is a pair  $(B, \nu_B)$ , where  $B$  is a finite (possibly empty) collection of points

$$B = \{p_1, \dots, p_m\} \subset \bigcup_{C \in \mathcal{B}} C$$

and  $\nu_B: B \rightarrow \text{ob}(\mathcal{C}) \times \{0, 1\}$ .

As in the previous section, we write simply  $\nu_B$  to refer to  $(\mathcal{B}, (B, \nu_B))$ .  $\diamond$

Fix a marked oriented surface  $(S, \mathcal{B})$ . Let  $\mathbb{k}$  be a commutative ring and  $\mathcal{C}$  a  $\mathbb{k}$ -linear pivotal category. We define  $\text{Graph}_{\mathcal{C}}^{\text{cyc}}(S, \mathcal{B})$  as the  $\mathbb{k}$ -linear space spanned by  $\mathcal{C}$ -coloured locally cyclic graphs on  $S$ , with boundary condition  $\nu_B$ . Null graphs are defined exactly as in Definition 1.3.9 using the evaluation functor for pivotal categories and we set  $\text{nGraph}_{\mathcal{C}}^{\text{cyc}}(S, \eta_B) \subset \text{Graph}_{\mathcal{C}}^{\text{cyc}}(S, \eta_B)$  for the  $\mathbb{k}$ -linear subspace spanned by null graphs.

**Definition 1.3.16.** The *oriented  $\mathcal{C}$ -framed string-net space* with boundary condition  $\nu_B$  is the quotient

$$\mathrm{SN}_{\mathcal{C}}^{\mathrm{or}}(S, \nu_B) := \mathrm{Graph}_{\mathcal{C}}^{\mathrm{cyc}}(S, \nu_B) / \mathrm{nGraph}_{\mathcal{C}}^{\mathrm{cyc}}(S, \nu_B). \quad \diamond$$

As with monoidal and rigid categories, the graphical calculus provides us with an equivalence

$$\mathcal{C} \simeq \mathrm{Piv}(\mathcal{C}) / \ker \mathrm{ev}_{\mathcal{C}}^{\mathrm{Piv}}$$

for every pivotal category, and we can define oriented analogues of the framed cylinder categories as follows:

**Definition 1.3.17.** For a pivotal category  $\mathcal{C}$ , the *oriented cylinder category*  $\mathrm{Cyl}(\mathcal{C})$  is the category with

- objects: finite configurations  $B = \{p_1, \dots, p_k\} \subset \mathbb{S}^1$  together with a labelling  $\nu: B \rightarrow \mathrm{ob}(\mathcal{C}) \times \{0, 1\}$ ;
- morphisms: given two objects  $(C_0, \nu_0)$  and  $(C_1, \nu_1)$ , we set

$$\mathrm{Hom}_{\mathrm{Cyl}(\mathcal{C})}(C_0, C_1) := \mathrm{SN}_{\mathcal{C}}^{\mathrm{or}}(\mathbb{A}, \nu_{01})$$

where the boundary component is supported on the set of points  $B_{01} := (B_0 \times \{0\}) \cup (B_1 \times \{1\})$  with value map  $\nu_{01}(-, i) := \nu_i$ .

The composition is given by stacking cylinders and rescaling in the second coordinate and the identity of an object is represented by a collection of vertical edges on the cylinder.  $\diamond$

The oriented string-net bimodule is defined exactly as in the framed case:

**Definition 1.3.18.** Given a marked oriented surface  $(S, \mathcal{B})$  with  $\mathcal{B} = \{C_1, \dots, C_m\}$ , the *oriented string-net bimodule* on  $S$  is the functor

$$\mathrm{SN}_{\mathcal{C}}^{\mathrm{or}}(S, \mathcal{B}): \bigotimes_{i=1}^m \mathrm{SN}_{\mathcal{C}}^{\mathrm{or}}(C_i) \rightarrow \mathrm{Vect}_{\mathbb{k}}$$

defined on objects by  $((B_1, \nu_1), \dots, (B_m, \nu_m)) \mapsto \mathrm{SN}_{\mathcal{C}}^{\mathrm{or}}(S, \nu_{1, \dots, m})$  where  $\nu_{1, \dots, m}: \bigcup B_i \times \{i\} \rightarrow \mathrm{ob}(\mathcal{C}) \times \{0, 1\}$  is defined by  $\nu_{1, \dots, m}(-, i) = \nu_i$ . The action of morphisms is defined by gluing rectangles and cylinders along tubular neighbourhoods of the components marked on the boundary.  $\diamond$

We obtain in this case an oriented  $(2, 1)$ -TQFT:

**Theorem 1.3.19** ([Wal06, Theorem 4.4.2]). *For every pivotal category  $\mathcal{C}$ , there is an oriented  $(2, 1)$ -TQFT*

$$\mathrm{SN}_{\mathcal{C}}^{\mathrm{or}}: \mathrm{Cob}_{2,1}^{\mathrm{mkd}} \rightarrow \mathrm{BIMOD}$$

assigning  $\mathcal{C}$  and  $\mathrm{Cyl}(\mathcal{C})$  to the positive interval and circle, respectively, and the oriented framed string-net bimodule to a marked oriented surface.

### 1.3.4 Skein theory

As we have seen in Section 1.2.4, a ribbon structure on a monoidal category yields a graphical calculus in  $\mathbb{R}^2 \times [0, 1]$  compatible with the topology of ribbon graphs. The extension of this graphical calculus to any oriented 3-manifold is what we call *skein theory*.

Let  $M$  be an oriented 3-manifold.

**Definition 1.3.20.** A *ribbon graph* on  $M$  is an oriented graph  $(\Gamma, \Gamma_0)$  together with the data of

- a smooth embedding  $\iota: \Gamma \rightarrow M$  whose image intersects  $\partial M$  exactly at its endpoints, i.e.,  $\iota(\Gamma) \cap \partial M = \iota(\partial\Gamma_0)$ ;
- a framing  $(x_v, y_v, z_v)$  for every vertex  $v \in \Gamma \setminus \partial\Gamma_0$ , i.e., a basis of  $T_v M$ , with the additional assumption that  $(x_v, z_v) \in T_v(\partial M)$  if  $v$  is an endpoint;
- a framing for every connected component  $c \subset \Gamma \setminus \partial\Gamma_0$ , that is, a trivialization of the normal bundle  $N(c) \subset TM$ ;

satisfying the following conditions:

- for every vertex  $v$  and every oriented edge  $e \in \Gamma_v$  adjacent to  $v$ , the velocity vector  $\partial_v(e)$  at  $v$  is contained in the  $(x_v, y_v)$ -plane and is not parallel to  $x_v$ ;
- for every vertex  $v$  and every oriented edge  $e \in \Gamma_v$  adjacent to  $v$ , the normal vector of  $e$  provided by the framing is in the positive direction of  $x_v$ .  $\diamond$

The set  $\Gamma_v$  of edges adjacent to an interior vertex  $v$  splits into two ordered sets of incoming and outgoing edges exactly as in  $\mathbb{R}^2 \times [0, 1]$ . Moreover, given a ribbon category  $\mathcal{C}$ , the notion of  $\mathcal{C}$ -colouring in  $\mathbb{R}^2 \times [0, 1]$  extends straightforwardly to yield a notion of  $\mathcal{C}$ -coloured ribbon graph in  $M$ .

**Definition 1.3.21.** A *generic cube* for a ribbon graph  $\Gamma$  in an oriented 3-manifold  $M$  is the image  $K$  of an embedding  $\Phi: \mathbb{R}^2 \times [0, 1] \rightarrow M$  such that  $\Phi^{-1}(K \cap \Gamma)$  is a ribbon graph in  $\mathbb{R}^2 \times [0, 1]$ .  $\diamond$

Fix a ribbon category  $\mathcal{C}$  and an oriented 3-manifold  $M$ . A boundary condition for ribbon graphs consists of a collection of  $\mathcal{C}$ -coloured, directed, framed points on the boundary of  $M$ . More precisely:

**Definition 1.3.22.** A *boundary condition* for ribbon graphs in  $M$  is a pair  $(B, \nu_B)$  where  $B \subset \partial M$  is a finite (possibly empty) collection of points on the boundary of  $M$  and

$$\nu_B: B \rightarrow \text{ob}(\mathcal{C}) \times \{0, 1\} \times T(\partial M)|_B$$

is such that  $\nu_B(p) \in \text{ob}(\mathcal{C}) \times \{0, 1\} \times (T_p(\partial M) \setminus \{0\})$ .  $\diamond$

Let  $\mathbb{k}$  be a commutative ring and suppose that the ribbon category  $\mathcal{C}$  is  $\mathbb{k}$ -linear. We write  $\text{Rib}_{\mathcal{C}}(M, \nu_B)$  for the  $\mathbb{k}$ -linear space spanned by ribbon graphs in  $M$  with boundary condition  $\nu_B$ .

**Definition 1.3.23. (i)** An element  $\Gamma := \sum_i \lambda_i \Gamma_i \in \text{Rib}_{\mathcal{C}}(M, \nu_B)$  is a *skein relation* if there exists an embedding  $\Phi: \mathbb{R}^2 \times [0, 1] \rightarrow M$  such that

- the image  $K$  of  $\Phi$  is a generic cube for all the  $\Gamma_i$ 's;
- all the  $\Gamma_i$ 's coincide outside  $K$  and

$$\sum_i \lambda_i \text{ev}_{\mathcal{C}}^{\text{Rib}}(\Phi^{-1}(\Gamma \cap K)) = 0.$$

We write  $\text{nRib}(M, \nu_B) \subset \text{Rib}(M, \nu_B)$  for the subspace of skein relations.

(ii) The *relative  $\mathcal{C}$ -skein module* of  $M$  with boundary condition  $\nu_B$  is the quotient

$$\text{SkMod}_{\mathcal{C}}(M, \nu_B) := \text{Rib}_{\mathcal{C}}(M, \nu_B) / \text{nRib}(M, \nu_B).$$

When  $B = \emptyset$ , we call it simply the  *$\mathcal{C}$ -skein module* and we set

$$\text{SkMod}_{\mathcal{C}}(M) := \text{SkMod}_{\mathcal{C}}(M, \emptyset).$$

◇

When the manifold  $M$  is a cylinder over an oriented surface, we have a natural composition induced by vertically stacking cylinders, so the skein module gets the structure of an algebra:

**Definition 1.3.24.** Let  $S$  be an oriented surface. The  *$\mathcal{C}$ -skein algebra*  $\text{SkAlg}_{\mathcal{C}}(S)$  has as its underlying  $\mathbb{k}$ -module  $\text{SkMod}_{\mathcal{C}}(S \times [0, 1])$ , and its multiplication is given by vertically stacking two copies of  $S \times [0, 1]$  and rescaling the second coordinate. ◇

Analogously, allowing open components in  $S \times [0, 1]$ , we can define a *composition* given again by stacking copies of the cylinder, so that ribbon graphs form the morphisms of the following category:

**Definition 1.3.25** ([Joh19; Wal06; Co023]). Let  $S$  be an oriented surface. The *skein category*  $\text{SkCat}_{\mathcal{C}}(S)$  is the category given by:

- objects: pairs  $(B, \nu_B)$  as above, consisting of finite configurations of framed directed  $\mathcal{C}$ -coloured points on  $S$ ;
- morphisms: given two objects  $(B_0, \nu_0)$  and  $(B_1, \nu_1)$ , we set  $\nu_{01}$  for the function on  $(B_0 \times \{0\}) \cup (B_1 \times \{1\})$  defined by  $\nu_{01}(-, i) = \nu_i$  and we define

$$\text{Hom}_{\text{SkCat}_{\mathcal{C}}(S)}((B_0, \nu_0), (B_1, \nu_1)) := \text{SkMod}_{\mathcal{C}}(S \times [0, 1], \nu_{01}).$$

The composition is given by vertically stacking two copies of  $S \times [0, 1]$  and rescaling the second coordinate. The identity morphism of an object is given by a collection of ribbons running vertically in the direction of the marked points.  $\diamond$

**Theorem 1.3.26** ([Wal06, Theorem 4.4.2]). *Skein theory defines an oriented  $(3, 2)$ -TQFT*

$$\text{Sk}_{\mathcal{C}}: \text{Cob}_{3,2}^{\text{or}} \rightarrow \text{BIMOD}$$

that assigns  $\text{SkCat}_{\mathcal{C}}(S)$  to an oriented surface  $S$ . To a cobordism  $S_0 \xrightarrow{M} S_1$ , it assigns a bimodule functor

$$\text{Sk}_{\mathcal{C}}(M): \text{SkCat}_{\mathcal{C}}(S_1) \boxtimes \text{SkCat}_{\mathcal{C}}(S_0)^{\text{op}} \rightarrow \text{VECT}_{\mathbb{k}}$$

whose components are

$$\text{Sk}_{\mathcal{C}}(M)((B, \nu_B), (B', \nu_{B'})) := \text{SkMod}_{\mathcal{C}}(M, \nu_{B \cup B'}),$$

where  $\nu_{B \cup B'}$  is defined in the obvious way. The bimodule is defined on morphisms by stacking cylinders on a tubular neighbourhood of  $\partial M \cong \overline{S}_0 \sqcup S_1$ .  $\blacksquare$



# Chapter 2

## Defect skein theory

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### 2.1 The algebraic setting

In this section, we describe the algebraic data needed to extend string-nets and skein theory to stratified 3-manifolds. We first recall the definition of bimodule categories over monoidal categories. Next, we review central algebras and centred bimodules, which will provide local models for surface and line defects, respectively. In the last subsection, we briefly explain how these algebraic structures are governed by the topology of disk embeddings. Most of the material in this section is taken from [BJS21, Section 3], to which we refer for additional details.

### 2.1.1 Bimodule categories

Let  $\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{1}_{\mathcal{C}}, \alpha, l, r)$  be a monoidal category. Module categories over  $\mathcal{C}$  are categories endowed with an action of  $\mathcal{C}$  that satisfy conditions that categorify the usual axioms in the definition of modules over an associative algebra:

**Definition 2.1.1. (i)** A *left module category* over  $\mathcal{C}$  is the data  $(\mathcal{M}, \triangleright, \gamma, \eta)$  of a category  $\mathcal{M}$  together with

- a functor  $- \triangleright - : \mathcal{C} \boxtimes \mathcal{M} \rightarrow \mathcal{M}$ ;
- a natural *associativity constraint*

$$\gamma_{c_1, c_2, m} : (c_1 \otimes c_2) \triangleright m \xrightarrow{\sim} c_1 \triangleright (c_2 \triangleright m), \quad c_1, c_2 \in \mathcal{C}, \quad m \in \mathcal{M};$$

- a natural isomorphism

$$\eta_m : \mathbf{1}_{\mathcal{C}} \triangleright m \xrightarrow{\sim} m, \quad m \in \mathcal{M};$$

such that the diagrams

$$\begin{array}{ccc} & ((c_1 \otimes c_2) \otimes c_3) \triangleright m & \\ \alpha_{c_1, c_2, c_3} \triangleright m \swarrow & & \searrow \gamma_{c_1 \otimes c_2, c_3, m} \\ (c_1 \otimes (c_2 \otimes c_3)) \triangleright m & & (c_1 \otimes c_2) \triangleright (c_3 \triangleright m) \\ \gamma_{c_1, c_2 \otimes c_3, m} \downarrow & & \downarrow \gamma_{c_1, c_2, c_3 \triangleright m} \\ c_1 \triangleright ((c_2 \otimes c_3) \triangleright m) & \xrightarrow{\text{id}_{c_1} \triangleright \gamma_{c_2, c_3, m}} & c_1 \triangleright (c_2 \triangleright (c_3 \triangleright m)) \end{array}$$

and

$$\begin{array}{ccc} (c \otimes \mathbf{1}_{\mathcal{C}}) \triangleright m & \xrightarrow{\gamma_{c, \mathbf{1}_{\mathcal{C}}, m}} & c \triangleright (\mathbf{1}_{\mathcal{C}} \triangleright m) \\ & \searrow r_c \triangleright m & \swarrow \text{id}_c \triangleright \eta_m \\ & c \triangleright m & \end{array}$$

commute for all  $c, c_1, c_2, c_3 \in \mathcal{C}$  and  $m \in \mathcal{M}$ .

(ii) A *right module category* over  $\mathcal{C}$  is a left module category over  $\mathcal{C}^{\text{mop}}$ . We denote the right action functor by  $- \triangleleft - : \mathcal{M} \boxtimes \mathcal{C} \rightarrow \mathcal{M}$ .  $\diamond$

Module functors and natural transformations are, roughly speaking, functors and natural transformations that are equivariant with respect to the action of  $\mathcal{C}$ . This is expressed by adding an additional constraint as made precise in the following definition:

**Definition 2.1.2. (i)** A *module functor* between two left  $\mathcal{C}$ -module categories  $\mathcal{M}$  and  $\mathcal{N}$  is a functor  $F : \mathcal{M} \rightarrow \mathcal{N}$  together with natural isomorphisms

$$s_{c, m} : F(c \triangleright m) \xrightarrow{\sim} c \triangleright F(m), \quad c \in \mathcal{C}, \quad m \in \mathcal{M},$$

such that the diagrams

$$\begin{array}{ccc}
& F((c_1 \otimes c_2) \triangleright m) & \\
F(\gamma_{c_1, c_2, m}^{\mathcal{M}}) \swarrow & & \searrow s_{c_1 \otimes c_2, m} \\
F(c_1 \triangleright (c_2 \triangleright m)) & & (c_1 \otimes c_2) \triangleright F(m) \\
s_{c_1, c_2 \triangleright m} \downarrow & & \downarrow \gamma_{c_1, c_2, F(m)}^{\mathcal{N}} \\
c_1 \triangleright F(c_2 \triangleright m) & \xrightarrow{\text{id}_{c_1} \triangleright s_{c_2, m}} & c_1 \triangleright (c_2 \triangleright F(m))
\end{array}$$

and

$$\begin{array}{ccc}
F(\mathbf{1}_{\mathcal{C}} \triangleright m) & \xrightarrow{F(\eta_m^{\mathcal{M}})} & F(m) \\
s_{\mathbf{1}_{\mathcal{C}}, m} \downarrow & \nearrow \eta_{F(m)}^{\mathcal{N}} & \\
\mathbf{1}_{\mathcal{C}} \triangleright F(m) & & 
\end{array}$$

are commutative for all  $c_1, c_2 \in \mathcal{C}$  and  $m \in \mathcal{M}$ .

(ii) Let  $(F, s), (G, t): \mathcal{M} \rightarrow \mathcal{N}$  be two module functors. A *natural transformation of module functors* is a natural transformation  $\delta: F \Rightarrow G$  such that

$$\begin{array}{ccc}
F(c \triangleright m) & \xrightarrow{\delta_{c \triangleright m}} & G(c \triangleright m) \\
s_{c, m} \downarrow & & \downarrow t_{c, m} \\
c \triangleright F(m) & \xrightarrow{\text{id}_c \triangleright \delta_m} & c \triangleright G(m)
\end{array}$$

is a commutative diagram for all  $c \in \mathcal{C}$  and  $m \in \mathcal{M}$ .  $\diamond$

We write  $\text{Mod}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$  for the category whose objects are  $\mathcal{C}$ -module functors from  $\mathcal{M}$  to  $\mathcal{N}$  and whose morphisms are natural transformations between them. These functors and natural transformations form the 1- and 2-morphisms of the following 2-category:

**Definition 2.1.3.** The *2-category of left  $\mathcal{C}$ -modules* is the 2-category  $\mathcal{C}\text{-mod}$  whose objects are left  $\mathcal{C}$ -module categories, whose 1-morphisms are  $\mathcal{C}$ -module functors and whose 2-morphisms are natural transformations of module functors. The 2-category *mod- $\mathcal{C}$*  of right  $\mathcal{C}$ -modules is defined in the same way.  $\diamond$

A bimodule category is a category equipped with both a left and a right module structure, together with an isomorphism identifying the two possible ways of iterating both actions. More precisely:

**Definition 2.1.4. (i)** Let  $\mathcal{C}$  and  $\mathcal{D}$  be monoidal categories. A  *$(\mathcal{C}, \mathcal{D})$ -bimodule category* is a category  $\mathcal{M}$  equipped with both a left module structure over  $\mathcal{C}$  and a right module structure over  $\mathcal{D}$ , together with a natural isomorphism

$$\phi_{c, m, d}: (c \triangleright m) \triangleleft d \xrightarrow{\sim} c \triangleright (m \triangleleft d), \quad c \in \mathcal{C}, \quad m \in \mathcal{M}, \quad d \in \mathcal{D},$$

making the diagrams

$$\begin{array}{ccc}
& ((c_1 \otimes c_2) \triangleright m) \triangleleft d & \\
\gamma_{c_1, c_2, m}^l \triangleleft \text{id}_d \swarrow & & \searrow \phi_{c_1 \otimes c_2, m, d} \\
(c_1 \triangleright (c_2 \triangleright m)) \triangleleft d & & (c_1 \otimes c_2) \triangleright (m \triangleleft d) \\
\downarrow \phi_{c_1, c_2 \triangleright m, d} & & \downarrow \gamma_{c_1, c_2, m \triangleleft d}^l \\
c_1 \triangleright ((c_2 \triangleright m) \triangleleft d) & \xrightarrow{\text{id}_{c_1} \triangleright \phi_{c_2, m, d}} & c_1 \triangleright (c_2 \triangleright (m \triangleleft d))
\end{array}$$

and

$$\begin{array}{ccc}
& c \triangleright (m \triangleleft (d_1 \otimes d_2)) & \\
\text{id}_c \triangleright \gamma_{c, d_1, d_2}^r \swarrow & & \searrow \phi_{c, m, d_1 \otimes d_2}^{-1} \\
c \triangleright ((m \triangleleft d_1) \triangleleft d_2) & & (c \triangleright m) \triangleleft (d_1 \otimes d_2) \\
\downarrow \phi_{c, m \triangleleft d_1, d_2}^{-1} & & \downarrow \gamma_{c \triangleright m, d_1, d_2}^r \\
(c \triangleright (m \triangleleft d_1)) \triangleleft d_2 & \xrightarrow{\phi_{c, m, d_1}^{-1} \triangleleft \text{id}_{d_2}} & ((c \triangleright m) \triangleleft d_1) \triangleleft d_2
\end{array}$$

commute for all  $c, c_1, c_2 \in \mathcal{C}$ ,  $d, d_1, d_2 \in \mathcal{D}$  and  $m \in \mathcal{M}$ .

(ii) A *pointed bimodule category* is a  $(\mathcal{C}, \mathcal{D})$ -bimodule category  $\mathcal{M}$  endowed with a distinguished object  $\text{Dist}_{\mathcal{M}}$ .  $\diamond$

**Remark 2.1.5.** Given a pointed  $(\mathcal{C}, \mathcal{D})$ -bimodule category  $\mathcal{M}$ , the pointing determines functors

$$\begin{array}{ccc}
\mathcal{C} & \rightarrow & \mathcal{M} \\
c & \mapsto & c \triangleright \text{Dist}_{\mathcal{M}}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathcal{D} & \rightarrow & \mathcal{M} \\
d & \mapsto & \text{Dist}_{\mathcal{M}} \triangleleft d
\end{array}$$

which will feature crucially in our definition of skein relations near a line defect.  $\diamond$

Modules over a monoidal category categorify the notion of modules over an associative algebra. In the non-categorical setting, left actions are also right actions whenever the underlying algebra is commutative. This has the following categorical analogue:

**Remark 2.1.6** (Modules over a braided category). Let  $\mathcal{A}$  be a braided monoidal category with braiding  $\beta$  and let  $\mathcal{M}$  be a left  $\mathcal{A}$ -module category. Then,  $\mathcal{M}$  is canonically a right  $\mathcal{A}$ -module with right action given by

$$m \triangleleft a := a \triangleright m.$$

The right associativity constraint and the unitor are, respectively,

$$m \triangleleft (a_1 \otimes a_2) = (a_1 \otimes a_2) \triangleright m \xrightarrow{\beta_{a_1, a_2} \triangleright \text{id}_m} (a_2 \otimes a_1) \triangleright m \xrightarrow{\gamma_{a_2, a_1, m}} a_2 \triangleright (a_1 \triangleright m) = (m \triangleleft a_1) \triangleleft a_2$$

and

$$m \triangleleft \mathbf{1}_{\mathcal{A}} = \mathbf{1}_{\mathcal{A}} \triangleright m \xrightarrow{\eta_m} m.$$

The compatibility between the left and right actions is given by the isomorphism:

$$\begin{aligned} (a_1 \triangleright m) \triangleleft a_2 &= a_2 \triangleright (a_1 \triangleright m) \xrightarrow{\gamma_{a_2, a_1, m}^{-1}} (a_2 \otimes a_1) \triangleright m \\ &\xrightarrow{\beta_{a_1, a_2}^{-1} \triangleright \text{id}_m} (a_1 \otimes a_2) \triangleright m \xrightarrow{\gamma_{a_1, a_2, m}} a_1 \triangleright (a_2 \triangleright m) = a_1 \triangleright (m \triangleleft a_2). \end{aligned}$$

The coherence axioms follow from naturality and the coherences of  $\beta$ ,  $\gamma$  and  $\eta$ .  $\diamond$

**Definition 2.1.7. (i)** Let  $\mathcal{M}$  be a  $(\mathcal{C}, \mathcal{C})$ -bimodule category and let  $\mathcal{E}$  be a linear category. A  $\mathcal{C}$ -balanced functor is a functor  $F: \mathcal{M} \rightarrow \mathcal{E}$  equipped with a natural isomorphism

$$b_{c,m}: F(c \triangleright m) \xrightarrow{\sim} F(m \triangleleft c), \quad c \in \mathcal{C}, \quad m \in \mathcal{M},$$

such that

$$\begin{array}{ccc} & F((c_1 \otimes c_2) \triangleright m) & \\ F(\gamma_{c_1, c_2, m}^l) \swarrow & & \searrow b_{c_1 \otimes c_2, m} \\ F(c_1 \triangleright (c_2 \triangleright m)) & & F(m \triangleleft (c_1 \otimes c_2)) \\ b_{c_1, c_2 \triangleright m} \downarrow & & \downarrow F(\gamma_{m, c_1, c_2}^r) \\ F((c_2 \triangleright m) \triangleleft c_1) & & F((m \triangleleft c_1) \triangleleft c_2) \\ F(\phi_{c_2, m, c_1}) \searrow & & \swarrow b_{c_2, m \triangleleft c_1} \\ & F(c_2 \triangleright (m \triangleleft c_1)) & \end{array}$$

and

$$\begin{array}{ccc} F(\mathbf{1}_{\mathcal{C}} \triangleright m) & \xrightarrow{F(\eta_m^l)} & F(m) \\ b_{\mathbf{1}_{\mathcal{C}}, m} \downarrow & \nearrow F(\eta_m^r) & \\ F(m \triangleleft \mathbf{1}_{\mathcal{C}}) & & \end{array}$$

are commutative diagrams for all  $c_1, c_2 \in \mathcal{C}$  and  $m \in \mathcal{M}$ .

(ii) Given two  $\mathcal{C}$ -balanced functors  $(F, b), (G, b'): \mathcal{M} \rightarrow \mathcal{E}$ , a natural transformation of  $\mathcal{C}$ -balanced functors between them is a natural transformation  $\delta: F \Rightarrow G$  such that the diagrams

$$\begin{array}{ccc} F(c \triangleright m) & \xrightarrow{\delta_{c \triangleright m}} & G(c \triangleright m) \\ b_{c,m} \downarrow & & \downarrow b'_{c,m} \\ F(m \triangleleft c) & \xrightarrow{\delta_{m \triangleleft c}} & G(m \triangleleft c) \end{array}$$

are commutative for all  $c \in \mathcal{C}$  and  $m \in \mathcal{M}$ .  $\diamond$

Let  $\text{Bal}_{\mathcal{C}}(\mathcal{M}, \mathcal{E})$  be the category of balanced functors between  $\mathcal{M}$  and  $\mathcal{E}$  and their natural transformations. The trace of a  $(\mathcal{C}, \mathcal{C})$ -bimodule category is defined by the following universal property:

**Definition 2.1.8.** Let  $\mathcal{M}$  be a  $(\mathcal{C}, \mathcal{C})$ -bimodule category. The *trace* of  $\mathcal{M}$  is a  $\mathbb{k}$ -linear category  $\mathrm{tr}_{\mathcal{C}}(\mathcal{M})$  together with a  $\mathcal{C}$ -balanced functor  $\mathcal{M} \rightarrow \mathrm{tr}_{\mathcal{C}}(\mathcal{M})$  inducing an equivalence of categories

$$\mathrm{Bal}_{\mathcal{C}}(\mathcal{M}, \mathcal{E}) \simeq \mathrm{Lin}(\mathrm{tr}_{\mathcal{C}}(\mathcal{M}), \mathcal{E}),$$

for every linear category  $\mathcal{E}$ , where  $\mathrm{Lin}(\mathcal{M}, \mathcal{E})$  is the category of linear functors from  $\mathcal{M}$  to  $\mathcal{E}$  and their natural transformations.  $\diamond$

The trace of a  $(\mathcal{C}, \mathcal{C})$ -bimodule category  $\mathcal{M}$  exists in  $\mathrm{Pr}$  and can be computed as the 2-colimit of the diagram

$$\mathcal{C} \boxtimes \mathcal{M} \begin{array}{c} \xrightarrow{\triangleleft} \\ \xrightarrow{\triangleright} \end{array} \mathcal{M}.$$

If  $\mathcal{M}$  is a right  $\mathcal{D}$ -module category and  $\mathcal{N}$  is a left  $\mathcal{C}$ -module category, then the Deligne–Kelly tensor product  $\mathcal{M} \boxtimes \mathcal{N}$  is canonically a  $(\mathcal{C}, \mathcal{D})$ -bimodule category with trivial compatibility constraint

$$(c \triangleright (m \boxtimes n)) \triangleleft d = (m \triangleleft d) \boxtimes (c \triangleright n) = c \triangleright ((m \boxtimes n) \triangleleft d),$$

for all  $c \in \mathcal{C}$ ,  $m \in \mathcal{M}$ ,  $n \in \mathcal{N}$  and  $d \in \mathcal{D}$ . In particular, when  $\mathcal{D} = \mathcal{C}$  we obtain a  $(\mathcal{C}, \mathcal{C})$ -bimodule category and we define:

**Definition 2.1.9.** The *relative Deligne–Kelly tensor product* of a right  $\mathcal{C}$ -module category  $\mathcal{M}$  and a left  $\mathcal{C}$ -module category  $\mathcal{N}$  is defined as the category

$$\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} := \mathrm{tr}_{\mathcal{C}}(\mathcal{M} \boxtimes \mathcal{N}).$$

$\diamond$

**Remark 2.1.10** (Modules over a braided category 2.0). As we have seen in Remark 2.1.6, every left module over a braided monoidal category  $\mathcal{A}$  is also a right module. Hence, if  $\mathcal{M}$  and  $\mathcal{N}$  are two objects in  $\mathcal{A}\text{-mod}$ , we can define the relative tensor product  $\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}$  using the right action of  $\mathcal{A}$  on  $\mathcal{M}$  induced by the left one. We obtain a functor

$$- \boxtimes_{\mathcal{A}} -: \mathcal{A}\text{-mod} \boxtimes \mathcal{A}\text{-mod} \rightarrow \mathcal{A}\text{-mod}$$

that turns  $\mathcal{A}\text{-mod}$  into a monoidal 2-category.  $\diamond$

### 2.1.2 Central algebras

The algebraic data needed to produce local relations around surface defects is provided by central algebras.

**Definition 2.1.11.** Let  $\mathcal{A}$  be a braided monoidal category. An  $\mathcal{A}$ -central algebra is an algebra object in the monoidal 2-category  $\mathcal{A}\text{-mod}$ .  $\diamond$

The algebraic data of a central algebra can be equivalently described in terms of central functors. Recall first the definition of the Drinfeld center of a monoidal category:

**Definition 2.1.12.** Let  $\mathcal{C}$  be a monoidal category. The *Drinfeld center* of  $\mathcal{C}$  is the category  $Z_{\text{Dr}}(\mathcal{C})$  given by:

- objects: pairs  $(c, \gamma^c)$ , where  $c \in \mathcal{C}$  and  $\gamma^c: c \otimes - \xrightarrow{\sim} - \otimes c$  is a natural isomorphism such that the diagram

$$\begin{array}{ccc} c \otimes x \otimes y & \xrightarrow{\gamma_x^c \otimes \text{id}_y} & x \otimes c \otimes y \\ & \searrow \gamma_{x \otimes y}^c & \downarrow \text{id}_x \otimes \gamma_y^c \\ & & x \otimes y \otimes c \end{array}$$

is commutative for all  $x, y \in \mathcal{C}$ ;

- morphisms: a morphism  $(c_1, \gamma^{c_1}) \rightarrow (c_2, \gamma^{c_2})$  is a morphism  $f: c_1 \rightarrow c_2$  of  $\mathcal{C}$  such that, for all  $x \in \mathcal{C}$ , the diagram

$$\begin{array}{ccc} c_1 \otimes x & \xrightarrow{f \otimes \text{id}_x} & c_2 \otimes x \\ \gamma_x^{c_1} \downarrow & & \downarrow \gamma_x^{c_2} \\ x \otimes c_1 & \xrightarrow{\text{id}_x \otimes f} & x \otimes c_2 \end{array}$$

is commutative.

