JONQUIÈRES MAPS AND SL(2;C)-COCYCLES

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(Communicated by Raphaël Krikorian)

ABSTRACT. We started the study of the family of birational maps \((f_{a,b})\) of \(\mathbb{P}^2_\mathbb{C}\) in [12]. For \((\alpha, \beta)\) well chosen of modulus 1, the centraliser of \(f_{a,b}\) is trivial, the topological entropy of \(f_{a,b}\) is 0, and there exist two domains of linearisation: in the first one the closure of the orbit of a point is a torus, in the other one the closure of the orbit of a point is the union of two circles. On \(\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}\), any \(f_{a,b}\) can be viewed as a cocyle; using recent results about SL(2;C)-cocycles ([1]), we determine the Lyapunov exponent of the cocyle associated to \(f_{a,b}\).

INTRODUCTION

In this article we deal with a family of birational maps \((f_{a,b})\) given by
\[
f_{a,b} : \mathbb{P}^2_\mathbb{C} \rightarrow \mathbb{P}^2_\mathbb{C} \quad (x : y : z) \rightarrow ((\alpha x + y)z : \beta y(x + z) : z(x + z)),
\]
where \(\alpha, \beta\) denote two complex numbers of modulus 1, a case for which we know almost nothing about the dynamics. Let us consider the set \(\Omega\) of pairs of complex numbers of modulus 1 that satisfy the Diophantine condition. The family \((f_{a,b})\) satisfies the following properties ([12]):
- For \((\alpha, \beta) \in \Omega\) the centraliser of \(f_{a,b}\), that is the set of birational maps of \(\mathbb{P}^2_\mathbb{C}\) that commutes with \(f_{a,b}\), is isomorphic to \(\mathbb{Z}\).
- The topological entropy of \(f_{a,b}\) is 0.
- Rotation domains of ranks 1 and 2 coexist: there is a domain of linearisation where the orbit of a generic point under \(f_{a,b}\) is a torus, and there is another domain of linearisation where the orbit of a generic point under \(f_{a,b}^2\) is a circle.

We can also view \(f_{a,b}\) on \(\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}\) (since all the computations of [12] have been done in an affine chart, they may all be carried on \(\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}\); the sets \(\mathbb{P}^1_\mathbb{C} \times \mathbb{S}^1_\rho\), where \(\mathbb{S}^1_\rho = \{y \in \mathbb{C} \mid |y| = \rho\}\), are invariant.

Let us define \(A^{a,\rho}_n : \mathbb{S}^1_\rho \rightarrow \text{M}(2;\mathbb{C})\), given in terms of \(A^{a,\rho}(y) = \begin{bmatrix} a & y \\ 1 & 1 \end{bmatrix}\), by
\[
A^{a,\rho}_n(\cdot) = A^{a,\rho}(\beta^n) \cdots A^{a,\rho}(\beta) A^{a,\rho}(\cdot).
\]

Received April 24, 2014; revised January 23, 2016.
2010 Mathematics Subject Classification: Primary: 37F10; Secondary: 14E07.
Key words and phrases: SL(2;C)-cocycle, Lyapunov exponent, birational map, Jonquières map.

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To compute $f_{a,\beta}^n(x, y)$ is equivalent to compute $A_{n,\rho}^{\alpha,\beta}(y)$ as soon as $f_{a,\beta}^k(x, y) \neq (-1, \alpha)$, for any $1 \leq k \leq n$.

Using [1] we are able to determine the Lyapunov exponent of the cocycle $(A^{\alpha,\beta}, \beta)$.

**Theorem A.** The Lyapunov exponent of $(A^{\alpha,\beta}, \beta)$ is positive if $\rho > 1$ and zero if $\rho \leq 1$. More precisely, $f_{a,\beta}$ is semi-conjugate to $\left(\frac{ax^2 + x^2}{x+1}, \beta^{1/2} y\right)$ and the Lyapunov exponent of the cocycle $(B^{\alpha,\beta}, \beta^{1/2})$, where

$$B^{\alpha,\beta}(y) = \begin{bmatrix} \alpha & y^2 \\ 1 & 1 \end{bmatrix},$$

is equal to $\max(0, \ln \rho)$.

