ON DEGREES OF BIRATIONAL MAPPINGS

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ABSTRACT. We prove that the degrees of the iterates $\deg(f^n)$ of a birational map satisfy $\lim \inf(\deg(f^n)) < +\infty$ if and only if the sequence $\deg(f^n)$ is bounded, and that the growth of $\deg(f^n)$ can not be arbitrarily slow, unless $\deg(f^n)$ is bounded.

1. Degree sequences

Let $k$ be a field. Consider a projective variety $X$, a polarization $H$ of $X$ (given by hyperplane sections of $X$ in some embedding $X \subset \mathbb{P}^N$), and a birational transformation $f$ of $X$, all defined over the field $k$. Let $k$ be the dimension of $X$. The degree of $f$ with respect to the polarization $H$ is the integer

$$\deg_H(f) = (f^*H) \cdot H^{k-1}$$

(1.1)

where $f^*H$ is the total transform of $H$, and $(f^*H) \cdot H^{k-1}$ is the intersection product of $f^*H$ with $k-1$ copies of $H$. The degree is a positive integer, which we shall simply denote by $\deg(f)$, even if it depends on $H$. When $f$ is a birational transformation of the projective space $\mathbb{P}^k$ and the polarization is given by $O_{\mathbb{P}^k}(1)$ (i.e. by hyperplanes $H \subset \mathbb{P}^k$), then $\deg(f)$ is the degree of the homogeneous polynomial formulas defining $f$ in homogeneous coordinates.

The degrees are submultiplicative, in the following sense:

$$\deg(f \circ g) \leq c_{X,H} \deg(f) \deg(g)$$

(1.2)

for some positive constant $c_{X,H}$ and for every pair of birational transformations. Also, if the polarization $H$ is changed into another polarization $H'$, there is a positive constant $c$ such that $\deg_H(f) \leq c \deg_{H'}(f)$ (see [7][11][13]).

The degree sequence of $f$ is the sequence $(\deg(f^n))_{n \geq 0}$; it plays an important role in the study of the dynamics and the geometry of $f$. There are infinitely, but only countably many degree sequences (see [14]); unfortunately, not much is known on these sequences when $\dim(X) \geq 3$. In this article, we obtain the following basic results.
• The sequence \((\deg(f^n))_{n \geq 0}\) is bounded if and only if it is bounded along an infinite subsequence (see Theorems A and B in §2 and 3).
• If the sequence \((\deg(f^n))_{n \geq 0}\) is unbounded, then its growth can not be arbitrarily small; for instance, \(\max_{0 \leq j \leq n} \deg(f^j)\) is asymptotically bounded from below by the inverse of the diagonal Ackermann function (see Theorem C in §4 for an effective result).

We focus on birational transformations because a rational dominant transformation which is not birational has a topological degree \(\delta > 1\), and this forces an exponential growth of the degrees: \(1 < \delta^{1/k} \leq \lim_{n \to \infty} (\deg(f^n)^{1/n})\) where \(k = \dim(X)\) (see [7] and [4], pages 120–126).

2. AUTOMORPHISMS OF THE AFFINE SPACE

We start with the simpler case of automorphisms of the affine space; the goal of this section is to introduce a \(p\)-adic method to study degree sequences.

**Theorem A (Urech).**—Let \(f\) be an automorphism of the affine space \(\mathbb{A}^k_k\). If \(\deg(f^n)\) is bounded along an infinite subsequence, then it is bounded.

In fact, Urech proves in [14] that \(\max_{0 \leq j \leq n} \deg(f^j)\) is bounded from below by \(\alpha n^{1/k}\) for some constant \(\alpha > 0\) when \((\deg(f^n))\) is unbounded. Here, we content ourselves with the simpler version stated in Theorem A.