The Drinfeld center  $Z_{\text{Dr}}(\mathcal{C})$  is a monoidal category with the tensor product defined by

$$(c_1, \gamma^{c_1}) \otimes (c_2, \gamma^{c_2}) := (c_1 \otimes c_2, \tilde{\gamma}^{c_1 \otimes c_2}),$$

where

$$\tilde{\gamma}_x^{c_1 \otimes c_2} := \left( c_1 \otimes c_2 \otimes x \xrightarrow{\text{id}_{c_1} \otimes \gamma_x^{c_2}} c_1 \otimes x \otimes c_2 \xrightarrow{\gamma_x^{c_1} \otimes \text{id}_{c_2}} x \otimes c_1 \otimes c_2 \right).$$

Moreover, it is braided, with braiding

$$\beta_{(c_1, \gamma^{c_1}), (c_2, \gamma^{c_2})}^{Z_{\text{Dr}}(\mathcal{C})} := \gamma_{c_2}^{c_1}.$$

◇

**Definition 2.1.13.** Let  $\mathcal{A}$  be a braided monoidal category and  $\mathcal{C}$  be a monoidal category. A  $\mathcal{A}$ -central functor on  $\mathcal{C}$  is a braided monoidal functor

$$(F, \gamma^F): \mathcal{A} \rightarrow Z_{\text{Dr}}(\mathcal{C}).$$

The notation means that we have an underlying monoidal functor  $F: \mathcal{A} \rightarrow \mathcal{C}$  and a half-braiding  $\gamma^{F(a)}: F(a) \otimes - \xrightarrow{\sim} - \otimes F(a)$  for every  $a \in \mathcal{A}$ . ◇

**Remark 2.1.14.** The functoriality forces the naturality of  $\gamma^F$  with respect to morphisms in the image of  $F$ . Indeed, if  $f: a \rightarrow b$  is a morphism of  $\mathcal{A}$ , its image by  $(F, \gamma^F)$  is a morphism in the Drinfeld center which means that

$$\begin{array}{ccc} F(a) \otimes x & \xrightarrow{\gamma_x^{F(a)}} & x \otimes F(a) \\ F(f) \otimes \text{id}_x \downarrow & & \downarrow \text{id}_x \otimes F(f) \\ F(b) \otimes x & \xrightarrow{\gamma_x^{F(b)}} & x \otimes F(b) \end{array}$$

for all  $x \in \mathcal{C}$ . ◇

It turns out that an  $\mathcal{A}$ -central algebra structure is equivalent to the data of an  $\mathcal{A}$ -central functor:

**Proposition 2.1.15** ([BJS21, Definition-Proposition 3.2]). *An  $\mathcal{A}$ -central algebra structure on a category  $\mathcal{C}$  is equivalent to the data of a monoidal structure on  $\mathcal{C}$  and a braided monoidal functor*

$$(F, \gamma^F) : \mathcal{A} \rightarrow Z_{\text{Dr}}(\mathcal{C}).$$
■

We introduce the notion of *dual central algebra*. Given a monoidal category  $\mathcal{C}$ , we have a monoidal equivalence

$$Z_{\text{Dr}}(\mathcal{C}) \rightarrow Z_{\text{Dr}}(\mathcal{C}^{\text{mop}}), (c, \gamma^c) \mapsto (c, (\gamma^c)^{-1})$$

that induces a braided equivalence  $Z_{\text{Dr}}(\mathcal{C})^{\text{bop}} \simeq Z_{\text{Dr}}(\mathcal{C}^{\text{mop}})$ .

**Definition 2.1.16.** Let  $\mathcal{C}$  be a central algebra with central functor  $(F, \gamma) : \mathcal{A} \rightarrow Z_{\text{Dr}}(\mathcal{C})$ . We define the *dual central algebra*  $\mathcal{C}^\vee$  as the  $\mathcal{A}^{\text{bop}}$ -central algebra with underlying category  $\mathcal{C}^{\text{mop}}$  and whose central functor  $(F^\vee, \gamma^\vee)$  is given by the composition

$$\mathcal{A}^{\text{bop}} \xrightarrow{(F, \gamma)} Z_{\text{Dr}}(\mathcal{C})^{\text{bop}} \simeq Z_{\text{Dr}}(\mathcal{C}^{\text{mop}}). \quad \diamond$$

**Lemma 2.1.17.** *Every  $\mathcal{A}$ -central algebra  $\mathcal{C}$  is canonically isomorphic to its double dual  $\mathcal{C}^{\vee\vee}$ .*

*Proof.* Let  $(F, \gamma^F) : \mathcal{A} \rightarrow Z_{\text{Dr}}(\mathcal{C})$  be the central functor on  $\mathcal{C}$ . Then,  $\mathcal{C}^{\vee\vee}$  is also an  $\mathcal{A}$ -central algebra with the same underlying functor  $F$  and half-braiding  $((\gamma^F)^{-1})^{-1} = \gamma^F$ . It remains to show that its monoidal structure has the same coherence isomorphism. First note that the monoidal structure on the central functor

$$(F, (\gamma^F)^{-1}) : \mathcal{A}^{\text{bop}} \rightarrow Z_{\text{Dr}}(\mathcal{C}^{\text{mop}})$$

is given by

$$F(x) \otimes^{\text{op}} F(y) = F(y) \otimes F(x) \xrightarrow{J_{y,x}^F} F(y \otimes x) \xrightarrow{F(\beta_{x,y}^{\mathcal{A}})^{-1}} F(x \otimes y), \quad (2.1.11)$$

where  $\beta^{\mathcal{A}}$  is the braiding of  $\mathcal{A}$  and  $J^F$  is the coherence isomorphism for the monoidal structure of  $F$ . Applying this formulation twice, we obtain  $(J^F)^{\vee\vee}$  as

$$F(x) \otimes F(y) = F(y) \otimes^{\text{op}} F(x) \xrightarrow{(J^F)_{y,x}^{\vee}} F(y \otimes x) \xrightarrow{F(\beta_{x,y}^{\mathcal{A}^{\text{bop}}})^{-1} = F(\beta_{y,x}^{\mathcal{A}})} F(x \otimes y).$$

Using the expression for  $(J^F)^{\vee}$  in (2.1.11), we obtain

$$(J^F)_{x,y}^{\vee\vee} = F(\beta_{y,x}^{\mathcal{A}}) \circ (J^F)_{y,x}^{\vee} = F(\beta_{y,x}^{\mathcal{A}}) \circ F(\beta_{y,x}^{\mathcal{A}})^{-1} \circ J_{x,y}^F = J_{x,y}^F,$$

so we get the initial central structure and we have an identity  $\mathcal{C} = \mathcal{C}^{\vee\vee}$ .  $\square$

In the skein theory with defects that we develop in the following sections, we will use central algebra to produce interfaces between the theories associated to two ribbon categories  $\mathcal{A}$  and  $\mathcal{B}$ . Concretely, we will use the following notion:

**Definition 2.1.18.** (i) Given two braided categories  $\mathcal{A}$  and  $\mathcal{B}$ , an  $(\mathcal{A}, \mathcal{B})$ -central algebra is a central algebra  $\mathcal{C}$  over the braided category  $\mathcal{A} \boxtimes \mathcal{B}^{\text{bop}}$ . Under the characterisation above, this corresponds to a monoidal category  $\mathcal{C}$  together with a braided monoidal functor

$$\mathcal{A} \boxtimes \mathcal{B}^{\text{bop}} \rightarrow Z_{\text{Dr}}(\mathcal{C}). \quad \diamond$$

(ii) A *pivotal*  $(\mathcal{A}, \mathcal{B})$ -central algebra is an  $(\mathcal{A}, \mathcal{B})$ -central algebra  $\mathcal{C}$  such that the underlying categories  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$ , and the underlying monoidal functor

$$F: \mathcal{A} \boxtimes \mathcal{B}^{\text{bop}} \rightarrow \mathcal{C}$$

are all pivotal.

### 2.1.3 Centred bimodules

Given two  $\mathcal{A}$ -central algebras  $\mathcal{C}$  and  $\mathcal{D}$ , a centred bimodule over them is a  $(\mathcal{C}, \mathcal{D})$ -bimodule category  $\mathcal{M}$  such that the actions of  $\mathcal{A}$  through  $\mathcal{C}$  and  $\mathcal{D}$  are compatible. More precisely:

**Definition 2.1.19.** (i) Let  $\mathcal{A}$  be a braided tensor category and let  $\mathcal{C}$  and  $\mathcal{D}$  be algebras in  $\mathcal{A}\text{-mod}$ . An  $\mathcal{A}$ -centred  $(\mathcal{C}, \mathcal{D})$ -bimodule is a  $(\mathcal{C}, \mathcal{D})$ -bimodule object  $\mathcal{L}$  in  $\mathcal{A}\text{-mod}$ .

(ii) A *pointed*  $\mathcal{A}$ -centred  $(\mathcal{C}, \mathcal{D})$ -bimodule is an  $\mathcal{A}$ -centred  $(\mathcal{C}, \mathcal{D})$ -bimodule  $\mathcal{L}$  with a distinguished object  $\text{Dist}_{\mathcal{L}}$ .  $\diamond$

As with central algebras, we can unpack the definition to obtain a more operational characterisation of centred bimodules:

**Proposition 2.1.20** ([BJS21, Proposition 3.10]). *Let  $\mathcal{A}$  be a braided tensor category and let  $\mathcal{C}$  and  $\mathcal{D}$  be  $\mathcal{A}$ -central algebras with central functors*

$$(F_1, \gamma^{F_1}) : \mathcal{A} \rightarrow Z_{\text{Dr}}(\mathcal{C}) \quad \text{and} \quad (F_2, \gamma^{F_2}) : \mathcal{A} \rightarrow Z_{\text{Dr}}(\mathcal{D}).$$

An  $\mathcal{A}$ -centred  $(\mathcal{C}, \mathcal{D})$ -bimodule is equivalent to the data of a  $(\mathcal{C}, \mathcal{D})$ -bimodule category  $\mathcal{L}$ , with associativity constraints  $m^L$  and  $m^R$ , equipped with natural isomorphisms

$$\eta_{a,\ell}: F_1(a) \triangleright \ell \xrightarrow{\sim} \ell \triangleleft F_2(a)$$

satisfying the coherence conditions expressed by the commutativity of the following diagrams:

(i) compatibility with the left half-braidings:

$$\begin{array}{ccc} (F_1(a) \otimes c) \triangleright \ell & \xrightarrow{m_{F_1(a),c,\ell}^L} & F_1(a) \triangleright c \triangleright \ell \xrightarrow{\eta_{a,c \triangleright \ell}} c \triangleright \ell \triangleleft F_2(a) \\ \gamma_c^{F_1(a)} \triangleright \text{id}_\ell \downarrow & & \nearrow \text{id}_c \triangleright \eta_{a,\ell} \\ (c \otimes F_1(a)) \triangleright \ell & \xrightarrow{m_{c,F_1(a),\ell}^L} & c \triangleright F_1(a) \triangleright \ell \end{array} \quad (2.1.12)$$

(ii) compatibility with the right half-braidings:

$$\begin{array}{ccc} \ell \triangleleft (d \otimes F_2(a)) & \xrightarrow{m_{\ell,d,F_2(a)}^R} & \ell \triangleleft d \triangleleft F_2(a) \xleftarrow{\eta_{a,\ell \triangleleft d}} F_1(a) \triangleright \ell \triangleleft d \\ \text{id}_\ell \triangleleft \gamma_d^{F_2(a)} \uparrow & & \nwarrow \eta_{a,\ell} \triangleleft \text{id}_d \\ \ell \triangleleft (F_2(a) \otimes d) & \xrightarrow{m_{\ell,F_2(a),d}^R} & \ell \triangleleft F_2(a) \triangleleft d \end{array} \quad (2.1.13)$$

(iii) compatibility with the monoidal structure:

$$\begin{array}{ccc} F_1(a \otimes b) \triangleright \ell & \xrightarrow{\eta_{a \otimes b, \ell}} & \ell \triangleleft F_2(a \otimes b) \xrightarrow{m^R \circ (\text{id}_\ell \triangleleft J_{a,b}^{F_2})} \ell \triangleleft F_2(a) \triangleleft F_2(b) \\ m^L \circ (J_{a,b}^{F_1} \triangleright \text{id}_\ell) \downarrow & & \nearrow \eta_{b,\ell \triangleleft F_2(a)} \\ F_1(a) \triangleright F_1(b) \triangleright \ell & \xrightarrow{\eta_{a,F_1(b) \triangleright \ell}} & F_1(b) \triangleright \ell \triangleleft F_2(a) \end{array} \quad (2.1.14)$$

■

As for central algebras, we have a notion of dual of a centred bimodule. First note that:

**Proposition 2.1.21.** *Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be two  $\mathcal{A}$ -central algebras and let  $\mathcal{L}$  be an  $\mathcal{A}$ -centred  $(\mathcal{C}_1, \mathcal{C}_2)$ -bimodule. Then  $\mathcal{L}$  is canonically an  $\mathcal{A}^{\text{bop}}$ -centred  $(\mathcal{C}_2^\vee, \mathcal{C}_1^\vee)$ -bimodule.*

*Proof.* The categories underlying  $\mathcal{C}_1^\vee$  and  $\mathcal{C}_2^\vee$  are  $\mathcal{C}_1^{\text{mop}}$  and  $\mathcal{C}_2^{\text{mop}}$ , respectively, so  $\mathcal{L}$  is a  $(\mathcal{C}_2^\vee, \mathcal{C}_1^\vee)$ -bimodule. We write

$$c_2 \blacktriangleright \ell \blacktriangleleft c_1 := c_1 \triangleright \ell \triangleleft c_2$$

for the corresponding actions. Let us check that

$$\eta_{a,\ell}^{-1}: F_2(a) \blacktriangleright \ell = \ell \triangleleft F_2(a) \rightarrow F_1(a) \triangleright \ell = \ell \blacktriangleleft F_1(a)$$

defines an  $\mathcal{A}^{\text{bop}}$ -centred structure. From the definition of a dual defect, the compatibility with left half-braiding amounts to showing that

$$\begin{array}{ccc} (F_2(a) \otimes^{op} c_2) \blacktriangleright \ell & \xrightarrow{m_{c_2, F_2(a), \ell}^R} & F_2(a) \blacktriangleright c_2 \blacktriangleright \ell \xrightarrow{\eta_{a, c_2}^{-1} \blacktriangleright \ell} c_2 \blacktriangleright \ell \blacktriangleleft F_1(a) \\ \downarrow (\gamma^{F_2(a)})_{c_2}^{-1} \blacktriangleright \text{id}_\ell & & \nearrow \text{id}_{c_2} \blacktriangleright \eta_{a, \ell} \\ (c_2 \otimes^{op} F_2(a)) \blacktriangleright \ell & \xrightarrow{m_{F_2(a), c_2, \ell}^R} & c_2 \blacktriangleright F_2(a) \blacktriangleright \ell \end{array}$$

commutes. But this diagram expresses exactly the compatibility of the  $\mathcal{A}$ -centred structure of  $\mathcal{L}$  with the right half-braiding, so the claim holds. The same is true for compatibility with the right half-braiding. For the monoidal structure, recall that the monoidal structure on  $(F_i, (\gamma^{F_i})^{-1})$  is given by (2.1.11). Therefore, we need to check the commutativity of outer square in the diagram

$$\begin{array}{ccccc} F_2(a) \blacktriangleright F_2(b) \blacktriangleright \ell & \xrightarrow{m_{F_2(b), F_2(a), \ell}^R} & F_2(b \otimes a) \blacktriangleright \ell & \xrightarrow{F_2(\beta_{b,a}^A)^{-1} \blacktriangleright \text{id}_\ell} & F_2(a \otimes b) \blacktriangleright \ell \\ \downarrow \eta_{a, F_2(b)}^{-1} \blacktriangleright \ell & & \downarrow \eta_{b \otimes a, \ell}^{-1} & & \downarrow \eta_{a \otimes b, \ell}^{-1} \\ F_2(b) \blacktriangleright \ell \blacktriangleleft F_1(a) & & & & \\ \downarrow \eta_{b, \ell \blacktriangleleft F_2(a)}^{-1} & & \downarrow \text{id}_\ell \blacktriangleleft F_1(\beta_{b,a}^A)^{-1} & & \downarrow \\ \ell \blacktriangleleft F_1(a) \blacktriangleleft F_1(b) & \xrightarrow{m_{F_1(a), F_1(b), \ell}^L} & \ell \blacktriangleleft F_1(b \otimes a) & \xrightarrow{\text{id}_\ell \blacktriangleleft F_1(\beta_{b,a}^A)^{-1}} & \ell \blacktriangleleft F_1(a \otimes b). \end{array}$$

The left interior rectangle is commutative by the compatibility between the centred and the monoidal structure, while the right one is commutative by the naturality of  $\eta$ , so the assertion follows.  $\square$

**Definition 2.1.22.** The *dual*  $\mathcal{L}^\vee$  of an  $\mathcal{A}$ -centred  $(\mathcal{C}_1, \mathcal{C}_2)$ -bimodule is the  $\mathcal{A}^{\text{bop}}$ -centred  $(\mathcal{C}_2^\vee, \mathcal{C}_1^\vee)$ -bimodule of the previous proposition.  $\diamond$

**Proposition 2.1.23.** The double dual  $\mathcal{L}^{\vee\vee}$  is canonically isomorphic to  $\mathcal{L}$ .

*Proof.* From the identifications  $(\mathcal{A}^{\text{bop}})^{\text{bop}} \simeq \mathcal{A}$  and  $\mathcal{C}_i^{\vee\vee} \simeq \mathcal{C}_i$ , we have that  $\mathcal{L}^{\vee\vee}$  is an  $\mathcal{A}$ -centred  $(\mathcal{C}_1, \mathcal{C}_2)$ -bimodule, with underlying category  $\mathcal{L}$ . It follows straightforwardly from the definition that this centred bimodule structure coincides with the original one.  $\square$

**Remark 2.1.24.** In particular, since  $(\mathcal{A} \boxtimes \mathcal{B}^{\text{bop}})^{\text{bop}} \simeq \mathcal{B} \boxtimes \mathcal{A}^{\text{bop}}$ , this implies that an  $(\mathcal{A}, \mathcal{B})$ -centred  $(\mathcal{C}_1, \mathcal{C}_2)$ -bimodule  $\mathcal{L}$  is canonically a  $(\mathcal{B}, \mathcal{A})$ -centred  $(\mathcal{C}_2^\vee, \mathcal{C}_1^\vee)$ -bimodule.  $\diamond$

A  $\mathcal{B} \boxtimes \mathcal{A}^{\text{bop}}$ -centred  $(\mathcal{C}, \mathcal{D})$ -bimodule is called a  $(\mathcal{B}, \mathcal{A})$ -centred  $(\mathcal{C}, \mathcal{D})$ -bimodule. These will provide interfaces between the theories associated to the central algebras  $\mathcal{C}$  and  $\mathcal{D}$ . Their structure can be characterised more compactly as follows:

**Proposition 2.1.25.** *A  $(\mathcal{B}, \mathcal{A})$ -centred  $(\mathcal{C}, \mathcal{D})$ -bimodule  $\mathcal{L}$  is equivalent to the data of a  $\text{tr}_{\mathcal{A}}(\mathcal{C} \boxtimes_{\mathcal{B}} \mathcal{D})$ -module.*

*Proof.* Note that the structure of a  $(\mathcal{B}, \mathcal{A})$ -centred  $(\mathcal{C}, \mathcal{D})$ -bimodule consists of a functor

$$\mathcal{C} \boxtimes \mathcal{D} \rightarrow \text{End}(\mathcal{L}), \quad c \boxtimes d \mapsto c \triangleright - \triangleleft d,$$

which is balanced with respect to the actions of  $\mathcal{A}$  and  $\mathcal{B}$ . By the universal property of the trace and the relative Deligne-Kelly tensor product, we have equivalences

$$\text{Bal}_{\mathcal{A}, \mathcal{B}}(\mathcal{C} \boxtimes \mathcal{D}, \text{End}(\mathcal{L})) \simeq \text{Bal}_{\mathcal{A}}(\mathcal{C} \boxtimes_{\mathcal{B}} \mathcal{D}, \text{End}(\mathcal{L})) \simeq \text{Lin}(\text{tr}_{\mathcal{A}}(\mathcal{C} \boxtimes_{\mathcal{B}} \mathcal{D}), \text{End}(\mathcal{L})).$$

Under these equivalences, the above functor corresponds to a  $\text{tr}_{\mathcal{A}}(\mathcal{C} \boxtimes_{\mathcal{B}} \mathcal{D})$ -module structure on  $\mathcal{L}$ .  $\square$

**Corollary 2.1.26.** *Let  $\mathcal{A}_1, \dots, \mathcal{A}_n$  be a family of braided categories. For each  $i = 1, \dots, n$  taken modulo  $n$ , let  $\mathcal{C}_i$  be an algebra in  $(\mathcal{A}_{i+1}, \mathcal{A}_i)$ -bimod. Then, a  $\text{tr}_{\mathcal{A}_1}(\mathcal{C}_1 \boxtimes_{\mathcal{A}_2} \dots \boxtimes_{\mathcal{A}_n} \mathcal{C}_n)$ -module is equivalent to a left  $\mathcal{C}_1 \boxtimes \dots \boxtimes \mathcal{C}_n$ -module with an  $\mathcal{A}_{i+1}$ -centred structure on the  $(\mathcal{C}_{i+1}, \mathcal{C}_i^\vee)$ -bimodule obtained by restriction and dualising  $\mathcal{C}_i$  (cf. Definition 2.1.16), for each  $i = 1, \dots, n$  taken modulo  $n$ .*

*Proof.* Let  $\mathcal{L}$  be a left  $\mathcal{C}_1 \boxtimes \dots \boxtimes \mathcal{C}_n$ -module category. By restriction of the actions, it is also a left  $\mathcal{C}_i \boxtimes \mathcal{C}_{i+1}$ -module or, equivalently, a  $(\mathcal{C}_{i+1}, \mathcal{C}_i^{\text{mop}})$ -bimodule. Since  $\mathcal{C}_i$  is an  $(\mathcal{A}_{i+1}, \mathcal{A}_i)$ -central algebra,  $\mathcal{C}_i^{\text{mop}}$  carries the dual  $(\mathcal{A}_i, \mathcal{A}_{i+1})$ -central algebra structure. As in Definition 2.1.16, we denote this central algebra by  $\mathcal{C}_i^\vee$ . If we further suppose that  $\mathcal{L}$  is  $\mathcal{A}_{i+1}$ -centred, we can iteratively apply the equivalences as in the previous proposition to obtain the equivalent  $\text{tr}_{\mathcal{A}_1}(\mathcal{C}_1 \boxtimes_{\mathcal{A}_2} \dots \boxtimes_{\mathcal{A}_n} \mathcal{C}_n)$ -module structure on  $\mathcal{L}$ .  $\square$

## 2.2 Local defects

In this section we introduce the local models for theories with defects. The algebraic structures described above admit a graphical calculus based on diagrams on stratified spaces, that is, manifolds equipped with a collection of distinguished embedded submanifolds called strata. We begin with the case of bimodule categories, whose graphical calculus is formulated in terms of diagrams on a stratified rectangle with a vertical line stratum. The line represents the linear category underlying the bimodule, while the two-dimensional regions encode the monoidal categories acting on it. This yields an extension

of the graphical calculus for monoidal, rigid, and pivotal categories developed in Sections 1.2.1, 1.2.2, and 1.2.3.

We then pass to three dimensions, where central algebras are described by diagrams on a stratified cube with a vertical surface stratum. Here, the surface represents the underlying monoidal category, while the three-dimensional regions encode the ribbon categories acting on it. This provides a diagrammatic description of morphisms that extends the graphical calculus for ribbon categories developed in Section 1.2.4.

Finally, combining these two constructions, we develop a graphical calculus for bi-module categories based on diagrams on a stratified cylinder with both two- and three-dimensional strata. These local models will be used in the next section to extend string-net and skein theory to stratified manifolds.

### 2.2.1 Stratified spaces

We fix here some terminology about stratified spaces that we will use throughout the section.

**Definition 2.2.1. (i)** Let  $P$  be a poset endowed with the topology whose open sets are generated by  $P_{>p} := \{x \in P \mid x > p\}$  for  $p \in P$ . A *stratified manifold indexed by  $P$*  is an oriented manifold  $M$  endowed with a continuous map

$$\phi: M \rightarrow P,$$

such that, for each  $p \in P$ ,  $M_p := \phi^{-1}(p)$  is a submanifold of  $M$  meeting  $\partial M$  transversally if at all. We call *strata* the connected components of  $M_p$ , for every  $p \in P$ .

**(ii)** A *morphism of stratified manifolds* between  $(M, \phi_1: M \rightarrow P)$  and  $(N, \phi_2: N \rightarrow Q)$  consists of a smooth map  $f: M \rightarrow N$  and a continuous map  $F: P \rightarrow Q$  such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \phi_1 \downarrow & & \downarrow \phi_2 \\ P & \xrightarrow{F} & Q \end{array}$$

is commutative. This means that  $f(M_p) \subseteq N_{F(p)}$ , for every  $p \in P$ .

**(iii)** A *stratified embedding* is a morphism  $(f, F): M \rightarrow N$  of stratified manifolds such that  $f$  is a smooth embedding.

**(iv)** A *stratified isotopy* between two stratified embeddings  $(f, F), (g, G): M \rightarrow N$  is a morphism  $(h, H): M \times I \rightarrow N$  of stratified manifolds such that  $h(-, t)$  is an embedding for every  $t \in I$ ,  $h(-, 0) = f$  and  $h(-, 1) = g$ . Here, the stratification on  $M \times I$  is given by  $\Phi(x, t) = \phi(x)$ , where  $\phi$  is the stratification of  $M$ .

**(v)** A *stratified isotopy of  $M$*  is a stratified isotopy of the form  $(h, \text{id}_P)$  between  $\text{id}_M$  and a stratified embedding  $f: M \rightarrow M$ . In other words, it is a smooth family of maps

$f_t: M \rightarrow M$ ,  $t \in I$ , such that

$$\begin{array}{ccc} M & \xrightarrow{f_t} & M \\ & \searrow \phi & \downarrow \phi \\ & & P \end{array}$$

commutes for all  $t \in I$ .

When some of the strata are endowed with a framing, we assume that morphisms, embeddings and isotopies preserve it up to homotopy.  $\diamond$

In order to define our theories, strata will be labelled with the algebraic data of some category with the appropriate structure. In that case, we call such a stratum a *defect*.

A stratified graph in a stratified manifold is an embedded graph intersecting the boundary and the strata nicely:

**Definition 2.2.2.** A *stratified graph* in a stratified manifold  $M$  is a graph  $(\Gamma, \Gamma_0)$  together with a smooth embedding  $\iota: \Gamma \rightarrow M$  such that:

- every edge or circle intersects a unique stratum;
- the image of  $\Gamma$  intersects the boundary of  $M$  exactly at its endpoints, i.e.,  $\iota(\Gamma) \cap (\partial M) = \iota(\partial \Gamma_0)$ ;
- every endpoint lies in the same region as its adjacent edge.

As usual, we refer to such a graph just by  $\Gamma$  when the context is clear.  $\diamond$

### 2.2.2 Planar line defects

This subsection is based on joint work in progress with Jennifer Brown, David Jordan and Matthias Vancraeynest [Bro+26].

The graphical calculus for  $\mathbb{k}$ -linear categories is one-dimensional: morphisms are represented by graphs on a line whose direction reflects the single available operation of composing morphisms. When the category is monoidal, this diagrammatic language naturally extends to two dimensions, with the additional dimension encoding the tensor product of objects and morphisms.

A bimodule category is a  $\mathbb{k}$ -linear category equipped with commuting left and right actions of two monoidal categories. In diagrammatic terms, this means that 2-dimensional diagrams act on diagrams supported on a line. A natural way to represent this situation is through the use of defects: we define a graphical calculus for bimodule categories by considering diagrams drawn inside a rectangle containing a distinguished vertical line. This line represents the underlying  $\mathbb{k}$ -linear category, while the planar regions on either side represent the two monoidal categories, which interact with the line through the bimodule structure.

From this perspective, the action of a monoidal category on a linear category can be viewed as passing from two dimensions to one, that is, as projecting planar diagrams onto the line. Building on this idea, we construct an evaluation functor from diagrams in the stratified rectangle to the underlying bimodule category, thereby extending the graphical calculus for monoidal, rigid, and pivotal categories developed in the first chapter.

From now on, we refer to the stratified space  $E := \mathbb{R} \times [0, 1]$  endowed with the stratification  $s: E \rightarrow (1 \leq 2)$  defined by

$$s(x, y) = \begin{cases} 1 & \text{if } x = 0, \\ 2 & \text{if } x \neq 0, \end{cases}$$

as the *stratified rectangle*. Thus,  $E$  consists of two 2-dimensional regions separated by the vertical segment  $\{0\} \times [0, 1]$ . The definition of progressive, polarised and cyclic graphs extend naturally to the stratified rectangle:

**Definition 2.2.3.** We say that a stratified planar graph  $\Gamma \hookrightarrow E$  is:

- (i) *progressive* if the restriction of the second coordinate function to each connected component of  $\Gamma \setminus \Gamma_0$  is injective;
- (ii) *polarised* if  $\Gamma$  is oriented, contains no cycles, and the restriction of the second coordinate function to every edge has only finitely many critical points, none of which occurs at a vertex;
- (iii) *cyclic* if  $\Gamma$  is oriented, every vertex is equipped with a framing  $(x_v, y_v)$  that is trivial for endpoints and for vertices lying on the vertical line, and, for every vertex  $v$  and every edge  $e \in \Gamma_v$ , the velocity vector of  $e$  at  $v$  is not parallel to  $x_v$ .  $\diamond$

For each of these graphs, the set  $\Gamma_v$  of vertices adjacent to a coupon  $v$  decomposes into two (possibly non-disjoint) ordered subsets,  $\Gamma_v^i$  and  $\Gamma_v^o$ , according to the same conventions as in the non-stratified setting (cf. Sections 1.2.1, 1.2.2 and 1.2.3). Observe also that progressive stratified planar graphs may contain finitely many cups and caps, all of which necessarily lie in one of the two-dimensional regions, since edges cannot cross the vertical line. The notions of *redirection points*, *turning numbers*, and *rotation numbers* are defined exactly as for polarised planar graphs.

**Definition 2.2.4.** An *isotopy of progressive (resp. polarised, cyclic) stratified planar graphs* of  $\Gamma$  is a stratified isotopy

$$H: E \times [0, 1] \rightarrow E$$

fixing  $\mathbb{R} \times 0, 1$  such that  $H(\Gamma, t)$  is a progressive (resp. polarised, cyclic) stratified planar graph for every  $t \in [0, 1]$ .  $\diamond$

We now define colourings for each class of stratified planar graphs. For progressive graphs, we use the data of a pointed  $(\mathcal{C}, \mathcal{D})$ -bimodule category  $\mathcal{M}$ , where  $\mathcal{C}$  and  $\mathcal{D}$  are

monoidal categories that are not assumed to be rigid or pivotal. We label the region of  $E$  to the left of the defect line by  $\mathcal{C}$ , the region to the right by  $\mathcal{D}$ , and the defect line itself by  $\mathcal{M}$ .

**Definition 2.2.5.** A  $\mathcal{M}$ -colouring for a progressive stratified planar graph  $\Gamma$  in  $E$  is the assignment of:

- an object for every edge  $e$ , chosen from the category labelling the region where  $e$  lies;
- a morphism from the corresponding monoidal category for every coupon lying in a 2-dimensional region, satisfying the same compatibility conditions as in the graphical calculus for monoidal categories;
- a morphism  $f$  from  $\mathcal{M}$  for every coupon  $v$  lying in the defect, chosen as follows. Let  $x_e$  (resp.  $y_e$ ) be the object decorating an incoming (resp. outgoing) edge  $e \in \Gamma_v$  in a 2-dimensional region, and  $m$  and  $m'$  are the objects decorating the incoming and outgoing adjacent defect edges. Then

$$f: \left( \bigotimes_{e \in \Gamma_v^i \cap E_{\mathcal{C}}} x_e \right) \triangleright m \triangleleft \left( \bigotimes_{e \in \Gamma_v^i \cap E_{\mathcal{D}}} x_e \right) \rightarrow \left( \bigotimes_{e \in \Gamma_v^o \cap E_{\mathcal{C}}} y_e \right) \triangleright m' \triangleleft \left( \bigotimes_{e \in \Gamma_v^o \cap E_{\mathcal{D}}} y_e \right),$$

where  $E_{\mathcal{C}}$  and  $E_{\mathcal{D}}$  are the regions decorated with  $\mathcal{C}$  and  $\mathcal{D}$ , respectively, and the tensor product is taken following the orders in  $\Gamma_v^i$  and  $\Gamma_v^o$ . If no incoming (resp. outgoing) edge lies in the defect, we take  $m$  (resp.  $m'$ ) to be the distinguished object  $\text{Dist}_{\mathcal{M}}$ .  $\diamond$

Assume, furthermore, that  $\mathcal{C}$  and  $\mathcal{D}$  are rigid categories. We obtain a notion of colouring for polarised stratified planar graphs:

**Definition 2.2.6.** A  $\mathcal{M}$ -colouring for a polarised stratified planar graph  $\Gamma$  in  $E$  is the assignment of:

- an object for every edge  $e$ , chosen from the category labelling the region where  $e$  lies;
- an integer  $N_0(e)$  for every edge  $e$  lying in a 2-dimensional region compatible with the orientation of the edge, as in Definition 1.2.8;
- a morphism in the corresponding rigid category for every coupon lying in a 2-dimensional region, satisfying the same compatibility conditions as in the graphical calculus for rigid categories;
- a morphism  $f$  from  $\mathcal{M}$  for every coupon  $v$  lying in the defect, chosen as follows. Let  $x_e$  (resp.  $y_e$ ) be the object decorating an incoming (resp. outgoing) edge  $e$

adjacent to  $v$  from a 2-dimensional region, and  $m$  and  $m'$  are the objects decorating the incoming and outgoing adjacent defect edges. Then

$$f: \left( \bigotimes_{e \in \Gamma_v^i \cap E_{\mathcal{C}}} x_e^{k(e)} \right) \triangleright m \triangleleft \left( \bigotimes_{e \in \Gamma_v^i \cap E_{\mathcal{D}}} x_e^{k(e)} \right) \rightarrow \left( \bigotimes_{e \in \Gamma_v^o \cap E_{\mathcal{C}}} y_e^{k(e)} \right) \triangleright m' \triangleleft \left( \bigotimes_{e \in \Gamma_v^o \cap E_{\mathcal{D}}} y_e^{k(e)} \right),$$

where  $E_{\mathcal{C}}$  and  $E_{\mathcal{D}}$  are the regions decorated with  $\mathcal{C}$  and  $\mathcal{D}$ , respectively, the tensor product is taken following the orders in  $\Gamma_v^i$  and  $\Gamma_v^o$ , and  $k(e)$  is the integer label of  $e$  at  $v$  (see Definition 1.2.9). If no incoming (resp. outgoing) edge lies in the defect, we take  $m$  (resp.  $m'$ ) to be the distinguished object  $\text{Dist}_{\mathcal{M}}$ .  $\diamond$

Finally, cyclic stratified planar graphs are coloured using the algebraic data of a  $(\mathcal{C}, \mathcal{D})$ -bimodule category  $\mathcal{M}$  where  $\mathcal{C}$  and  $\mathcal{D}$  are pivotal categories:

**Definition 2.2.7.** A  $\mathcal{M}$ -colouring for a cyclic stratified planar graph  $\Gamma$  in  $E$  is the assignment of:

- for every edge  $e$ , an object chosen from the category labelling the region where  $e$  lies;
- for every coupon  $v$  lying in a 2-dimensional region, a morphism from the underlying pivotal category that is compatible with the colours of the adjacent edges in the same way as for pivotal categories (cf. Definition 1.2.13);
- for every coupon  $v$  lying in the defect, a morphism  $f$  from the underlying pointed  $(\mathcal{C}, \mathcal{D})$ -bimodule  $\mathcal{M}$  chosen as follows (see Figure 2.1): if  $x_e$  (resp.  $y_e$ ) is the object decorating an incoming (resp. outgoing) edge  $e$  adjacent to the vertex from a 2-dimensional region and  $m$  and  $m'$  are the objects decorating the incoming and outgoing adjacent edges from the defect, then

$$f: \left( \bigotimes_{e \in \Gamma_v^i \cap E_{\mathcal{C}}} x_e^{\varepsilon(e)} \right) \triangleright m \triangleleft \left( \bigotimes_{e \in \Gamma_v^i \cap E_{\mathcal{D}}} x_e^{\varepsilon(e)} \right) \rightarrow \left( \bigotimes_{e \in \Gamma_v^o \cap E_{\mathcal{C}}} y_e^{\varepsilon(e)} \right) \triangleright m' \triangleleft \left( \bigotimes_{e \in \Gamma_v^o \cap E_{\mathcal{D}}} y_e^{\varepsilon(e)} \right),$$

where  $E_{\mathcal{C}}$  and  $E_{\mathcal{D}}$  are the regions decorated with  $\mathcal{C}$  and  $\mathcal{D}$ , respectively, and  $\varepsilon(e) = \emptyset$  if  $e$  is oriented upwards and  $\varepsilon(e) = *$  if  $e$  is oriented downwards at  $v$ . If no incoming (resp. outgoing) edge lies in the defect, we take  $m$  (resp.  $m'$ ) to be the distinguished object  $\text{Dist}_{\mathcal{M}}$ .  $\diamond$

Let  $\mathcal{M}$  be a  $(\mathcal{C}, \mathcal{D})$ -bimodule category. We let  $\text{Free}(\mathcal{C})$  and  $\text{Free}(\mathcal{D})$  be the free monoidal, free rigid or free pivotal categories on  $\mathcal{C}$  and  $\mathcal{D}$ , depending on whether  $\mathcal{C}$  and  $\mathcal{D}$  are both monoidal, rigid or pivotal.

