Constraints on automorphism groups of higher dimensional manifolds

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\begin{abstract}
In this note, we prove, for instance, that the automorphism group of a rational manifold $X$ which is obtained from $P^k(C)$ by a finite sequence of blow-ups along smooth centers of dimension at most $r$ with $k > 2r + 2$ has finite image in $GL^+(H^*(X, \mathbb{Z}))$. In particular, every holomorphic automorphism $f : X \to X$ has zero topological entropy.
\end{abstract}

\section{Introduction}

\subsection{Dimensions of indeterminacy loci}

Recall that a rational map admitting a rational inverse is called birational. Birational transformations are, in general, not defined everywhere. The domain of definition of a birational map $f : M \to N$ is the largest Zariski-open subset on which $f$ is locally a well defined morphism. Its complement is the indeterminacy set Ind$(f)$; its codimension is always larger than, or equal to, 2. The following statement shows that the dimension of Ind$(f)$ and Ind$(f^{-1})$ cannot be too small simultaneously unless $f$ is an automorphism. This result is inspired by a nice argument of Nessim Sibony concerning the degrees of regular automorphisms of the complex affine space $\mathbb{C}^k$ (see [13]). It may be considered as an extension of a theorem due to Matsusaka and Mumford (see [10], and [7, Exercise 5.6]).

\begin{theorem}
Let $k$ be a field. Let $M$ be a smooth connected projective variety defined over $k$. Let $f$ be a birational transformation of $M$. Assume that the following two properties are satisfied.
\begin{enumerate}
  \item The Picard number of $M$ is equal to 1;
  \item The indeterminacy sets of $f$ and its inverse satisfy
  \[\dim(\text{Ind}(f)) + \dim(\text{Ind}(f^{-1})) < \dim(M) - 2.\]
\end{enumerate}

Then $f$ is an automorphism of $M$.
\end{theorem}

Moreover, $\text{Aut}(M)$ is an algebraic group because the Picard number of $M$ is equal to 1. As explained below, this statement provides a direct proof of the following corollary, which was our initial motivation.

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Corollary 1.2. Let $M_0$ be a smooth, connected, projective variety with Picard number 1. Let $m$ be a positive integer, and let $\pi_i: M_{i+1} \to M_i$, $i = 0, \ldots, m-1$, be a sequence of blow-ups of smooth irreducible subvarieties of dimension at most $r$. If $\dim(M_0) > 2r + 2$ then the number of connected components of $\text{Aut}(M_m)$ is finite; moreover, the projection $\pi: M_m \to M_0$ conjugates $\text{Aut}(M_m)$ to a subgroup of the algebraic group $\text{Aut}(M_0)$.

For instance, if $M_0$ is the projective space (respectively a cubic hypersurface of $\mathbb{P}^4_k$) and if one modifies $M_0$ by a finite sequence of blow-ups of points, then $\text{Aut}(M_0)$ is isomorphic to a linear algebraic subgroup of $\text{PGL}_4(k)$ (respectively is finite). This provides a sharp (and strong) answer to a question of Eric Bedford. In Section 3, we provide a second, simpler proof of this last statement.

Remark 1.3. The initial question of E. Bedford concerned the existence of automorphisms of compact Kähler manifolds with positive topological entropy in dimension $\geq 2$. This link with dynamical systems is described, for instance, in [4]. If a compact complex surface $S$ admits an automorphism with positive entropy, then $S$ is Kähler and is obtained from the projective plane $\mathbb{P}^2$, a torus, a $K3$ surface or an Enriques surface, by a finite sequence of blow-ups (see [5,6,12]). Examples of automorphisms with positive entropy are easily constructed on tori, $K3$ surfaces, or Enriques surfaces. Examples of automorphisms with positive entropy on rational surfaces are given in [2,3,11]; these examples are obtained from birational transformations $f$ of the plane by a finite sequence of blow-ups that resolves all indeterminacies of $f$ and its iterates simultaneously. These results suggest looking for birational transformations of $\mathbb{P}^n_k$, $n \geq 3$, that can be lifted to automorphisms with a nice dynamical behavior after a finite sequence of blow-ups; the above result shows that at least one center of the blow-ups must have dimension $\geq n/2 - 1$.

Remark 1.4. Recently, Tuyen Truong obtained results which are similar to Corollary 1.2, but with hypothesis on the Hodge structure and nef classes of $M_0$ that replace our strong hypothesis on the Picard number (see [14,15]).