2.1. Urech’s proof. Assume \(\deg(f^{n_i}) \leq B\) for some sequence \(n_1 < n_2 < \ldots < n_\ell\) of positive integers. The iterates \(f^{n_i}\) are in the vector space \(\text{End}_B(\mathbb{A}^k_k)\) of endomorphisms of \(\mathbb{A}^k_k\) of degree \(\leq B\). This vector space has dimension
\[
\dim(\text{End}_B(\mathbb{A}^k_k)) = k \left( \frac{k + B}{B} \right). \tag{2.1}
\]
Thus, if \(\ell > \dim(\text{End}_B(\mathbb{A}^k_k))\) there is a non-trivial linear relation between the \(f^{n_i}\) in the vector space \(\text{End}_B(\mathbb{A}^k_k)\), which we can write
\[
f^n = \sum_{m=1}^{n-1} a_m f^m \tag{2.2}
\]
for some integer \(n \leq n_\ell\) and some coefficients \(a_m \in k\). Then, every iterate \(f^N\) of \(f\) with \(N \geq n\) is a linear combination of the automorphisms \(f^m\) with \(m < n\), and so \(\deg(f^N)\) is bounded from above by the maximum of the degrees of \(f^m\) for \(0 \leq m \leq n - 1\). This shows that the sequence \((\deg(f^N))_{N \geq 0}\) is bounded.
2.2. The $p$-adic argument. Let us give a second proof when $\text{char}(k) = 0$, which will be generalized in §3 to treat the case of birational transformations.

2.2.1. Tate diffeomorphisms. Let $p$ be a prime number. Let $K$ be a field of characteristic 0 which is complete with respect to an absolute value $| \cdot |$ satisfying $|p| = 1/p$; such an absolute value is automatically ultrametric (see [9], Ex. 2 and 3, Chap. I.2). Let $R = \{ x \in K; |x| \leq 1 \}$ be the valuation ring of $K$; in the vector space $K^k$, the unit polydisk is the subset $U = R^k$.

Fix a positive integer $k$, and consider the ring $R[x] = R[x_1, \ldots, x_k]$ of polynomial functions in $k$ variables with coefficients in $R$. For $f$ in $R[x]$, define the norm $\| f \|$ to be the supremum of the absolute values of the coefficients of $f$:

$$\| f \| = \sup_I |a_I|$$

(2.3)

where $f = \sum_{I=(i_1,\ldots,i_k)} a_I x^I$. By definition, the Tate algebra $R\langle x \rangle$ is the completion of $R[x]$ with respect to this norm. It coincides with the set of formal power series $f = \sum_I a_I x^I$ converging (absolutely) on the closed unit polydisk $R^k$. Moreover, the absolute convergence is equivalent to $|a_I| \to 0$ as $\text{length}(I) \to \infty$. Every element $g$ in $R\langle x \rangle^k$ determines a Tate analytic map $g: U \to U$.

For $f$ and $g$ in $R\langle x \rangle$ and $c$ in $\mathbb{R}_+$, the notation $f \in p^c R\langle x \rangle$ means $\| f \| \leq |p|^c$ and the notation $f \equiv g \mod (p^c)$ means $\| f - g \| \leq |p|^c$; we then extend such notations component-wise to $(R\langle x \rangle)^m$ for all $m \geq 1$.

For indeterminates $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_m)$, the composition $R\langle y \rangle \times R\langle x \rangle^m \to R\langle x \rangle$ is well defined, and coordinatewise we obtain

$$R\langle y \rangle^n \times R\langle x \rangle^m \to R\langle x \rangle^n.$$

(2.4)

When $m = n = k$, we get a semigroup $R\langle x \rangle^k$. The group of (Tate) analytic diffeomorphisms of $U$ is the group of invertible elements in this semigroup; we denote it by $\text{Diff}^a(U)$. Elements of $\text{Diff}^a(U)$ are bijective transformations $f: U \to U$ given by $f(x) = (f_1, \ldots, f_k)(x)$ where each $f_i$ is in $R\langle x \rangle$ with an inverse $f^{-1}: U \to U$ that is also defined by power series in the Tate algebra.

The following result is due to Jason Bell and Bjorn Poonen (see [2,12]).

**Theorem 2.1.** Let $f$ be an element of $R\langle x \rangle^k$ with $f \equiv \text{id} \mod (p^c)$ for some real number $c > 1/(p-1)$. Then $f$ is a Tate diffeomorphism of $U = R^k$ and there exists a unique Tate analytic map $\Phi: R \times U \to U$ such that

1. $\Phi(n, x) = f^n(x)$ for all $n \in \mathbb{Z}$;
2. $\Phi(s + t, x) = \Phi(s, \Phi(t, x))$ for all $t, s$ in $R$. 