**Definition 2.2.8.** The *free bimodule category* on  $\mathcal{M}$  is the category  $\text{Bimod}(\mathcal{M})$  with:

- objects: triples of the form  $(X, m, Y)$  where  $X \in \text{Free}(\mathcal{C})$ ,  $m \in \mathcal{M}$  and  $Y \in \text{Free}(\mathcal{D})$ ;

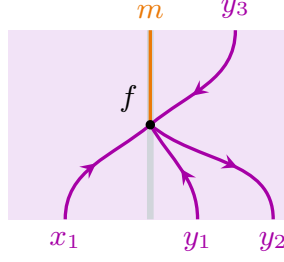


Figure 2.1: Here,  $f: x_1 \triangleright \text{Dist}_{\mathcal{M}} \triangleleft (y_1 \otimes y_2) \rightarrow \mathbf{1}_{\mathcal{C}} \triangleright m \triangleleft y_3$ .

- morphisms: a morphism from  $(X_1, m_1, Y_1)$  to  $(X_2, m_2, Y_2)$  is a  $\mathbb{k}$ -linear combinations of isotopy classes of  $\mathcal{M}$ -coloured progressive, polarised or cyclic (depending on the algebraic data) stratified planar graphs in  $E$  whose source and target are  $(X_1, m_1, Y_1)$  and  $(X_2, m_2, Y_2)$ , respectively.

As usual, composition is given by vertically stacking diagrams.  $\diamond$

This category is naturally a bimodule over  $\text{Free}(\mathcal{C})$  and  $\text{Free}(\mathcal{D})$ :

**Proposition 2.2.9.** *Let  $\mathcal{M}$  be a  $(\mathcal{C}, \mathcal{D})$ -bimodule category. Then, the category  $\text{Bimod}(\mathcal{M})$  has a canonical  $(\text{Free}(\mathcal{C}), \text{Free}(\mathcal{D}))$ -bimodule category structure.*

*Proof.* The left and right actions are given by horizontally stacking diagrams to the left and right, respectively. The associativity and unit constraints are the identity.  $\square$

Next, we construct an evaluation functor

$$\text{ev}_{\mathcal{M}}^{\text{Bimod}}: \text{Bimod}(\mathcal{M}) \rightarrow \mathcal{M}.$$

On objects, it is given by

$$\text{ev}_{\mathcal{M}}^{\text{Bimod}}(X, m, Y) = \text{ev}_{\mathcal{C}}^{\text{Free}}(X) \triangleright m \triangleleft \text{ev}_{\mathcal{D}}^{\text{Free}}(Y),$$

where  $\text{ev}_{\mathcal{C}}^{\text{Free}}$  and  $\text{ev}_{\mathcal{D}}^{\text{Free}}$  is the evaluation functor for  $\text{Free}(\mathcal{C})$  and  $\text{Free}(\mathcal{D})$  (monoidal, rigid or pivotal, depending on their algebraic structure). Given a planar graph  $\Gamma$  representing a morphism in  $\text{Bimod}(\mathcal{M})$ , we construct  $\text{ev}_{\mathcal{M}}^{\text{Bimod}}(\Gamma)$  as follows (see Figure 2.2):

1. replace  $\Gamma$  with a stratified isotopically equivalent planar graph which is *generic* with respect to the height function: this means again that the restriction of the second coordinate to any edge has (at most) a finite number of critical points, that they not coincide with vertices, and that two vertices/critical points never lie at the same height;

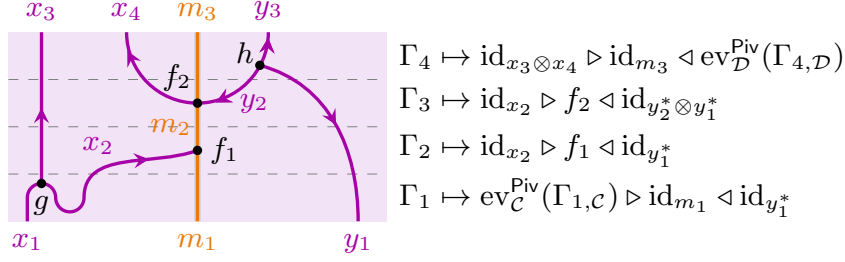


Figure 2.2: Evaluation of a morphism  $\Gamma$  of the free bimodule category  $\text{Bimod}(\mathcal{M})$ .

2. factor  $\Gamma = \Gamma_k \circ \dots \circ \Gamma_1$ , where each  $\Gamma_i$  is either in:

- (A) a small strip containing a vertex landing on the defect (a *mixed vertex*) with edges potentially coming from any of the three regions;
- (B) a small strip with no such vertices, so that the three regions do not interact at all;

3. evaluate  $\Gamma_i$  according to its type:

- (A) in the first case,  $\Gamma_i$  consists (possibly up to an isotopy supported inside the  $i$ th strip) of a vertex on the defect, labelled with a morphism  $f$ , and a bunch of vertical lines labelled with  $x_1, \dots, x_m$  and  $y_1, \dots, y_n$  in the  $\mathcal{C}$ - and  $\mathcal{D}$ -regions, so

$$\text{ev}_{\mathcal{M}}^{\text{Bimod}}(\Gamma_i) = \text{id}_{x_1^{k_1} \otimes \dots \otimes x_m^{k_m}} \triangleright f \triangleleft \text{id}_{y_1^{s_1} \otimes \dots \otimes y_n^{s_n}},$$

where the  $k_i$ 's and  $s_j$ 's are the integer labels of the lines;

- (B) in the second case,  $\Gamma_i$  decomposes into three isolated graphs:  $\Gamma_{i,\mathcal{C}}$  in the  $\mathcal{C}$ -region,  $\Gamma_{i,\mathcal{M}}$  in the line defect and  $\Gamma_{i,\mathcal{D}}$  in the  $\mathcal{D}$ -region. The evaluation of  $\Gamma_i$  is now given by

$$\text{ev}_{\mathcal{M}}^{\text{BIMOD}}(\Gamma_i) = \text{ev}_{\mathcal{C}}^{\text{Free}}(\Gamma_{i,\mathcal{C}}) \triangleright \text{ev}_{\mathcal{M}}^{\text{Lin}}(\Gamma_{i,\mathcal{M}}) \triangleleft \text{ev}_{\mathcal{D}}^{\text{Free}}(\Gamma_{i,\mathcal{D}});$$

4. finally, compose the individual pieces to get the evaluation

$$\text{ev}_{\mathcal{M}}^{\text{Bimod}}(\Gamma) = \text{ev}_{\mathcal{M}}^{\text{Bimod}}(\Gamma_k) \circ \dots \circ \text{ev}_{\mathcal{M}}^{\text{Bimod}}(\Gamma_1)$$

on generators and extend  $\mathbb{k}$ -linearly to the full hom-spaces of  $\text{Bimod}(\mathcal{M})$ .

In what follows, we will write  $\Gamma_1 \sim \Gamma_2$  whenever  $\text{ev}_{\mathcal{M}}^{\text{Bimod}}(\Gamma_1) = \text{ev}_{\mathcal{M}}^{\text{Bimod}}(\Gamma_2)$ .

**Proposition 2.2.10.** *The  $\mathbb{k}$ -linear evaluation functor*

$$\text{ev}_{\mathcal{M}}^{\text{Bimod}}: \text{Bimod}(\mathcal{M}) \rightarrow \mathcal{M}$$

*obtained by the previous procedure is well-defined, i.e., the value of a given graph  $\Gamma$  (progressive, polarised or cyclic) is invariant under the proper notion of stratified isotopy and the choice of decomposition.*

*Proof.* We give the proof for the case where  $\mathcal{C}$  and  $\mathcal{D}$  are pivotal, hence for cyclic graphs, since it is the more general. A proof for progressive and polarised graphs is obtained by just restricting our attention to the class of isotopies allowed in that cases. First note that  $\text{ev}_{\mathcal{M}}^{\text{Bimod}}$  is functorial. Indeed, if  $\Gamma^0 = \Gamma_m^0 \circ \dots \circ \Gamma_1^0$  and  $\Gamma^1 = \Gamma_n^1 \circ \dots \circ \Gamma_1^1$  are two composable coloured planar graphs with given decompositions, then we get a decomposition

$$\Gamma^0 \circ \Gamma^1 = \Gamma_m^0 \circ \dots \circ \Gamma_1^0 \circ \Gamma_n^1 \circ \dots \circ \Gamma_1^1$$

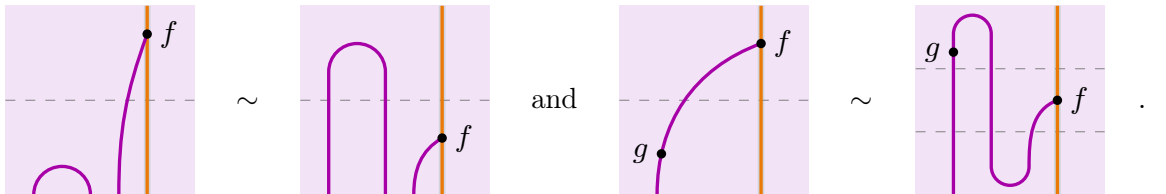
for  $\Gamma^0 \circ \Gamma^1$ . Since the evaluation is given by evaluating each of the pieces and composing them in  $\mathcal{M}$ , we obtain the functoriality.

For the choice of the decomposition, it suffices to note that two decompositions of a given coloured planar diagram  $\Gamma$  can always be obtained from each other by subdividing strips of type (B) or by fusing two consecutive such strips together. The functoriality implies that these two operations do not alter the value of the diagram, so the evaluation functor is independent of the choice of a decomposition.

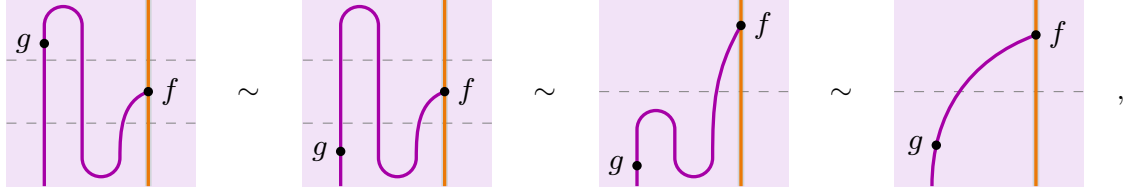
Finally, we check the isotopy invariance. We need to consider two types of isotopies:

- (i) isotopies preserving the ordering (by height) of the vertices, cups and caps;
- (ii) isotopies swapping a mixed vertex with either a cup/cap or a vertex lying on the planar region.

The first kind of isotopies are supported in strands containing no mixed vertices, i.e., where the three regions do not interact. Thus, the invariance follows from the coherence of the graphical calculus for pivotal categories, explained in Section 1.2.3. For the second type, we need to check the invariance under relations of the form



The first equality is just a consequence of the functoriality of the action of  $\mathcal{C}$  on  $\mathcal{M}$ . For the second one, we have



again by the functoriality of the action and the zig-zag identity. The rest of the cases are treated similarly.  $\square$

### 2.2.3 Pivotal surface defects

We now turn to central algebras over ribbon categories. As we saw in the first chapter, the graphical calculus for ribbon categories is based on a class of graphs embedded in three dimensions. By contrast, a central algebra consists of a monoidal category  $\mathcal{C}$  together with a braided central functor

$$F: \mathcal{A} \boxtimes \mathcal{B}^{\text{op}} \rightarrow Z_{\text{Dr}}(\mathcal{C}),$$

where  $\mathcal{A}$  is a ribbon category. Since the graphical calculus of  $\mathcal{C}$  is inherently 2-dimensional, defects once again provide a natural way to encode the additional central structure. Accordingly, the graphical calculus for central algebras is defined using diagrams in a stratified 3-dimensional cube whose stratification consists of a single vertical wall. As in the planar case, the wall represents the underlying monoidal category  $\mathcal{C}$ , while the two 3-dimensional regions on either side represent the ribbon categories  $\mathcal{A}$  and  $\mathcal{B}$ .

The interaction between the 2- and 3-dimensional parts of the calculus is governed by the central structure. Since  $\mathcal{C}$  is not braided in general, edges lying on the wall are not allowed to cross. However, objects in the image of  $F$  are equipped with half-braidings, which describe how objects of  $\mathcal{A}$  or  $\mathcal{B}$  commute with arbitrary objects of  $\mathcal{C}$ . Diagrammatically, these half-braidings can be interpreted as allowing an edge on the wall, provided it is labelled by an object in the image of  $F$ , to leave the wall temporarily, move into the surrounding 3-dimensional region, pass over any other edge on the wall, and then return to the surface. Conversely, a graph embedded in three dimensions can be projected onto the wall. Such a projection may create new crossings, but these have a natural interpretation, since at least one of the strands is labelled by an object of  $\mathcal{A}$  or  $\mathcal{B}$ : they represent precisely the half-braidings determined by the central structure.

Building on these ideas, we describe in this section the graphical calculus for central algebras developed in [BJ25] by Brown–Jordan in the pivotal case. In the next section, we explain how this calculus can be modified in order to drop the assumption that the underlying central structure is pivotal.

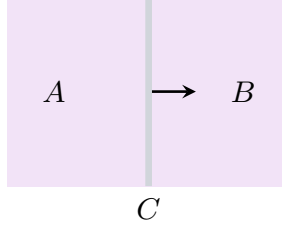


Figure 2.3: In our pictures we always represent the chosen normal vector field on the defect. It points towards the target region. For instance, we read the defect in this picture as going from  $A$  to  $B$ , so the corresponding algebraic data is a  $(\mathcal{B}, \mathcal{A})$ -central algebra.

Throughout this section, we refer to the 3-dimensional cube  $E := \mathbb{R}^2 \times [0, 1]$  equipped with the stratification

$$s(x, y, z) = \begin{cases} 1 & \text{if } x = 0, \\ 2 & \text{if } x \neq 0, \end{cases}$$

as the *walled or stratified cube*. Thus,  $E$  decomposes into two 3-dimensional regions separated by a vertical wall. We assume that the wall is oriented, i.e., equipped with a chosen normal direction. The region towards which this normal vector points is called the *source region*, while the other is called the *target region* (see Figure 2.3).

We consider the following class of stratified graphs in  $E$ , which look like ribbon graphs in the 3-dimensional regions and like cyclic graphs in the wall:

**Definition 2.2.11.** A *ribbon-cyclic stratified graph* is an oriented stratified graph  $\iota: \Gamma \rightarrow E$  together with the following data:

- a framing  $(x_v, y_v, z_v)$  for every vertex  $v \in \Gamma_0$ , such that
  1. the framing is trivial if  $v$  is an endpoint,
  2.  $(y_v, z_v)$  are tangent to the defect if  $v$  lies on the wall. Interpreting these vertices as boxes (cf. Figure 1.8), this amounts to saying that they are contained in the wall;
- a framing for every connected component  $c \in \pi_0(\Gamma \setminus \Gamma_0)$  of  $\Gamma \setminus \Gamma_0$  lying in a 3-dimensional region, meaning a non-vanishing section of the normal bundle  $N(c) \subset T(\mathbb{R}^2 \times [0, 1])$ ;

satisfying the following conditions:

- the orthogonal projection onto the wall is injective on the set of endpoints of  $\Gamma$ ;
- for every vertex  $v$  and every oriented edge  $e \in \Gamma_v$  adjacent to  $v$ , the velocity vector  $\partial_v(e)$  at  $v$  lies in the  $(y_v, z_v)$ -plane and is not parallel to  $y_v$ ;

- for every vertex  $v$  and every oriented edge  $e \in \Gamma_v$ , the normal vector  $n_v(e)$  provided by the framing is parallel to  $y_v$ , and  $(n_v(e), \partial_v(e))$  is a positive basis of the  $(y_v, z_v)$ -plane.  $\diamond$

As with ribbon graphs, we will use diagrams to represent ribbon-cyclic stratified graphs, assuming that they are endowed with the blackboard framing. Also, using the same rules as for ribbon and cyclic graphs, the set  $\Gamma_v$  of edges adjacent to a given vertex decomposes into two (possibly non-disjoint) ordered subsets  $\Gamma_v^i$  and  $\Gamma_v^o$ . We consider ribbon-cyclic stratified graphs up to isotopy:

**Definition 2.2.12.** An *isotopy of ribbon-cyclic stratified graphs* of  $\Gamma$  is a stratified isotopy

$$H: E \times [0, 1] \rightarrow E$$

such that  $H(\Gamma, t)$  is a ribbon-cyclic stratified graph, for every  $t \in [0, 1]$ .  $\diamond$

We colour ribbon-cyclic stratified graphs using the algebraic data of a pivotal central algebra. Let  $\mathcal{C}$  be a  $(\mathcal{B}, \mathcal{A})$ -central algebra and suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are ribbon. We label the regions of  $E$  as in Figure 2.3, i.e.,  $\mathcal{A}$  corresponds to the source,  $\mathcal{B}$  corresponds to the target and  $\mathcal{C}$  corresponds to the defect.

**Definition 2.2.13.** A  $\mathcal{C}$ -colouring of a ribbon-cyclic stratified graph  $\Gamma$  is the assignment of:

- an object for every edge  $e$ , chosen from the category labelling the region where  $e$  lies;
- a morphism from the corresponding ribbon category for every coupon  $v$  lying in a 3-dimensional region, chosen following the same rules as for ribbon categories (cf. Section 1.2.4);
- for every coupon  $v$  lying on a surface defect, a morphism  $f$  from the underlying category  $\mathcal{C}$  chosen as follows. If  $x_e$  (resp.  $y_e$ ) is the object decorating an incoming (resp. outgoing) edge  $e$  adjacent to the vertex, then

$$f: F \left( \bigotimes_{e \in \Gamma_v^i \cap E_{\mathcal{A}}} x_e^{\varepsilon(e)} \boxtimes \bigotimes_{e \in \Gamma_v^i \cap E_{\mathcal{B}}} x_e^{\varepsilon(e)} \right) \otimes \bigotimes_{e \in \Gamma_v^i \cap E_{\mathcal{C}}} x_e^{\varepsilon(e)} \longrightarrow F \left( \bigotimes_{e \in \Gamma_v^o \cap E_{\mathcal{A}}} y_e^{\varepsilon(e)} \boxtimes \bigotimes_{e \in \Gamma_v^o \cap E_{\mathcal{B}}} y_e^{\varepsilon(e)} \right) \otimes \bigotimes_{e \in \Gamma_v^o \cap E_{\mathcal{C}}} y_e^{\varepsilon(e)},$$

where  $F$  is the central functor;  $E_{\mathcal{A}}$ ,  $E_{\mathcal{B}}$  and  $E_{\mathcal{C}}$  are the regions labelled with  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$ , respectively; and  $\varepsilon(e) = 0$  if  $e$  is oriented upwards and  $\varepsilon(e) = 1$  if  $e$  is oriented downwards at  $v$  (see Figure 2.4).  $\diamond$

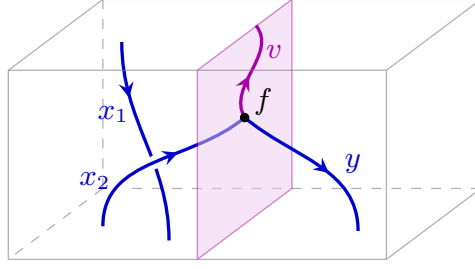


Figure 2.4: A  $\mathcal{C}$ -coloured ribbon-cyclic stratified graph in  $E$ , with  $f: F(x_2 \boxtimes y^*) \rightarrow v$ .

Given a  $\mathcal{C}$ -coloured ribbon-cyclic stratified graph  $\Gamma$ , the condition that two endpoints are never identified by the orthogonal projection into the defect wall induces an ordering on the sets of top and bottom endpoints in each region. This order is obtained by projecting the endpoints onto a plane parallel to the wall and using the induced linear order on that plane. In this way we obtain two triples  $(X, V, Y)$ , one associated with the bottom face and one with the top face, called respectively the *source* and the *target*.

**Definition 2.2.14.** The *free pivotal central algebra on  $\mathcal{C}$*  is the category  $\text{CENTRAL}(\mathcal{C})$  with:

- objects: triples  $(X, V, Y)$  where  $X$  is an object of  $\text{Rib}(\mathcal{A})$ ,  $V$  is an object of  $\text{Piv}(\mathcal{C})$  and  $Y$  is an object of  $\text{Rib}(\mathcal{B})$ ;
- morphisms: a morphism between  $(X_1, V_1, Y_1)$  and  $(X_2, V_2, Y_2)$  is a linear combination of isotopy classes of  $\mathcal{C}$ -coloured ribbon-cyclic stratified graphs in  $E$  whose endpoints match  $(X_1, V_1, Y_1)$  and  $(X_2, V_2, Y_2)$  on the bottom and top bases, respectively (see Figure 2.4).

Composition is defined by vertically stacking two copies of  $E$  and rescaling the vertical coordinate. The tensor product is given by stacking diagrams along the second coordinate, with the monoidal unit given by the empty configuration.  $\diamond$

The category  $\text{CENTRAL}(\mathcal{C})$  is rigid, with duals obtained by reversing the orientation, and it has a pivotal structure given by the identity isomorphism.

**Proposition 2.2.15.** *The category  $\text{CENTRAL}(\mathcal{C})$  has a canonical pivotal  $(\text{Rib}(\mathcal{B}), \text{Rib}(\mathcal{A}))$ -central algebra structure.*

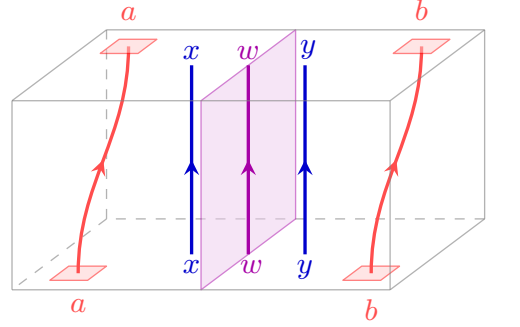
*Proof.* There is a canonical strict monoidal functor

$$F^{\text{Rib}}: \text{Rib}(\mathcal{A}) \boxtimes \text{Rib}(\mathcal{B})^{\text{mop}} \rightarrow \text{CENTRAL}(\mathcal{C})$$

given by stacking diagrams on the left and right of an empty walled cube. The half-braidings

$$F^{\text{Rib}}(a \boxtimes b) \otimes (x, w, y) \rightarrow (x, w, y) \otimes F^{\text{Rib}}(a \boxtimes b)$$

are represented by diagrams of the form



where the red squares represent the image of  $F^{\text{Rib}}$  in  $\text{CENTRAL}(\mathcal{C})$ . It is straightforward to check that they define a lift to the Drinfeld center. Moreover, distinguished dualities are preserved, so the central functor is pivotal since all the pivotal structures are trivial.  $\square$

**Theorem 2.2.16** ([BJ25, Definition-Proposition 2.16]). *There is a well-defined full pivotal monoidal functor*

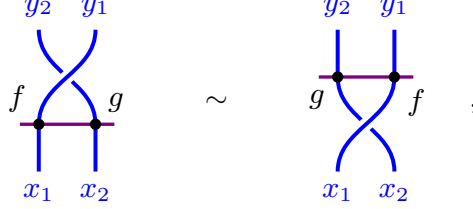
$$\text{ev}_{\mathcal{C}}^{\text{CENTRAL}}: \text{CENTRAL}(\mathcal{C}) \rightarrow \mathcal{C}$$

determined by the following procedure:

1. given a  $\mathcal{C}$ -coloured ribbon-cyclic stratified graph, choose an isotopically equivalent graph  $\Gamma$  which is in generic position with respect to the orthogonal projection onto the defect, so that all crossings are transverse, coupons are not identified by the projection and the framing of every edge and circle is nowhere orthogonal to the wall;
2. project  $\Gamma$  onto the defect and adjust the decoration as follows:
  - (i) replace the labels of the edges coming from a 3-dimensional region by their images under the central structure  $F$ ;
  - (ii) replace all the crossings appearing with the corresponding half-braidings in  $Z_{\text{Dr}}(\mathcal{C})$  using the central structure on  $\mathcal{C}$ ;
3. the resulting planar diagram  $\widehat{\Gamma}$  represents a morphism in  $\text{Piv}(\mathcal{C})$ , so we set

$$\text{ev}_{\mathcal{C}}^{\text{CENTRAL}}(\Gamma) = \text{ev}_{\mathcal{C}}^{\text{Piv}}(\widehat{\Gamma}).$$

*Proof.* We give some ideas of the proof, see the original reference [BJ25] for more details. In order to prove the invariance under isotopy, one first shows that the evaluation functor is invariant under deformations of ribbon graphs that make crossings go through the defect. These are isotopies of the form



where we have represented the defect horizontally to simplify the pictures. Let  $(F, \gamma^F): \mathcal{A} \boxtimes \mathcal{B}^{\text{bop}} \rightarrow Z_{\text{Dr}}(\mathcal{C})$  be the underlying central functor and write

$$F^{\mathcal{A}}(x) := F(x \boxtimes \mathbf{1}_{\mathcal{B}}) \quad \text{and} \quad F^{\mathcal{B}}(y) := F(\mathbf{1}_{\mathcal{A}} \boxtimes y).$$

The first graph evaluates to

$$F^{\mathcal{A}}(x_1) \otimes F^{\mathcal{A}}(x_2) \xrightarrow{f \otimes g} F^{\mathcal{B}}(y_1) \otimes F^{\mathcal{B}}(y_2) \xrightarrow{F^{\mathcal{B}}(\beta_{y_1, y_2}^{\mathcal{B}})} F^{\mathcal{B}}(y_2) \otimes F^{\mathcal{B}}(y_1).$$

Since  $F$  is a braided functor, we have

$$F^{\mathcal{B}}(\beta_{y_1, y_2}^{\mathcal{B}}) = \beta_{F^{\mathcal{A}}(x_1), F^{\mathcal{B}}(y_2)}^{Z_{\text{Dr}}(\mathcal{C})} = \gamma_{y_1}^{F^{\mathcal{B}}(y_2)},$$

which is natural in the first factor (since we are considering  $\mathcal{B}^{\text{bop}}$  instead of  $\mathcal{B}$ ). Hence, naturality implies that

$$F^{\mathcal{B}}(\beta_{y_1, y_2}^{\mathcal{B}}) \circ (f \otimes g) = (\text{id}_{y_2} \otimes f) \circ \beta_{F^{\mathcal{A}}(x_1), F^{\mathcal{B}}(y_2)}^{Z_{\text{Dr}}(\mathcal{C})} \circ (\text{id}_{x_1} \otimes g). \quad (2.2.15)$$

Now, note that

$$\beta_{F^{\mathcal{A}}(x_1), F^{\mathcal{B}}(y_2)}^{Z_{\text{Dr}}(\mathcal{C})}: F^{\mathcal{A}}(x_1) \otimes F^{\mathcal{B}}(y_2) \rightarrow F^{\mathcal{B}}(y_2) \otimes F^{\mathcal{A}}(x_1)$$

and that

$$F^{\mathcal{A}}(x_1) \otimes F^{\mathcal{B}}(y_2) = F(x_1 \boxtimes y_2) = F^{\mathcal{B}}(y_2) \otimes F^{\mathcal{A}}(x_1).$$

This implies that

$$\beta_{F^{\mathcal{A}}(x_1), F^{\mathcal{B}}(y_2)}^{Z_{\text{Dr}}(\mathcal{C})} = \gamma_{x_1}^{F^{\mathcal{B}}(y_2)} = \gamma_{y_2}^{F^{\mathcal{A}}(x_1)}.$$

The latter is natural in the first factor by definition of the braiding of  $Z_{\text{Dr}}$ . Thus, so is  $\beta_{F^{\mathcal{A}}(x_1), F^{\mathcal{B}}(y_2)}^{Z_{\text{Dr}}(\mathcal{C})}$  and equation (2.2.15) yields

$$F^{\mathcal{B}}(\beta_{y_1, y_2}^{\mathcal{B}}) \circ (f \otimes g) = (g \otimes f) \circ \beta_{F^{\mathcal{A}}(x_1), F^{\mathcal{A}}(x_2)}^{Z_{\text{Dr}}(\mathcal{C})} = (g \otimes f) \circ F^{\mathcal{A}}(\beta_{x_1, x_2}^{\mathcal{A}}),$$

where the last equality uses again the fact that  $F$  is braided. The right-most term in the equality is the evaluation of the second graph above, so the evaluation functor is invariant under isotopies making a crossing pass through the defect wall.

On the other hand, we need to prove the compatibility with the framing of edges. Representing edges as ribbons, this comes to show the invariance under isotopies of the form

performing a  $2\pi$ -rotation in plane of the coupon to which the ribbon is attached. These graphs evaluate to

$$F^{\mathcal{B}}(\theta^{\mathcal{B}}) \circ f \quad \text{and} \quad f \circ F^{\mathcal{A}}(\theta^{\mathcal{A}}),$$

respectively, where  $\theta^{\mathcal{A}}$  and  $\theta^{\mathcal{B}}$  are the twists of  $\mathcal{A}$  and  $\mathcal{B}$ . The equality between both morphisms follows from the fact that  $\theta$  is defined using the braidings and the pivotal structures, and that  $F$  is both braided and pivotal.

The rest of the proof consists of showing the invariance under stratified versions of Reidemeister moves. We refer to [BJ25] for the details.  $\square$

**Remark 2.2.17.** As explained in Section 2.3.1, the algebraic data of the defect is determined by its orientation and the algebraic data of the adjacent bulk regions. In particular, by reversing the orientation and replacing  $\mathcal{C}$  with  $\mathcal{C}^{\vee}$  (cf. Definition 2.1.16), we get another well-defined surface defect which is equivalent to the initial one, i.e., the induced evaluation morphism is the same in both cases. Indeed, when we reverse the orientation, positive crossings become negative, and vice versa. For a positive crossing, the evaluation morphism for  $\mathcal{C}$  will assign the corresponding half-braiding  $\gamma^{\mathcal{C}}$ . Reversing the orientation, the crossing becomes negative, so the evaluation morphism for  $\mathcal{C}^{\vee}$  will send it to  $(\gamma^{\mathcal{C}^{\vee}})^{-1} = \gamma^{\mathcal{C}}$ .  $\diamond$

#### 2.2.4 Rigid surface defects

In this subsection, we adapt the Brown–Jordan construction to central algebras whose underlying category or central functor is not pivotal. Diagrammatically, removing the pivotal structure requires a more rigid class of graphs and isotopies. Namely, we consider graphs in the stratified cube whose 2-dimensional part looks like a polarised graph. Consequently, we need also to consider a more restrictive class of isotopies, preserving the canonical framing of the defect wall.

**Definition 2.2.18.** A *ribbon-polarised stratified graph* is a stratified graph  $\iota: \Gamma \rightarrow E$  together with

- a framing  $(x_v, y_v, z_v)$  for every vertex  $v \in \Gamma_0$ , which is the canonical one if  $v$  is an endpoint or a coupon lying in the defect;
- a framing for every connected component  $c \in \pi_0(\Gamma \setminus \Gamma_0)$  lying in a 3-dimensional region, meaning a non-vanishing section of the normal bundle  $N(c) \subset T(\mathbb{R}^2 \times [0, 1])$ ;

satisfying the following conditions:

- the orthogonal projection onto the wall is injective on the set of endpoints of  $\Gamma$ ;
- the restriction of the third coordinate function to any edge or circle has finitely many critical points, none of which is a vertex;
- for every vertex  $v$  and every oriented edge  $e \in \Gamma_v$ , the velocity vector  $\partial_v(e)$  is contained in the  $(y_v, z_v)$ -plane and is not parallel to  $y_v$ ;
- for every vertex  $v$  and every oriented edge  $e \in \Gamma_v$ , the normal vector  $n_v(e)$  provided by the framing is parallel to  $y_v$  and  $(n_v(e), \partial_v(e))$  is a positive basis of the  $(y_v, z_v)$ -plane.  $\diamond$

The sets of incoming and outgoing edges of a given vertex is defined as in all the previous cases and we consider graphs up to stratified isotopies in the class of ribbon-polarised stratified graphs. Given a  $(\mathcal{B}, \mathcal{A})$ -central algebra whose underlying category  $\mathcal{C}$  is rigid, we have also a notion of  $\mathcal{C}$ -colouring for this class of graphs, assigning the extra data of an integer to the edges in the defect. The assignment is done following the same rules as in the previous sections. We omit all these technical definitions here, as they follow directly from the cases treated above. We obtain a free central algebra defined exactly as in the pivotal case:

**Definition 2.2.19.** The *free central algebra on  $\mathcal{C}$*  is the category  $\text{CENTRAL}(\mathcal{C})$  with:

- objects: triples  $(X, V, Y)$  where  $X$  is an object of  $\text{Rib}(\mathcal{A})$ ,  $V$  is an object of  $\text{Rig}(\mathcal{C})$  and  $Y$  is an object of  $\text{Rib}(\mathcal{B})$ ;
- morphisms: a morphism between  $(X_1, V_1, Y_1)$  and  $(X_2, V_2, Y_2)$  is a linear combination of stratified isotopy classes of  $\mathcal{C}$ -coloured ribbon-polarised stratified graphs in  $E$  whose source and target are  $(X_1, V_1, Y_1)$  and  $(X_2, V_2, Y_2)$ , respectively.