In the next section we introduce the family $(f_{a,\beta})$ and its properties (§1). Then we deal with the recent works of Avila on SL(2; $\mathbb{C}$)-cocycles. In the last section we give the proof of Theorem A (see §2). Let us explain the sketch of it. We associate to $(B^{\alpha,\beta}, \beta^{1/2})$ a cocycle $(\tilde{B}^{\alpha,\beta}, \beta^{1/2})$ that belongs to SL(2; $\mathbb{C}$). We first determine

$$\lim_{\rho \to 0} L(\tilde{B}^{\alpha,\beta}, \beta^{1/2})$$

and, then,

$$\lim_{\rho \to +\infty} L(\tilde{B}^{\alpha,\beta}, \beta^{1/2}),$$

where $L(C, \gamma)$ denotes the Lyapunov exponent of the SL(2; $\mathbb{C}$)-cocycle $(C, \gamma)$. In both cases, we get 0. Using [1, Theorem 5] we obtain that $L(\tilde{B}^{\alpha,\beta}, \beta^{1/2})$ vanishes everywhere; it allows us to determine $L(A^{\alpha,\beta}, \beta)$ since

$$L(B^{\alpha,\beta}(y), \beta^{1/2}) = L(\tilde{B}^{\alpha,\beta}(y), \beta^{1/2}) + \max(0, \ln \rho),$$

and since $(A^{\alpha,\beta}, \beta)$ and $(\beta^{1/2}, B^{\alpha,\beta})$ are conjugate.

1. SOME PROPERTIES OF THE FAMILY $(f_{a,\beta})$

A rational map $\phi$ from $\mathbb{P}^2_{\mathbb{C}}$ into itself is a map of the form

$$(x : y : z) \mapsto (\phi_0(x, y, z) : \phi_1(x, y, z) : \phi_2(x, y, z)),$$

where the $\phi_i$’s are some homogeneous polynomials of the same degree without common factor; $\phi$ is birational if it admits an inverse of the same type. We will denote by Bir$(\mathbb{P}^2_{\mathbb{C}})$ the group of birational maps of $\mathbb{P}^2_{\mathbb{C}}$, also called the Cremona group. The degree of $\phi$, denoted $\deg \phi$, is the degree of the $\phi_i$’s. The degree is not a birational invariant: $\deg \psi \phi \psi^{-1} \neq \deg \phi$ for generic birational maps $\phi$ and $\psi$. The first dynamical degree of $\phi$ given by

$$\lambda(\phi) = \lim_{n \to +\infty} \left(\deg \phi^n\right)^{1/n}$$

is a birational invariant; it is strongly related to the topological entropy $h_{\text{top}}(\phi)$ of $\phi$ (see [17, 20]),

$$h_{\text{top}}(\phi) \leq \log \lambda(\phi).$$
Any birational map $\phi$ admits a resolution

$$
\begin{array}{c}
\pi_1 \\
\downarrow \\
\mathbb{P}^2 \overset{\phi}{\longrightarrow} \mathbb{P}^2 \\
\pi_2 \uparrow \\
S
\end{array}
$$

where $\pi_1, \pi_2: S \to \mathbb{P}^2_\mathbb{C}$ are sequences of blow-ups (see [3], for example). The resolution is \textit{minimal} if and only if no $(-1)$-curve of $S$ is contracted by both $\pi_1$ and $\pi_2$. The \textit{base-points} of $\phi$ are the points blown-up in $\pi_1$, which can be points of $\mathbb{P}^2_\mathbb{C}$ or infinitely near points. We denote by $b(\phi)$ the number of such points, which is also equal to the difference of the ranks of $\text{Pic}(S)$ and $\text{Pic}(\mathbb{P}^2_\mathbb{C})$, and thus is equal to $b(\phi^{-1})$.