2. Dimensions of Indeterminacy loci

In this section, we prove Theorem 1.1 under a slightly more general assumption. Indeed, we replace assumption (i) with the following assumption

(i') There exists an ample line bundle $L$ such that $f^*(L) \cong L^\otimes d$ for some $d > 1$.

This property is implied by (i). Indeed, if $M$ has Picard number 1, the torsion-free part of the Néron–Severi group of $M$ is isomorphic to $\mathbb{Z}$, and is generated by the class $[H]$ of an ample divisor $H$. Thus, $[f^*H]$ must be a multiple of $[H]$.

In what follows, we assume that $f$ satisfies property (i’) and property (ii). Replacing $H$ by a large enough multiple, we may and do assume that $H$ is very ample. Thus, the complete linear system $|H|$ provides an embedding of $M$ into some projective space $\mathbb{P}^d_k$, and we identify $M$ with its image in $\mathbb{P}^d_k$. With such a convention, members of $|H|$ correspond to hyperplane sections of $M$.

2.1. Degrees

Denote by $k$ the dimension of $M$, and by $\deg(M)$ its degree, i.e. the number of intersections of $M$ with a generic subspace of dimension $n - k$.

If $H_1, \ldots, H_k$ are hyperplane sections of $M$, and if $f^*(H_1)$ denotes the total transform of $H_1$ under the action of $f$, one defines the degree of $f$ by the following intersection of divisors of $M$

$$\deg(f) = \frac{1}{\deg(M)} f^*(H_1) \cdot H_2 \cdots H_k.$$ 

Since $M$ has Picard number 1, we know that divisor class $[f^*(H_1)]$ is proportional to $[H]$. Our definition of $\deg(f)$ implies that $f^*[H_1] = \deg(f)[H_1]$. As a consequence,

$$f^*(H_1) \cdot f^*(H_2) \cdots f^*(H_j) \cdot H_{j+1} \cdots H_k = \deg(f)^j \deg(M)$$

for all $0 \leq j \leq k$.

2.2. Degree bounds

Assume that the sum of the dimension of $\text{Ind}(f)$ and of $\text{Ind}(f^{-1})$ is at most $k - 3$. Then there exist at least two integers $l \geq 1$ such that

$$\dim(\text{Ind}(f)) \leq k - l - 1;$$

$$\dim(\text{Ind}(f^{-1})) \leq l - 1.$$ 

Let $H_1, \ldots, H_l$ and $H_1', \ldots, H_{k-1}'$ be generic hyperplane sections of $M$; by Bertini’s theorem,
(a) $H_1, \ldots, H_l$ intersect transversally the algebraic variety $\text{Ind}(f^{-1})$ (in particular, $H_1 \cap \cdots \cap H_l$ does not intersect $\text{Ind}(f^{-1})$ because $\dim(\text{Ind}(f^{-1})) < l$;
(b) $H'_1, \ldots, H'_{k-l}$ intersect transversally the algebraic variety $\text{Ind}(f)$ (in particular, $H'_1 \cap \cdots \cap H'_{k-l}$ does not intersect $\text{Ind}(f)$ because $\dim(\text{Ind}(f)) < k-l$).

For $j \leq l$, consider the variety $V_j = f^*(H_1 \cap \cdots \cap H_j)$: In the complement of $\text{Ind}(f)$, $V_j$ is smooth, of dimension $k-j$; since $j \leq l$ and $\dim(\text{Ind}(f)) < k-l$, $V_j$ extends in a unique way as a subvariety of dimension $k-j$ in $M$. The varieties $V_j$ are reduced and irreducible.

Since each $H_i$, $1 \leq i \leq l$, intersects $\text{Ind}(f^{-1})$ transversally, $f^*(H_i)$ is an irreducible hypersurface (it does not contain any component of the exceptional locus of $f$). Thus

$$
\text{deg}(f)^j \text{deg}(M) = f^*(H_1 \cap \cdots \cap H_i) \cdot (H'_1 \cap \cdots \cap H'_{k-l}).
$$

More precisely, since the $H'_i$ are generic, this intersection is transversal and $V_j \cdot (H'_1 \cap \cdots \cap H'_{k-l})$ is made of $\text{deg}(f)^j \text{deg}(M)$ points, all of them with multiplicity 1, all of them in the complement of $\text{Ind}(f)$ (see property (b) above).