Composition is given by vertically stacking two copies of  $E$  and rescaling the vertical coordinate. The tensor product is given by stacking diagrams along the second coordinate, with the monoidal unit given by the empty configuration.  $\diamond$

The category  $\text{CENTRAL}(\mathcal{C})$  carries a  $(\text{Rib}(\mathcal{B}), \text{Rib}(\mathcal{A}))$ -central structure defined exactly as in Proposition 2.2.15.

We define an evaluation functor for ribbon-polarised stratified graphs. We define it by forgetting the pivotal structures of the ribbon categories  $\mathcal{A}$  and  $\mathcal{B}$ . Since multiple duals

were identified under pivotal structure, we need to choose representatives, in a compatible way, for this identification. This amounts to assigning an integer to every edge lying in a 3-dimensional region, accordingly to its orientation. We choose this integer to be 0 at every initial point, and we determine the integer of the endpoint using the rotation number. Next, we also choose representatives for the morphisms. Recall that the pivotal structures of  $\mathcal{A}$  or  $\mathcal{B}$  yield an identification

$$\tilde{\iota}_x: F(x) \xrightarrow{F(\iota_x)} F(x^{**}) \xrightarrow{\eta_x^r} F(x)^{**}, \quad (2.2.16)$$

in  $\mathcal{C}$ , where  $\eta_x^r$  is the isomorphism in (1.1.5). This identification is not natural in general, but it is for morphisms lying in the image of  $F$  (just because of functoriality). For every vertex, we replace its label  $f$  with the only morphism  $\widetilde{F(f)}$  equivalent to  $F(f)$  under the equivalence induced by (2.2.16) and compatible with the integer labels introduced in the first step. Projecting onto the defect and replacing crossings by half-braidings, we obtain a well-defined coloured graph in the defect, that we evaluate using the graphical calculus for rigid categories. Finally, we precompose and postcompose with (2.2.16) in order to reduce multiple applications of the duality functor in the source and target to either the identity or  $(-)^*$ , so that both source and target agree with the representatives chosen for the objects. The rest of the section is devoted to making this precise.

**Definition 2.2.20.** (i) A ribbon-polarised stratified graph is *in parallel position* with respect to the defect wall if the framing of every vertex coincides with the canonical framing of  $\mathbb{R}^2 \times [0, 1]$  and the framing of every edge is nowhere orthogonal to the defect wall. This means, roughly, that both ribbons and coupons are parallel to the defect wall.

(ii) Given an edge or a circle  $c$  of a graph  $\Gamma$  in parallel position, we define the *rotation number* of  $c$  as the rotation number  $\text{rot}(e)$  of its projection into the defect, that is, the sum of all the turning numbers of the cups and caps appearing after projection.  $\diamond$

For every edge  $e \in \Gamma$  of a graph in parallel position, we define set

$$N_0(e) = \begin{cases} 0 & \text{if } e \text{ is upwards,} \\ 1 & \text{if } e \text{ is downwards.} \end{cases}$$

We also set  $N_1(e) := N_0(e) + \text{rot}(e)$ . For a circle  $c$ , we define  $N_0, N_1$  in the same way, by choosing arbitrarily a point  $p$  on  $c$  and treating the circle as an edge with  $c(0) = c(1) = p$ .

**Theorem 2.2.21.** Let  $\mathcal{C}$  be a  $(\mathcal{B}, \mathcal{A})$ -central algebra whose underlying category is rigid but not necessarily pivotal. There is a well-defined full monoidal functor

$$\text{ev}_{\mathcal{C}}^{\text{CENTRAL}}: \text{CENTRAL}(\mathcal{C}) \rightarrow \mathcal{C}$$

determined by the following procedure:

**On objects:** project points lying in a 3-dimensional region to the defect, apply the central functor  $F$  to their label by an object of  $\mathcal{A}$  or  $\mathcal{B}$  and keep the integer label 0 or 1 unchanged;

**On morphisms:**

1. given a  $\mathcal{C}$ -coloured ribbon-polarised stratified graph, choose an isotopically equivalent graph  $\Gamma$  which is in parallel generic position with respect to the orthogonal projection onto the defect;
2. modify the decoration according to the following rules:
  - (i) given an edge or circle, label their initial and endpoints using  $N_0$  and  $N_1$ , respectively, and replace the object colouring it by its image under  $F$ ;
  - (ii) using the pivotal structures of  $\mathcal{A}$  and  $\mathcal{B}$ , modify the label of every vertex in order to make its image compatible with the integer labels, that is, if  $v$  was initially coloured with

$$f: \bigotimes_{e \in \Gamma_v^i} x_e^{\varepsilon(e)} \rightarrow \bigotimes_{e' \in \Gamma_v^o} y_{e'}^{\varepsilon(e')},$$

then we switch its decoration to

$$\widetilde{F(f)}: \bigotimes_{e \in \Gamma_v^i} F(x_e)^{k(e)} \rightarrow \bigotimes_{e' \in \Gamma_v^o} F(y_{e'})^{k(e')},$$

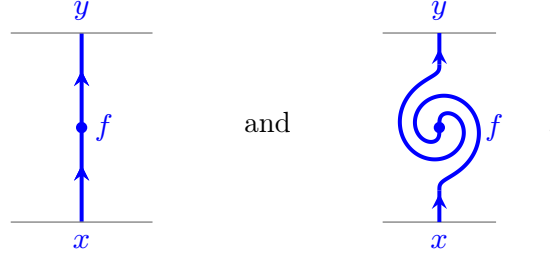
where  $k(e)$  stands for the integer label of  $e$  at  $v$ , and the morphism  $\widetilde{F(f)}$  is determined by pre-/post-composing with the images (2.2.16) of the pivotal structures;

3. project the graph into the defect and replace the crossing appearing by the half-braiding provided by the central structure;
4. apply  $\text{ev}_C^{\text{Rig}}$  to the planar diagram obtained and use (2.2.16) factorwise to identify the image of multiple duals  $F(x^n)$  with either  $F(x)$  or  $F(x^*)$ , depending on the parity of  $n$ .

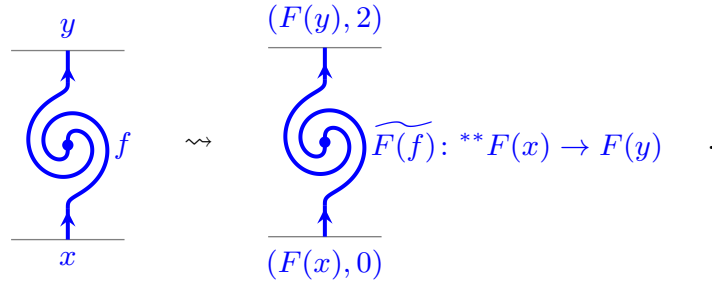
*Proof.* The invariance under isotopies is proven exactly in the same way as in the proof of Theorem 2.2.16. Note that in this case the central functor is not pivotal but also that coupons in the defect can not be rotated, so the invariance under isotopies making a twist pass through the defect does not need to be checked.

The only point that needs to be verified in this case is the independence of the chosen parallel generic representative of a given ribbon-polarised stratified graph. Any two such representatives differ by an even number of twists along each edge, resulting in different labelings of the vertices. These labelings are related by conjugation with (2.2.16). The resulting discrepancies between the coloured planar graphs obtained are corrected in the final step by precomposing and postcomposing with the images of the pivotal structure, which remain natural for morphisms in the image of  $F$  (see the example below).  $\square$

**Example 2.2.22.** Consider the two ribbon-polarised stratified graphs represented by the diagrams



where we are supposing that the diagrams lie entirely in a 3-dimensional region and that the defect wall is behind them. Both graphs represent the same morphism in  $\text{CENTRAL}(\mathcal{C})$  and are in parallel position with respect to the wall. The first one evaluates directly to  $F(f)$ . Applying the evaluation process to the second one, it is modified as follows:



Applying the graphical calculus for rigid categories and post-composing with the image of the pivotal structure, this yields

$$(\tilde{\iota}_y)^{-1} \circ \widetilde{F(f)}^{**} = (\tilde{\iota}_y)^{-1} \circ (**\tilde{\iota}_y \circ **F(f))^{**} = (\tilde{\iota}_y)^{-1} \circ \tilde{\iota}_y \circ F(f) = F(f),$$

so the evaluation for both diagrams is the same.  $\diamond$

### 2.2.5 Line defects

This subsection is based on joint work in progress with Jennifer Brown, David Jordan and Matthias Vancraeynest [Bro+26].

We describe a graphical calculus for centred bimodules. Recall that a centred bimodule  $\mathcal{L}$  is a category endowed with commuting left and right actions of two central algebras  $\mathcal{C}$  and  $\mathcal{D}$ . As explained in Section 2.2.2, the actions of  $\mathcal{C}$  and  $\mathcal{D}$  as simply monoidal categories are modelled topologically by planar line defects. Unlike the setting considered there, however, the categories  $\mathcal{C}, \mathcal{D}$  are not merely monoidal categories: they

are equipped with compatible central structures, that is, compatible actions of a ribbon category  $\mathcal{A}$ . These actions were modelled in Sections 2.2.3 and 2.2.4 by means of surface defects.

We now combine these two pictures to obtain a topological model for centred bimodules. The resulting graphical calculus involves three types of regions. The 1-dimensional region represents the linear category underlying the bimodule  $\mathcal{L}$ . This line lies at the interface between two 2-dimensional regions decorated by the central algebras acting on  $\mathcal{L}$ , and their actions are represented diagrammatically by projecting planar diagrams onto the line, exactly as in the planar case. The 3-dimensional regions adjacent to the surface defects are decorated by the ribbon category  $\mathcal{A}$ , whose action on the centred bialgebras is again encoded by projection from the 3-dimensional regions onto the corresponding 2-dimensional surface defects. These projections model the  $\mathcal{A}$ -central structures of the surface defects. Consequently, there are two induced actions of the 3-dimensional regions on the 1-dimensional region, obtained by first projecting onto either the  $\mathcal{C}$ - or the  $\mathcal{D}$ -surface defect and then onto the line. The centred structure precisely expresses the compatibility between these two actions.

We treat this construction in greater generality by allowing lines to lie at the intersection of an arbitrary number of surface defects. Let  $C_n = \mathbb{R}^2 \times [0, 1]$  be a cylinder endowed with the stratification given, in cylindrical coordinates, by

$$s(r, \theta, z) = \begin{cases} 1 & \text{if } r = 0; \\ 2 & \text{if } r \neq 0 \text{ and } \theta = \frac{2\pi k}{n}, \text{ for } k = 1, \dots, n; \\ 3 & \text{otherwise.} \end{cases}$$

That is,  $C_n$  is a cylinder with  $n$  “radial” surface defects meeting at a vertical line defect, as in left-hand side of Figure 2.6. We will refer to this space as the *stratified cylinder*. Its top and bottom basis are stratified disks as in the right-hand side of Figure 2.6, that we denote by  $\mathbb{D}_n$ . The orientation of the line stratum induces a partial ordering of both the bulk regions and the surface strata. We suppose that the surface strata are oriented in such a way that the outward normals point anticlockwise with respect to the orientation of the line stratum. We label the bulk regions with ribbon categories  $\mathcal{A}_1, \dots, \mathcal{A}_n$ ; the surface regions with  $(\mathcal{A}_{i+1}, \mathcal{A}_i)$ -central algebras  $\mathcal{D}_i$ , with  $i \in \{1, \dots, n\}$ ; and the defect itself with a pointed  $\text{tr}_{\mathcal{A}_1}(\mathcal{D}_1 \boxtimes_{\mathcal{A}_2} \boxtimes \dots \boxtimes_{\mathcal{A}_n} \mathcal{D}_n)$ -module  $\mathcal{L}$ , that we fix for the rest of the section.

Fix a deformation retraction  $p$  of  $C_n$  onto the union of its surface and line defects. We consider the following class of stratified graphs, depending on whether the central algebras are pivotal or not:

**Definition 2.2.23.** (i) A *ribbon-cyclic-linear stratified graph* is a stratified graph  $\Gamma \hookrightarrow C_n$  such that:

- the restriction of the retraction  $r$  to the endpoints is injective;
- the edges lying in the 2- and 3-dimensional regions are oriented, but not those lying in the defect;

- the edges adjacent to a coupon  $v$  belong to a region whose codimension differs by at most one from that of the region containing  $v$ ;
- edges, circles and vertices lying in the 2- and 3-dimensional regions are endowed with framing, satisfying the same conditions as in Definition 2.2.11, with the exception that the canonical framing here is given by the vector fields  $(r, \theta, z)$  determined by the polar coordinates.

(ii) A *ribbon-polarised-linear stratified graph* is defined analogously, by asking the framings and the edges on the 2- and 3-dimensional regions to satisfy the conditions from Definition 2.2.18.  $\diamond$

To simplify the exposition, we will focus for the rest of the section on ribbon-cyclic-linear graphs. The theory for ribbon-polarised-linear graphs is completely analogous. We consider these graphs up to stratified isotopies of the stratified cylinder in the class of ribbon-cyclic linear stratified graphs.

**Definition 2.2.24.** The *free centred bimodule category on  $\mathcal{L}$*  is the category  $\text{CENTRED}(\mathcal{L})$  given by:

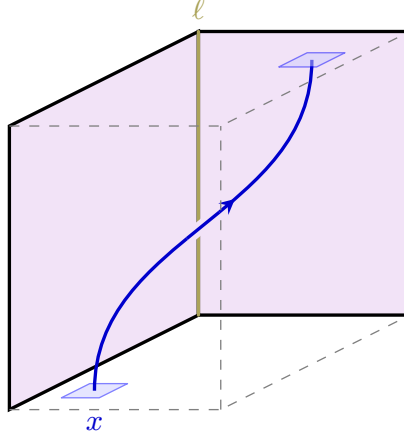
- objects: finite collections  $\{(p_i, \varepsilon_i, x_i)\}_i$ , where  $p_i$  is a framed point in  $\mathbb{D}_n$ ,  $\varepsilon_i \in \{\emptyset, \uparrow, \downarrow\}$  is an orientation (which is trivial if and only if  $p_i$  lies in the defect line) and  $x_i$  is an object from the category labelling the region where  $p_i$  lies;
- morphisms: a morphism between  $a = \{(p_i, \varepsilon_i, x_i)\}_i$  and  $b = \{(q_j, \delta_j, y_j)\}_j$  is a linear combination of isotopy classes of  $\mathcal{L}$ -coloured ribbon-cyclic-linear graphs in  $C_n$  whose endpoints match  $a$  and  $b$  on the bottom and top bases, respectively.

As usual, the composition is defined by stacking diagrams and rescaling the vertical coordinate. The distinguished object of  $\text{CENTRED}(\mathcal{L})$  is the empty configuration.  $\diamond$

**Proposition 2.2.25.** *The category  $\text{CENTRED}(\mathcal{L})$  has a canonical structure of left module over  $\text{tr}_{\text{Rib}(\mathcal{A}_1)}(\text{Piv}(\mathcal{D}_1) \boxtimes_{\text{Rib}(\mathcal{A}_2)} \boxtimes \cdots \boxtimes_{\text{Rib}(\mathcal{A}_n)} \text{Piv}(\mathcal{D}_n))$ .*

*Proof.* The category  $\text{CENTRED}(\mathcal{L})$  is canonically a pointed  $\text{Piv}(\mathcal{D}_1) \boxtimes \cdots \boxtimes \text{Piv}(\mathcal{D}_n)$ -module with action given by inserting planar diagrams radially on the surface defects. Moreover, this action is centred with respect to the action of the bulk regions on the surfaces (given, again, by inserting 3-dimensional diagrams) with centred structure rep-

resented by the diagram



It follows from Corollary 2.1.26 that  $\text{CENTRED}(\mathcal{L})$  is a left module over the category  $\text{tr}_{\text{Rib}(\mathcal{A}_1)}(\text{Piv}(\mathcal{D}_1) \boxtimes_{\text{Rib}(\mathcal{A}_2)} \boxtimes \cdots \boxtimes_{\text{Rib}(\mathcal{A}_n)} \text{Piv}(\mathcal{D}_n))$ .  $\square$

We define an evaluation functor

$$\text{ev}_{\mathcal{L}}^{\text{CENTRED}}: \text{CENTRED}(\mathcal{L}) \rightarrow \mathcal{L}$$

by the following process. Fix a deformation retraction  $p$  of the stratified cylinder  $C_n$  onto its surface and line defects. Given a coloured ribbon graph  $\Gamma$  in  $C_n$ , we evaluate it in  $\mathcal{L}$  in several steps:

- S0. Generic representative.** Replace  $\Gamma$  with a stratified isotopically equivalent graph which is in generic position with respect to  $p$ . This means that the retraction of  $\Gamma$  along  $p$  contains no triple points, that double points correspond to transverse intersections and that pairs of vertices are never identified and never sent to the defect line. For this purpose, the line defect is considered as an edge.
- S1. Retract the graph.** Retract the ribbon graph along  $p$  onto the union of surface and line defects. Consider the image  $p(\Gamma)$  as a graph in  $\bigsqcup_i d_i \sqcup \bigsqcup_j l_j$  by treating each introduced double point and each intersection with the line defect as a new vertex. We denote by  $p(\Gamma)_0$  and  $p(\Gamma)_1$  the new sets of edges and vertices, respectively.
- S2. Update edges' decoration.** For each  $e \in p(\Gamma)_1$  that has been pushed from a bulk region to a surface defect, let  $x_e$  be the decoration of the corresponding edge  $p^{-1}(e)$  in  $\Gamma$ . We label  $e$  with  $F_i(x_e)$ , where  $F_i$  is the central functor associated with the corresponding surface defect. Edges which were originally in one of the defects do not have to be updated.
- S3. Update vertices' decorations.** We distinguish several cases:

- (A) vertices in a surface defect coming from the intersection between two edges, at least one of them coming from a bulk region, are decorated with the corresponding half-braiding, using the central structure of the underlying central algebra;
- (B) vertices coming from the intersection of an edge from the bulk region and an edge running along the line defect are labelled with the underlying central structure (cf. Corollary 2.1.26). Namely, if these edges are labelled, respectively, with  $x_e$  and  $\ell$ , we decorate the intersection between them with

$$\eta_{x_e, \ell}: F_{i+1}(x_e) \triangleright \ell \rightarrow \ell \triangleleft F_i(x_e)$$

or its inverse, depending on the relative position of the vertices. When there is no edge running along the defect line, we will treat the defect itself as an edge coloured with  $\text{Dist}_{\mathcal{L}}$ .

- (C) vertices which were originally in one of the defects do not have to be updated, even if they involved bulk edges.

**S4. Separate vertices and evaluate.** Apply a planar isotopy so that each vertex of  $p(\Gamma)$  lies at a different height, even those on distinct surfaces. Finish by evaluating  $p(\Gamma)$  following the same rules as in Proposition 2.2.10, resulting in a morphism of  $\mathcal{L}$ . Here we are using that the evaluation extends from 2 to  $n$  adjacent surface defects because the actions of each of the defect wall factors commute.

In what follows, we will write  $\Gamma_1 \sim \Gamma_2$  whenever  $\text{ev}_{\mathcal{L}}^{\text{CENTRED}}(\Gamma_1) = \text{ev}_{\mathcal{L}}^{\text{CENTRED}}(\Gamma_2)$ , for some choice of isotopy representatives and retractions.

The rest of the section is devoted to proving that this evaluation functor is well-defined. Namely, we need to show that  $\text{ev}_{\mathcal{L}}^{\text{CENTRED}}(\Gamma)$  does not depend on the choice of isotopy or retraction. We start by considering isotopies away from the defect line:

**Lemma 2.2.26.** *The evaluation functor  $\text{ev}_{\mathcal{L}}^{\text{CENTRED}}$  is invariant under isotopies supported in a ball whose retraction does not intersect the line defect.*

*Proof.* These are precisely isotopies happening on one side of the defect line. Therefore, the result follows directly from Proposition 2.2.10 and the isotopy invariance of both the Reshetikhin–Turaev evaluation functor and of the evaluation functor for surface defects defined in [BJ25].  $\square$

Proving isotopy invariance when there is some interaction with the defect is more involved. The following identities are the diagrammatic counterpart of the axioms for centred bimodule categories.

**Lemma 2.2.27.** *With the same notations as above, the following identities hold:*

1. *Left half-braiding:*

$$(2.2.17)$$

2. *Right half-braiding:*

$$(2.2.18)$$

3. *Monoidal structure:*

$$(2.2.19)$$

4. *Naturality in the first variable:*

$$\sim \quad (2.2.20)$$

5. *Naturality in the second variable:*

$$\sim \quad (2.2.21)$$

*Proof.* Applying the evaluation process defined above to each of the identities yields the corresponding commutative diagram in  $\mathcal{L}$  (cf. Proposition 2.1.20).  $\square$

We also need the following version of the Reidemeister moves II and III with one of the strands running along the line defect:

**Lemma 2.2.28.** *The evaluation process defined above is invariant under the following version of Reidemeister moves II and III:*

1. *Defect Reidemeister move II:*

$$\begin{array}{c} H_1(x) \quad \ell \\ \text{[Diagram: Two vertical lines, blue and orange]} \\ H_1(x) \quad \ell \end{array} \sim \begin{array}{c} H_1(x) \quad \ell \\ \text{[Diagram: Wavy blue line crossing orange line at } \eta_{x,\ell}^{-1} \text{ and } \eta_{x,\ell}] \\ H_1(x) \quad \ell \end{array} . \quad (2.2.22)$$

2. *Defect Reidemeister move III:*

$$\begin{array}{c} \ell \quad H_2(y) \quad H_2(x) \\ \text{[Diagram: Blue line crossing orange line l at } \eta_{y,\ell} \text{ and } \eta_{x,\ell}] \\ H_1(x) \quad H_1(y) \quad \ell \end{array} \sim \begin{array}{c} \ell \quad H_2(y) \quad H_2(x) \\ \text{[Diagram: Wavy blue line crossing orange line l at } \eta_{x,\ell} \text{ and } \eta_{y,\ell}] \\ H_1(x) \quad H_1(y) \quad \ell \end{array} . \quad (2.2.23)$$

*Proof.* The first claim follows from the invertibility of  $\eta_{x,\ell}$ . To pass a braiding through the line defect, we use the monoidal structure to reduce the argument to naturality as follows:

$$\begin{array}{c} \ell \quad H_2(y) \quad H_2(x) \\ \text{[Diagram: Blue line crossing orange line l at } \eta_{y,\ell} \text{ and } \eta_{x,\ell}] \\ H_1(x) \quad H_1(y) \quad \ell \end{array} \sim \begin{array}{c} \ell \quad H_2(y) \quad H_2(x) \\ \text{[Diagram: Blue line with loop and crossing, labeled } (J_1)_{y,x}, \eta_{y \otimes x, \ell}, (J_2)_{y,x}^{-1}] \\ H_1(x) \quad H_1(y) \quad \ell \end{array}$$

$$\begin{array}{ccc}
\sim & \begin{array}{c} \text{Diagram 1: A blue strand enters from the bottom left, passes through a crossing labeled } (J_1)_{y,x}, \text{ then crosses a vertical orange line labeled } \ell. \text{ After the line, it passes through a crossing labeled } \eta_{y \otimes x, \ell} \text{ and ends at a crossing labeled } (J_2)_{y,x}^{-1}. \text{ The top boundary is labeled } H_2(y) H_2(x) \text{ and the bottom boundary is labeled } H_1(x) H_1(y). \end{array} & \sim & \begin{array}{c} \text{Diagram 2: A blue strand enters from the bottom left, passes through a crossing labeled } (J_1)_{x,y}, \text{ then crosses a vertical orange line labeled } \ell. \text{ After the line, it passes through a crossing labeled } \eta_{x \otimes y, \ell} \text{ and ends at a crossing labeled } (J_2)_{x,y}^{-1}. \text{ The top boundary is labeled } H_2(y) H_2(x) \text{ and the bottom boundary is labeled } H_1(x) H_1(y). \end{array} & \sim
\end{array}$$

where the first equality is the compatibility of  $\eta$  with the monoidal structure, the second and the last are planar (non-stratified) skein relations and the third is a consequence of naturality.  $\square$

**Corollary 2.2.29.** *The following equalities hold:*

1. **Left half-braiding 2.0:**

$$\begin{array}{ccc}
\begin{array}{c} \text{Diagram 1: A blue strand enters from the top left, crosses a vertical orange line labeled } \ell. \text{ After the line, it passes through a crossing labeled } \eta_{x, d_1 \triangleright \ell}^{-1} \text{ and ends at a crossing labeled } \text{id}_{d_1 \triangleright \ell}. \text{ The top boundary is labeled } H_1(x) \text{ and the bottom boundary is labeled } H_2(x). \end{array} & \sim & \begin{array}{c} \text{Diagram 2: A blue strand enters from the top left, crosses a vertical orange line labeled } \ell. \text{ After the line, it passes through a crossing labeled } \eta_{x, \ell}^{-1} \text{ and ends at a crossing labeled } \text{id}_{d_1 \triangleright \ell}. \text{ The top boundary is labeled } H_1(x) \text{ and the bottom boundary is labeled } H_2(x). \end{array} & (2.2.24)
\end{array}$$

2. **Right half-braiding 2.0:**

$$\begin{array}{ccc}
\begin{array}{c} \text{Diagram 1: A blue strand enters from the bottom left, crosses a vertical orange line labeled } \ell. \text{ After the line, it passes through a crossing labeled } \eta_{x, \ell \triangleleft d_1} \text{ and ends at a crossing labeled } \text{id}_{\ell \triangleleft d_1}. \text{ The top boundary is labeled } H_2(x) \text{ and the bottom boundary is labeled } H_1(x). \end{array} & \sim & \begin{array}{c} \text{Diagram 2: A blue strand enters from the bottom left, crosses a vertical orange line labeled } \ell. \text{ After the line, it passes through a crossing labeled } \eta_{x, \ell} \text{ and ends at a crossing labeled } \text{id}_{\ell \triangleleft d_1}. \text{ The top boundary is labeled } H_2(x) \text{ and the bottom boundary is labeled } H_1(x). \end{array} & (2.2.25)
\end{array}$$

*Proof.* This follows by pre- and post-composing the equalities (2.2.17) and (2.2.18) and applying Reidemeister move II.  $\square$

**Lemma 2.2.30.** *The morphism  $\eta_{x,\ell}$  is natural in  $\mathcal{L}$ , so that*

(2.2.26)

*Proof.* By the construction of the evaluation morphism, we can factor the coupon labelled  $f$  as follows:

The result then follows from (2.2.21), (2.2.17) and (2.2.25).  $\square$

Note that (2.2.26) holds regardless of the orientation of the bulk defect.

**Proposition 2.2.31.** *The evaluation functor*

$$\text{ev}_{\mathcal{L}}^{\text{CENTRED}} : \text{CENTRED}(\mathcal{L}) \rightarrow \mathcal{L}$$

*is well-defined.*

*Proof.* First note that Remark 2.2.17 implies that replacing a surface defect by its dual and switching orientations does not affect the evaluation, so there is no loss of generality in assuming that all the outward normals point anticlockwise. Next, we show that the evaluation is invariant under isotopy.

Invariance under isotopies passing cups, caps and crossings through surface defects follows from [BJ25, Prop. 2.6]. For invariance under isotopies impacting the line defect, this follows directly from Lemmas 2.2.27, 2.2.28, 2.2.30 and Corollary 2.2.29.  $\square$

## 2.3 Defect theories

The formalism developed in the previous section allows us to extend the string-net and skein theories introduced in Chapter 1 to stratified surfaces and 3-manifolds. We extend the graphical calculus for bimodules, central algebras and centred bimodules to decorated manifolds, and we use the evaluation functors to produce skein relations around defects. Based on these spaces, we define categorified TQFTs assigning defect versions of string-net and skein categories to closed manifolds, and bimodule functors to cobordisms between them.

### 2.3.1 Decorated stratified manifolds

We define *decorated* manifolds as stratified manifolds whose strata are labelled with algebraic data. A basis for the topology of these surfaces is determined by non-stratified cubes, together with the stratified cubes and cylinders supporting the graphical calculus for central algebras and centred bimodules. In order to determine the algebraic data attached to such a manifold, orientations play an important role, so we define:

**Definition 2.3.1.** An *oriented stratified manifold* is a stratified manifold  $M$  together with a choice of orientation for each stratum. When  $\dim(M_p) = \dim(M)$ , we will always suppose that the orientation on  $M_p$  is the one induced by  $M$ . For lower dimensional strata, these orientations may be different.  $\diamond$

The simplest case that we will treat is that of *walled surfaces*. These are surfaces whose strata consist of a collection of mutually non-intersecting smooth lines on them:

**Definition 2.3.2. (i)** A *walled surface* is an oriented stratified surface  $(S, \phi)$  with

$$\phi: S \rightarrow \{1 \leq 2\},$$

such that  $S_2 = \bigsqcup_{i=1}^k c_i$  is a collection of mutually disjoint connected open surfaces and

$S_1 = \bigsqcup_{i=1}^n m_i$  is a collection of mutually disjoint smooth curves, each being a connected component of either the common boundary of two distinct 2-dimensional strata, or the boundary of a single 2-dimensional stratum (see Figure 2.5).

**(ii)** A *pivotal walled surface* is an oriented walled surface endowed with a framing compatible with its orientation.  $\diamond$

Given a walled (resp. framed walled) surface  $S$ , the algebraic data associated with its strata is determined as follows. To each 2-dimensional stratum  $c_i$  we associate a pivotal

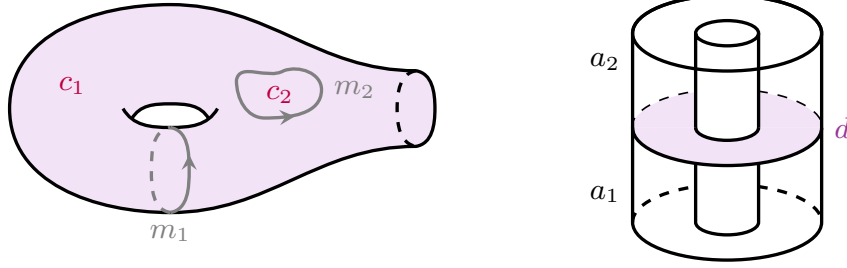


Figure 2.5: **Left:** a walled surface with two 2-dimensional strata  $c_1$  and  $c_2$  and two 1-dimensional strata  $m_1$  and  $m_2$ . The algebraic data attached to this surface consists of two pivotal categories  $\mathcal{C}_1$  and  $\mathcal{C}_2$  together with a  $(\mathcal{C}_1, \mathcal{C}_1)$ -bimodule  $\mathcal{M}_1$  and a  $(\mathcal{C}_2, \mathcal{C}_1)$ -bimodule  $\mathcal{M}_2$ . **Right:** a walled 3-manifold with a vertical surface defect and two bulk regions. The algebraic label corresponding to this manifold is an  $(\mathcal{A}_2, \mathcal{A}_1)$ -central algebra  $\mathcal{D}$ .

(resp. rigid) category  $\mathcal{C}_i$ . The algebraic data attached to the 1-dimensional strata depend on their orientation as well as the orientation and decoration of the adjacent surfaces in the following way. Let  $m_i$  be an oriented smooth curve lying on the common boundary between  $c_{i_1}$  and  $c_{i_2}$ . We decorate this curve with a pointed bimodule category  $\mathcal{M}_i$  over  $\mathcal{C}_{i_1}$  and  $\mathcal{C}_{i_2}$ . To determine whether the actions are on the left or the right, we use the following criterion: if the orientation of  $m_i$  coincides with the orientation induced by  $c_{i_1}$  using the outward vector normal to its boundary, then  $\mathcal{C}_{i_1}$  acts on the left on  $\mathcal{M}_i$ ; otherwise, it acts on the right. The same applies for  $c_{i_2}$ .

**Definition 2.3.3.** A *decorated walled surface* is a walled surface with the additional data of a rigid or pivotal category for each 2-dimensional stratum and a bimodule category, with actions determined as above, for each 1-dimensional stratum.  $\diamond$

Analogously, a walled 3-manifold is a stratified 3-manifold whose strata consists of a collection of non-intersecting smooth surfaces:

**Definition 2.3.4. (i)** A *walled 3-manifold* is an oriented stratified 3-manifold  $(M, \phi)$  with

$$\phi: M \rightarrow \{2 \leq 3\},$$

such that  $M_3 = \bigsqcup_{i=1}^k a_i$  is a collection of mutually disjoint connected open 3-submanifolds,

and  $M_2 = \bigsqcup_{i=1}^n d_i$  is a collection of mutually disjoint smooth surfaces, each being a connected component of either the common boundary of two distinct 3-dimensional strata, or the boundary of a single 3-dimensional stratum (see Figure 2.5).

(ii) A *framed walled 3-manifold* is a walled 3-manifold whose 2-dimensional strata are endowed with a framing.

(iii) A marked (framed) walled surface is a (framed) walled surface endowing with a marking  $\mathcal{B}$ . Note that the marked components may intersect different strata.  $\diamond$

Let  $M$  be a walled 3-manifold. We label each 3-dimensional stratum  $a_i$  with a ribbon category  $\mathcal{A}_i$ . If  $d_i$  is a 2-dimensional stratum lying in the common boundary between  $\mathcal{A}_{i_1}$  and  $\mathcal{A}_{i_2}$ , then we decorate  $d_i$  with a central algebra  $\mathcal{D}_i$  whose source and target are determined as in the 2-dimensional case. Namely, if the orientation of  $d_i$  coincides with the orientation induced by  $a_{i_1}$ , then  $\mathcal{A}_{i_1}$  is in the source of  $\mathcal{D}_i$ ; otherwise it is in the target. Since we are assuming that all the 3-dimensional strata have the same orientation, this amounts to saying that the outward normal vector field of  $d_i$  (determined by its orientation) points from the source to the target region (see Figure 2.3). The central algebras are supposed to be pivotal if  $M$  is a walled manifold, and rigid if it is framed.

**Definition 2.3.5.** A *decorated walled 3-manifold* is a walled 3-manifold with the additional data of a ribbon category for each 3-dimensional stratum and a central algebra, with source and target determined as above, for each 2-dimensional stratum.  $\diamond$

In the most general case, we will consider more complicated stratifications on 3-manifolds, consisting of several surface defects meeting at defect lines:

**Definition 2.3.6. (i)** A *walled-and-edged 3-manifold* is an oriented stratified 3-manifold  $(M, \phi)$  with

$$\phi: M \rightarrow \{1 \leq 2 \leq 3\},$$

such that

- $M_3 = \bigsqcup_{i=1}^k a_i$  is a collection of mutually disjoint connected open 3-dimensional submanifolds, which we refer to as *bulk regions*;
- $M_2 = \bigsqcup_{i=1}^m d_i$  is a collection of mutually disjoint connected embedded surfaces, each one lying either at the common boundary of two distinct 3-dimensional strata, or the boundary of a single 3-dimensional stratum;
- $M_1 = \bigsqcup_{i=1}^r l_i$  is a collection of mutually disjoint smooth curves, each one being a connected component of either the common boundary of two distinct 2-dimensional strata, or the boundary of a single 2-dimensional stratum.