2.2.2. Second proof of Theorem A. Denote by $S$ the finite set of all the coefficients that appear in the polynomial formulas defining $f$. Let $R_S \subset k$ be the ring generated by $S$ over $\mathbb{Z}$, and let $K_S$ be its fraction field:

$$\mathbb{Z} \subset R_S \subset K_S \subset k.$$ (2.5)

Since $\text{char}(k) = 0$, there exists a prime $p > 2$ such that $R_S$ embeds into $\mathbb{Z}_p$ (see [10], §4 and 5, and [2], Lemma 3.1). We apply this embedding to the coefficients of $f$ and get an automorphism of $A^k_\mathbb{Q}$ which is defined by polynomial formulas in $\mathbb{Z}_p[x_1, \ldots, x_k]$; for simplicity, we keep the same notation $f$ for this automorphism (embedding $R_S$ in $\mathbb{Z}_p$ does not change the value of the degrees $\deg(f^n)$).

Since $f$ is a polynomial automorphism with coefficients in $\mathbb{Z}_p$, it determines an element of $\text{Diff}^m(U)$, the group of analytic diffeomorphisms of the polydisk $U = \mathbb{Z}_p^k$.

There exists a positive integer $m$ such that $f^m$ fixes the origin $0 \in U$ modulo $p^2$: $f^m(0) \equiv 0 \mod (p^2)$. Taking some further iterate, we may also assume that the differential $Df^m_0$ satisfies $Df^m_0 \equiv \text{Id} \mod (p)$. We fix such an integer $m$ and replace $f$ by $f^m$. The following lemma follows from the submultiplicativity of degrees (see Equation (1.2) in Section 1). It shows that replacing $f$ by $f^m$ is armless if one wants to bound the degrees of the iterates of $f$.

**Lemma 2.2.** If the sequence $\deg(f^{mn})$ is bounded for some $m > 0$, then the sequence $\deg(f^n)$ is bounded too.

Denote by $x = (x_1, \ldots, x_k)$ the coordinate system of $A^k$, and by $m_p$ the multiplication by $p$: $m_p(x) = px$. Change $f$ into $g := m_p^{-1} \circ f \circ m_p$; then $g \equiv \text{Id} \mod (p)$ in the sense of Section 2.2.1. Since $p \geq 3$, Theorem 2.1 gives a Tate analytic flow $\Phi: \mathbb{Z}_p \times A^k(\mathbb{Z}_p) \to A^k(\mathbb{Z}_p)$ which extends the action of $g$: $\Phi(n, x) = g^n(x)$ for every integer $n \in \mathbb{Z}$. Since $\Phi$ is analytic, one can write

$$\Phi(t, x) = \sum_J A_J(t)x^J$$ (2.6)

where $J$ runs over all multi-indices $(j_1, \ldots, j_k) \in (\mathbb{Z}_{\geq 0})^k$ and each $A_J$ defines a $p$-adic analytic curve $\mathbb{Z}_p \to A^k(\mathbb{Q}_p)$. By submultiplicativity of the degrees, there is a constant $C > 0$ such that $\deg(g^n) \leq CB^m$. Thus, we obtain $A_J(n_i) = 0$ for all indices $i$ and all multi-indices $J$ of length $|J| > CB^m$. The $A_J$ being analytic functions of $t \in \mathbb{Z}_p$, the principle of isolated zeros implies that

$$A_J = 0 \text{ in } \mathbb{Z}_p(t), \forall J \text{ with } |J| > CB^m.$$ (2.7)
Thus, $\Phi(t, x)$ is a polynomial automorphism of degree $\leq CB^m$ for all $t \in \mathbb{Z}_p$, and $g^n(x) = \Phi(n, x)$ has degree at most $CB^m$ for all $n$. By Lemma 2.2, this proves that $\deg(f^n)$ is a bounded sequence.

3. Birational transformations

We now extend Theorem A to the case of birational transformations.

Theorem B.-- Let $k$ be a field of characteristic 0. Let $X$ be a projective variety and $f : X \to X$ be a birational transformation of $X$, both defined over $k$. If the sequence $(\deg(f^n))_{n \geq 0}$ is not bounded, then it goes to $+\infty$ with $n$:

$$\lim\inf_{n \to +\infty} \deg(f^n) = +\infty.$$ 

Urech’s argument does not apply to this context, because the dimension of the space of rational transformations of $\mathbb{A}_k^r$ of degree $\leq B$ is infinite. We shall therefore apply the $p$-adic method, adapting the proof given in Section 2.2.2.