The \textit{dynamical number of base-points} of $\phi$ introduced in [8] is by definition

$$
\mu(\phi) = \lim_{n \to +\infty} \frac{b(\phi^n)}{n}.
$$

It is a real positive number that satisfies $\mu(\psi \phi \psi^{-1}) = \mu(\phi)$ and, for any $n \in \mathbb{Z}$, $\mu(\phi^n) = |n\mu(\phi)|$. It allows us to give a characterization of birational maps conjugate to automorphisms.

\textbf{Theorem 1.1} ([8]). \textit{Let $S$ be a smooth projective surface; the birational map $\phi \in \text{Bir}(S)$ is conjugate to an automorphism of a smooth projective surface if and only if $\mu(\phi) = 0$.}

The behavior of $\phi \in \text{Bir}(\mathbb{P}^2_\mathbb{C})$ is strongly related to the behavior of $\{\deg \phi^n\}_{n \in \mathbb{N}}$ (see [16, 15, 8]); up to birational conjugacy exactly one of the following holds:

1. The sequence $\{\deg \phi^n\}_{n \in \mathbb{N}}$ is bounded and either $\phi$ is of finite order, or $\phi$ is an automorphism of $\mathbb{P}^2_\mathbb{C}$.
2. There exists an integer $k$ such that

$$
\lim_{n \to +\infty} \frac{\deg \phi^n}{n} = k^2 \frac{\mu(\phi)}{2}
$$

and $\phi$ is not an automorphism.
3. There exists an integer $k \geq 3$ such that

$$
\lim_{n \to +\infty} \frac{\deg \phi^n}{n^2} = k^2 \frac{\kappa(\phi)}{9},
$$

where $\kappa(\phi) \in \mathbb{Q}$ is a birational invariant, and $\phi$ is an automorphism.
4. The sequence $\{\deg \phi^n\}_{n \in \mathbb{N}}$ grows exponentially (see [15] for more precise dynamical properties).

In the first three cases $\lambda(\phi) = 1$, in the last one $\lambda(\phi) > 1$. In case 2 (respectively, 3) the map $\phi$ preserves a unique fibration which is rational (respectively, elliptic).

In case 1 (respectively 2, 3, and 4) we say that $\phi$ is \textit{elliptic} (respectively a \textit{Jonquières twist}, an \textit{Halphen twist}, and \textit{hyperbolic}).
Let us give some examples. Let 
\[ \phi(x, y) = \begin{pmatrix} a(y)x + b(y) & \alpha y + \beta \\ c(y)x + d(y) & \gamma y + \delta \end{pmatrix} \]
be an element of the Jonquières group \( \text{PGL}(2; \mathbb{C}(y)) \times \text{PGL}(2; \mathbb{C}) \); either \( \phi \) is elliptic (for instance, \( \phi: (x : y : z) \rightarrow (yz : xz : xy) \)), or \( \phi \) is a Jonquières twist (for example, \( \phi: (x : y : z) \rightarrow (xz : xy : z^2) \) for which the unique invariant fibration is \( y/z = \text{constant} \)). The map 
\[ \phi: \mathbb{P}^2_{\mathbb{C}} \rightarrow \mathbb{P}^2_{\mathbb{C}} \quad (x : y : z) \rightarrow ((2y + z)(y + z) : x(2y - z) : 2z(y + z)) \]
is an Halphen twist ([15, Proposition 9.5]). Hénon automorphisms give by homogenization examples of hyperbolic maps.

Clearly, elliptic birational maps have a poor dynamical behavior contrary to hyperbolic ones. The study of automorphisms of positive entropy is strongly related with birational maps of \( \mathbb{P}^2_{\mathbb{C}} \).