(ii) A *framed walled-and-edged 3-manifold* is a walled-and-edged 3 manifold whose 2-dimensional strata are endowed with a framing.  $\diamond$

The algebraic data attached to a walled-and-edged 3-manifold  $M$  is determined as follows. We associate a ribbon category  $\mathcal{A}_i$  to each bulk region  $a_i$ . To every 2-dimensional

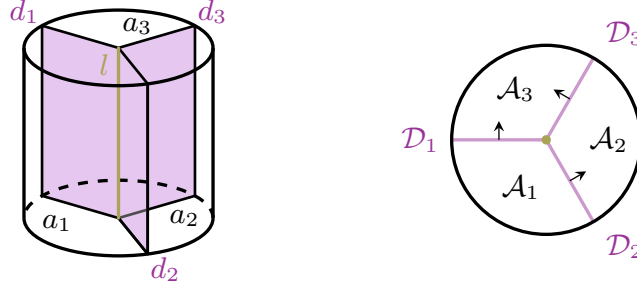


Figure 2.6: **Left:** a walled-and-edged cylinder. **Right:** since the normal vector of  $d_1$  points clockwise and the normal vectors of  $d_2$  and  $d_3$  point anticlockwise, the algebraic data of the 1-dimensional stratum is a pointed  $\text{tr}_{\mathcal{A}_1}(\mathcal{D}_1^\vee \boxtimes_{\mathcal{A}_2} \mathcal{D}_2 \boxtimes_{\mathcal{A}_3} \mathcal{D}_3)$ -module.

stratum  $d_i$ , we associate a central algebra  $\mathcal{D}_i$  whose source and target are determined by the labels of the adjacent bulk regions, following the same rule as in the case of walled 3-manifolds. Finally, let  $l_i$  be a 1-dimensional stratum. The orientation of  $l_i$  induces a cyclic ordering of the 2- and 3-dimensional strata adjacent to  $l_i$ , say  $(a_{i_1}, \dots, a_{i_s})$  and  $(d_{i_1}, \dots, d_{i_s})$ , so that  $\mathcal{D}_{i_k}$  is either an  $(\mathcal{A}_{i_k}, \mathcal{A}_{i_{k+1}})$ -central algebra or an  $(\mathcal{A}_{i_{k+1}}, \mathcal{A}_{i_k})$ -central algebra, depending on the orientations, for each  $k = 1, \dots, s$ . Using the right-hand screw rule, set

$$\tilde{\mathcal{D}}_{i_k} = \begin{cases} \mathcal{D}_{i_k} & \text{if the outward normal to } d_{i_k} \text{ points anticlockwise,} \\ \mathcal{D}_{i_k}^\vee & \text{otherwise.} \end{cases}$$

Then  $\tilde{\mathcal{D}}_{i_k}$  is an  $(\mathcal{A}_{i_{k+1}}, \mathcal{A}_{i_k})$ -central algebra for each  $k$ , and we label  $l_i$  with the data of a pointed  $\text{tr}_{\mathcal{A}_{i_1}}(\tilde{\mathcal{D}}_{i_1} \boxtimes_{\mathcal{A}_{i_2}} \dots \boxtimes_{\mathcal{A}_{i_s}} \tilde{\mathcal{D}}_{i_s})$ -module (see Figure 2.6).

**Definition 2.3.7.** A *decorated 3-manifold* is a walled-and-edged 3-manifold with the additional data of a ribbon category for each bulk region, a central algebra for each 2-dimensional stratum and a trace-module for each 1-dimensional stratum, chosen as explained above.  $\diamond$

### 2.3.2 Defect string-net theories

We extend the framed string-net theory introduced in Section 1.3.2 to framed stratified surfaces. A similar construction can be carried out in the case of oriented stratified surfaces, using bimodule categories over pivotal categories instead of rigid ones. We leave the details to the reader since the construction is completely analogous.

**Definition 2.3.8.** Let  $S$  be a framed walled surface. A *polarised stratified graph* in  $S$  is

a stratified graph  $\Gamma \hookrightarrow S$  whose vertices and edges lying in 2-dimensional regions satisfy the conditions of Definition 1.3.5.  $\diamond$

Roughly, a polarised stratified graph on a framed walled surface consists of a collection of polarised graphs lying on the 2-dimensional strata whose edges may be attached to vertices in the defect line.

Let  $(S, f, \mathcal{B})$  be a marked framed walled surface and write  $f = (X, Y)$  for its framing. Generic stratified rectangles are defined exactly as in Definition 1.3.6, replacing  $\mathbb{R} \times [0, 1]$  by  $E$  and considering stratified embeddings instead. For a marked surface, we define colourings of graphs by assigning objects from the underlying categories to edges, integer labels to edges lying in the 2-dimensional strata, and morphisms to vertices, following the rules explained in the previous section. More precisely, a colouring is well defined if the pullback along every generic rectangle and every generic stratified rectangle is a well-defined coloured polarised graph in  $E$ . A boundary condition is defined as in the non-stratified case as a (possibly empty) finite collection

$$B \subset \bigcup_{C \in \mathcal{B}} C \subset \partial S,$$

together with an assignment  $\nu_B$  of an object and an integer to each point lying in a 2-dimensional stratum, and an object to each point lying on a defect line, chosen from the category labelling that stratum.

Write  $\mathcal{M} = \{\mathcal{M}_i, \mathcal{C}_k\}_{i,k}$  for the algebraic data attached to  $S$ . Given a commutative ring  $\mathbb{k}$ , we set  $\text{StrGraph}_{\mathcal{M}}(S, f, \nu_B)$  for the  $\mathbb{k}$ -linear space spanned by  $\mathcal{M}$ -coloured polarised stratified graphs on  $S$  with boundary condition  $\nu_B$ . Let

$$\text{nStrGraph}_{\mathcal{M}}(S, f, \nu_B) \subset \text{StrGraph}_{\mathcal{M}}(S, f, \nu_B)$$

be the subspace of null-graphs. These are linear combinations of graphs coinciding outside a generic rectangle or a generic stratified rectangle and such that the evaluation of the restriction to such rectangle is 0 in the corresponding rigid category (for rectangles) or bimodule category (for stratified rectangles).

**Definition 2.3.9.** The *defect framed  $\mathcal{M}$ -string-net space* of  $(S, f)$  with boundary condition  $\nu_B$  is the quotient

$$\text{SN}_{\mathcal{M}}^{\text{def,fr}}(S, f, \nu_B) = \text{StrGraph}_{\mathcal{M}}(S, f, \nu_B) / \text{nStrGraph}_{\mathcal{M}}(S, f, \nu_B). \quad \diamond$$

Let  $I_n$  and  $\mathbb{S}_n^1$  denote the  $n$ -pointed interval and  $n$ -pointed circle, respectively, that is, the interval and the circle equipped with marked points  $q_1, \dots, q_n$ . A 2-framing on either of these spaces consists of a 2-framing of the complement of the marked points, which is a disjoint union of intervals whenever  $n > 0$ . Let  $K_n$  denote either  $I_n$  or  $\mathbb{S}_n^1$ . We label each connected component of  $K_n \setminus \{q_1, \dots, q_n\}$  by a rigid monoidal category, and each marked point  $q_i$  by a bimodule category over the rigid categories labelling the adjacent components. We denote this collection of algebraic data by  $\mathcal{M}$ .

**Definition 2.3.10.** The *stratified cylinder category*  $\text{Cyl}_{\mathcal{M}}(K_n, f)$  is the category with:

- objects: finite configurations of points  $B = \{p_1, \dots, p_k\}$  together with a labelling  $\nu$  assigning to each point  $p_i$  an object of the underlying bimodule category if  $p_i$  is a marked point, or an object of the underlying rigid category together with an integer label if it lies in a 2-dimensional stratum;
- morphisms: given two objects  $(B_0, \nu_0)$  and  $(B_1, \nu_1)$ , we set

$$\text{Hom}_{\text{Cyl}_{\mathcal{M}}(K_n, f)}((B_0, \nu_0), (B_1, \nu_1)) := \text{SN}_{\mathcal{M}}^{\text{def, fr}}(K_n, f, \nu_{0,1}),$$

where the boundary condition is supported on the set  $B_{01} := (B_0 \times \{0\}) \cup (B_1 \times \{1\})$ , with value map  $\nu_{0,1}(-, i) := \nu_i$ .  $\diamond$

As in Section 1.3.2, we set

$$\text{SN}_{\mathcal{M}}^{\text{def, fr}}(K_n, f) := \text{Cyl}_{\mathcal{M}}(K_n, f).$$

Let  $(S, f, \mathcal{B})$  be a marked framed walled surface. Write  $\mathcal{B} = \{C_1, \dots, C_m\}$  for the intervals and circles marked on the boundary. The intersection of each  $C_i$  with the defect line meeting the boundary determines both a pointing and a colouring for  $C_i$ . Moreover, the framing  $f$  determines a framing  $f_i$  on it. Under these considerations, we have:

**Definition 2.3.11.** The *defect framed string-net bimodule* on  $S$  is the functor

$$\text{SN}_{\mathcal{M}}^{\text{def, fr}}(S, f, \nu_B): \bigotimes_{i=1}^m \text{SN}_{\mathcal{M}}^{\text{def, fr}}(C_i, f_i) \rightarrow \text{Vect}_{\mathbb{k}}$$

defined on objects by

$$((B_1, \nu_1), \dots, (B_m, \nu_m)) \mapsto \text{SN}_{\mathcal{M}}^{\text{def, fr}}(S, f, \nu_{1, \dots, m}),$$

where  $\nu_{1, \dots, m}$  is defined by  $\nu_{1, \dots, m}(-, i) = \nu_i$ . The action of morphisms is defined by gluing rectangles and cylinders along tubular neighbourhoods of the marked components.  $\diamond$

Note that, while the framed string-net bimodule of Section 1.3.2 is a bimodule functor over the string-net spaces associated with a single rigid category  $\mathcal{C}$ , the defect framed string-net bimodule relates the theories associated with different categories  $\mathcal{C}_i$  through line defects.

**Definition 2.3.12.** Let  $\mathcal{M} = \{\mathcal{M}_i, \mathcal{C}_i\}_{i=1, \dots, n}$  be a collection of categories such that the  $\mathcal{C}_i$ 's are rigid categories and, for each  $i$ ,  $\mathcal{M}_i$  is a  $(\mathcal{C}_i, \mathcal{C}_{i+1})$ -bimodule category. The category  $\text{Cob}_{2,1}^{\text{fr, mkd, dec}}$  of *framed decorated marked cobordisms* has

- objects: disjoint unions of 2-framed pointed intervals and circles labelled with the  $\mathcal{C}_i$ 's and the  $\mathcal{M}_i$ 's as explained above;

- morphisms: a morphism between  $X$  and  $Y$  is an equivalence class of framed walled cobordisms decorated with  $\mathcal{M}$  and such that

$$\bigcup_{C \in \mathcal{B}} C \cong \overline{X} \sqcup Y$$

as framed manifolds.

The composition and the monoidal structure defined as in the non-decorated case (cf. Definition 1.3.4).  $\diamond$

The proof of [Wal06, Theorem 4.4.2] extends to walled cobordisms easily, yielding:

**Theorem 2.3.13.** *The defect framed string-net construction provides a framed  $(2,1)$ -TQFT*

$$\text{SN}_{\mathcal{M}}^{\text{def,fr}} : \text{Cob}_{2,1}^{\text{fr,mkd,dec}} \rightarrow \text{BIMOD}$$

assigning the stratified cylinder categories to pointed intervals and circles, and the defect framed string-net bimodule to a marked framed surface.  $\blacksquare$

### 2.3.3 Defect skein theory

We complete the story by extending skein theory to 3-dimensional stratified manifolds. We obtain notions of defect skein modules, algebras and categories generalizing the non-defect versions. These are defined, as usual, by ribbon graphs embedded in an oriented stratified 3-manifold modulo skein relations, produced using some algebraic data. In particular, we have now three different types of skein relations, depending on whether we apply them on cubes intersecting only the 3-dimensional strata, or intersecting surface or line defects. Relations in the bulk regions are induced by ribbon categories, while surface defects and line defects are modelled locally by central algebras and centred bimodules, respectively, as explained in the previous sections.

Let  $M$  be a decorated walled-and-edged oriented or framed 3-manifold and write  $\mathcal{S}$  for its decoration. That is,  $\mathcal{S}$  is a collection of ribbon categories, central algebras and centred bimodules labelling the strata of  $M$ , following the rules given in Section 2.3.1. The notions of ribbon-polarised-linear graphs and ribbon-cyclic-linear graphs from Section 2.2.5 extends in an obvious way to walled-and-edged 3-manifolds. We refer to graphs in any of these classes simply as *stratified ribbon graphs*. We have a well-defined notion of  $\mathcal{C}$ -colouring assigning objects and morphisms to edges and vertices as in the previous cases. Boundary conditions and stratified generic cubes are defined in exactly the same way as before, so we omit all these definitions here.

**Definition 2.3.14. (i)** The *relative defect  $\mathcal{S}$ -skein module*  $\text{SkMod}_{\mathcal{S}}^{\text{def}}(M, \nu_B)$  with boundary condition  $\nu_B$  is the  $\mathbb{k}$ -module spanned by coloured stratified ribbon graphs embedded in  $M$  and compatible with  $\nu_B$ , modulo skein relations. These relations are defined as in Definition 1.3.23, using the evaluation functors from Sections 2.2.3, 2.2.4 and 2.2.5 whenever the corresponding generic cube intersects a surface or a line defect.

(ii) When  $B = \emptyset$ , we set

$$\text{SkMod}_S^{\text{def}}(M) := \text{SkMod}_S^{\text{def}}(M, \nu_\emptyset)$$

and we call this space the *defect  $S$ -skein module* of  $M$ .  $\diamond$

Recall that, when dealing with stratified manifolds, we allow the strata to intersect the boundary transversally. In particular, if  $M$  is a walled-and-edged 3-manifold, the intersection of its strata with  $\partial M$  induces a stratification on  $\partial M$  itself, consisting of a collection of embedded smooth lines whose closures intersect at some meeting points. Lines correspond to the intersection of surface defects with  $\partial M$ , whereas the meeting points are the points where defect lines hit the boundary. We define:

**Definition 2.3.15.** A *decorated surface* is a surface  $S$  with the additional data of

- a stratification of  $S$  consisting of a collection of non-intersecting smoothly embedded oriented (or 2-framed) curves whose closures meet at a finite collection of oriented points forming the 1-strata, and may intersect the boundary transversally;
- the data of a ribbon category for each 2-dimensional region, a pivotal or rigid central algebra for each line, and a centred bimodule (in the form of Corollary 2.1.26) for each point, chosen in a compatible way.

Compatibility means the cylinder  $S \times I$  is an oriented or framed walled-and-edged 3-manifold, with orientation or framing induced by those of  $S$ , and we can apply the same rules as in Section 2.3.1 to determine the algebraic data.  $\diamond$

Note that decorated surfaces in the sense of the previous definition are more general than decorated walled-surfaces, since in the latter case the stratification contains no points and defect lines cannot meet.

**Definition 2.3.16. (i)** The *defect skein algebra* of a decorated surface  $S$  with underlying algebraic data  $\mathcal{S}$  is the algebra  $\text{SkAlg}_S^{\text{def}}(S)$  whose underlying module is the defect skein module  $\text{SkMod}_S^{\text{def}}(S \times I)$  with multiplication given by vertically stacking two copies of  $S \times I$  and rescaling the second coordinate.

**(ii)** The *defect skein category* of a decorated surface  $S$  with underlying algebraic data  $\mathcal{S}$  is the category  $\text{SkCat}_S^{\text{def}}(S)$  with:

- objects: pairs  $(B, \nu)$  where  $B$  is a finite configuration of framed directed points on  $S$ , and  $\nu$  is a value function assigning objects to these points, chosen from the region decorating the strata where they lie;
- morphisms: given two objects  $(B_0, \nu_0)$  and  $(B_1, \nu_1)$ , we set

$$\text{Hom}_{\text{SkCat}_S^{\text{def}}(S)}((B_0, \nu_0), (B_1, \nu_1)) := \text{SkMod}_S^{\text{def}}(S \times [0, 1], \nu_{0,1}),$$

where  $\nu_{0,1}$ , defined by  $\nu_{0,1}(-, i) = \nu_i$ , is supported in the set  $(B_0 \times \{0\}) \cup (B_1 \times \{1\})$ .

The composition is given by vertically stacking two copies of  $S \times [0, 1]$  and rescaling the second coordinate. The identity morphism of a given object  $X$  is given by a collection of strands running vertically in the direction of the marked points.  $\diamond$

**Definition 2.3.17.** Let  $M$  be a decorated 3-manifold with boundary  $\partial M \cong \bar{S}_1 \sqcup \Sigma_2$  and decoration  $\mathcal{S}$ . The *defect skein bimodule* of  $M$  is the functor

$$\begin{aligned} \text{Sk}_{\mathcal{S}}^{\text{def}}(M): \quad \text{SkCat}_{\mathcal{S}}^{\text{def}}(S_2) \boxtimes \text{SkCat}_{\mathcal{S}}^{\text{def}}(S_1)^{\text{op}} &\rightarrow \text{Vect} \\ ((B, \nu), (B', \nu')) &\mapsto \text{SkMod}_{\mathcal{S}}^{\text{def}}(M, \nu_{B \cup B'}). \end{aligned}$$

The action on morphisms is defined by stacking in a neighbourhood of  $\partial M$  diffeomorphic to  $(\bar{S}_1 \sqcup S_2) \times [0, 1]$ .  $\diamond$

The following theorem generalises Theorem 1.3.26 to decorated surfaces. The proof is essentially the same as in [Wal06].

**Theorem 2.3.18.** *Skein bimodules and skein categories define a symmetric monoidal functor*

$$\text{Sk}^{\text{def}}: \text{Cob}_{3,2}^{\text{dec}} \rightarrow \text{BIMOD},$$

i.e., a categorified 3-TQFT with defects.  $\blacksquare$

### 2.3.4 Defect skein theory and factorisation homology

Finally, we interpret the algebraic structures given in Section 2.1 and the topological theories from Sections 2.2 and 2.3 using the language of disk algebras and factorisation homology.

We will consider categories whose objects are disjoint unions of disks, whose 1-morphisms are embeddings between them, and whose 2-morphisms are isotopies between them. These categories will take different variants depending on the decoration and the extra structures that we put on our disks, but in all cases they come with a symmetric monoidal structure given by disjoint unions. Let  $\mathbb{D}\text{isk}$  be any of such categories (more precise definitions are given below).

**Definition 2.3.19.** A *disk algebra* in a higher symmetric monoidal category  $\mathcal{C}$  is a symmetric monoidal functor  $\mathcal{F}: \mathbb{D}\text{isk} \rightarrow \mathcal{C}$ .  $\diamond$

We are interested here in the case where  $\mathcal{C}$  is a category of linear categories. Let  $\text{CAT}_{\mathbb{k}}$  be the category of  $\mathbb{k}$ -linear categories. We study first the non-stratified setting:

**Definition 2.3.20. (i)** Let  $\mathbb{D}\text{isk}_n^{\text{or}}$  be the symmetric monoidal  $(2, 1)$ -category with:

- objects: disjoint unions of  $n$ -dimensional oriented disks;
- 1-morphisms: orientation-preserving embeddings;
- 2-morphisms: isotopies of orientation-preserving embeddings.

(ii) The category  $\mathbb{D}\mathrm{isk}_n^{\mathrm{fr}}$  is defined analogously, but requiring disks to be framed and embeddings and isotopies to preserve the framings.  $\diamond$

$$\begin{array}{c} \text{1} \quad \text{2} \\ \hline \end{array} \xleftarrow{l_1} \begin{array}{c} \text{1} \quad \text{2} \\ \hline \end{array} \xrightarrow{l_2} \begin{array}{c} \text{2} \quad \text{1} \\ \hline \end{array} .$$
$$\mathcal{F}(\iota_1): \mathcal{F}(\mathbb{R}) \boxtimes \mathcal{F}(\mathbb{R}) \rightarrow \mathcal{F}(\mathbb{R}).$$
$$\alpha_1: \iota_1 \circ (\iota_1 \sqcup \mathbb{R}) \cong \iota_1 \circ (\mathbb{R} \sqcup \iota_1), \quad \lambda_r: \iota_1 \boxtimes \emptyset \cong \iota_1 \quad \text{and} \quad \lambda_l: \emptyset \boxtimes \iota_1 \cong \iota_1,$$

**Proposition 2.3.21.** *A disk algebra  $\mathcal{F}: \mathbb{D}\text{isk}_1 \rightarrow \text{CAT}_{\mathbb{k}}$  is equivalent to the data of a monoidal category in  $\text{CAT}_{\mathbb{k}}$ .  $\blacksquare$*

$$\begin{pmatrix} \bar{1} & \bar{2} \end{pmatrix} \simeq \begin{pmatrix} \bar{2} & \bar{1} \end{pmatrix} \quad (2.3.27)$$

performing a  $\pi$ -rotation that exchanges the image of two copies of  $\mathbb{R}^2$ . Applying a disk algebra  $\mathcal{F}$  to such an isotopy induces a natural isomorphism

$$- \otimes - \cong - \otimes^{\text{op}} -: \mathcal{F}(\mathbb{R}^2) \boxtimes \mathcal{F}(\mathbb{R}^2) \rightarrow \mathcal{F}(\mathbb{R}^2) \boxtimes \mathcal{F}(\mathbb{R}^2),$$

that is, a braiding.

On the other hand, if disks are oriented, we also have an isotopy of the identity embedding  $\mathbb{R}^2 \hookrightarrow \mathbb{R}^2$  given by performing a  $2\pi$ -rotation of the disk. Applying  $\mathcal{F}$  to this rotation, we get an isomorphism of the identity functor which is nothing but a twist. Note that there is no such rotation in the framed case, since embeddings must preserve the framing.

**Proposition 2.3.22.** (i) *A disk algebra  $\mathcal{F}: \text{Disk}_2^{\text{fr}} \rightarrow \text{CAT}_{\mathbb{k}}$  is equivalent to the data of a braided monoidal category in  $\text{CAT}_{\mathbb{k}}$ .*

(ii) *A disk algebra  $\mathcal{F}: \text{Disk}_2^{\text{or}} \rightarrow \text{CAT}_{\mathbb{k}}$  is equivalent to the data of a balanced braided monoidal category in  $\text{CAT}_{\mathbb{k}}$ .* ■

Note that the isotopy obtained by applying (2.3.27) twice cannot be deformed into the identity. Indeed, a continuous deformation between these two isotopies necessarily collapses the two embedded copies of disks at some instant, so it is not in the class of embeddings. This can be solved again by increasing the dimension: when  $n = 3$ , we can use the extra dimension to continuously deform the  $2\pi$ -rotation of the disk into the identity, without producing any intersection. Since higher morphisms in  $\text{CAT}_{\mathbb{k}}$  are trivial, this implies that the induced braiding is in fact a symmetry when  $n = 3$ .

Factorisation homology ([AF15; AF19; Gin13]) allows to extend disk algebras to arbitrary oriented or framed manifolds. Roughly, a manifold is constructed by gluing together oriented or framed disks. Using the gluing pattern provided by a given manifold, one can glue together the algebraic data attached to disks by a disk algebra, resulting in an categorical invariant of the manifold. Let us make this precise:

**Definition 2.3.23.** (i) Let  $\text{Surf}_n^{\text{or}}$  be the symmetric monoidal  $(2, 1)$  category with:

- objects:  $n$ -dimensional oriented manifolds;
- 1-morphisms: orientation-preserving embeddings;
- 2-morphisms: isotopies of orientation preserving embeddings.

The disjoint union of manifolds endows  $\text{Surf}_n^{\text{or}}$  with a symmetric monoidal structure whose monoidal unit is the empty manifold.

(ii) The category  $\text{Surf}_n^{\text{fr}}$  is defined analogously, but requiring manifolds to be framed and embeddings and isotopies to preserve the framings. ◇

We write  $\text{Surf}_n$  for either the category of oriented or framed surfaces, and we will not distinguish them unless it is strictly necessary. The category  $\text{Disk}_n$  is a full monoidal subcategory of  $\text{Surf}_n$ .

**Definition 2.3.24.** Given a disk algebra  $\mathcal{F}: \mathbb{D}\text{isk}_n \rightarrow \mathcal{C}$ , the *factorisation homology with coefficients in  $\mathcal{F}$*  is the functor

$$\int_{(-)} \mathcal{F}: \text{Surf}_n \rightarrow \mathcal{C}$$

define as the left Kan extension of  $\mathcal{F}$  along the embedding  $\mathbb{D}\text{isk}_n \hookrightarrow \text{Surf}_n$ .  $\diamond$

Factorisation homology exists under the assumption that  $\mathcal{C}$  contains a certain class of colimits. It can be computed as the homotopy colimit

$$\int_X \mathcal{F} = \text{colim}_{(\mathbb{R}^n)^{\sqcup k} \hookrightarrow X} \mathcal{F}(X).$$

In the  $(2, 1)$ -categorical setting of  $\text{Pr}$ , it has been extensively studied and computed in [BBJ18a; BBJ18b].

A key property characterising factorisation homology is *excision*: the invariant associated with the gluing of two surfaces is obtained by “gluing” the invariants of each of the single pieces. More precisely:

**Theorem 2.3.25** ([AF15]). *Each decomposition of a stratified manifold as*

$$X \simeq X_1 \sqcup_{C \times (0,1)} X_2$$

*induces an equivalence of categories*

$$\int_S \mathcal{F} \simeq \int_{X_1} \mathcal{F} \boxtimes_{\int_{C \times (0,1)} \mathcal{F}} \int_{X_2} \mathcal{F}.$$

■

Ribbon categories are a special kind of 2-dimensional disk algebra. It was shown in [Coo23] that skein categories satisfy factorisation homology, so that (see also [BH24]):

**Theorem 2.3.26** ([Coo23; BH24]). *Let  $\mathcal{A}$  be a ribbon and  $S$  an oriented surface. There is an equivalence of categories*

$$\text{SkCat}_{\mathcal{A}}(S) \simeq \int_S \mathcal{A}$$

*between skein categories and  $\mathbb{k}$ -linear factorisation homology with coefficients in  $\mathcal{A}$ . Taking free cocompletions recovers locally finitely presentable factorisation homology:*

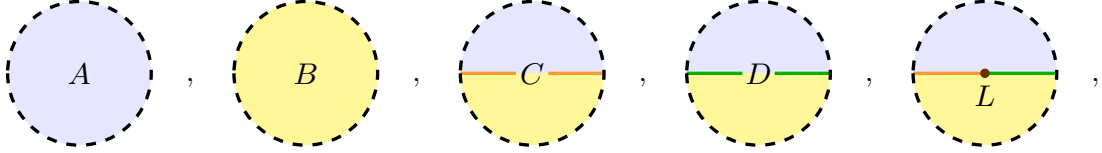
$$\widehat{\text{SkCat}_{\mathcal{A}}}(S) \simeq \int_{\mathcal{A}}^{\text{Pr}} \mathcal{A}.$$

■

We extend the story to incorporate defects. The following category can be defined either in the oriented or the framed case. In both cases, we demand embeddings and isotopies to preserve either the orientation or the framing:

**Definition 2.3.27.** Let  $\mathbb{D}isk_2^{\text{str}}$  be the symmetric monoidal  $(2, 1)$ -category with

- objects: disjoint unions of the following decorated stratified disks

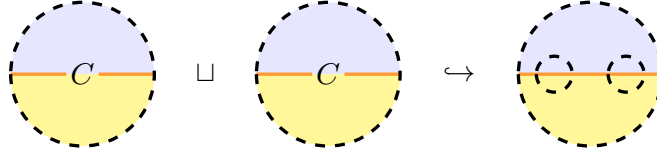


that we denote by  $\mathbb{D}_A, \mathbb{D}_B, \mathbb{D}_C, \mathbb{D}_D$  and  $\mathbb{D}_L$ , respectively;

- 1-morphisms: stratified embeddings preserving the decorations;
- 2-morphisms: isotopies of stratified embeddings preserving the decorations.

The tensor product is the disjoint union, with the empty manifold as the unit object.  $\diamond$

Let  $\mathcal{F}: \mathbb{D}isk_2^{\text{str}} \rightarrow \text{CAT}_{\mathbb{k}}$  be a disk algebra. As we have seen above, applying  $\mathcal{F}$  to either the  $A$ - or the  $B$ -disk, one recovers braided (or braided balanced) monoidal categories  $\mathcal{F}(\mathbb{D}_A)$  and  $\mathcal{F}(\mathbb{D}_B)$ . Let us explore the algebraic structures induced by the embeddings of stratified disks. The image under  $\mathcal{F}$  of the embedding

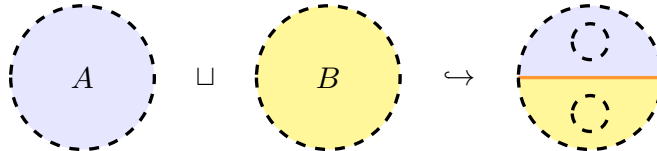


yields a monoidal structure

$$\otimes: \mathcal{F}(\mathbb{D}_C) \boxtimes \mathcal{F}(\mathbb{D}_C) \rightarrow \mathcal{F}(\mathbb{D}_C)$$

on  $\mathcal{F}(\mathbb{D}_C)$ , so  $\mathcal{F}(\mathbb{D}_C)$  is also a monoidal category. The same is true for  $\mathcal{F}(\mathbb{D}_D)$ .

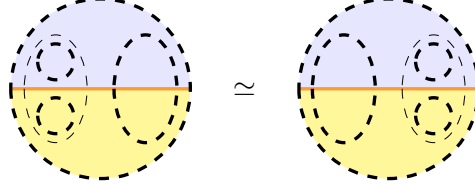
On the other hand, we have an embedding



yielding a functor

$$H: \mathcal{F}(\mathbb{D}_A) \boxtimes \mathcal{F}(\mathbb{D}_B)^{\text{bop}} \rightarrow \mathcal{F}(\mathbb{D}_C).$$

Turning clockwise in the blue region and anticlockwise in the yellow one, we have an isotopy

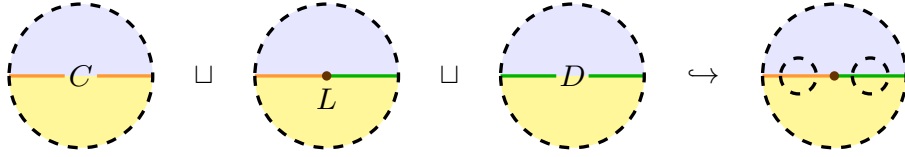


between the embeddings

$$\mathbb{D}_A \sqcup \mathbb{D}_B \sqcup \mathbb{D}_C \hookrightarrow \mathbb{D}_C \sqcup \mathbb{D}_C \begin{matrix} \xrightarrow{\iota_{12}} \\ \xleftarrow{\iota_{21}} \end{matrix} \mathbb{D}_C,$$

where  $\iota_{12}$  and  $\iota_{21}$  are the embeddings inducing the monoidal structure of  $\mathcal{F}(\mathbb{D}_C)$  and its opposite, respectively. This isotopy evaluates to a natural transformation  $H(-) \otimes - \Rightarrow - \otimes H(-)$  that defines a central structure on  $H$ . Therefore,  $\mathcal{F}(\mathbb{D}_C)$  is a  $(\mathcal{F}(\mathbb{D}_A), \mathcal{F}(\mathbb{D}_B))$ -central algebra. The same is true for  $\mathcal{F}(\mathbb{D}_D)$ .

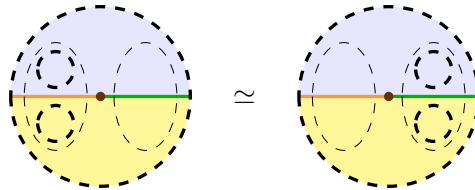
Finally, we consider the stratified disk  $\mathbb{D}_L$ . The image of the embeddings



is a functor

$$- \triangleright - \triangleleft -: \mathcal{F}(\mathbb{D}_A) \boxtimes \mathcal{F}(\mathbb{D}_L) \boxtimes \mathcal{F}(\mathbb{D}_D) \rightarrow \mathcal{F}(\mathbb{D}_L)$$

turning  $\mathcal{F}(\mathbb{D}_L)$  into a  $(\mathcal{F}(\mathbb{D}_C), \mathcal{F}(\mathbb{D}_D))$ -bimodule category. Now the isotopy



between the embeddings

$$\mathbb{D}_A \sqcup \mathbb{D}_B \begin{matrix} \xrightarrow{\iota_C} \\ \xleftarrow{\iota_D} \end{matrix} \mathbb{D}_C \sqcup \mathbb{D}_D \hookrightarrow \mathbb{D}_L,$$

where  $\iota_C$  and  $\iota_D$  are the embeddings into the  $\mathbb{D}_C$  and  $\mathbb{D}_D$  disks, respectively, induces a natural transformation

$$H(-) \triangleright - \triangleleft \mathbf{1}_{\mathcal{F}(\mathbb{D}_D)} \Rightarrow \mathbf{1}_{\mathcal{F}(\mathbb{D}_C)} \triangleright - \triangleleft H(-),$$

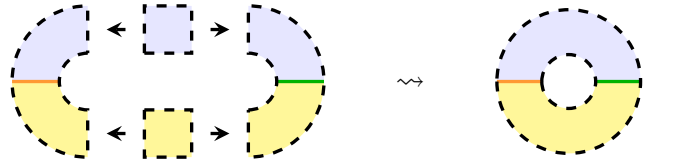
i.e., a  $(\mathcal{F}(\mathbb{D}_A), \mathcal{F}(\mathbb{D}_B))$ -centred  $(\mathcal{F}(\mathbb{D}_C), \mathcal{F}(\mathbb{D}_D))$ -bimodule structure on  $\mathcal{F}(\mathbb{D}_L)$ .