Note that Theorem B can be combined with a theorem of Weil to obtain the following: if $f$ is a birational transformation of the projective variety $X$, over an algebraically closed field of characteristic 0, and if the degrees of its iterates are bounded along an infinite subsequence $f^{n_i}$, then there exist a birational map $\psi : Y \to X$ and an integer $m > 0$ such that $f_Y := \psi^{-1} \circ f \circ \psi$ is in $\text{Aut}(Y)$, and $f_Y^m$ is in the connected component $\text{Aut}(Y)^0$ (see [3] and references therein).

In what follows, $f$ and $X$ are as in Theorem B; we also assume, without loss of generality, that $k = \mathbb{C}$ and that $X$ is smooth. We suppose that there is an infinite sequence of integers $n_1 < n_2 < \ldots < n_j < \ldots$ and a positive number $B$ such that $\deg(f^{n_j}) \leq B$ for all $j$. We fix a finite set $S$ of complex numbers such that $X$ and $f$ are defined by equations and formulas with coefficients in $S$, and we embed the ring $R_S \subset \mathbb{C}$ generated by $S$ in some $\mathbb{Z}_p$, for some prime number $p > 2$. According to [5], Section 3, we may assume that $X$ and $f$ have good reduction modulo $p$.

3.1. The Hrushovski’s theorem and $p$-adic polydisks. According to a theorem of Hrushovski (see [8]), there is a periodic point $z_0$ of $f$ in $X(F)$ for some finite field extension $F$ of the residue field $F_p$, the orbit of which does not intersect the indeterminacy points of $f$ and $f^{-1}$. If $\ell$ is the period of $z_0$, then $f^\ell(z_0) = z_0$ and $Df_{z_0}^\ell$ is an element of the finite group $\text{GL}(\text{TX}_{F_q}(z_0)) \simeq \text{GL}(k, F_q)$. Thus, there is an integer $m > 0$ such that $f^m(z_0) = z_0$ and $Df_{z_0}^m = \text{Id}.$
Replace $f$ by its iterate $g = f^m$. Then, $g$ fixes $z_0$ in $X(F)$, $g$ is an isomorphism in a neighborhood of $z_0$, and $Dg_{z_0} = \text{Id}$. According to [1] and [5] Section 3, this implies that there is

- a finite extension $K$ of $Q_p$, with valuation ring $R \subset K$;
- a point $z$ in $X(K)$ and a polydisk $V_z \simeq R^k \subset X(K)$ which is $g$-invariant and such that $g_{V_z} \equiv \text{Id} \mod (p)$ (in the coordinate system $(x_1, \ldots, x_k)$ of the polydisk).

When the point $z_0$ is in $X(F_p)$ and is the reduction of a point $z \in X(Z_p)$, the polydisk $V_z$ is the set of points $w \in X(Z_p)$ with $|z - w| < 1$; one identifies this polydisk to $U = (Z_p)^k$ via some $p$-adic analytic diffeomorphism $\phi: U \to V_z$; changing $\phi$ into $\phi \circ m_p$ if necessary, we obtain $g_{V_z} \equiv \text{Id} \mod (p)$ (see Section 2.2.2 and [5], Section 3). In full generality, a finite extension $K$ of $Q_p$ is needed because $z_0$ is a point in $X(F)$ for some extension of the residue field.

3.2. Controlling the degrees. As in Section 2.2.1 denote by $U$ the polydisk $R^k \simeq V_z$; thus, $U$ is viewed as the polydisk $R^k$ and also as a subset of $X(K)$. Applying Theorem 2.1 to $g$, we obtain a $p$-adic analytic flow $\Phi: R \times U \to U$, $(t, x) \mapsto \Phi(t, x)$ (3.1)

such that $\Phi(n, x) = g^n(x)$ for every integer $n$. In other words, the action of $g$ on $U$ extends to an analytic action of the additive compact group $(R, +)$.