**Theorem 1.2** ([9]). Let \( S \) be a compact complex surface that carries an automorphism \( \phi \) of positive topological entropy. Then, either

- the Kodaira dimension of \( S \) is zero and \( \phi \) is conjugate to an automorphism on the unique minimal model of \( S \) that necessarily is a torus, a K3 surface, or an Enriques surface; or
- the surface \( S \) is a non-minimal rational one, isomorphic to \( \mathbb{P}^2_{\mathbb{C}} \) blown up at \( n \) points, \( n \geq 10 \), and \( \phi \) is conjugate to a birational map of \( \mathbb{P}^2_{\mathbb{C}} \).

This yields many examples of hyperbolic birational maps for which we can establish many dynamical properties ([18, 4, 5, 6, 7, 14, 13]).

Another way to measure chaos is to look at the size of centralisers. Let us give two examples. The polynomial automorphisms of \( \mathbb{C}^2 \) having rich dynamics are Hénon maps; furthermore, a polynomial automorphism of \( \mathbb{C}^2 \) is a Hénon one if and only if its centraliser is countable. Let us now consider rational maps on \( \mathbb{S}^1 \); if the centraliser of such maps is not trivial\(^1\), then the Julia set is “special”. The centraliser of an elliptic birational map of infinite order is uncountable ([8]). The centralisers of Halphen twists are described in [16]. The centraliser of an hyperbolic map is countable ([10]). In [11] we end the story by studying centralisers of Jonquières twists. If the fibration is fiberwise invariant, then the centraliser is uncountable; but if it isn’t, then generically the centraliser is isomorphic to \( \mathbb{Z} \). We don’t know much about the dynamics of these maps, thus in this article we will focus on a family of such maps. We consider the Jonquières maps 
\[ f_{\alpha, \beta}: \mathbb{P}^2_{\mathbb{C}} \rightarrow \mathbb{P}^2_{\mathbb{C}} \quad (x : y : z) \rightarrow ((ax + y)z : \beta y(x + z) : z(x + z)), \]
where \( \alpha, \beta \) denote two complex numbers of modulus 1. The base-points of \( f_{\alpha, \beta} \) are 
\[ (1 : 0 : 0), \quad (0 : 1 : 0), \quad (1 : \alpha : 1). \]

---

\(^1\) The centraliser of a map \( \phi \) is trivial if it coincides with the iterates of \( \phi \).
Any \( f_{\alpha, \beta} \) preserves a rational fibration (the fibration \( y = \text{constant} \) in the affine chart \( z = 1 \)). Each element of the family \( (f_{\alpha, \beta}) \) has first dynamical degree 1, hence topological entropy zero (1.1); more precisely, one has ([8, Example 4.3])

\[
\mu(f_{\alpha, \beta}) = \frac{1}{2},
\]

so \( f_{\alpha, \beta} \) is not conjugate to an automorphism (Theorem 1.1). The centralizer of \( f_{\alpha, \beta} \) is isomorphic to \( \mathbb{Z} \) (see [12, Theorem 1.6]). The idea of the proof is as follows: the point \( p = (1 : \alpha : 1) \) is blown-up onto a fiber of the fibration \( y = \text{constant} \). Let \( \psi \) be an element of

\[
\text{Cent}(f_{\alpha, \beta}) = \{ g \in \text{Bir}(\mathbb{P}^2_{\mathbb{C}}) \mid g \circ f_{\alpha, \beta} = f_{\alpha, \beta} \circ g \}.
\]

Since \( \psi \) blows down a finite number of curves, there exists a positive integer \( k \) \((\text{chosen minimal})\) such that \( f_{\alpha, \beta}^k(p) \) is not blown down by \( \psi \). Replacing \( \psi \) by \( \tilde{\psi} = \psi f_{\alpha, \beta}^{-1} \), one gets that \( \tilde{\psi}(p) \) is an indeterminacy point of \( f_{\alpha, \beta} \). In other words, \( \tilde{\psi} \) permutes the indeterminacy points of \( f_{\alpha, \beta} \). A more precise statement allows us to establish that \( p \) is fixed by \( \tilde{\psi} \). The pair \((\alpha, \beta)\) being in \( \Omega \), the closure of the negative orbit of \( p \) under the action of \( f_{\alpha, \beta} \) is Zariski dense; since \( \tilde{\psi} \) fixes any element of the orbit of \( p \), one obtains \( \tilde{\psi} = \text{id} \).