In summary, we have the following:

**Proposition 2.3.28.** *The following data uniquely determine a disk algebra in  $\text{CAT}_{\mathbb{K}}$ :*

- ribbon categories  $\mathcal{F}(\mathbb{D}_A)$  and  $\mathcal{F}(\mathbb{D}_B)$ ;
- $(\mathcal{F}(\mathbb{D}_B), \mathcal{F}(\mathbb{D}_A))$ -central algebras  $\mathcal{F}(\mathbb{D}_C)$  and  $\mathcal{F}(\mathbb{D}_D)$ ;
- a  $(\mathcal{F}(\mathbb{D}_B), \mathcal{F}(\mathbb{D}_A))$ -centred  $(\mathcal{F}(\mathbb{D}_C), \mathcal{F}(\mathbb{D}_D))$ -bimodule  $\mathcal{F}(\mathbb{D}_L)$ . ■

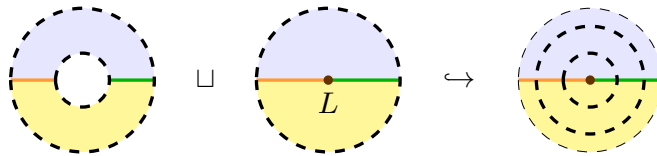
A stratified version of factorisation homology has been constructed in [AFT17], and it is also characterised by excision. For instance, the content of Lemma 2.1.25 can be interpreted topologically in terms of factorization homology as follows. The stratified annulus can be obtained by gluing two stratified disks in the following way:



Excision then yields an equivalence

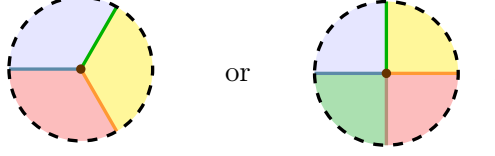
$$\begin{array}{c} \text{Stratified Annulus} \end{array} \xrightarrow{\int_{(-)}^{\mathcal{F}}} \mathcal{F}(\mathbb{D}_C) \boxtimes_{\mathcal{F}(\mathbb{D}_A) \boxtimes \mathcal{F}(\mathbb{D}_B)} \mathcal{F}(\mathbb{D}_D) \simeq \text{tr}_{\mathcal{F}(\mathbb{D}_A)} (\mathcal{F}(\mathbb{D}_C) \boxtimes_{\mathcal{F}(\mathbb{D}_B)} \mathcal{F}(\mathbb{D}_D)).$$

Hence, applying the factorization homology functor to the embedding



induces the  $\text{tr}_{\mathcal{F}(\mathbb{D}_A)} (\mathcal{F}(\mathbb{D}_C) \boxtimes_{\mathcal{F}(\mathbb{D}_B)} \mathcal{F}(\mathbb{D}_D))$ -module structure on  $\mathcal{F}(\mathbb{D}_L)$  described in Proposition 2.1.25.

The category  $\mathbb{Disk}_2^{\text{str}}$  can be extended to a more general category containing decorated disks with an arbitrary number of marked lines meeting at a central point, such as, for instance,



The topology of the embeddings of these disks controls the algebraic structure appearing in Corollary 2.1.26. The results of [Coo23; BH24] can be extended to the stratified case. Firstly, we have the following equivalences:

**Proposition 2.3.29.** *Let  $\mathcal{F}: \mathbb{Disk}_2^{\text{str}} \rightarrow \text{CAT}_{\mathbb{k}}$  be a disk algebra, determined a collection of ribbon categories, central algebras and centred bimodules that we denote as usual. We have equivalences*

$$\text{SkCat}_{\mathcal{F}}^{\text{def}} \left( \left( \text{disk with 2 marked lines} \right) \right) \simeq \mathcal{C} \quad \text{and} \quad \text{SkCat}_{\mathcal{F}}^{\text{def}} \left( \left( \text{disk with 4 marked lines} \right) \right) \simeq \mathcal{L}$$

as  $(\mathcal{A}, \mathcal{B})$ -central algebras and  $\text{tr}_{\mathcal{A}_1}(\mathcal{D}_1 \boxtimes_{\mathcal{A}_2} \cdots \boxtimes_{\mathcal{A}_n} \mathcal{D}_n)$ -modules, respectively.

*Proof.* The first case is [BJ25, Lemma 2.22]. Namely, the evaluation functor

$$\text{ev}_{\mathcal{C}}^{\text{CENTRAL}}: \text{CENTRAL}(\mathcal{C}) \rightarrow \mathcal{C}$$

is surjective on objects and full. Moreover,

$$\text{SkMod}_{\mathcal{S}}^{\text{def}}(\mathbb{D}_{\mathcal{C}}) = \text{CENTRAL}(\mathcal{C}) / \ker \text{ev}_{\mathcal{C}}^{\text{CENTRAL}},$$

so we get an equivalence  $\text{SkMod}_{\mathcal{S}}^{\text{def}}(\mathbb{D}_{\mathcal{C}}) \simeq \mathcal{C}$  as  $\mathbb{k}$ -linear categories. The  $(\text{Piv}(\mathcal{A}), \text{Piv}(\mathcal{B}))$ -central structure of  $\text{CENTRAL}(\mathcal{C})$  descends to a  $(\text{SkCat}_{\mathcal{A}}(\mathbb{D}), \text{SkCat}_{\mathcal{B}}(\mathbb{D}))$ -central structure on  $\text{SkCat}_{\mathcal{S}}^{\text{def}}(\mathbb{D}_{\mathcal{C}})$ , and we obtain a diagram

$$\begin{array}{ccc} \text{SkCat}_{\mathcal{A}}(\mathbb{D}) \boxtimes \text{SkCat}_{\mathcal{B}}(\mathbb{D})^{\text{bop}} & \xrightarrow{(F^{\text{SkMod}}, \sigma^{\text{SkMod}})} & Z_{\text{Dr}}(\text{SkMod}_{\mathcal{S}}^{\text{def}}(\mathbb{D}_{\mathcal{C}})) \\ \downarrow \text{ev}_{\mathcal{A}} \boxtimes \text{ev}_{\mathcal{B}} & & \downarrow \text{ev}_{\mathcal{C}} \\ \mathcal{A} \boxtimes \mathcal{B}^{\text{bop}} & \xrightarrow{(H, \sigma)} & Z_{\text{Dr}}(\mathcal{C}), \end{array}$$

where the vertical arrows are induced by the evaluation functors and the horizontal ones are the central algebra structures. It follows directly from the definitions that it is commutative, yielding the equivalence of central algebras.

For the second case, we have an equivalence of categories  $\text{SkMod}_S^{\text{def}}(\mathbb{D}_L) \simeq \mathcal{L}$  which is also induced by the evaluation functor. Again, we obtain a commutative diagram

$$\begin{array}{ccc} \text{SkCat}_S^{\text{def}}(\mathbb{D}_{C_1}) \boxtimes \cdots \boxtimes \text{SkCat}_S^{\text{def}}(\mathbb{D}_{C_n}) \boxtimes \text{SkCat}_S^{\text{def}}(\mathbb{D}_L) & \xrightarrow{-\triangleright-} & \text{SkCat}_S^{\text{def}}(\mathbb{D}_L) \\ \text{ev}_{C_1} \boxtimes \cdots \boxtimes \text{ev}_{C_n} \boxtimes \text{ev}_{\mathcal{L}} \downarrow & & \downarrow \text{ev}_{\mathcal{L}} \\ \mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n \boxtimes \mathcal{L} & \xrightarrow{-\triangleright-} & \mathcal{L}, \end{array}$$

which is commutative, lifting the previous equivalence to an equivalence of module categories. Moreover, for every  $i$ , the diagram

$$\begin{array}{ccc} \text{ev}_{\mathcal{L}}(A \triangleright_i L) & \xrightarrow{\eta_{A,L}^{\text{SkMod}}} & \text{ev}_{\mathcal{L}}(A \triangleright_{i+1} L) \\ \downarrow = & & \downarrow = \\ \text{ev}_{\mathcal{A}_i}(A) \triangleright \text{ev}_{\mathcal{L}}(L) & \xrightarrow{\eta_{\text{ev}_{\mathcal{A}_i}(A), \text{ev}_{\mathcal{L}}(L)}} & \text{ev}_{\mathcal{A}_{i+1}}(L) \triangleright \text{ev}_{\mathcal{L}}(L) \end{array}$$

is also commutative by the definition of the evaluation functors, showing that the equivalence is an equivalence of centred bimodule categories.  $\square$

And more generally:

**Claim 2.3.30.** *Given a decorated surface  $S$  and a disk algebra  $\mathcal{F}: \text{Disk}_2^{\text{str}} \rightarrow \text{CAT}_{\mathbb{k}}$  determined by a collection of ribbon categories, central algebras and centred bimodules, we have an equivalence of categories*

$$\text{SkCat}_{\mathcal{F}}^{\text{def}}(S) \simeq \int_S \mathcal{F}$$

*between stratified skein categories and  $\mathbb{k}$ -linear factorisation homology with coefficients in  $\mathcal{F}$ .*  $\blacksquare$



# Chapter 3

## HOMFLY parabolic restriction

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### 3.1 Parabolic restriction

In this section, we recall the formalism of parabolic restriction. We refer to [KS97] for complete details of the material covered here. We follow the notation and some of the conventions of [Bru17, Section 3]. Throughout the rest of the section, we fix a field  $\mathbb{k}$  of characteristic 0 and  $q \in \mathbb{k}^\times$ , and we assume that  $q$  is not a root of unity.

#### 3.1.1 The quantum algebra $U_q(\mathfrak{gl}_N)$

Let  $N \in \mathbb{N}$  be a positive integer.

**Definition 3.1.1.** The *quantum group*  $U_q(\mathfrak{gl}_N)$  is the algebra defined by generators

$$e_i, f_i, d_j, d_j^{-1}, \quad i = 1, \dots, N-1, \quad j = 1, \dots, N,$$

and relations

$$\begin{aligned} d_i d_j &= d_j d_i, & d_i d_i^{-1} &= d_i^{-1} d_i = 1, \\ d_i e_j d_i^{-1} &= q^{\delta_{ij} - \delta_{i,j+1}} e_j, & d_i f_j d_i^{-1} &= q^{-\delta_{ij} + \delta_{i,j+1}} f_j, \\ e_i f_j - f_j e_i &= \delta_{ij} \frac{d_i d_{i+1}^{-1} - d_i^{-1} d_{i+1}}{q - q^{-1}}, \\ e_i e_j &= e_j e_i, & f_i f_j &= f_j f_i, & |i - j| &\geq 2, \\ e_i^2 e_{i\pm 1} - (q + q^{-1}) e_i e_{i\pm 1} e_i + e_{i\pm 1} e_i^2 &= 0, \\ f_i^2 f_{i\pm 1} - (q + q^{-1}) f_i f_{i\pm 1} f_i + f_{i\pm 1} f_i^2 &= 0. \end{aligned}$$

In the trivial case  $N = 0$ , we interpret  $U_q(\mathfrak{gl}_0)$  as  $\mathbb{k}$ . ◇

The algebra  $U_q(\mathfrak{gl}_N)$  admits a Hopf algebra structure whose comultiplication  $\Delta$ , counit  $\varepsilon$  and antipode  $S$  are determined by the formulas

$$\begin{aligned} \Delta(e_i) &= d_i^{-1} d_{i+1} \otimes e_i + e_i \otimes 1, & \Delta(f_i) &= 1 \otimes f_i + f_i \otimes d_i d_{i+1}^{-1}, \\ \Delta(d_i^{\pm 1}) &= d_i^{\pm 1} \otimes d_i^{\pm 1}, & \varepsilon(e_i) &= \varepsilon(f_i) = 0, & \varepsilon(d_i^{\pm 1}) &= 1, \\ S(e_i) &= -d_i d_{i+1}^{-1} e_i, & S(f_i) &= -f_i d_i^{-1} d_{i+1}, & S(d_i^{\pm 1}) &= d_i^{\mp 1}. \end{aligned}$$

In particular, the antipode is invertible, so the category of finite-dimensional representations of  $U_q(\mathfrak{gl}_N)$  is rigid.

**Definition 3.1.2. (i)** The *natural representation*  $V_N$  of  $U_q(\mathfrak{gl}_N)$  is the  $U_q(\mathfrak{gl}_N)$ -module with basis  $\{v_i \mid 1 \leq i \leq N\}$  and action given by

$$f_i v_j = \delta_{i,j} v_{i+1}, \quad e_i v_j = \delta_{i+1,j} v_i, \quad d_i v_j = q^{\delta_{i,j}} v_j.$$

**(ii)** The *dual natural representation*  $V_N^-$  is the  $U_q(\mathfrak{gl}_N)$ -module with basis  $\{\psi_i \mid 1 \leq i \leq N\}$  and action given by

$$f_i \psi_j = -\delta_{i+1,j} \psi_i, \quad e_i \psi_j = -\delta_{i,j} \psi_{i+1}, \quad d_i \psi_j = q^{-\delta_{i,j}} \psi_j.$$

◇

There is, up to scalar, a unique non-degenerate pairing

$$\langle -, - \rangle: V_N \times V_N^- \rightarrow \mathbb{k}$$

such that  $\langle S^{-1}(u)v, \psi \rangle = \langle v, u\psi \rangle$  which we may assume to be given by  $\langle v_i, \psi_j \rangle = q^{-i}\delta_{i,j}$ . This exhibits  $V_N^-$  as a right dual of  $V_N$ , with evaluation and coevaluation determined by the formulas

$$\begin{aligned} \text{ev}_N: V_N \otimes V_N^- &\rightarrow \mathbb{k}, & \text{coev}_N: \mathbb{k} &\rightarrow V_N^- \otimes V_N, \\ v_i \otimes \psi_j &\mapsto q^{-i}\delta_{i,j}, & 1 &\mapsto \sum_{j=1}^N q^j \psi_j \otimes v_j. \end{aligned} \quad (3.1.28)$$

In what follows, we choose  $V_N^-$  as distinguished right dual for  $V_N$ , so we set  $V_N^* := V_N^-$ .

The Hopf algebra  $U_q(\mathfrak{gl}_N)$  is *pivotal*: for every  $u \in U_q(\mathfrak{gl}_N)$ , we have

$$S^2(u) = g_N u g_N^{-1},$$

where  $g_N := d_1^{N-1} d_2^{N-3} \dots d_N^{1-N}$ . This group-like element is called a *pivot*. Pivotal structures on the category of finite-dimensional  $U_q(\mathfrak{gl}_N)$ -modules are in bijection with pivots in  $U_q(\mathfrak{gl}_N)$ . The pivotal structure corresponding to  $g_N$  is given by

$$\iota_V: V \rightarrow V^{**}, \quad v \mapsto \text{can}_V(g_N^{-1}v),$$

where  $\text{can}_V$  is the canonical identification  $V \cong V^{**}$  as vector spaces. The action of  $g_N^{-1}$  on the basis elements of  $V_N$  is given by

$$g_N^{-1}v_i = q^{-N+2i-1}v_i,$$

hence, from the distinguished right duality, we get a left evaluation and coevaluation

$$\begin{aligned} \text{ev}'_N: V_N^- \otimes V_N &\rightarrow \mathbb{k}, & \text{coev}'_N: \mathbb{k} &\rightarrow V_N \otimes V_N^-, \\ \psi_i \otimes v_j &\mapsto q^{i-N-1}\delta_{i,j}, & 1 &\mapsto \sum_{j=1}^N q^{N-j+1}v_j \otimes \psi_j, \end{aligned} \quad (3.1.29)$$

exhibiting  $V_N^-$  also as a left dual of  $V_N$ .

**Definition 3.1.3.** We denote by  $\text{Rep}_q \text{GL}_N$  the category of finite-dimensional representations of  $U_q(\mathfrak{gl}_N)$  that are isomorphic to finite direct sums of summands of the representations obtained by taking tensor products of  $V_N$  and  $V_N^-$ .  $\diamond$

The category  $\text{Rep}_q \text{GL}_N$  is braided. The braiding  $c$  is determined by the action on  $V_N \otimes V_N$  of the universal  $R$ -matrix of  $U_q(\mathfrak{gl}_N)$ . Explicitly, in the basis  $\{v_i \mid i = 1, \dots, N\}$ , it can be written as

$$\beta_N := c_{V_N, V_N} = q \sum_i (E_{i,i} \otimes E_{i,i}) + \sum_{i \neq j} (E_{j,i} \otimes E_{i,j}) + (q - q^{-1}) \sum_{i > j} (E_{i,i} \otimes E_{j,j}). \quad (3.1.30)$$

### 3.1.2 The specialised restriction functor

For  $m, n \in \mathbb{N}$ , we define a quantum analogue of the Levi subalgebra  $\mathfrak{l}$  of two-block diagonal matrices of  $\mathfrak{gl}_{m+n}$  as follows:

**Definition 3.1.4.** The *quantum Levi subgroup*  $U_q(\mathfrak{l})$  is the subalgebra of  $U_q(\mathfrak{gl}_{m+n})$  generated (as an algebra) by  $\{e_i, f_i, d_j, d_j^{-1} \mid i \neq m\}$ .  $\diamond$

The mapping

$$\begin{aligned} d_i &\mapsto \begin{cases} d_i \otimes 1 & \text{if } 1 \leq i \leq m \\ 1 \otimes d_{i-m} & \text{if } m+1 \leq i \leq m+n, \end{cases} \\ e_j, f_j &\mapsto \begin{cases} e_j \otimes 1, f_j \otimes 1 & \text{if } 1 \leq j \leq m-1, \\ 1 \otimes e_{j-m}, 1 \otimes f_{j-m} & \text{if } m+1 \leq j \leq m+n-1, \end{cases} \end{aligned}$$

yields an isomorphism

$$U_q(\mathfrak{l}) \cong U_q(\mathfrak{gl}_m) \otimes U_q(\mathfrak{gl}_n) \quad (3.1.31)$$

thus providing representations  $V_m \equiv V_m \otimes \mathbb{k}$  and  $V_n \equiv \mathbb{k} \otimes V_n$  (and their duals) of  $U_q(\mathfrak{l})$  obtained by tensoring the natural representations of  $U_q(\mathfrak{gl}_m)$  and  $U_q(\mathfrak{gl}_n)$  with the trivial representations and acting diagonally.

**Definition 3.1.5.** We denote by  $\text{Rep}_q \mathfrak{L}$  the category of finite-dimensional representations of  $U_q(\mathfrak{l})$  that are isomorphic to finite direct sums of summands of the representations obtained by taking tensor products of  $V_m$ ,  $V_n$  and their duals.  $\diamond$

Note that, by definition,

$$\text{Rep}_q \mathfrak{L} \simeq \text{Rep}_q \text{GL}_m \boxtimes \text{Rep}_q \text{GL}_n$$

and we have pivotal and braided structures given component-wise by the pivotal and braided structures of  $\text{Rep}_q \text{GL}_m$  and  $\text{Rep}_q \text{GL}_n$ . Restricting along the inclusion  $U_q(\mathfrak{l}) \subset U_q(\mathfrak{gl}_{m+n})$  we obtain a functor

$$\text{res}_{m,n} : \text{Rep}_q \text{GL}_{m+n} \rightarrow \text{Rep}_q \text{GL}_m \boxtimes \text{Rep}_q \text{GL}_n$$

sending  $V_{m+n}$  to the direct sum  $V_m \oplus V_n$ . This functor is strict monoidal, but it does not preserve the distinguished dualities, the pivotal structure, or the braiding. Indeed, the right evaluation and coevaluation of  $V_m \oplus V_n$  are given by

$$\text{ev}_{m \oplus n} = \text{ev}_m + \text{ev}_n, \quad \text{coev}_{m \oplus n} = \text{coev}_m + \text{coev}_n,$$

respectively. Yet,

$$\text{res}_{m,n}(\text{ev}_{m+n}) = \text{ev}_m + q^{-m} \text{ev}_n, \quad \text{res}_{m,n}(\text{coev}_{m+n}) = \text{coev}_m + q^m \text{coev}_n, \quad (3.1.32)$$

so the restriction does not preserve the distinguished right duals. The isomorphism

$$\varphi_{m+n} : \text{res}(V_{m+n}^*) \rightarrow V_m^* \oplus V_n^*$$

identifying the dualities is thus given by

$$\varphi_{m+n} = \text{id}_{V_m^*} + q^{-m} \text{id}_{V_n^*}. \quad (3.1.33)$$

On the other hand, the pivotal structure of  $\text{Rep}_q \text{GL}_m \boxtimes \text{Rep}_q \text{GL}_n$  is induced by the pivot

$$g_m \otimes g_n \in U_q(\mathfrak{gl}_m) \otimes U_q(\mathfrak{gl}_n).$$

Under the identification (3.1.31), this corresponds to the element

$$g' = \prod_{\substack{j=1 \\ j \neq m}}^{m+n} d_j^{m+n-2j+1} \in U_q(\mathfrak{l}) \subset U_q(\mathfrak{gl}_{m+n}),$$

which differs from  $g_{m+n}$ , so  $\text{res}_{m,n}$  is not pivotal. Finally, applying the restriction functor to the braiding of  $\text{Rep}_q \text{GL}_{m+n}$ , we obtain

$$\text{res}_{m,n}(\beta_{m+n}) = \left( \begin{array}{c|c|c|c} \beta_m & 0 & 0 & 0 \\ \hline 0 & 0 & \sigma_{m,n} & 0 \\ \hline 0 & \sigma_{m,n} & (q - q^{-1}) \text{id}_{m,n} & 0 \\ \hline 0 & 0 & 0 & \beta_n \end{array} \right) : (V_m \oplus V_n)^{\otimes 2} \rightarrow (V_m \oplus V_n)^{\otimes 2}, \quad (3.1.34)$$

where  $\sigma_{m,n}$  is the trivial flip, so  $\text{res}_{m,n}$  is not braided.

### 3.1.3 The parabolic restriction central algebra

As in the previous section, let  $m, n \in \mathbb{N}$ . We define a quantum analogue of the parabolic subalgebra  $\mathfrak{p}$  of block triangular matrices of  $\mathfrak{gl}_{m+n}$  as follows:

**Definition 3.1.6.** The *parabolic quantum group*  $U_q(\mathfrak{p})$  is the subalgebra of  $U_q(\mathfrak{gl}_{m+n})$  generated by  $\{e_i, f_j, d_k, d_k^{-1} \mid j \neq m\}$ .  $\diamond$

The quantum Levi subalgebra  $U_q(\mathfrak{l})$  is both a subalgebra and a quotient of  $U_q(\mathfrak{p})$ . We set

$$j_{m,n} : U_q(\mathfrak{l}) \hookrightarrow U_q(\mathfrak{p})$$

for the inclusion map sending each generator of  $U_q(\mathfrak{l})$  to the same generator of  $U_q(\mathfrak{p})$ . On the other hand,

$$U_q(\mathfrak{l}) \cong U_q(\mathfrak{p}) / \langle e_m \rangle$$

and we set

$$\pi_{m,n} : U_q(\mathfrak{p}) \twoheadrightarrow U_q(\mathfrak{l})$$

for the projection sending  $e_m$  to 0 and preserving the rest of generators. Let

$$\iota_{m,n} : U_q(\mathfrak{p}) \hookrightarrow U_q(\mathfrak{gl}_{m+n})$$

be the inclusion of  $U_q(\mathfrak{p})$  into  $U_q(\mathfrak{gl}_{m+n})$ .

**Definition 3.1.7.** Let  $\text{Rep}_q \mathfrak{p}$  be the category of finite-dimensional representations of  $U_q(\mathfrak{p})$  that are isomorphic to finite direct sums of summands of the representations obtained by taking tensor products of the restrictions of  $V_{m+n}$  and its dual along  $\iota_{m,n}$ , and the restrictions of  $V_m, V_n$  and their duals along  $j_{m,n}$ .  $\diamond$

We have four strict monoidal restriction functors

$$\begin{array}{ccc}
 & \text{Rep}_q \mathbf{P} & \\
 \iota_{m,n}^* \nearrow & & \nwarrow \pi_{m,n}^* \\
 \text{Rep}_q \text{GL}_{m+n} & \xrightarrow{\text{res}_{m,n}} & \text{Rep}_q \mathbf{L}
 \end{array}
 \quad \begin{array}{c} \text{---} j_{m,n}^* \text{---} \end{array}$$

We will abuse notation and call  $V_{m+n}$ ,  $V_m$  and  $V_n$  both the generating objects of  $\text{Rep}_q \text{GL}_{m+n}$  and  $\text{Rep}_q \mathbf{L}$  and their images in  $\text{Rep}_q \mathbf{P}$ .

**Remark 3.1.8. (i)** The notation  $\text{Rep}_q \mathbf{P}_t$  is slightly abusive, since there are *a priori* many other finite-dimensional  $U_q(\mathfrak{p})$ -modules. For our purposes, however, it suffices to consider this smaller subcategory, as it contains the essential image of the restriction functors  $\iota_{m,n}^*$  and  $\pi_{m,n}^*$  that induce the parabolic central algebra structure (cf. Definition 3.1.10 below).

**(ii)** The category  $\text{Rep}_q \mathbf{P}$  is not semisimple: the short exact sequence

$$0 \rightarrow V_m \hookrightarrow V_{m+n} \twoheadrightarrow V_n \rightarrow 0$$

does not split in  $\text{Rep}_q \mathbf{P}$ . ◇

Transporting the dualities of  $\text{Rep}_q \text{GL}_{m+n}$  and  $\text{Rep}_q \mathbf{L}$  along the restriction functors, the category  $\text{Rep}_q \mathbf{P}$  becomes a rigid category with distinguished left and right dualities. It is, however, not a pivotal category: as explained in the previous sections, the pivot of  $U_q(\mathfrak{l})$  is not the restriction of that of  $U_q(\mathfrak{gl}_{m+n})$ . Thus, the restrictions to  $\text{Rep}_q \mathbf{P}$  of the pivotal structures of  $\text{Rep}_q \text{GL}_{m+n}$  and  $\text{Rep}_q \mathbf{L}$  are not natural in  $\text{Rep}_q \mathbf{P}$ .

Moreover,  $\text{Rep}_q \mathbf{P}$  is not braided: the universal  $R$ -matrix of  $U_q(\mathfrak{gl}_{m+n})$  belongs to (a completion of)  $U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^-)$ , where  $\mathfrak{b}^\pm$  are the positive and negative Borel subalgebras of  $\mathfrak{gl}_{m+n}$ . Since  $U_q(\mathfrak{b}^-) \not\subset U_q(\mathfrak{p})$ , the action of this  $R$ -matrix does not commute with morphisms of  $\text{Rep}_q \mathbf{P}$ , hence the restriction of the braidings of  $\text{Rep}_q \text{GL}_{m+n}$  and  $\text{Rep}_q \mathbf{L}$  are not natural in the second factor. Yet,  $U_q(\mathfrak{b}^+) \subset U_q(\mathfrak{p})$ , so they are natural in the first one, and hence yield lifts of the functors  $\iota_{m,n}^*$  and  $\pi_{m,n}^*$  to the Drinfeld center of  $\text{Rep}_q \mathbf{P}$ :

**Proposition 3.1.9.** *The restriction functors  $\iota_{m,n}^*$  and  $\pi_{m,n}^*$  lift to a central functor*

$$(\iota_{m,n}^* \otimes \pi_{m,n}^*, \gamma): \text{Rep}_q \text{GL}_{m+n} \boxtimes \text{Rep}_q \mathbf{L}^{\text{bop}} \rightarrow Z_{\text{Dr}}(\text{Rep}_q \mathbf{P}),$$

with half-braiding  $\gamma$  given by

$$\gamma_W^{U,V} = \left( \iota_{m,n}^*(U) \otimes \pi_{m,n}^*(V) \otimes W \xrightarrow{\sigma_{(1,2,3)} \circ (\mathcal{R}_{13}^{\mathfrak{gl}_{m+n}})^{-1} \circ (\mathcal{R}_{32}^{\mathfrak{l}})^{-1}} W \otimes \iota_{m,n}^*(U) \otimes \pi_{m,n}^*(V) \right),$$

where  $\sigma_{(1,2,3)}$  is a cyclic permutation of the tensor factors and  $\mathcal{R}^{\mathfrak{gl}_{m+n}}$  and  $\mathcal{R}^{\mathfrak{l}}$  are the universal  $R$ -matrices of  $U_q(\mathfrak{gl}_{m+n})$  and  $U_q(\mathfrak{l})$ , respectively. ■

**Definition 3.1.10.** We call the central structure in the previous proposition the *parabolic restriction central algebra*. ◇

## 3.2 HOMFLY categories

In this section, we recall the definition of the HOMFLY category  $\text{Rep}_q \text{GL}_t$ , a quantum analogue of the Deligne category  $\text{Rep GL}_t$  [Del07; CW11] that has been extensively studied in [Bru17]. The HOMFLY category can be understood as a universal version of the category  $\text{Rep}_q \text{GL}_N$ , in the sense that, for any  $N \in \mathbb{N}$ , the latter can be recovered as (a quotient of the) specialisation of  $\text{Rep}_q \text{GL}_t$  to  $t = N$ . In the second and third part of the section, we construct a two-colour version  $\text{Rep}_q \text{L}_t$  of the HOMFLY category together with a restriction functor  $\text{res}_t$  interpolating both the category  $\text{Rep}_q \text{L}_t$  and the functor  $\text{res}_{m,n}$  described in the previous section.

For the rest of the chapter, we fix the ground ring

$$\mathbb{k} = \mathbb{C}(q)[q^{\pm t}, \delta] \Big/ \langle (q - q^{-1}) \delta = q^t - q^{-t} \rangle.$$

### 3.2.1 The HOMFLY category $\text{Rep}_q \text{GL}_t$

Following [Bru17], we recall the definition of the HOMFLY category  $\text{Rep}_q \text{GL}_t$ . This category is a quotient of the following category of framed oriented tangles:

**Definition 3.2.1.** A *framed oriented tangle* is a ribbon graph (cf. Section 1.2.4) embedded in  $\mathbb{R}^2 \times [0, 1]$  containing no coupons.  $\diamond$

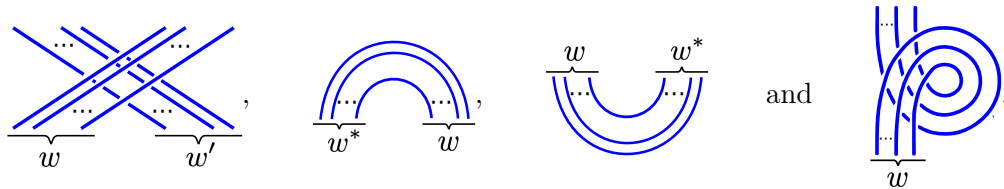
Considering tangles up to isotopy, we have:

**Definition 3.2.2.** The *category of framed oriented tangles* is the  $\mathbb{k}$ -linear category  $\mathcal{T}$  with:

- objects: finite words over  $\{\uparrow, \downarrow\}$ ;
- morphisms:  $\mathbb{k}$ -linear combinations of isotopy classes of framed oriented tangles, matching the source and target objects in  $\mathbb{R}^2 \times \{0\}$  and  $\mathbb{R}^2 \times \{1\}$ , respectively.

The composition is defined by vertically stacking diagrams and rescaling the second coordinate. The identity of an object  $w \in \langle \uparrow, \downarrow \rangle$  is the diagram consisting of a bunch of vertical strands whose orientations match  $w$ .  $\diamond$

We endow the category  $\mathcal{T}$  with the structure of a ribbon category. The tensor product is defined on objects by the concatenation of words and, on morphisms, by horizontally stacking diagrams. The unit object is the empty diagram  $\mathbf{1}_{\mathcal{T}}$ . The left dual of an object  $w = w_1 \cdots w_k$  is the object  $w^* := w_k^* \cdots w_1^*$ , where  $(-)^*$  interchanges  $\uparrow$  and  $\downarrow$ . The braiding, evaluation, coevaluation and twists are represented by the diagrams



where the directions are determined by the depicted words. Exchanging  $w$  and  $w^*$  in the evaluation and coevaluation maps, we can exhibit  $w^*$  as a left dual of  $w$ , so that  $\mathcal{T}$  has a trivial pivotal structure.

**Definition 3.2.3.** The category  $\widetilde{\text{Rep}}_q \text{GL}_t$  is the quotient of  $\mathcal{T}$  by the relations

$$\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} - \begin{array}{c} \nwarrow \nearrow \\ \nearrow \searrow \end{array} = (q - q^{-1}) \begin{array}{c} \uparrow \\ \downarrow \end{array}, \quad (3.2.35)$$

$$\begin{array}{c} \uparrow \\ \text{loop} \end{array} = q^t \begin{array}{c} \uparrow \\ \downarrow \end{array}, \quad (3.2.36)$$

$$\begin{array}{c} \circlearrowleft \end{array} = \delta \mathbf{1}_{\mathcal{T}}. \quad (3.2.37)$$

We call them the *HOMFLY skein*, *twist* and *dimension relation*, respectively.  $\diamond$

This category inherits a ribbon structure from  $\mathcal{T}$  and it admits the following presentation by generators and relations:

**Proposition 3.2.4** ([Bru17, Theorem 1.1]). *The category  $\widetilde{\text{Rep}}_q \text{GL}_t$  is generated, as a  $\mathbb{k}$ -linear strict monoidal category, by the objects  $\uparrow$  and  $\downarrow$  and the morphisms*

$$\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array}, \quad \begin{array}{c} \nwarrow \nearrow \\ \nearrow \searrow \end{array}, \quad \text{and} \quad \begin{array}{c} \text{cup} \end{array}$$

subject to the following relations:

$$\begin{array}{c} \text{crossing} \end{array} = (q - q^{-1}) \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} + \begin{array}{c} \uparrow \\ \downarrow \end{array}, \quad (3.2.38)$$

$$\begin{array}{c} \text{wavy up} \end{array} = \begin{array}{c} \uparrow \end{array}, \quad \begin{array}{c} \text{wavy down} \end{array} = \begin{array}{c} \downarrow \end{array}, \quad (3.2.39)$$

$$\left( \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right)^{-1} = \begin{array}{c} \nwarrow \nearrow \\ \nearrow \searrow \end{array}, \quad (3.2.40)$$

$$q^t \begin{array}{c} \text{figure 8} \end{array} = \delta \mathbf{1}_{\mathcal{T}}, \quad (3.2.41)$$

$$\begin{array}{c} \text{crossing} \end{array} = \begin{array}{c} \text{crossing} \end{array}. \quad (3.2.42)$$

■

**Definition 3.2.5.** The *HOMFLY category*  $\text{Rep}_q \text{GL}_t$  is the Cauchy completion of the category  $\widetilde{\text{Rep}_q \text{GL}_t}$ .  $\diamond$

The HOMFLY category interpolates between the categories  $\text{Rep}_q \text{GL}_N$  of finite representations of  $U_q(\mathfrak{gl}_N)$  in the following sense:

**Remark 3.2.6.** For any integer  $N \in \mathbb{N}$ , consider the evaluation map

$$\varphi_N : \mathbb{k} \rightarrow \mathbb{C}(q), \quad q^t \mapsto q^N, \quad \delta \mapsto \frac{q^N - q^{-N}}{q - q^{-1}}.$$

There exists a  $\varphi_N$ -linear full ribbon functor

$$\text{ev}_N^G : \text{Rep}_q \text{GL}_t \rightarrow \text{Rep}_q \text{GL}_N$$

sending the object  $\uparrow$  to the natural representation  $V_N$ . The functor is, however, not faithful: its kernel is the ideal of negligible morphisms of  $\text{Rep}_q \text{GL}_t$  (see [Bru17, Theorem 1.3] for a precise statement).  $\diamond$

### 3.2.2 The HOMFLY category $\text{Rep}_q L_t$

Next, we describe diagrammatically the *two-coloured HOMFLY category*  $\text{Rep}_q L_t$ , interpolating the categories  $\text{Rep}_q L$  of finite  $U_q(\mathfrak{l})$ -modules.