Let $\pi_1: X \times X \to X$ denote the projection onto the first factor. Denote by $\text{Bir}_D(X)$ the set of birational transformations of $X$ of degree $D$; once birational transformations are identified to their graphs, this set becomes naturally a finite union of irreducible, locally closed algebraic subsets in the Hilbert scheme of $X \times X$ (see [3], Section 2.2, and references therein). Taking a subsequence, there is a positive integer $D$, an irreducible component $B_D$ of $\text{Bir}_D(X)$, and a strictly increasing, infinite sequence of integers $(n_j)$ such that

$$g^n_j \in B_D$$

(3.2) for all $j$. Denote by $\overline{B_D}$ the Zariski closure of $B_D$ in the Hilbert scheme of $X \times X$. To every element $h \in \overline{B_D}$ corresponds a unique algebraic subset $G_h$ of $X \times X$ (the graph of $h$, when $h$ is in $B_D$). Our goal is to show that, for every $t \in R$, the graph of $\Phi(t, \cdot)$ is the intersection $G_h \cap U^2$ for some element $h_t \in \overline{B_D}$; this will conclude the proof because $g^n(x) = \Phi(n, x)$ for all $n \geq 0$.

We start with a simple remark, which we encapsulate in the following lemma.
Lemma 3.1. There is a finite subset $E \subset U \subset X(K)$ with the following property. Given any subset $\tilde{E}$ of $U \times U$ with $\pi_1(\tilde{E}) = E$, there is at most one element $h \in \overline{B_D}$ such that $\tilde{E} \subset G_h$.

Fix such a set $E$, and order it to get a finite list $E = (x_1, \ldots, x_{\ell_0})$ of elements of $U$. Let $E' = (x_1, \ldots, x_{\ell_0}, x_{\ell_0+1}, \ldots, x_\ell)$ be any list of elements of $U$ which extends $E$.

For every element $h$ in $\overline{B_D}$, the variety $G_h$ determines a correspondence $G_h \subset X \times X$. The subset of elements $(h, (x_i, y_i))_{1 \leq i \leq \ell}$ in $\overline{B_D} \times (X \times X)^{\ell}$ defined by the incidence relation

$$(x_i, y_i) \in G_h$$

for every $1 \leq i \leq \ell$ is an algebraic subset of $\overline{B_D} \times (X \times X)^{\ell}$. Add one constraint, namely that the first projection $(x_i)_{1 \leq i \leq \ell}$ coincides with $E'$, and project the resulting subset on $(X \times X)^{\ell}$: we get a subset $G(E')$ of $(X \times X)^{\ell}$.

Then, define a $p$-adic analytic curve $\Lambda: R \to (X \times X)^{\ell}$ by

$$\Lambda(t) = (x_i, \Phi(t, x_i))_{1 \leq i \leq \ell}.$$ (4.4)

If $t = n_j$, $g^{n_j}$ is an element of $B_D$ and $\Lambda(n_j)$ is contained in the graph of $g^{n_j}$; hence, $\Lambda(n_j)$ is an element of $G(E')$. By the principle of isolated zeros, the analytic curve $t \mapsto \Lambda(t) \subset (X \times X)^{\ell}$ is contained in $G(E')$ for all $t \in R$. Thus, for every $t$ there is an element $h_t \in \overline{B_D}$ such that $\Lambda(t)$ is contained in the subset $G_{h_t}$ of $(X \times X)^{\ell}$. From the choice of $E$ and the inclusion $E \subset E'$, we know that $h_t$ does not depend on $E'$. Thus, the graph of $\Phi(t, \cdot)$ coincides with the intersection of $G_{h_t}$ with $U \times U$. This implies that the graph of $g^n(\cdot) = \Phi(n, \cdot)$ coincides with $G_{h_n}$, and that the degree of $g^n$ is at most $D$ for all values of $n$.

4. Lower bounds on degree growth

We now prove that the growth of $(\deg(f^n))$ can not be arbitrarily slow unless $(\deg(f^n))$ is bounded. For simplicity, we focus on birational transformations of the projective space; there is no restriction on the characteristic of $k$.