Similarly, one defines the subvarieties $V'_j = f_*(H'_1 \cap \cdots \cap H'_i)$ with $j \leq k-l$; as above, these subvarieties have dimension $k-j$, are smooth in the complement of $\text{Ind}(f^{-1})$, and uniquely extend to varieties of dimension $k-j$ through $\text{Ind}(f^{-1})$. Each of them is equal to the intersection of the $j$ irreducible divisors $f_*(H_i)$, $1 \leq i \leq j$. Hence,

$$(H_1 \cap \cdots \cap H_i) \cdot V'_{k-l} = \text{deg}(f^{-1})^{k-l} \text{deg}(M).$$

If one applies the transformation $f : M \setminus \text{Ind}(f) \to M$ to $V_l$ and to $(H'_1 \cap \cdots \cap H'_{k-l})$, one deduces that $\text{deg}(f)^j \text{deg}(M) \leq \text{deg}(f^{-1})^{k-l} \text{deg}(M)$, because all points of intersection of $V_l$ with $(H'_1 \cap \cdots \cap H'_{k-l})$ are contained in the complement of $\text{Ind}(f)$. Applied to $f^{-1}$, the same argument provides the opposite inequality. Thus,

$$\text{deg}(f)^l = \text{deg}(f^{-1})^{k-l}.$$ 

Since there are at least two distinct values of $l$ for which this equation is satisfied, one concludes that

$$\text{deg}(f) = \text{deg}(f^{-1}) = 1.$$ 

As a consequence, $f$ has degree 1 if it satisfies assumptions (i') and (ii).

2.3. From birational transformations to automorphisms

To conclude the proof of Theorem 1.1, one applies the following lemma.

**Lemma 2.1.** Let $M$ be a smooth projective variety and $f$ a birational transformation of $M$. If there exists an ample divisor $H$ such that $f^*H$ and $f_*(H)$ are numerically equivalent to $H$, then $f$ is an automorphism.

**Proof.** Taking multiples, we assume that $H$ is very ample. Consider the graph $Z$ of $f$ in $M \times M$, together with its two natural projections $\pi_1$ and $\pi_2$ onto $M$.

The complete linear system $|H|$ is mapped by $f^*$ to a linear system $|H'|$ with the same numerical class, and vice versa if one applies $f^{-1}$ to $|H'|$. Thus, $|H'|$ is also a complete linear system, of the same dimension. Both of them are very ample (but they may differ if the dimension of $\text{Pic}^0(M)$ is positive).

Assume that $\pi_2$ contracts a curve $C$ to a point $q$. Take a generic member $H_0$ of $|H|$: It does not intersect $q$, and $\pi_2^*(H_0)$ does not intersect $C$. The projection $\langle \pi_1, \pi_2^*(H_0) \rangle$ is equal to $f^*(H_0)$; since $f^*$ maps the complete linear system $|H|$ to the complete linear system $|H'|$ and $H_0$ is generic, we may assume that $f^*(H_0)$ is a generic member of $|H'|$. As such, it does not intersect the finite set $\pi_1(C) \cap \text{Ind}(f)$. Thus, there is no fiber of $\pi_1$ that intersects simultaneously $C$ and $(\pi_2)^*(H_0)$, and $(\pi_1)_*(\pi_2^*(H_0))$ does not intersect $C$. This contradicts the fact that $f^*(H_0)$ is ample. \hfill $\square$

2.4. Conclusion, and Kähler manifolds

Under the assumptions of Theorem 1.1, Section 2.2 shows that $f^*H$ is numerically equivalent to $H$. Lemma 2.1 implies that $f$ is an automorphism. This concludes the proof of Theorem 1.1.

This proof is inspired by an argument of Sibony in [13] (see Proposition 2.3.2 and Remark 2.3.3); which makes use of complex analysis: the theory of closed positive current, and intersection theory. With this viewpoint, one gets the following statement.

**Theorem 2.2.** Let $M$ be a compact Kähler manifold and $f$ a bi-meromorphic transformation of $M$. Assume that
(i) there exists a Kähler form $\omega$ such that the cohomology class of $f^* \omega$ is proportional to the cohomology class of $\omega$;

(ii) the indeterminacy locus of $f$ and its inverse satisfy

$$\dim(\text{Ind}(f)) + \dim(\text{Ind}(f^{-1})) < \dim(M) - 2.$$ 

Then $f$ is an automorphism of $M$ that fixes the cohomology class of $\omega$.

Moreover, Lieberman's theorem (see [8]) implies that a positive iterate $f^m$ of $f$ is contained in the connected component of the identity of the complex Lie group $\text{Aut}(M)$.