**Definition 3.2.7.** The *two-coloured HOMFLY category*  $\text{Rep}_q L_t$  is the Cauchy completion of  $\widetilde{\text{Rep}_q \text{GL}_t} \boxtimes_{\mathbb{C}(q)} \widetilde{\text{Rep}_q \text{GL}_t}$ .  $\diamond$

In the previous definition, the symbol  $\boxtimes_{\mathbb{C}(q)}$  means that the tensor product is taken in the category of  $\mathbb{C}(q)$ -linear categories (instead of  $\mathbb{k}$ -linear categories). Therefore,  $\text{Rep}_q L_t$  is linear over  $\mathbb{k} \otimes_{\mathbb{C}(q)} \mathbb{k}$ . We will set

$$q^{t_1} := q^t \otimes 1, \quad q^{t_2} := 1 \otimes q^t, \quad \delta_1 := \delta \otimes 1, \quad \text{and} \quad \delta_2 := 1 \otimes \delta$$

for the generators of this ring. The following is straightforward:

**Proposition 3.2.8.**  $\text{Rep}_q L_t$  has a ribbon structure given factor-wise by that of  $\text{Rep}_q \text{GL}_t$ .  $\blacksquare$

We will represent this category diagrammatically (cf. Section 1.2.4) using the following colour conventions:

$$\blacksquare := (\blacksquare, \emptyset), \quad \blacksquare := (\emptyset, \blacksquare) \quad \text{and} \quad \blacksquare := \blacksquare \oplus \blacksquare.$$

The coupons representing the inclusions and projections between the green/red colours and the orange one will be represented by

$$\begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array}, \quad \begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array}, \quad \begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array} \quad \text{and} \quad \begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array}.$$

With this notation, the following relations hold in  $\text{Rep}_q L_t$  by definition:

- the *one-colour* skein, twist and dimension relations:

$$\begin{array}{c} \nearrow \searrow \\ \searrow \nearrow \end{array} - \begin{array}{c} \searrow \nearrow \\ \nearrow \searrow \end{array} = (q - q^{-1}) \begin{array}{c} \uparrow \\ \uparrow \end{array}, \quad (3.2.43)$$

$$\begin{array}{c} \uparrow \\ \text{p} \end{array} = q^{t_{\blacksquare}} \begin{array}{c} \uparrow \\ \uparrow \end{array}, \quad (3.2.44)$$

$$\begin{array}{c} \circlearrowright \end{array} = \delta_{\blacksquare} \mathbf{1}_{\mathcal{T}} \quad (3.2.45)$$

with all the strands carrying the same colour  $\blacksquare \in \{\blacksquare = 1, \blacksquare = 2\}$ ;

- the *two-colour* crossing relations

$$\begin{array}{c} \nearrow \searrow \\ \searrow \nearrow \end{array} = \begin{array}{c} \searrow \nearrow \\ \nearrow \searrow \end{array} \quad \text{and} \quad \begin{array}{c} \searrow \nearrow \\ \nearrow \searrow \end{array} = \begin{array}{c} \nearrow \searrow \\ \searrow \nearrow \end{array}; \quad (3.2.46)$$

- the relations

$$\begin{array}{c} \uparrow \\ \uparrow \end{array} = \begin{array}{c} \uparrow \\ \uparrow \end{array} + \begin{array}{c} \uparrow \\ \uparrow \end{array}, \quad (3.2.47)$$

$$\begin{array}{c} \uparrow \\ \uparrow \end{array} = \begin{array}{c} \uparrow \\ \uparrow \end{array}, \quad \begin{array}{c} \uparrow \\ \uparrow \end{array} = 0, \quad \begin{array}{c} \uparrow \\ \uparrow \end{array} = 0 \quad \text{and} \quad \begin{array}{c} \uparrow \\ \uparrow \end{array} = \begin{array}{c} \uparrow \\ \uparrow \end{array}, \quad (3.2.48)$$

expressing the fact that  $\blacksquare = \blacksquare \oplus \blacksquare$ .

Relation (3.2.46) identifies positive and negative crossings between green and red strands so, from now on, we will suppress the crossing data and simply draw

$$\begin{array}{c} \nearrow \searrow \\ \searrow \nearrow \end{array} \quad \text{and} \quad \begin{array}{c} \searrow \nearrow \\ \nearrow \searrow \end{array}$$

in our diagrams. Moreover, note that the HOMFLY, twist and dimension relations do not hold for the orange object. Instead, we have:

**Lemma 3.2.9.** *The following relations hold in  $\text{Rep}_q \mathcal{L}_t$ :*

$$\begin{array}{c} \nearrow \searrow \\ \searrow \nearrow \end{array} - \begin{array}{c} \searrow \nearrow \\ \nearrow \searrow \end{array} = (q - q^{-1}) \left( \begin{array}{c} \uparrow \\ \uparrow \end{array} + \begin{array}{c} \uparrow \\ \uparrow \end{array} \right), \quad (3.2.49)$$

$$\text{orange loop} = q^{t_1} \text{green strand} + q^{t_2} \text{red strand}, \quad (3.2.50)$$

$$\text{orange circle} = \delta_1 + \delta_2. \quad (3.2.51)$$

*Proof.* These relations are the diagrammatic counterpart of the fact that the ribbon structure of  $\text{Rep}_q L_t$  is the extension under direct sums and summands of that of  $\widetilde{\text{Rep}_q \text{GL}_t} \boxtimes_{\mathbb{C}(q)} \widetilde{\text{Rep}_q \text{GL}_t}$ . They follow by applying relation (3.2.47) to every strand, then using naturality to slide dots along crossings and finally applying relations (3.2.43)-(3.2.46) to simplify the resulting expressions.  $\square$

**Remark 3.2.10.** Again,  $\text{Rep}_q L_t$  is universal among the categories  $\text{Rep}_q L$ . Specialising  $t_1$  and  $t_2$  at integer values  $m, n \in \mathbb{N}$ , we get an evaluation morphism

$$\varphi_{m,n} : \mathbb{k} \otimes_{\mathbb{C}(q)} \mathbb{k} \rightarrow \mathbb{C}(q),$$

and there is a canonical evaluation functor

$$\text{ev}_{m,n}^L : \text{Rep}_q L_t \rightarrow \text{Rep}_q L$$

mapping  $\uparrow$  and  $\uparrow$  to the natural representations of  $U_q(\mathfrak{gl}_m)$  and  $U_q(\mathfrak{gl}_n)$ , respectively, and  $\uparrow$  to their direct sum.  $\diamond$

**Lemma 3.2.11.** *For any oriented surface  $S$ , we have an equivalence of  $(\mathbb{k} \otimes_{\mathbb{C}(q)} \mathbb{k})$ -linear categories*

$$\text{SkCat}_{L_t}(S) \simeq \text{SkCat}_{\text{GL}_t} \boxtimes_{\mathbb{C}(q)} \text{SkCat}_{\text{GL}_t}(S).$$

*In particular, this induces an algebra isomorphism*

$$\text{SkAlg}_{L_t}(S) \cong \text{SkAlg}_{\text{GL}_t}(S) \otimes_{\mathbb{C}(q)} \text{SkAlg}_{\text{GL}_t}(S).$$

*Proof.* We define a functor  $\text{Rib}_{\text{Rep}_q L_t}(S) \rightarrow \text{SkCat}_{\text{GL}_t}(S) \boxtimes \text{SkCat}_{\text{GL}_t}(S)$  by separating the green and the red colours on the first and second factors, respectively. By the definition of  $\text{Rep}_q L_t$ , it is compatible with the skein relations, so it factors through the skein category, yielding the desired equivalence.  $\square$

### 3.2.3 The restriction functor

The restriction functor

$$\text{res}_{m,n} : \text{Rep}_q \text{GL}_{m+n} \rightarrow \text{Rep}_q L$$

also admits a HOMFLY version that we construct diagrammatically in this section. We first define a coproduct  $\Delta : \mathbb{k} \rightarrow \mathbb{k} \otimes_{\mathbb{C}(q)} \mathbb{k}$  by setting

$$\Delta(q^t) = q^{t_1} q^{t_2}, \quad \Delta(\delta) = \delta_1 q^{t_2} + q^{-t_1} \delta_2 \quad (3.2.52)$$

on generators and extending  $\mathbb{C}(q)$ -linearly.

**Lemma 3.2.12.** *The map  $\Delta: \mathbb{k} \rightarrow \mathbb{k} \otimes_{\mathbb{C}(q)} \mathbb{k}$  is well-defined and coassociative.*

*Proof.* This is an easy computation. Firstly,

$$\begin{aligned} \Delta(q^t - q^{-t}) &= q^{t_1} q^{t_2} - q^{-t_1} q^{-t_2} \\ &= (q^{t_1} - q^{-t_1}) q^{t_2} + q^{-t_1} (q^{t_2} - q^{-t_2}) \\ &= (q - q^{-1}) (\delta_1 q^{t_2} + q^{-t_1} \delta_2) = (q - q^{-1}) \Delta(\delta), \end{aligned}$$

so the map is well-defined. For the coassociativity, we have

$$(1 \otimes \Delta) \circ \Delta(q^t) = q^{t_1} q^{t_2} q^{t_3} = (\Delta \otimes 1) \circ \Delta(q^t),$$

and

$$\begin{aligned} (1 \otimes \Delta) \circ \Delta(\delta) &= \delta_1 (q^{t_2} q^{t_3}) + q^{-t_1} (\delta_2 q^{t_3} + q^{-t_2} \delta_3) \\ &= (\delta_1 q^{t_2} + q^{-t_1} \delta_2) q^{t_3} + (q^{-t_1} q^{-t_2}) \delta_3 = (\Delta \otimes 1) \circ \Delta(\delta). \end{aligned}$$

□

**Remark 3.2.13.** Specialising formulas (3.2.52) at  $t = m + n$ ,  $t_1 = m$  and  $t_2 = n$ , with  $m, n \in \mathbb{N}$ , we recover the identities

$$q^{m+n} = q^m q^n, \quad [m+n]_q = q^n [m]_q + q^{-m} [n]_q,$$

expressing the quantum dimension of the fundamental representation of  $U_q(\mathfrak{gl}_{m+n})$  in terms of the quantum dimensions of the fundamental representations of  $U_q(\mathfrak{gl}_m)$  and  $U_q(\mathfrak{gl}_n)$ . Again, we may think of  $\Delta$  as interpolating these formulas. ◇

We introduce the following notation:

$$\begin{array}{c} \text{cross with blue square} := \text{cross} + (q - q^{-1}) \begin{array}{c} \uparrow \text{red} \\ \downarrow \text{green} \end{array}, \end{array} \quad (3.2.53)$$

$$\begin{array}{c} \text{cross with blue square} := \text{cross} - (q - q^{-1}) q^{-t_2} \begin{array}{c} \text{red arc} \\ \text{green arc} \end{array}, \end{array} \quad (3.2.54)$$

$$\begin{array}{c} \text{vertical line with orange square} := \begin{array}{c} \uparrow \text{green} \\ \downarrow \text{green} \end{array} + q^{-t_1} \begin{array}{c} \uparrow \text{red} \\ \downarrow \text{red} \end{array}, \end{array} \quad (3.2.55)$$

$$\begin{array}{c} \text{vertical line with white square} = \left( \begin{array}{c} \uparrow \text{orange} \\ \downarrow \text{orange} \end{array} \right)^{-1} = \begin{array}{c} \uparrow \text{green} \\ \downarrow \text{green} \end{array} + q^{t_1} \begin{array}{c} \uparrow \text{red} \\ \downarrow \text{red} \end{array}. \end{array} \quad (3.2.56)$$

These morphisms are interpolations of the morphisms (3.1.33) and (3.1.34) of  $\text{Rep}_q L_t$ . The following theorem provides us with an interpolating version of the restriction of representations from  $U_q(\mathfrak{gl}_{m+n})$  to  $U_q(\mathfrak{l})$ :

**Theorem 3.2.14.** *There exists a unique strict monoidal  $\Delta$ -linear functor*

$$\text{res}_t: \text{Rep}_q \text{GL}_t \rightarrow \text{Rep}_q \text{L}_t \quad (3.2.57)$$

determined by

$$\uparrow \mapsto \uparrow, \quad \downarrow \mapsto \downarrow$$

and

*Proof.* We check that the mapping in the statement yields a uniquely determined well-defined functor

$$\widetilde{\text{Rep}_q \text{GL}_t} \rightarrow \widetilde{\text{Rep}_q \text{GL}_t} \boxtimes_{\mathbb{C}(q)} \widetilde{\text{Rep}_q \text{GL}_t}.$$

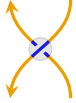
The restriction functor  $\text{res}_t$  is then obtained by extending to the Cauchy completions by universal property.

Proposition 3.2.4 provides us with a presentation by generators and relations of the category  $\widetilde{\text{Rep}_q \text{GL}_t}$ . Since  $\text{res}_t$  is defined on generators, it will be unique and strict monoidal if it is well-defined. We check that it preserves relations (3.2.38)-(3.2.42). For relation (3.2.38), we have

$$\begin{aligned} & \text{(3.2.53)} \quad \text{Diagram} = \text{Diagram} + (q - q^{-1}) \left( \text{Diagram} + \text{Diagram} \right) + (q - q^{-1})^2 \text{Diagram} \\ & \text{(3.2.47)+(3.2.46)+(3.2.38)} \quad (q - q^{-1}) \left( \text{Diagram} + \text{Diagram} + \text{Diagram} + \text{Diagram} \right) \\ & \quad + \left( \text{Diagram} + \text{Diagram} + \text{Diagram} + \text{Diagram} \right) + (q - q^{-1})^2 \text{Diagram} \\ & \text{(3.2.47)} \quad (q - q^{-1}) \left( \text{Diagram} + (q - q^{-1}) \text{Diagram} \right) + \text{Diagram} \\ & = (q - q^{-1}) \text{Diagram} + \text{Diagram}. \end{aligned}$$

For the zig-zag relations (3.2.39), there is nothing to check. Applying  $\text{res}_t$  to the left-hand side of (3.2.40), we have

$$\text{res}_t \left( \text{Diagram} \right) = \text{Diagram} + (q - q^{-1}) q^{t_1} \text{Diagram}.$$

Composing with  and applying the one-colour twist relations, we obtain the identity, so (3.2.40) is preserved. On the other hand,

$$q^{t_1} q^{t_2} \text{ (orange twist) } = q^{t_1} q^{t_2} \text{ (green twist) } + q^{t_1} q^{t_2} \text{ (red twist) } - (q - q^{-1}) \text{ (green and red twist) } = (q^{t_2} \delta_1 + q^{-t_1} \delta_2) \mathbf{1}_{\mathcal{T}},$$

so relation (3.2.41) is preserved. Finally,

$$\begin{aligned} \text{res}_t \left( \text{blue crossing} \right) &= \text{orange crossing} + (q - q^{-1}) \left( \text{orange and green crossing} + \text{orange and red crossing} \right) \\ &\quad + (q - q^{-1})^2 \left( \text{orange and green and red crossing} + \text{orange and red and green crossing} \right) = \text{res}_t \left( \text{blue crossing} \right), \end{aligned}$$

so the functor is well-defined.  $\square$

**Remark 3.2.15.** Along the same lines as Remarks 3.2.10 and 3.2.13, the functor  $\text{res}_t$  interpolates the restriction functors  $\text{res}_{m,n}: \text{Rep}_q \text{GL}_{m+n} \rightarrow \text{Rep}_q \text{L}$ . That is, the diagram

$$\begin{array}{ccc} \text{Rep}_q \text{GL}_t & \xrightarrow{\text{res}_t} & \text{Rep}_q \text{L}_t \\ \text{ev}_{m+n}^G \downarrow & & \downarrow \text{ev}_{m,n}^L \\ \text{Rep}_q \text{GL}_{m+n} & \xrightarrow{\text{res}_{m,n}} & \text{Rep}_q \text{L} \end{array}$$

commutes for any  $m, n \in \mathbb{N}$ .  $\diamond$

**Remark 3.2.16.** The restriction functor  $\text{res}_t$  is strictly monoidal, but it does not preserve the distinguished dualities. The natural isomorphism (1.1.3) identifying duals is determined here by

$$\varphi_{\uparrow}^r = \text{orange strand with blue dot} = \text{green strand} + q^{-t_1} \text{red strand}.$$

Moreover,  $\text{res}_t$  is not pivotal: the categories  $\text{Rep}_q \text{GL}_t$  and  $\text{Rep}_q \text{L}_t$  are endowed with trivial pivotal structures, but the isomorphism

$$\eta_{\uparrow}^r : \uparrow = \text{res}(\uparrow)^{**} \xrightarrow{(\varphi_{\uparrow}^r)^*} \text{res}(\uparrow)^* \xrightarrow{(\varphi_{\downarrow}^r)^{-1}} \text{res}(\uparrow^{**}) \xrightarrow{\text{res}(\text{id}_{\downarrow})} \text{res}(\uparrow) = \uparrow$$

defined in (1.1.5) is represented by

$$\eta_{\uparrow}^r = q^{t_2} \text{ (green strand) } + q^{-t_1} \text{ (red strand) },$$

which is not the identity of  $\uparrow$  in  $\text{Rep}_q L_t$ . Similarly, a direct computation shows that

$$\begin{aligned} \text{res}_t \left( \text{blue arc} \right) &= q^{-t_2} \text{green arc} + \text{red arc}, \\ \text{res}_t \left( \text{blue arc} \right) &= q^{t_2} \text{green arc} + \text{red arc}. \end{aligned}$$

Therefore, the isomorphism identifying the restriction of the left duality with the distinguished one is given by

$$\varphi_{\uparrow}^1 = q^{-t_2} \text{green line} + \text{red line} =: 1$$

and we obtain an isomorphism

$$\eta_{\uparrow}^1 = q^{-t_2} \text{green line} + q^{t_1} \text{red line}$$

between the distinguished and the restricted double left duals. This is a diagrammatic counterpart of the properties of the specialised restriction functor  $\text{res}_{m,n}$ , as explained in Section 3.1.2.  $\diamond$

**Remark 3.2.17.** It easily follows from the definition of the restriction functor that

$$\begin{aligned} \text{blue crossing} &= \text{green crossing}, & \text{blue crossing} &= \text{red crossing}, & \text{blue crossing} &= \text{red crossing} \quad \text{and} \quad \text{blue crossing} &= \text{green crossing}, \end{aligned}$$

but, for instance,

$$\text{blue crossing} \neq \text{green crossing} \quad \text{and} \quad \text{blue crossing} = \text{red crossing}.$$

Again, this is a diagrammatic counterpart of the fact that the  $R$ -matrix of  $U_q(\mathfrak{gl}_{m+n})$  belongs to (a completion of)  $U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^-)$ , where  $\mathfrak{b}^\pm$  are the corresponding positive and negative Borel subalgebras, respectively. Consequently, the restriction of the braiding of  $\text{Rep}_q \text{GL}_{m+n}$  to  $\text{Rep}_q L$  is natural in the first factor (resp. in the second) for morphisms of  $U_q(\mathfrak{l})$ -modules commuting with the action of  $U_q(\mathfrak{p})$  (resp.  $U_q(\mathfrak{p}^-)$ ).  $\diamond$

### 3.3 The HOMFLY parabolic central algebra

We construct next an interpolating version of the formalism of parabolic restriction. We define first a category  $\text{Rep}_q P_t$  interpolating the categories of quantum representations of the parabolic group  $P \subset \text{GL}_{m+n}$  of block-triangular matrices. Next, we construct HOMFLY analogues of the restriction functors in diagram 3.3.59, and we show that these functors endow  $\text{Rep}_q P_t$  with a  $(\text{Rep}_q \text{GL}_t, \text{Rep}_q L_t)$ -central algebra structure. Finally, we exhibit a  $\text{Rep}_q P_t$ -centred  $(\text{Rep}_q \text{GL}_t, \text{Rep}_q L_t)$ -bimodule structure on  $\text{Rep}_q L_t$ .

### 3.3.1 The planar category $\text{Rep}_q P_t$

Based on the graphical calculus for rigid categories, we define in this section a category  $\text{Rep}_q P_t$  interpolating the categories  $\text{Rep}_q P_{m,n}$  generated by the restriction of the fundamental representations of  $\text{Rep}_q \text{GL}_{m+n}$  and  $\text{Rep}_q \text{L}_{m,n}$ . The restriction of the pivotal structures of these two categories are not natural, so  $\text{Rep}_q P_{m,n}$  is not naturally a pivotal category (or the restriction functors are not pivotal). Therefore, we define the interpolating category  $\text{Rep}_q P_t$  using the planar graphical calculus for rigid categories. More precisely, we define first a category  $\text{Rig}(P)$  whose morphisms are planar graphs as in Section 1.2.2. In particular, edges will be coloured and labelled with integers, and we will introduce a collection of allowed vertices representing the restriction of structural morphisms from both  $\text{Rep}_q \text{GL}_t$  and  $\text{Rep}_q \text{L}_t$ . This category comes with a natural functor

$$\text{Rig}(P) \rightarrow \text{Rep}_q \text{L}_t$$

and we define  $\text{Rep}_q P_t$  as the quotient of  $\text{Rig}(P)$  by its kernel. In particular,  $\text{Rep}_q P_t$  embeds into the rigid category underlying  $\text{Rep}_q \text{L}_t$ .

**Definition 3.3.1.** The category  $\text{Rig}(P)$  is the category whose:

- objects are finite words on  $\{\uparrow, \downarrow, \uparrow, \downarrow, \uparrow, \downarrow\}$ ;
- morphisms are  $\mathbb{k}$ -linear combinations of polarised planar graphs (not up to isotopy!) embedded in  $\mathbb{R} \times [0, 1]$ , as defined in Section 1.2.2, whose edges are coloured with  $\blacksquare$ ,  $\blacksquare$  or  $\blacksquare$  and whose coupons admit the following decorations:

$$\begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array}, \quad \begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array}, \quad \begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array}, \quad \begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array}, \quad \begin{array}{c} \diagup \\ \diagdown \end{array}, \quad \begin{array}{c} \diagup \\ \diagdown \end{array}, \quad (3.3.58)$$

for all the possible combinations of orientations and colours in the last two vertices.

◇

We define an evaluation functor

$$\text{ev}^P : \text{Rig}(P) \rightarrow \text{Rep}_q \text{L}_t$$

as follows. On objects, it switches  $\blacksquare \rightsquigarrow \blacksquare$  and preserves all the orientations and the green and red colour. On morphisms, it is defined by modifying the decoration of the graphs locally around coupons, cups and caps in the following way:

- for the first four morphisms in (3.3.58), it just switches  $\blacksquare \rightsquigarrow \blacksquare$ , so that

$$\begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array}, \quad \begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array}, \quad \begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array}, \quad \begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array} \mapsto \begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array}, \quad \begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array}, \quad \begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array}, \quad \begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array};$$

- for the last two morphisms in (3.3.58), we set

$$\begin{array}{c} \text{X} \end{array} \mapsto \text{res}_t \left( \begin{array}{c} \text{X} \end{array} \right)$$

if both strands are coloured in blue, and we map the rest of those morphisms to the underlying crossing;

- for blue cups and caps with any orientation, we have

$$\begin{array}{c} \text{cup} \end{array}, \begin{array}{c} \text{cap} \end{array} \mapsto \text{res}_t \left( \begin{array}{c} \text{cup} \end{array} \right), \text{res}_t \left( \begin{array}{c} \text{cap} \end{array} \right),$$

while the rest of cups and caps remain unmodified.

**Lemma 3.3.2.** *The evaluation functor  $\text{ev}^P: \text{Rig}(\mathcal{P}) \rightarrow \text{Rep}_q \mathcal{L}_t$  is well-defined.*

*Proof.* This is straightforward since applying the process above always yields a well-formed graph representing a morphism of  $\text{Rep}_q \mathcal{L}_t$ . Moreover, it is functorial since we only modify graphs locally around coupons, cups and caps.  $\square$

**Definition 3.3.3.** The category  $\text{Rep}_q \mathcal{P}_t$  is the Cauchy completion of the quotient of  $\text{Rig}(\mathcal{P})$  by the kernel of  $\text{ev}^P$ .  $\diamond$

The category  $\text{Rep}_q \mathcal{P}_t$  is monoidal, with tensor product induced by concatenating words and horizontally stacking graphs. By definition, the evaluation functor induces a strong monoidal faithful functor

$$j_t^*: \text{Rep}_q \mathcal{P}_t \rightarrow \text{Rep}_q \mathcal{L}_t,$$

which is an interpolation of the restriction functors  $j_{m,n}^*: \text{Rep}_q \mathcal{P} \rightarrow \text{Rep}_q \mathcal{L}$  described in Section 3.1.3. This functor is, however, not compatible with the extra structure (braided and pivotal) of  $\text{Rep}_q \mathcal{L}_t$ , as we explain in the following remark.

**Remark 3.3.4.** Isotopy is not a relation in  $\text{Rep}_q \mathcal{P}_t$ . For instance,

$$\text{ev}^P \left( \begin{array}{c} \text{blue strand} \\ \text{green strand} \end{array} \right) = \begin{array}{c} \text{orange strand} \end{array} \quad \text{and} \quad \text{ev}^P \left( \begin{array}{c} \text{blue strand} \\ \text{green strand} \end{array} \right) = q^{t_1} \begin{array}{c} \text{orange strand} \end{array},$$

while the graphs are related by a planar isotopy.  $\diamond$

Yet, the category  $\text{Rep}_q \mathcal{P}_t$  still has both left and right duals:

**Lemma 3.3.5.** *The category  $\text{Rep}_q P_t$  is rigid, with distinguished duals given by reversing orientations and evaluations/coevaluations represented by caps/cups.*

*Proof.* This is straightforward. Indeed, even though isotopy is not a relation, the image by  $\text{ev}^P$  of the zigzag isomorphisms are precisely the zigzags of  $\text{Rep}_q L_t$ , so the zigzag identities are actual relations in  $\text{Rep}_q P_t$ .  $\square$

Note that, by the definition of the evaluation functor  $\text{ev}^P$ , the distinguished duality of  $\text{Rep}_q P_t$  is a lift of the dualities of  $\text{Rep}_q GL_t$  and  $\text{Rep}_q L_t$  (precisely, the functors  $\iota_t^*$  and  $\pi_t^*$  defined in the next section preserve the distinguished dualities). While for every object  $U \in \text{Rep}_q P_t$ , we still have  $U^{**} = U$ , the incompatibility with isotopy (cf. Remark 3.3.4) is a diagrammatic counterpart of the fact that the restriction to  $\text{Rep}_q P$  of the pivotal structures of  $\text{Rep}_q GL_{m+n}$  and  $\text{Rep}_q L$  are not natural.

**Remark 3.3.6.** The presentation of  $\text{Rep}_q P_t$  given here differs from the one in [G6m26]. Indeed, the evaluation functor defined there assigns a different value to blue cups and caps, resulting in a different distinguished duality in  $\text{Rep}_q P_t$ . For that duality, the identification  $U^{**} \cong U$  induced in  $\text{Rep}_q P_t$  by the pivotal structure of  $\text{Rep}_q GL_t$  is not trivial and must be included explicitly in the presentation of the planar category. By replacing that duality with a lift of the duality of  $\text{Rep}_q GL_t$  (as we do here), the identification becomes trivial so we do not need to represent it anymore and can drop the integer labels appearing in [G6m26], obtaining a cleaner presentation of  $\text{Rep}_q P_t$ .  $\diamond$

We finish the section with the following proposition, which is an analogue for  $\text{Rep}_q P_t$  of Remarks 3.2.6 and 3.2.10:

**Proposition 3.3.7.** *Let  $m, n \in \mathbb{N}$  and let  $\varphi_{m,n}: \mathbb{k} \otimes_{\mathbb{C}(q)} \mathbb{k} \rightarrow \mathbb{C}(q)$  be the evaluation morphism defined in Remark 3.2.10. Then, there exists a monoidal  $\varphi_{m,n}$ -linear functor*

$$\text{ev}_{m,n}^P: \text{Rep}_q P_t \rightarrow \text{Rep}_q P$$

mapping

$$\uparrow \mapsto \iota_{m,n}^*(V_{m+n}), \quad \uparrow \mapsto \pi_{m,n}^*(V_m), \quad \uparrow \mapsto \pi_{m,n}^*(V_n),$$

where  $\iota_{m,n}^*$  and  $\pi_{m,n}$  are the restriction functors defined in Section 3.1.3.

*Proof.* The restriction functor  $j_{m,n}^*: \text{Rep}_q P \rightarrow \text{Rep}_q L$  is faithful so we may describe  $\text{Rep}_q P$  as a subcategory of  $\text{Rep}_q L$ . By construction, the image of the composition

$$\text{Rep}_q P \xrightarrow{j_t^*} \text{Rep}_q L_t \xrightarrow{\text{ev}_{m,n}^L} \text{Rep}_q L$$

lies in that subcategory, so it lifts to a monoidal functor  $\text{ev}_{m,n}^P$  as in the statement.  $\square$

We obtain a commutative diagram of functors

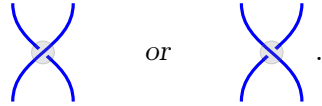
$$\begin{array}{ccc} \mathrm{Rep}_q \mathbf{P}_t & \xrightarrow{j_t^*} & \mathrm{Rep}_q \mathbf{L}_t \\ \mathrm{ev}_{m,n}^{\mathbf{P}} \downarrow & & \downarrow \mathrm{ev}_{m,n}^{\mathbf{L}} \\ \mathrm{Rep}_q \mathbf{P} & \xrightarrow{j_{m,n}^*} & \mathrm{Rep}_q \mathbf{L} \end{array}$$

making precise the claim that  $j_t^*$  is an interpolation version of the functors  $j_{m,n}^*$ .

### 3.3.2 The central algebra structure

In this section, we interpolate the restriction functors  $\iota_{m,n}^*$  and  $\pi_{m,n}^*$  defined in Section 3.1.3 and show that they lift to the Drinfeld centre of  $\mathrm{Rep}_q \mathbf{P}_t$ , yielding a central structure which is a universal version of the central algebra structure of  $\mathrm{Rep}_q \mathbf{P}_t$ .

**Proposition 3.3.8.** *Given a morphism  $f$  of  $\mathrm{Rep}_q \mathbf{GL}_t$  represented by a ribbon diagram  $\Gamma_f$ , let  $\widehat{\Gamma}_f$  be the planar diagram obtained by replacing every crossing of  $\Gamma_f$  by either*



*The assignment  $\uparrow \mapsto \uparrow, \downarrow \mapsto \downarrow$  and  $f \mapsto \widehat{\Gamma}_f$  yields a strict monoidal functor*

$$\iota_t^*: \mathrm{Rep}_q \mathbf{GL}_t \rightarrow \mathrm{Rep}_q \mathbf{P}_t.$$

Moreover,  $\mathrm{res}_t = j_t^* \circ \iota_t^*$ .

*Proof.* Let  $\Gamma_f^1$  and  $\Gamma_f^2$  be two diagrams representing the same morphism of  $\mathrm{Rep}_q \mathbf{GL}_t$ . By definition of the evaluation functor  $\mathrm{ev}^{\mathbf{P}}$  and the restriction functor  $\mathrm{res}_t$ , we have

$$\mathrm{ev}^{\mathbf{P}} \left( \widehat{\Gamma}_f^1 \right) = \mathrm{res}_t(f) = \mathrm{ev}^{\mathbf{P}} \left( \widehat{\Gamma}_f^2 \right)$$

since  $\mathrm{res}_t$  is well-defined. This proves that  $\iota_t^*$  is well-defined and that  $\mathrm{res}_t = j_t^* \circ \iota_t^*$ . The assignment is clearly functorial and preserves the tensor product.  $\square$

**Proposition 3.3.9.** *With the same notation as in the previous proposition, we have a strict monoidal functor*

$$\pi_t^*: \mathrm{Rep}_q \mathbf{L}_t \rightarrow \mathrm{Rep}_q \mathbf{P}_t$$

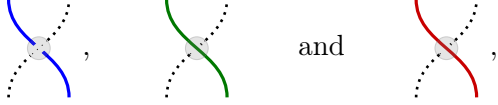
*which is the identity on objects and maps  $f \mapsto \widehat{\Gamma}_f$  on morphisms. We have  $j_t^* \circ \pi_t^* = \mathrm{id}_{\mathrm{Rep}_q \mathbf{L}_t}$ .*

*Proof.* This follows straightforwardly from the definition of  $\mathrm{Rep}_q \mathbf{P}_t$ .  $\square$

We have now four functors

$$\begin{array}{ccc}
 & \text{Rep}_q \mathbf{P}_t & \\
 \iota_t^* \nearrow & & \nwarrow j_t^* \\
 \text{Rep}_q \mathbf{GL}_t & \xrightarrow{\text{res}_t} & \text{Rep}_q \mathbf{L}_t \\
 & \nwarrow \pi_t^* &
 \end{array} \tag{3.3.59}$$

interpolating the restriction functors presented in Section 3.1.2. We show next that the functor  $\iota_t^* \otimes \pi_t^*$  lifts to the Drinfeld centre, endowing  $\text{Rep}_q \mathbf{P}_t$  with the structure of a  $(\text{Rep}_q \mathbf{GL}_t, \text{Rep}_q \mathbf{L}_t)$ -central algebra. Consider the family of isomorphisms represented by the diagrams



where the coloured strands may have any orientation and the dotted one can be replaced by any collection of parallel coloured strands. By Remark 3.2.17, these isomorphisms are natural, hence define a strict monoidal functor

$$Z: \text{Rep}_q \mathbf{P}_t \rightarrow Z_{\text{Dr}}(\text{Rep}_q \mathbf{P}_t).$$

Let

$$\begin{array}{lll}
 F := \iota_t^* \otimes \pi_t^*: & \text{Rep}_q \mathbf{GL}_t \boxtimes \text{Rep}_q \mathbf{L}_t & \rightarrow \text{Rep}_q \mathbf{P}_t, \\
 & (\mathbf{u}, \mathbf{v}, \mathbf{w}) & \mapsto \mathbf{uvw} \\
 & (\mathbf{f}, \mathbf{g}, \mathbf{h}) & \mapsto \iota_t^*(\mathbf{f}) \otimes \pi_t^*(\mathbf{g}) \otimes \pi_t^*(\mathbf{h}).
 \end{array}$$

**Theorem 3.3.10.** *The functor  $Z \circ F$  lifts to a braided monoidal functor*

$$(Z \circ F, J): \text{Rep}_q \mathbf{GL}_t \boxtimes \text{Rep}_q \mathbf{L}_t^{\text{bop}} \rightarrow Z_{\text{Dr}}(\text{Rep}_q \mathbf{P}_t),$$

turning  $\text{Rep}_q \mathbf{P}_t$  into a  $(\text{Rep}_q \mathbf{GL}_t, \text{Rep}_q \mathbf{L}_t)$ -central algebra.