4.1. A family of integer sequences. Fix two positive integers $k$ and $d$: later on, $k$ will be the dimension of the projective space $\mathbb{P}^k_k$, and $d$ will be the degree of $f: \mathbb{P}^k \to \mathbb{P}^k$. Set

$$m = (d-1)(k+1).$$ (4.1)

Then, consider an auxiliary integer $D \geq 1$, which will play the role of the degree of an effective divisor in the next paragraphs, and define

$$q = (dD)^m(D+1).$$ (4.2)
Thus, $q$ depends on $k$, $d$ and $D$ because $m$ depends on $k$ and $d$. Then, set

$$a_0 = \binom{k+D}{k} - 1, \quad b_0 = 1, \quad c_0 = D + 1. \quad (4.3)$$

Starting from the triple $(a_0, b_0, c_0)$, we define a sequence $((a_j, b_j, c_j))_{j \geq 0}$ inductively by

$$(a_{j+1}, b_{j+1}, c_{j+1}) = (a_j, b_j - 1, qc_j^2) \quad (4.4)$$

if $b_j \geq 2$, and by

$$(a_{j+1}, b_{j+1}, c_{j+1}) = (a_j - 1, qc_j^2, qc_j^2) = (a_j - 1, c_{j+1}, c_{j+1}) \quad (4.5)$$

if $b_j = 1$. By construction, $(a_1, b_1, c_1) = (a_0 - 1, qc_0^2, qc_0^2)$.

Define $\Phi : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$\Phi(c) = qc^2. \quad (4.6)$$

**Lemma 4.1.** Define the sequence of integers $(F_i)_{i \geq 1}$ recursively by $F_1 = q(D + 1)^2$ and $F_{i+1} = \Phi^{F_i}(F_i)$ for $i \geq 1$ (where $\Phi^{F_i}$ is the $F_i$-iterate of $\Phi$). Then

$$(a_1 + F_1 + \cdots + F_i, b_1 + F_1 + \cdots + F_i, c_1 + F_1 + \cdots + F_i) = (a_0 - i - 1, F_i + 1, F_i + 1).$$

The proof is straightforward. Now, define the function $S : \mathbb{Z}^+ \to \mathbb{Z}^+$ as the sum of the $F_i$:

$$S(j) = 1 + F_1 + F_2 + \cdots + F_j \quad (4.7)$$

for all $j \geq 1$. The function $S$ is increasing and goes to $+\infty$ extremely fast with $j$. Then, set

$$\chi_{d,k}(n) = \max \left\{ D \geq 0 \mid S\left( \binom{k+D}{k} - 2 \right) < n \right\}. \quad (4.8)$$

**Lemma 4.2.** The function $\chi_{d,k} : \mathbb{Z}^+ \to \mathbb{Z}^+$ is non-decreasing and goes to $+\infty$ with $n$.

**Remark 4.3.** The function $S$ is primitive recursive (see [6], Chapters 3 and 13). In other words, $S$ is obtained from the basic functions (the zero function, the successor $s(x) = x + 1$, and the projections $(x_i)_{1 \leq i \leq m} \mapsto x_i$) by a finite sequence of compositions and recursions. Equivalently, one can write a program that computes $S$, all of whose instructions are limited to (1) the zero initialization $V \leftarrow 0$, (2) the increment $V \leftarrow V + 1$, (3) the assignment $V \leftarrow V'$, and (4) loops of definite length. Writing such a program is an easy exercise. Now, consider the diagonal Ackermann function $A(n)$ (see [6], Section 13.3). It
grows asymptotically faster than any primitive recursive function; hence, the
inverse of the Ackermann diagonal function
\[
\alpha(n) = \max\{D \geq 0 \mid \text{Ack}(D) \leq n\}.
\] (4.9)
is, asymptotically, a lower bound for \(\chi_{d,k}(n)\). A better lower bound is obtained
by showing that \(\chi_{d,k}\) is in the \(L_6\) hierarchy of [6], Chapter 13; this gives an
asymptotic lower bound by the inverse of the function \(f_7\) of [6], independent
on the values of \(d\) and \(k\), but this is a very week bound too.

4.2. **Statement of the lower bound.** We can now state the result that will be
proved in the next paragraphs.