Let us recall that if \( \psi \) is an automorphism on a compact complex manifold \( \mathcal{M} \), then the Fatou set \( \mathcal{F}(\psi) \) of \( \psi \) is the set of points that have a neighborhood \( \mathcal{V} \) such that \( \{ f_{\psi}^n \mid n \in \mathbb{N} \} \) is a normal family. Set

\[
\mathcal{U}(\mathcal{V}) = \{ \phi : \mathcal{V} \to \mathcal{V} \mid \phi = \lim_{n \to +\infty} \psi^n \}.
\]

We say that \( \mathcal{U} \) is a rotation domain if \( \mathcal{U}(\mathcal{V}) \) is a subgroup of \( \text{Aut}(\mathcal{V}) \). An equivalent definition is the following: a component \( \mathcal{U} \) of \( \mathcal{F}(\psi) \) which is invariant by \( \psi \) is a rotation domain if \( \psi \mid \mathcal{U} \) is conjugate to a linear rotation. If \( \mathcal{U} \) is a rotation domain, then \( \mathcal{U}(\mathcal{V}) \) is a compact Lie group, and the action of \( \mathcal{U}(\mathcal{V}) \) on \( \mathcal{V} \) is analytic real. Since \( \mathcal{U}(\mathcal{V}) \) is a compact, infinite, abelian Lie group, the connected component of the identity of \( \mathcal{U}(\mathcal{V}) \) is a torus of dimension \( 0 \leq d \leq \dim_{\mathbb{C}} \mathcal{M} \). The integer \( d \) is the rank of the rotation domain. The rank coincides with the dimension of the closure of a generic orbit of a point in \( \mathcal{U} \).

We can also view \( f_{\alpha, \beta} \) on \( \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \) and that is what we will do in the sequel (since all the computations of [12] have been done in an affine chart, they may all be carried on \( \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \); the sets \( \mathbb{P}^1_{\mathbb{C}} \times \mathbb{S}^1_{\mathbb{R}} \) are invariant. In [12] we show that there are two rotation domains for \( f_{\alpha, \beta}^2 \), one of rank 1, and the other one of rank 2; for the first case, we give below a more precise statement than in [12].

**Theorem 1.3.** Assume that \((\alpha, \beta)\) belongs to \( \Omega \). There exists a strictly positive real number \( r \) such that \( f_{\alpha, \beta} \) is conjugate to \((\alpha x, \beta y)\) on \( \mathbb{P}^1_{\mathbb{C}} \times \mathbb{D}(0, r) \), where \( \mathbb{D}(0, r) \) denotes the disk centered at the origin with radius \( r \). There exists a strictly positive real number \( \bar{r} \) such that \( f_{\alpha, \beta}^2 \) is conjugate to \((\frac{\bar{z}}{\bar{p}}, \frac{\bar{z}}{\bar{p}})\) on \( \mathbb{P}^1_{\mathbb{C}} \times \mathbb{D}(0, \bar{r}) \).

2. There already exists an example of automorphism of positive entropy with rotation domains of rank 1 and 2 (see [5]), but \( f_{\alpha, \beta} \) is not conjugate to an automorphism on a rational surface.
**Remark 1.4.** The point \((\alpha - 1, 0)\) is also a fixed point of \(f_{\alpha, \beta}\), where the behavior of \(f_{\alpha, \beta}\) is the same as near \((0, 0)\).