*Proof.* The family of isomorphisms

$$J = \left( \begin{array}{c} \text{Diagram with 6 strands: } \mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1, \mathbf{u}_2, \mathbf{v}_2, \mathbf{w}_2 \\ \text{Strands } \mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1 \text{ are blue, green, red respectively. Strands } \mathbf{u}_2, \mathbf{v}_2, \mathbf{w}_2 \text{ are blue, green, red respectively.} \end{array} \right) : F(\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1) \otimes F(\mathbf{u}_2, \mathbf{v}_2, \mathbf{w}_2) \rightarrow F(\mathbf{u}_1 \mathbf{u}_2, \mathbf{v}_1 \mathbf{v}_2, \mathbf{w}_1 \mathbf{w}_2)$$

defines a monoidal structure on  $Z \circ F$ . Checking that  $(Z \circ F, J)$  is braided is an easy computation.  $\square$

### 3.3.3 The centred bimodule structure

Besides the central algebra structure described in Theorem 3.3.10, the formalism described in this chapter provides us with the following two (trivial) central algebras:

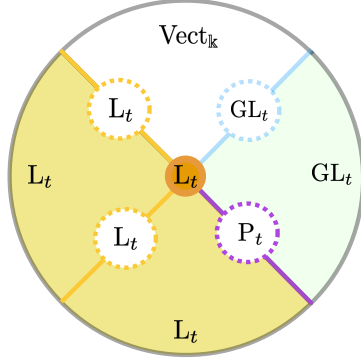


Figure 3.1: Local model for the centred bimodule structure of  $\text{Rep}_q L_t$ .

- $\text{Rep}_q GL_t$  has a  $(\text{Vect}_{\mathbb{k}}, \text{Rep}_q GL_t)$ -central structure induced by the functor

$$\text{Vect}_{\mathbb{k}} \boxtimes \text{Rep}_q GL_t \simeq \text{Rep}_q GL_t \xrightarrow{\text{id}} \text{Rep}_q GL_t;$$

- $\text{Rep}_q L_t$  has a  $(\text{Vect}_{\mathbb{k}}, \text{Rep}_q L_t)$ -central structure induced by the functor

$$\text{Vect}_{\mathbb{k}} \boxtimes \text{Rep}_q L_t \simeq \text{Rep}_q L_t \xrightarrow{\text{id}} \text{Rep}_q L_t.$$

On the other hand, the functors

$$\text{res}_t: \text{Rep}_q GL_t \rightarrow \text{Rep}_q L_t \quad \text{and} \quad j_t^*: \text{Rep}_q P_t \rightarrow \text{Rep}_q L_t$$

induce actions of  $\text{Rep}_q GL_t$  and  $\text{Rep}_q P_t$  on  $\text{Rep}_q L_t$ . Moreover,  $\text{Rep}_q L_t$  acts trivially on itself through the identity functor (see Figure 3.1).

**Theorem 3.3.11.** *The previous actions induce a left module structure over*

$$\text{tr}_{\text{Rep}_q L_t} \left( \text{Rep}_q P_t \boxtimes_{\text{Rep}_q GL_t} \text{Rep}_q GL_t \boxtimes_{\text{Vect}_{\mathbb{k}}} \text{Rep}_q L_t \right)$$

on  $\text{Rep}_q L_t$ .

*Proof.* By Corollary 2.1.26, this amounts to showing that the actions of  $\text{Rep}_q GL_t$ ,  $\text{Rep}_q L_t$  and  $\text{Vect}_{\mathbb{k}}$  through the three central algebras are centred. The action of  $\text{Vect}_{\mathbb{k}}$  is always trivial so there is nothing to prove. For the action of  $\text{Rep}_q GL_t$ , the commutativity of the outer triangle in (3.3.59) implies that both the actions through  $\text{Rep}_q GL_t$  and  $\text{Rep}_q P_t$  of an object  $x \in \text{Rep}_q GL_t$  are given by tensoring with  $\text{res}_t(x)$  on the left or right, respectively. A centred structure is then provided by the braiding of  $\text{Rep}_q L_t$ . The same argument holds for the action of  $\text{Rep}_q L_t$  using the fact that  $j_t^* \circ \pi_t^* = \text{id}_{\text{Rep}_q L_t}$ .  $\square$



# Chapter 4

## The Turaev coproduct

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### 4.1 HOMFLY defect theories

We describe the spaces assigned by the HOMFLY skein theory to some framed surfaces with a well-chosen stratification.

Let  $(S, f)$  be a connected framed surface with non-empty boundary  $\partial S$ . Let  $\mathcal{B} \neq \emptyset$  be a marking on  $S$ . We consider decorations  $\phi$  of  $S$  of the following form:

- $(S, \phi)$  is a walled surface whose line defects are decorated with  $\text{Rep}_q L_t$  and whose 2-dimensional strata are decorated with  $\text{Rep}_q L_t$  and  $\text{Rep}_q P_t$ ;
- near the boundary, the decoration consists of a collection of discs and cylinders inserted along some of the marked boundary components, as in Figure 4.1.

We write  $S^{L_t}$  and  $S^{P_t}$  for the two dimensional strata labelled with  $\text{Rep}_q L_t$  and  $\text{Rep}_q P_t$ , respectively.

We extend the decoration on  $S$  to a decoration of the cylinder  $S \times [-1, 1]$  in the following way. First,  $S \times \{0\}$  is copy of the walled  $(S, f, \phi)$ , that is, the cylinder contains a horizontal wall identified with the decorated walled surface obtained above by inserting disks and cylinders along the chosen boundary components. Next, we add vertical  $\text{Rep}_q GL_t$ -walls over each defect line on  $S \times \{0\}$ : if  $c$  is such a line, then  $c \times (0, 1]$  is a 2-dimensional strata labelled with  $\text{Rep}_q GL_t$ . Finally, we decorate the 3-dimensional regions. The region  $S \times [-1, 0)$  under the wall is labelled with  $\text{Rep}_q L_t$ ; the regions over

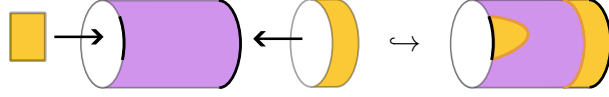


Figure 4.1: We modify the stratification by inserting  $\text{Rep}_q L_t$ -coloured disks and cylinders along some of the marked boundary components. The  $\text{Rep}_q P_t$ -region (in violet) is diffeomorphic to the original surface.

a  $\text{Rep}_q L_t$ -region of  $S \times \{0\}$  are trivial regions labelled with  $\text{Vect}_{\mathbb{k}}$ ; the regions over a  $\text{Rep}_q P_t$ -region of  $S \times \{0\}$  are labelled with  $\text{Rep}_q \text{GL}_t$  (see Figure 4.2).

We will restrict our attention to skeins whose endpoints lie either on the top or bottom basis  $S \times \{\pm 1\}$  of the boundary components marked on the boundary, i.e. we restrict to collections of points

$$B = \{p_1, \dots, p_m\} \subset \bigcup_{C \in \mathcal{B}} C \times \{0\} \cup (S \times \{\pm 1\}) \subset \partial(S \times [-1, 1]).$$

Given a boundary condition  $\nu_B$ , we write

$$\text{SkMod}_H^{\text{def}}(S, f, \nu)$$

for the *HOMFLY defect skein module* of  $S$ , that is, the defect skein module induced by the algebraic data of the HOMFLY parabolic restriction.

For each  $C \in \mathcal{B}$  with the 2-framing  $f_C$  induced by the framing of the surface, write  $\mathcal{H}(C)$  for either  $\text{Rep}_q L_t$  or  $\text{Rep}_q P_t$ , depending on the label of the region where  $C$  lies, if  $C$  is an interval; or the corresponding cylinder category if  $C$  is a circle.

**Definition 4.1.1.** The *HOMFLY skein bimodule* of  $(S, f, \mathcal{B}, \phi)$  is the bimodule functor

$$\mathcal{H}(S, f, \mathcal{B}, \phi): \left( \bigotimes_{C \in \mathcal{B}} \mathcal{H}(C) \right) \boxtimes \text{SkCat}_{\text{GL}_t}(S^{P_t}) \boxtimes \text{SkCat}_{L_t}(S^{L_t})^{\text{op}} \rightarrow \text{Vect},$$

defined on objects by sending a boundary condition  $\nu$  to the HOMFLY defect skein module  $\text{SkMod}_H^{\text{def}}(S, f, \nu)$ . On morphisms, it is defined by gluing rectangles and cylinders along the components marked on the boundary of the vertical wall, and by stacking copies of  $S \times [0, 1]$  and rescaling the second coordinate.  $\diamond$

For the rest of the section we fix a marked walled framed surface  $(S, f, \phi, \mathcal{B})$ , that we will denote simply by  $S$ . We give first an explicit proof that the construction is functorial, following the proof of [Wal06, Theorem 4.4.2]:

**Lemma 4.1.2.** Let  $\text{gl}_C(S)$  be obtained by gluing  $S$  along a boundary component  $C$ , where  $C, \bar{C} \in \mathcal{B}$  and are labelled with the same category. Then,

$$\mathcal{H}(\text{gl}_C(S))(-) = \int^{U \in \mathcal{H}(C)} \mathcal{H}(S)(-, U, U),$$

where the object  $U$  on the right-hand side lie in the components  $C$  and  $\bar{C}$ .

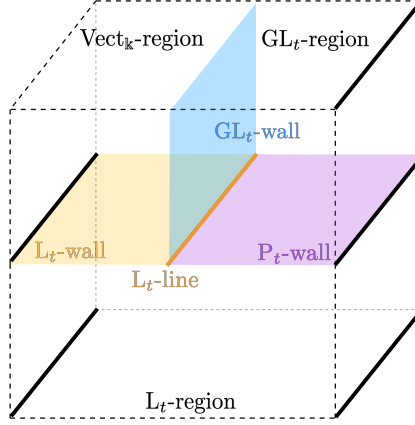


Figure 4.2: Local model around line defect for stratified cylinder.

*Proof.* Gluing along  $C$  induces morphisms

$$\mathrm{gl}_U: \mathcal{H}(S)(-, U, U) \rightarrow \mathcal{H}(\mathrm{gl}_C(S))(-)$$

for every  $U \in \mathcal{H}(C)$ . Moreover, if  $f: U \rightarrow V$  is a morphism in  $\mathcal{H}(C)$ ,  $\Omega$  is a stratified ribbon graph representing an element of  $\mathcal{H}(S)(-, U, V)$  and  $f \star \Omega$  (resp.  $\Omega \star f$ ) are the graphs obtained by gluing  $f$  to  $\Omega$  along  $C$  (resp.  $\bar{C}$ ), then the graphs  $\mathrm{gl}_U(f \star \Omega)$  and  $\mathrm{gl}_V(\Omega \star f)$  are related by an isotopy supported on a tubular neighbourhood of  $\partial S$ . Therefore, they define the same element of  $\mathcal{H}(\mathrm{gl}_C(S))(-)$  and the diagram

$$\begin{array}{ccccc}
 & & \mathcal{H}(S)(-, U, U) & & \\
 & \nearrow^{-\star f} & & \searrow^{\mathrm{gl}_U} & \\
 \mathcal{H}(S)(-, U, V) & & & & \mathcal{H}(\mathrm{gl}_C(S))(-) \\
 & \searrow_{f \star -} & & \nearrow_{\mathrm{gl}_V} & \\
 & & \mathcal{H}(S)(-, V, V) & & 
 \end{array} \tag{4.1.60}$$

commutes. We shall show that  $\mathcal{H}(\mathrm{gl}_C(S))(-)$  is universal for this property. Let  $E$  be a vector space together with linear maps

$$\varphi_U: \mathcal{H}(S)(-, U, U) \rightarrow E$$

making the equivalent diagrams commute. We have to prove that there is a unique linear map

$$\varphi: \mathcal{H}(\mathrm{gl}_C(S))(-) \rightarrow E$$

such that  $\varphi_U = \varphi \circ \mathrm{gl}_U$  for every  $U \in \mathcal{H}(C)$ . Let  $\Omega$  be a stratified ribbon graph in  $\mathrm{gl}_C(S) \times I$  representing an element of  $\mathcal{H}(\mathrm{gl}_C(S))(-)$ . Applying isotopy and stratified skein relations, we may locally project  $\Omega$  onto the defect, so that we may suppose that there is a neighborhood of  $C \times I$  where  $\Omega$  is represented by a planar graph contained in

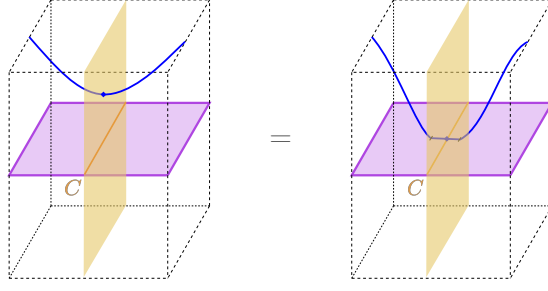


Figure 4.3: Applying isotopy and skein relations, we may always cut a graph along a planar component.

the defect (see Figure 4.3). Let  $U$  be the intersection of  $\Omega$  with  $C \times \{0\}$ , which determines an object of  $\mathcal{H}(C)$ . Cutting along  $C \times I$ , we get a ribbon graph  $\Omega_U$  in  $S \times I$  defining an element of  $\mathcal{H}(S)(-, U, U)$ . If  $\varphi$  exists, then  $\varphi(\Omega) = \varphi_U(\Omega_U)$  so it is uniquely defined.

To prove the existence, suppose first that  $\tilde{\Omega}$  is another stratified ribbon graph representing the same element of  $\mathcal{H}(\text{gl}_C(S))(-)$  and related to  $\Omega$  by an isotopy shifting a collar neighbourhood through  $C \times I$ . Then, cutting along  $C \times I$  yields a ribbon graph  $\tilde{\Omega}_U$  in  $S \times I$  and, since we may suppose that it is planar around  $C \times I$ , the commutativity of (4.1.60) implies that  $\text{gl}_U(\Omega_U) = \text{gl}_U(\tilde{\Omega}_U)$ . On the other hand, if  $\Omega$  and  $\tilde{\Omega}$  are related by an isotopy of  $\text{gl}_C(S)$  supported in a cube not intersecting  $C \times I$ , then  $\Omega_U$  and  $\tilde{\Omega}_U$  are related by the same isotopy of  $S$ , so that they represent the same element of  $\mathcal{H}(S)(-, U, U)$  and  $\varphi_U(\Omega_U) = \varphi_U(\tilde{\Omega}_U)$ . Therefore,  $\varphi$  is compatible with isotopies. Finally, if  $\Omega$  and  $\tilde{\Omega}$  are related by a skein relation on a cube not intersecting  $C \times I$ , then  $\Omega_U$  and  $\tilde{\Omega}_U$  are related by the same relation and they represent the same element. If the skein relation occurs in a cube intersecting  $S$ , we can apply an isotopy moving it off  $C \times I$ . We get stratified ribbon graphs  $\Omega'$  and  $\tilde{\Omega}'$  related by a skein relation on a cube not intersecting  $C \times I$ , so  $\text{gl}_{U'}(\Omega'_{U'}) = \text{gl}_{U'}(\tilde{\Omega}'_{U'})$ . Moreover, the isotopy invariance of  $\varphi$  implies that  $\text{gl}_U(\Omega_U) = \text{gl}_{U'}(\Omega'_{U'})$  and  $\text{gl}_U(\tilde{\Omega}) = \text{gl}_{U'}(\tilde{\Omega}_{U'})$ . This proves that  $\varphi$  is compatible with skein relations.  $\square$

When all the boundary components are coloured with  $\text{Rep}_q L_t$  and the boundary conditions are empty on the top or bottom basis,  $\mathcal{H}$  coincides with the  $\text{Rep}_q L_t$ -framed string-net theory on  $S$ :

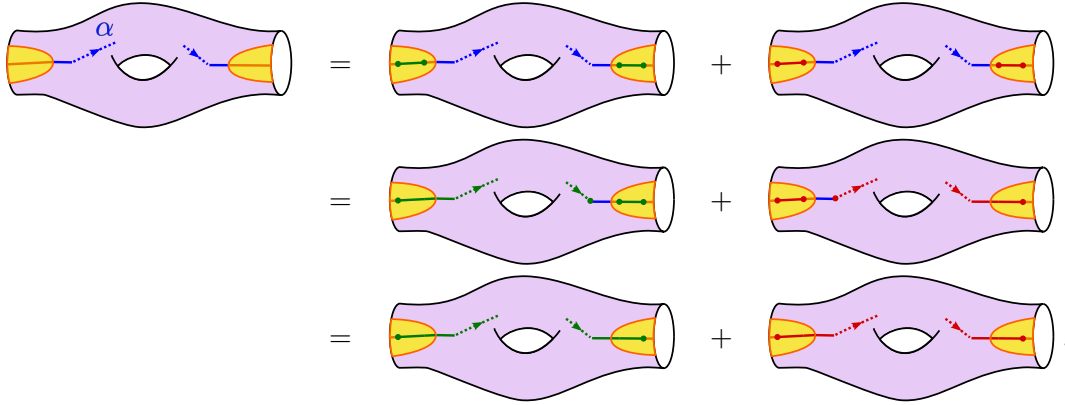
**Proposition 4.1.3.** *There is a natural family of isomorphisms of vector spaces*

$$\mathcal{H}(S)(-, \emptyset, \emptyset) \cong \text{SN}_{L_t}^{\text{fr}}(S)(-),$$

where  $\text{SN}_{L_t}^{\text{def, fr}}(S)(-)$  is the framed defect string-net theory associated with  $\text{Rep}_q L_t$ .

*Proof.* Fix boundary conditions  $U = (U_C)_{C \in \mathcal{B}}$  and let  $\Omega$  be a stratified ribbon graph representing an element of  $\mathcal{H}(S)(U, \emptyset, \emptyset)$ . Since there is no strand attached to  $S \times \{\pm 1\}$ , the 3-dimensional stratified skein relations allow to project  $\Omega$  entirely onto the defect,

so that it can be represented by a planar graph on  $S \times \{0\}$ , coloured by  $\text{Rep}_q P_t$  in the  $P_t$ -region and by  $\text{Rep}_q L_t$  in the  $L_t$ -region. We first prove that this graph can be written as a linear combination of diagrams with no strand coloured in  $\blacksquare$ . Indeed, suppose that  $\alpha$  is a component coloured in blue. If  $\alpha$  is an open strand, then it is attached to one of the intervals/circles marked on the boundary, hence it traverses the 1-dimensional  $L_t$ -defect (through an identity coupon). Crossing these defect lines switches its colour to  $\blacksquare$  near its endpoints. We can then apply relation (3.2.47) inside the  $L_t$ -disks/cylinders. Then, by applying a skein relation around the defect line, one of the inclusions/projections appearing on each term cross the defect, and slide along  $\alpha$  (by naturality, cf. Remark 3.2.17) to reach the opposite endpoint (see figure below). The part of  $\alpha$  lying in the  $P_t$  region is now entirely coloured with  $\blacksquare$  and  $\blacksquare$ :



Similarly, if  $\alpha$  is a closed strand, we can push it close to a marked component by applying isotopy. Then, skein relations around the line defect allows us to project it partially into the  $L_t$ -region where we can apply the same trick to write it as a combination of a green and a red component.

Let us call  $\tilde{\Omega}$  a diagram obtained by  $\Omega$  by applying the previous transformations. Since it is entirely coloured by  $\blacksquare$ ,  $\blacksquare$  and  $\blacksquare$ , it represents an element of  $\text{SN}_{L_t}^{\text{fr}}(S)(U)$ . This element is well-defined, since all the relations applied to switch  $\blacksquare$  to  $\blacksquare$  are local relations from  $\text{Rep}_q L_t$ , hence they hold in  $\text{SN}_{L_t}^{\text{fr}}(S)(U)$ . We get a linear map

$$\mathcal{H}(S)(U, \emptyset, \emptyset) \rightarrow \text{SN}_{L_t}^{\text{fr}}(S)(U), \quad \Omega \mapsto \tilde{\Omega}.$$

An inverse of this map is constructed as follows. Given a planar diagram  $\Gamma$  on  $S$  coloured by  $\text{Rep}_q L_t$ , we can apply isotopies and naturality to make all the “forbidden” morphisms (that is, the inclusions/projection not lying in the image of the restriction functor  $\pi^*$ ) slide along open strands and lie arbitrarily close to the marked components of the boundary, so that they are all inside one of the embedded disks/cylinders defining an  $L_t$ -region. Note that this can be done in a unique way so that the intersection of the strand with the  $P_t$ -region is coloured in either  $\blacksquare$  or  $\blacksquare$ . We get this way a well-defined graph  $\hat{\Gamma}$  on the stratified defect wall and it is straightforward to see that  $\Gamma \mapsto \hat{\Gamma}$  defines an inverse for the linear map above. Naturality follows from the fact that these maps do not modify diagrams in a tubular neighborhood of the marked boundary components.  $\square$

Therefore, the presence of regions coloured by  $\text{Rep}_q L_t$  near the boundary of the defect wall allows one to apply relation (3.2.47) to components coloured with  $\blacksquare$ , yielding an identification  $\blacksquare = \blacklozenge$ . In the case where there are open strands attached to  $S \times \{\pm 1\}$ , this identification still holds, but only on closed components:

**Proposition 4.1.4.** *Let  $\mathcal{B}$  and  $\phi$  be as in the previous proposition and fix empty boundary conditions on each of the components marked on the defect. Then,*

$$\mathcal{H}(S)(\emptyset, -, -) \cong \text{SkMod}_H^{\text{def}}(S, \nu_{\emptyset \sqcup - \sqcup -}) \Big/ \langle \blacksquare = \blacklozenge \text{ on closed components on the wall} \rangle.$$

*Proof.* The idea of the proof is the same as in the previous proposition: we can construct linear maps between both spaces by pushing a part of the diagram near one of the  $L_t$ -regions on the boundary, then use skein relations to project into the  $L_t$ -region and make morphisms slide along closed components. This allows us to decompose closed components coloured with  $\blacksquare$  into linear combinations of green and red diagrams, yielding the identification  $\blacksquare = \blacklozenge$ . On open strands, this induces no new relations: since they are all attached to  $S \times \{\pm 1\}$ , “forbidden” morphisms cannot be simplified, so making them appear just yields a different way of writing the same morphism, but they induce no relation.  $\square$

Combining both propositions, we see that  $\mathcal{H}(S)(\emptyset)$  is isomorphic as a linear space to  $\text{SkAlg}_{L_t}(S)$  and, by functoriality of  $\mathcal{H}(S)$ , we have now actions of both  $\text{SkAlg}_{\text{GL}_t}(S)$  and  $\text{SkAlg}_{L_t}(S)$ :

**Theorem 4.1.5.** *Let  $(S, f, \mathcal{B})$  be a marked framed surface with  $\partial S \neq \emptyset$  and  $\mathcal{B} \neq \emptyset$  as above. Then, the  $(\text{SkAlg}_{\text{GL}_t}(S), \text{SkAlg}_{L_t}(S))$ -bimodule  $\mathcal{H}(S)(\emptyset)$  is isomorphic to  $\text{SkAlg}_{L_t}(S)$  as a linear space. In particular,  $\text{SkAlg}_{L_t}(S)$  becomes a left  $\text{SkAlg}_{\text{GL}_t}(S)$ -module under this identification.*  $\blacksquare$

We treat now the case of the torus  $S = \mathbb{T}^2$ . Let  $(\mathbb{T}^2, f, \phi)$  be a framed torus endowed with a stratification  $\phi$  whose  $L_t$ -region consists of only a disk embedded in  $\mathbb{T}^2$ . By the same argument as in the proof of Propositions 4.1.3 and 4.1.4,

$$\mathcal{H}(\mathbb{T}^2, f, \phi)(\emptyset) \cong \text{SkAlg}_{L_t}(\mathbb{T}^2).$$

Moreover, the  $P_t$ -region is homeomorphic to  $\mathbb{T}^2 \setminus \mathbb{D}^2$ , so we have an action

$$\text{SkAlg}_{\text{GL}_t}(\mathbb{T}^2 \setminus \mathbb{D}^2) \otimes \text{SkAlg}_{L_t}(\mathbb{T}^2) \rightarrow \text{SkAlg}_{L_t}(\mathbb{T}^2)$$

of the  $\text{GL}_t$ -skein algebra of this punctured torus.

**Proposition 4.1.6.** *The action above descends to an action of the  $\text{GL}_t$ -skein algebra of the torus.*

*Proof.* A loop  $\alpha$  around the puncture acts by multiplication by

$$\pi_t^* \circ \text{res}_t \left( \text{Diagram of a blue circle with a clockwise arrow} \right)$$

on the  $L_t$ -disk, hence the action descends to the quotient

$$\mathrm{SkAlg}_{\mathrm{GL}_t}(\mathbb{T}^2 \setminus \mathbb{D}^2) \Big/ \left\langle \alpha = \text{blue circle} \right\rangle \cong \mathrm{SkAlg}_{\mathrm{GL}_t}(\mathbb{T}^2).$$

□

## 4.2 The coproduct map

Let  $(S, f)$  be any framed surface and choose  $\mathcal{B} \neq \emptyset$  if  $\partial S \neq \emptyset$ . Let  $\phi$  be a stratification of  $S$  as in Propositions 4.1.3 and 4.1.6, so that  $\mathrm{SkAlg}_{\mathrm{GL}_t}(S)$  acts on  $\mathrm{SkAlg}_{L_t}(S)$ :

**Corollary 4.2.1.** *The morphism  $\mathrm{SkAlg}_{\mathrm{GL}_t}(S) \rightarrow \mathcal{H}(S)(\emptyset, \emptyset)$  defined by acting on the empty diagram induces a  $\mathbb{C}(q)$ -algebra homomorphism*

$$\tilde{\Delta}_f: \mathrm{SkAlg}_{\mathrm{GL}_t}(S) \rightarrow \mathrm{SkAlg}_{L_t}(S).$$

■

Composing  $\tilde{\Delta}_f$  with the isomorphism in Lemma 3.2.11, we obtain a  $\mathbb{C}(q)$ -algebra morphism

$$\Delta_f: \mathrm{SkAlg}_{\mathrm{GL}_t}(S) \rightarrow \mathrm{SkAlg}_{\mathrm{GL}_t}(S) \otimes_{\mathbb{C}(q)} \mathrm{SkAlg}_{\mathrm{GL}_t}(S) \quad (4.2.61)$$

whose restriction to  $\mathbb{k} \cong \mathbb{k} \cdot \mathbf{1}_\emptyset$  is given by (3.2.52). By construction, it can be computed on links by taking a diagram on the surface where every crossing, cup and cap has been replaced by their image by the restriction functor (3.2.57), and applying relations (3.2.53)–(3.2.56) and (3.2.47) to split the diagram into two coloured diagrams, which we consider as lying in two different copies of  $\mathrm{SkAlg}_{\mathrm{GL}_t}(S)$ . Here by a cup and a cap we mean a portion of the diagram where the rotation number increases or decreases by 1.

**Proposition 4.2.2.** *The coproduct in (4.2.61) is coassociative.*

*Proof.* The map

$$(\Delta_f \otimes \mathrm{id}) \circ \Delta_f: \mathrm{SkAlg}_{\mathrm{GL}_t}(S) \rightarrow \mathrm{SkAlg}_{\mathrm{GL}_t}(S) \otimes_{\mathbb{C}(q)} \mathrm{SkAlg}_{\mathrm{GL}_t}(S) \otimes_{\mathbb{C}(q)} \mathrm{SkAlg}_{\mathrm{GL}_t}(S)$$

can be computed as follows. Given a link diagram  $\Gamma$ , we first compute  $\tilde{\Delta}_f(\Gamma) \in \mathrm{SkAlg}_{L_t}(S)$  as in the previous paragraph. Then, we split the green part into two colours (green again for the first copy of  $\mathrm{SkAlg}_{\mathrm{GL}_t}(S)$  and violet for the second one) using the same rules: replacing all green crossings, cups and caps by their images by the restriction functor (3.2.57) and applying relations (3.2.53)–(3.2.56) and (3.2.47) with a proper choice of colours. Since applying  $\tilde{\Delta}$  does not produce new crossings, we can do all this at once from the initial diagram using the following relations:

$$\text{blue crossing} \mapsto \sum_{\text{green, red, violet}} \text{split} + (q - q^{-1}) \left( \text{green vertical} + \text{red vertical} + \text{violet vertical} \right), \quad (4.2.62)$$

$$\begin{array}{c} \text{crossing} \mapsto \sum_{\substack{\text{green}, \text{red}, \text{purple}}} \text{crossing} + (q - q^{-1}) \left( q^{-t_2} q^{-t_3} \begin{array}{c} \text{red arc} \\ \text{green arc} \end{array} + q^{-t_2} \begin{array}{c} \text{purple arc} \\ \text{green arc} \end{array} + \begin{array}{c} \text{red arc} \\ \text{purple arc} \end{array} \right), \end{array} \quad (4.2.63)$$

$$\begin{array}{c} \text{arc with } r \text{ box} \mapsto \text{green arc} + q^{-t_1} \text{purple arc} + q^{-t_1} q^{-t_2} \text{red arc}, \end{array} \quad (4.2.64)$$

$$\begin{array}{c} \text{arc with } r \text{ box} \mapsto \text{green arc} + q^{-t_1} \text{purple arc} + q^{-t_1} q^{-t_2} \text{red arc}, \end{array} \quad (4.2.65)$$

where the sums are taken over all the possible combinations of colours for the involved strands and

$$q^{t_1} = q^t \otimes 1 \otimes 1, \quad q^{t_2} = 1 \otimes q^t \otimes 1, \quad q^{t_3} = 1 \otimes 1 \otimes q^t,$$

in  $\mathbb{k} \otimes_{\mathbb{C}(q)} \mathbb{k} \otimes_{\mathbb{C}(q)} \mathbb{k}$ . The same formulas compute  $(\text{id} \otimes \Delta_f) \circ \Delta_f$ , hence the coproduct is coassociative.  $\square$

**Remark 4.2.3.** The coproduct  $\Delta_f$  depends on the choice of the framing  $f$ . For instance, let  $S = \mathbb{A}$  be the annulus, which can be equipped with either a radial framing or the framing induced by  $\mathbb{R}^2$ . Consider first the case where  $f$  is the radial framing. We have

$$\begin{aligned} \text{cylinder with blue loop} &= \text{annulus with blue loop} = \text{annulus with green loop} + \text{annulus with red loop} \\ &= \text{annulus with green loop} + \text{annulus with red loop} \\ &= \text{annulus with green loop} + \text{annulus with red loop} = \text{cylinder with green loop} + \text{cylinder with red loop}. \end{aligned}$$

On the other hand, if  $f$  is the framing induced by  $\mathbb{R}^2$ , the local relations projecting the link into the defect make use of the pivotal structure of  $\text{Rep}_q \text{GL}_t$ :

$$\text{cylinder with blue loop} = \text{annulus with blue loop} = q^{t_2} \text{cylinder with green loop} + q^{-t_1} \text{cylinder with red loop},$$

where the last equality follows from the definition of the pivotal structure of  $\text{Rep}_q \text{GL}_t$  by doing the same manipulations as in the previous paragraph.  $\diamond$

### 4.3 The counit map

Let  $\overline{\text{Rep}}_q \text{GL}_n$  be the  $\mathbb{C}$ -linear category obtained by specialising

$$q^t \rightsquigarrow q^n, \quad \delta \rightsquigarrow \frac{q^n - q^{-n}}{q - q^{-1}}$$

in Definition 3.2.3, with  $q \in \mathbb{C}^\times$  and  $q^2 \neq 1$ . This is the category  $\mathcal{OS}(q - q^{-1}, q^n)$  from [Bru17]. Let  $\varphi_n: \mathbb{k} \rightarrow \mathbb{C}(q)$  be the evaluation morphism in Remark 3.2.6. Then, the evaluation functor  $\text{ev}_{q,n}^G$  factors as

$$\text{Rep}_q \text{GL}_t \rightarrow \overline{\text{Rep}}_q \text{GL}_n \rightarrow \text{Rep}_q \text{GL}_n,$$

where the first arrow is a  $\varphi_n$ -linear functor and the second one is  $\mathbb{C}$ -linear.

**Definition 4.3.1.** ([Del07]) Let  $\mathcal{C}$  be a ribbon category. A morphism  $f: X \rightarrow Y$  is *negligible* if  $\text{Tr}_{\mathcal{C}}(f \circ g) = 0$  for all  $g: Y \rightarrow X$  in  $\mathcal{C}$ .  $\diamond$

**Theorem 4.3.2.** ([Del07; Bru17]) The functor  $\overline{\text{Rep}}_q \text{GL}_n \rightarrow \text{Rep}_q \text{GL}_n$  above induces a monoidal equivalence

$$\Phi: \overline{\text{Rep}}_q \text{GL}_n / \mathcal{N} \xrightarrow{\sim} \text{Rep}_q \text{GL}_n,$$

where  $\mathcal{N}$  is the tensor ideal of  $\overline{\text{Rep}}_q \text{GL}_n$  consisting of negligible morphisms.

In particular,  $\text{GL}_0$  is the trivial group, so that representations are just vector spaces and we have an equivalence

$$\text{Rep}_q \text{GL}_0 \simeq \text{Vect}.$$

Moreover, if  $n = 0$ , then  $\delta = 0$  and the dimension relation

$$\bigcirc = 0$$

in  $\overline{\text{Rep}}_q \text{GL}_0$  implies that the identity  $\uparrow$  is a negligible morphism. Therefore, all nonempty

diagrams are zero in  $\overline{\text{Rep}}_q \text{GL}_n / \mathcal{N}$  and we obtain a  $\mathbb{C}(q)$ -linear functor

$$\mathcal{E}: \text{Rep}_q \text{GL}_t \rightarrow \overline{\text{Rep}}_q \text{GL}_0 / \mathcal{N} \simeq \text{Vect}$$

annihilating all non-empty diagrams. If  $S$  is any oriented surface, this clearly extends to a functor

$$\mathcal{E}^S: \text{SkCat}_{\text{GL}_t}(S) \rightarrow \text{Vect}.$$

**Proposition 4.3.3.** The linear map

$$\varepsilon: \text{SkAlg}_{\text{GL}_t}(S) \rightarrow \mathbb{C}(q)$$

obtained by restricting  $\mathcal{E}^S$  to  $\text{SkAlg}_{\text{GL}_t}(S)$  provides a counit for the Turaev coproduct (4.2.61).

*Proof.* Recall the local relations (3.2.53)–(3.2.56). Since  $\varepsilon$  annihilates non-empty diagrams and maps  $q^t$  to 1, the map  $(1 \otimes \varepsilon) \circ \Delta_f$  just switches  $\blacksquare$  with  $\blacksquare$ . The same happens when  $\varepsilon$  is applied to the second factor, so it is a counit for  $\Delta_f$ .  $\square$

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