**Theorem C.**— *Let \(f\) be a birational transformation of the complex projective
space \(\mathbb{P}^k_C\). If the sequence \((\max_{0 \leq j \leq n} (\deg(f^j)))_{n \geq 0}\) is unbounded, then it is
bounded from below by the sequence of integers \((\chi_{d,k}(n))_{n \geq 0}\).*

**Remark 4.4.** There are infinitely, but only countably many sequences of de-
grees \((\deg(f^n))_{n \geq 0}\) (see [14]). Consider the countably many sequences
\[
\left(\max_{0 \leq j \leq n} (\deg(f^j))\right)_{n \geq 0}
\] (4.10)
restricted to the family of birational maps for which \((\deg(f^n))\) is unbounded.
We get a countable family of non-decreasing, unbounded sequences of inte-
gers. Now, let \((u_i)_{i \in \mathbb{Z}_0}\) be any countable family of non-decreasing and un-
bounded sequences of integers \((u_i(n))\). Define a sequence \(w(n)\) as follows.
First, set \(v_j = \min\{u_0, u_1, \ldots, u_j\}\); this defines a new family of sequences, with
the same limit \(+\infty\), but now \(v_j(n) \geq v_{j+1}(n)\) for every pair of non-negative in-
tegers. Then, set \(m_0 = 0\), and define \(m_{n+1}\) recursively to be the first positive
integer such that \(v_{n+1}(m_{n+1}) \geq v_n(m_n) + 1\). We have \(m_{n+1} \geq m_n + 1\) for all
\(n \in \mathbb{Z}_0\). Set \(w(n) := v_{r_n}(m_{r_n})\) where \(r_n\) is the unique non-negative integer satis-
fying \(m_{r_n} \leq n \leq m_{r_n+1} - 1\). By construction, \(w(n)\) goes to \(+\infty\) with \(n\) and
\(u_i(n)\) is asymptotically bounded from below by \(w(n)\).

In Theorem C, the result is more explicit. Firstly, the lower bound is ex-
plicitly given by the sequence \((\chi_{d,k}(n))_{n \geq 0}\). Secondly, the lower bound is not
asymptotic: it works for every value of \(n\). In particular, if \(\deg(f^j) < \chi_{d,k}(n)\)
for \(0 \leq j \leq n\) and \(\deg(f) = d\), then the sequence \((\deg(f^n))\) is bounded.
4.3. Divisors and strict transforms. To prove Theorem C, we consider the action of $f$ by strict transform on effective divisors. As above, $d = \deg(f)$ and $m = (d - 1)(k + 1)$ (see Section 4.1).

4.3.1. Exceptional locus. Let $X$ be a smooth projective variety and $\pi_1$ and $\pi_2 : X \to \mathbb{P}^k$ be two birational morphisms such that $f = \pi_2 \circ \pi_1^{-1}$; then, consider the exceptional locus $\text{Exc}(\pi_2) \subset X$, project it by $\pi_1$ into $\mathbb{P}^k$, and list its irreducible components of codimension 1: we obtain a finite number

$$E_1, \ldots, E_{m(f)}$$

of irreducible hypersurfaces, contained in the zero locus of the jacobian determinant of $f$. Since this critical locus has degree $m$, we obtain:

$$m(f) \leq m, \quad \text{and} \quad \deg(E_i) \leq m \quad (\forall i \geq 1).$$

4.3.2. Effective divisors. Denote by $M$ the semigroup of effective divisors of $\mathbb{P}^k$; every element of $M$ is a finite sum of irreducible hypersurfaces with non-negative integer coefficients. There is a partial ordering $\leq$ on $M$, which is defined by $E \leq E'$ if and only if the divisor $E' - E$ is effective.

We denote by $\deg : M \to \mathbb{Z}_{\geq 0}$ the degree function. For every degree $D \geq 1$, we denote by $M_D$ the set $\mathbb{P}(H^0(\mathbb{P}^k, O_{\mathbb{P}^k}(D)))$ of effective divisors of degree $D$; thus, $M$ is the disjoint union of all the $M_D$, and each of these components will be endowed with the Zariski topology of $\mathbb{P}(H^0(\mathbb{P}^k, O_{\mathbb{P}^k}(D)))$. The dimension of $M_D$ is equal to the integer $a_0 = a_0(D, k)$ from Section 4.1:

$$\dim(M_D) = \left( \binom{k + D}{k} \right) - 1.$$ (4.13)

Let $G \subset M$ be the semigroup generated by the $E_i$:

$$G = \bigoplus_{i=1}^{m(f)} \mathbb{Z}_{\geq 0}E_i.$$ (4.14)