**Proof:** The first assertion is proved in [12].

Let us consider the map \(\psi(x, z) = \left(\frac{a(z)x + b(z)}{c(z)x + 1}, z\right)\). The equation

\[
\psi^{-1} f_{\alpha, \beta}^2 \psi = \left(\frac{x}{\beta}, \frac{z}{\beta^2}\right)
\]

yields

\[
(\alpha + 1) z + b(z) - \beta b(\beta^2 z) - \alpha^2 z b(\beta^2 z) + z b(z) - (\alpha + \beta) b(\beta^2 z) b(z) = 0.
\]

Let us set

\[
a(z) = \sum_{i \geq 0} a_i z^i, \quad b(z) = \sum_{i \geq 0} b_i z^i, \quad c(z) = \sum_{i \geq 0} c_i z^i.
\]

We easily get \(a_0 = 1 - \beta, b_0 = 0\) and \(c_0 = \alpha + \beta\).

Relation (1.4) implies that

\[
b_1 = \frac{\beta (1 + \alpha)}{1 - \beta} \quad \text{and} \quad \beta b_v \left(1 - \beta^{1-2v}\right) + F_i(b_i \mid i < v) = 0, \forall v > 1,
\]

(1.3) yields

\[
a_v \left(\beta^{1-2v} - \beta\right) + b_v \left((\alpha + \beta)a_0 \left(1 + \beta^{1-2v}\right) + c_0 \left(\beta^{2-2v} - 1\right)\right) + G_i(a_i, b_i, c_i \mid i < v) = 0,
\]

and (1.2) implies

\[
c_v a_0 \left(\beta - \beta^{-2v}\right) + a_v \left((\alpha + \beta)a_0 \left(1 + \beta^{-2v}\right) + c_0 \left(\beta^{1-2v} - 1\right)\right) + H_i(a_i, b_i, c_i \mid i < v) = 0,
\]

where the \(F_i\)’s, \(G_i\)’s and \(H_i\)’s denote universal polynomials; this allows to compute \(b_v, a_v\) and \(c_v\). Thus we get a formal conjugacy of \(f_{\alpha, \beta}^2\) to its linear part. Since this linear part satisfies a Rüssmann condition (see [19, Theorem 2.1], condition (2)), according to [19, Theorem 2.1], any formal linearizing map conjugating \(f_{\alpha, \beta}^2\) to its linear part is convergent on a polydisc. \(\square\)
2. About SL(2; C)-cocycles

A (one-frequency, analytic) quasiperiodic SL(2; C)-cocycle is a pair $(A, \beta)$, where $\beta \in \mathbb{R}$ and

$$A: \mathbb{S}_1^1 \rightarrow \text{SL}(2; \mathbb{C})$$

is analytic, and defines a linear skew product acting on $\mathbb{C}^2 \times \mathbb{S}_1^1$ by

$$(x, y) \mapsto (A(y) \cdot x, \beta y).$$

The iterates of the cocycle are given by $(A_n, n\beta)$ where $A_n$ is given by

$$A_n(y) = A(\beta^{n-1}y) \cdots A(y), \quad n \geq 1, \quad A_0(y) = \text{id}, \quad A_{-n}(y) = A_n(\beta^{-n}y)^{-1}.$$  

The Lyapunov exponent $L(A, \beta)$ of a quasiperiodic SL(2; C)-cocycle $(A, \beta)$ is given by

$$\lim_{n \to +\infty} \frac{1}{n} \int_{\mathbb{S}_1^1} \ln ||A_n(y)|| \, dy.$$  

A quasiperiodic SL(2; C)-cocycle $(A, \beta)$ is uniformly hyperbolic if there exist analytic functions $u, s: \mathbb{S}_1^1 \rightarrow \mathbb{P}_2^2$, called the unstable and stable directions, and $n \geq 1$ such that for any $y \in \mathbb{S}_1^1$,

$$A(y) \cdot u(y) = u(\beta y), \quad A(y) \cdot s(y) = s(\beta y),$$

and for any unit vector $x \in s(y)$ (respectively, $x \in u(y)$) we have $||A_n(y) \cdot x|| < 1$ (respectively, $||A_n(y) \cdot x|| > 1$). The unstable and stable directions are uniquely characterized by those properties, and clearly $u(y) \neq s(y)$ for any $y \in \mathbb{S}_1^1$. If $(A, \beta)$ is uniformly hyperbolic, then $L(A, \beta) > 0$. Let us denote by