2.5. Proof of Corollary 1.2

Since $M_m$ is obtained from $M_0$ by a sequence of blow-ups of centers of dimension $< \dim(M_m)/2 - 1$, all automorphisms $f$ of $M_m$ are conjugate, through the obvious birational morphism $\pi : M_m \to M_0$, to birational transformations of $M_0$ that satisfy

$$\dim(\text{Ind}(f)) < \dim(M_0)/2 - 1 \quad \text{and} \quad \dim(\text{Ind}(f^{-1})) < \dim(M_0)/2 - 1.$$ 

Thus, by Theorem 1.1 $\pi$ conjugates $\text{Aut}(M)$ to a subgroup of $\text{Aut}(M_0)$. Moreover, given any polarization of $M_0$ by a very ample class, all elements of $\text{Aut}(M_0)$ have degree 1 with respect to this polarization. Hence, $\text{Aut}(M_0)$ is a linear algebraic group, and the kernel of the action of $\text{Aut}(M_0)$ on $\text{Pic}^0(M_0)$ is a linear algebraic group; if $\text{Pic}^0(M_0)$ is trivial, there is a projective embedding of $\Theta : M_0 \to \mathbb{P}_k^3$ that conjugates $\text{Aut}(M_0)$ to the group of linear projective transformations $G \subset \text{PGL}_{n+1}(k)$ that preserve $\Theta(M)$.

3. Constraints on automorphisms from the structure of the intersection form

Let $X$ be a smooth projective variety of dimension $k$ over a field $k$. Denote by $\text{NS}(X)$ the Néron–Severi group of $X$, i.e. the group of classes of divisors for the numerical equivalence relation. We consider the multi-linear forms

$$Q_d : \text{NS}(X)^d \to \mathbb{Z}$$

which are defined by

$$Q_d(u_1, u_2, \ldots, u_d) = u_1 \cdot u_2 \cdots u_d \cdot k_X^{k-d}.$$ 

These forms are invariant under $\text{Aut}(X)^*$ and we shall derive new constraints on the size of $\text{Aut}(X)^*$ from this invariance.

**Theorem 3.1.** Let $X$ be a smooth projective variety of dimension $k \geq 3$, defined over a field $k$. Let $d$ be an integer that satisfies $3 \leq d \leq k$. If the projective variety

$$W_d(X) := \{ u \in \mathbb{P}(\text{NS}(X) \otimes \mathbb{Z} \cdot \mathbb{C}) | Q_d(u, u, \ldots, u) = 0 \}$$

is smooth, then $\text{Aut}(X)^*$ is finite.

**Proof.** The group $\text{Aut}(X)^*$ acts by linear projective transformations on the projective space $\mathbb{P}(\text{NS}(X) \otimes \mathbb{Z} \cdot \mathbb{C})$ and preserves the smooth hypersurface $W_d$. Since $d \geq 3$ it follows from [9] that the group of linear projective transformations preserving a smooth hypersurface of degree $d$ is finite. Hence, there is a finite index subgroup $\Gamma$ of $\text{Aut}(X)^*$ which is contained in the center of $\text{GL}(\text{NS}(X))$; since the later is a finite group of homotheties, this finishes the proof.

As a corollary, let us state the following one, already obtained in the previous sections:

**Corollary 3.2.** Let $X$ be a smooth projective variety of dimension $k \geq 3$. Assume that there exists a birational morphism $\pi : X \to V$ such that

- the Picard number of $V$ is equal to 1
- $\pi^{-1}$ is the blow-up of $l$ distinct points of $V$.

Then $\text{Aut}(X)^*$ is a finite group.

**Proof.** We identify $\text{NS}(V)$ with $Ne$ where $e_0$ is the class of an ample divisor. Let $a := e_0^k$. Since $X$ is obtained from $V$ by blowing up $l$ distinct points $p_1, \ldots, p_l$ we have

$$\text{NS}(X) = Ne_0 + \bigoplus_{1 \leq i \leq l} \mathbb{Z} e_i$$

where $e_i$ is the class of the exceptional divisor $E_i := \pi^{-1}(p_i)$. Then the form $Q_k$ is given by

$$Q_k(u) = a(X_0)^k + (-1)^{k+1} \sum_{i=1}^l (X_i)^k$$

where $u = X_0 e_0 + \sum_i X_i e_i$ and $[X_0 : \cdots : X_l]$ denotes the homogeneous coordinates on $\mathbb{P}(\text{NS}(X) \otimes \mathbb{Z} \cdot \mathbb{C})$. Hence, the projective variety defined by $Q_k$ is $\mathbb{P}(\text{NS}(X) \otimes \mathbb{Z} \cdot \mathbb{C})$ is smooth and $\text{Aut}(X)^*$ is finite.
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References