$$\mathcal{UH} \subset C^\omega(\text{SL}(2; \mathbb{C}), \mathbb{S}_1^1)$$

the set of $A$ such that $(A, \beta)$ is uniformly hyperbolic. Uniform hyperbolicity is a stable property: $\mathcal{UH}$ is open, and $A \mapsto L(A, \beta)$ is analytic over $\mathcal{UH}$ (regularity properties of the Lyapunov exponent are consequence of the regularity of the unstable and stable directions which depend smoothly on both variables).

**Definition.** Let $(A, \beta)$ be a quasiperiodic SL(2; C)-cocycle. If $L(A, \beta) > 0$ but $(A, \beta) \notin \mathcal{UH}$, then $(A, \beta)$ is nonuniformly hyperbolic.

If $A \in C^\omega(\text{SL}(2; \mathbb{C}), \mathbb{S}_1^1)$ admits a holomorphic extension to $|\text{Im } y| < \delta$, then for $|\epsilon| < \delta$ we can define $A_\epsilon \in C^\omega(\text{SL}(2; \mathbb{C}), \mathbb{S}_1^1)$ by

$$A_\epsilon(y) = A(y + i\epsilon).$$

The Lyapunov exponent $L(A_\epsilon, \beta)$ is a convex function of $\epsilon$. We can thus introduce the following notion. The acceleration of a quasiperiodic SL(2; C)-cocycle $(A, \beta)$ is given by

$$\omega(A, \beta) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} (L(A_\epsilon, \beta) - L(A, \beta)).$$

**Remark 2.1.** The convexity of the Lyapunov exponent as function of $\epsilon$ implies that the acceleration is decreasing.
Since the Lyapunov exponent is a convex and continuous function, the acceleration is an upper semi-continuous function in \( \mathbb{R} \sim \mathbb{Q} \times C^0(\text{SL}(2; \mathbb{C}), S^1) \). The acceleration is quantized.

**Theorem 2.2** ([1]). If \((A, \beta)\) is a \(\text{SL}(2; \mathbb{C})\)-cocycle with \(\beta \in \mathbb{R} \setminus \mathbb{Q}\), then \(\omega(A, \beta)\) is always an integer.

A direct consequence is the following.

**Corollary 2.3.** The function \(\varepsilon \mapsto L(A_\varepsilon, \beta)\) is a piecewise affine function of \(\varepsilon\).

It is thus natural to introduce the notion of regularity. A cocycle

\[(A, \beta) \in C^0(\text{SL}(2; \mathbb{C}), S^1) \times \mathbb{R} \sim \mathbb{Q}\]

is regular if \(L(A_\varepsilon, \beta)\) is affine for \(\varepsilon\) in a neighborhood of 0. In other words, \((A, \beta)\) is regular if the equality

\[L(A_\varepsilon, \beta) - L(A, \beta) = 2\pi \varepsilon \omega(A, \beta)\]

holds for all \(\varepsilon\) small, and not only for the positive ones. Regularity is equivalent to the acceleration being locally constant near \((A, \beta)\). It is an open condition in \(C^0(\text{SL}(2; \mathbb{C}), S^1) \times \mathbb{R} \sim \mathbb{Q}\). The following statement gives a characterization of the dynamics of regular cocycles with positive Lyapunov exponent.

**Theorem 2.4** ([1]). Let \((A, \beta)\) be a \(\text{SL}(2; \mathbb{C})\)-cocycle with \(\beta \in \mathbb{R} \setminus \mathbb{Q}\). Assume that \(L(A, \beta) > 0\); then \((A, \beta)\) is regular if and only if \((A, \beta)\) is in \(\mathcal{U}_A\).

One striking consequence is the following:

**Corollary 2.5** ([1]). For any \((A, \beta)\) in \(C^0(\text{SL}(2; \mathbb{C}), S^1) \times \mathbb{R} \sim \mathbb{Q}\), there exists \(\varepsilon_0\) such that

- \(L(A_\varepsilon, \beta) = 0\) (and \(\omega(A, \beta) = 0\)) for every \(0 < \varepsilon < \varepsilon_0\), or
- \((A_\varepsilon, \beta)\) \(\in \mathcal{U}_A\) for every \(0 < \varepsilon < \varepsilon_0\).

**Remark 2.6.** Let us mention that there is a link between \(\text{SL}(2; \mathbb{C})\)-cocycles and Schrödinger operators (see [1] for more details).

3. **Proof of Theorem A**

Suppose that \(\rho \neq 1\), and let us consider the cocycle \((B^{a, \rho}, \beta^{1/2})\), where

\[B^{a, \rho}(y) = \begin{pmatrix} a & y^2 \\ 1 & 1 \end{pmatrix}.
\]

Since

\[
\left( \frac{ax + y}{x + 1}, \beta \right) \left( x, y^2 \right) = \left( x, y^2 \right) \left( \frac{ax + y^2}{x + 1}, \beta^{1/2} y \right),
\]

the cocycles \((A^{a, \rho}, \beta)\) and \((B^{a, \rho}, \beta^{1/2})\) have the same behavior. Using two different arguments of monodromy (one for \(\rho < 1\), and the other one for \(\rho > 1\)) we see that there is a continuous determination for the square root of \(\det B^{a, \rho}(y) = a - y^2\). Let us set

\[\tilde{B}^{a, \rho}(y) = \frac{1}{\sqrt{a - y^2}} B^{a, \rho}(y) \in \text{SL}(2; \mathbb{C})\]
that is thus defined on two different domains of analyticity. According to Theorem 1.3 one has \( L(\Bar{B}^{a,\rho}, \beta^{1/2}) = 0 \) when \( \rho \) is close to both 0 and \( \infty \).

Assume that \( L(\Bar{B}^{a,\rho}, \beta^{1/2}) \) is nonconstant. When \( \Bar{B}^{a,\rho} \) is holomorphic, in particular, when \( \rho < 1 \) and \( \rho > 1 \), the acceleration is decreasing (Remark 2.1); furthermore, the acceleration is positive for \( \rho < 1 \) and negative for \( \rho > 1 \) (because \( L \) is continuous). Theorem 2.2 thus implies

\[
\omega(\Bar{B}^{a,1+}, \beta^{1/2}) - \omega(\Bar{B}^{a,1-}, \beta^{1/2}) \leq -2.
\]

By definition of \( \Bar{B}^{a,\rho} \) we have

\[
L(\Bar{B}^{a,\rho}(y), \beta^{1/2}) = L(\Bar{B}^{a,\rho}(y), \beta^{1/2}) - \int_{s_k^\rho} \ln \sqrt{\alpha - y^2} \, dy
\]

\[
= L(\Bar{B}^{a,\rho}(y), \beta^{1/2}) - \max(0, \ln \rho).
\]

Even though \( (\Bar{B}^{a,\rho}(y), \beta^{1/2}) \) is not a \( \text{SL}(2; \mathbb{C}) \)-cocycle, the Lyapunov exponent is still a convex function of \( \log \rho \) (see for example [2]). The jump of \( \omega(\Bar{B}^{a,\rho}(y), \beta^{1/2}) \) is thus \( \geq 0 \), and the jump for the second term of the right member is \( -1 \). Therefore the jump of \( L(\Bar{B}^{a,\rho}(y), \beta^{1/2}) \) is \( \geq -1 \), contradiction.

\[\square\]

**Acknowledgments.** I would like to thank Artur Avila for very helpful discussions, Dominique Cerveau for his constant support, and Serge Cantat for his remarks. Thanks also to the referee whose comments helped me improve the text.

**References**


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