The elements of $G$ are the effective divisors which are supported by the exceptional locus of $f$. For every $E \in G$, there is a translation operator $T_E : M \to M$ which is defined by $T_E : E' \mapsto E + E'$; it is a linear projective embedding of the projective space $M_D$ into the projective space $M_{D+\deg(E)}$. We define

$$M_D^o = M_D \setminus \bigcup_{E \in G \setminus \{0\}, \deg(E) \leq D} T_E(M_{D-\deg(E)}).$$ (4.15)

Thus, $M_D^o$ is an open subset of $M_D$; it is the complement of finitely many proper linear projective subspaces. Also, $M_0^o = M_0$ and $M_1^o$ is obtained from
by removing finitely many points, corresponding to the $E_i$ of degree 1 (the hyperplanes contracted by $f$). Set $M^o = \bigcup_{D \geq 0} M^o_D$. This is the set of effective divisors without any component in the exceptional locus of $f$. The inclusion of $M^o$ in $M$ will be denoted by $\imath : M^o \to M$.

There is a natural projection $\pi_G : M \to G$; namely, $\pi_G(E)$ is the maximal element such that $E - \pi_G(E)$ is effective. We denote by $\pi_o : M \to M^o$ the projection $\pi_o = \text{id} - \pi_G$; this homomorphism removes the part of an effective divisor $E$ which is supported on the exceptional locus of $f$.

**Remark 4.5.** The restriction of the map $\pi_o$ to the projective space $M_D$ is piecewise linear, in the following sense. Consider the subsets $U_{E,D}$ of $M_D$ which are defined for every $E \in G$ with $\deg(E) \leq D$ by

$$U_{E,D} = T_E(M_{D-\deg(E)}) \setminus \bigcup_{E' > E, E' \in G, \deg(E') \leq D} T_{E'}(M_{D-\deg(E')}).$$

They define a stratification of $M_D$ by (open subsets of) linear subspaces, and $\pi_o$ coincides with the of the linear map inverse of $T_E$ on each $U_{E,D}$.

**4.3.3. Strict transform.** First, we consider the total transform $f^* : M \to M$, which is defined by $f^*(E) = (\pi_1)_* \pi_2^*(E)$ for every divisor $E \in M$. This is an injective homomorphism of semigroups. Let $[x_0, \ldots, x_k]$ be homogeneous coordinates on $\mathbb{P}^k$. If $f = [f_0 : \cdots : f_k]$ is defined by homogeneous polynomial functions $f_i \in \mathbb{k}[x_0, \ldots, x_k]$ of degree $d$, and if $E$ is defined by the homogeneous equation $P(x_0, \ldots, x_k) = 0$, then $f^*(E)$ is defined by $P \circ f = P(f_0, \ldots, f_k) = 0$. Thus, $f^*$ induces a linear projective embedding of $M_D$ into $M_{dD}$ for every $D$.

Then, we denote by $f^o : M^o \to M^o$ the strict transform. It is defined by

$$f^o(E) = (\pi_o \circ f^* \circ \imath)(E). \quad \text{(4.16)}$$

This is a homomorphism of semigroups. Removing the exceptional locus $(\pi_1)_*(E(\pi_2))$ from $\mathbb{P}^k_\mathbb{k}$, one gets a variety $Y$, and an induced birational transformation $f_Y : Y \dashrightarrow Y$. Then, every divisor $E \in M^o$ intersects $Y$ on a divisor $E_Y$ of the same degree: this provides a bijection between effective divisors of $Y$ and elements of $M^o$ that conjugates $(f_Y)^*$ to $f^o$. In particular, $(f^o)^n = (f^n)^o$.

**4.4. Proof of Theorem C.** Let $\eta$ be the generic point of $M^o_1$ ($\eta$ corresponds to a generic hyperplane of $\mathbb{P}^k_\mathbb{k}$). The degree of $f^*(\eta)$ is equal to the degree of $f$, and since $\eta$ is generic, $f^*(\eta)$ coincides with $f^o(\eta)$. Thus, $\deg(f) = \deg(f^o(\eta))$ and more generally

$$\deg(f^n) = \deg((f^o)^n(\eta)) \quad (\forall n \geq 1). \quad \text{(4.17)}$$
Fix an integer $D \geq 0$. Write $M^0_{\leq D}$ for the union of the $M^0_D$ with $D' \leq D$, and define recursively $Z_D(0) = M^0_{\leq D}$ and

$$Z_D(i + 1) = \{ E \in Z_D(i) \mid f^e(E) \in Z_D(i) \}$$

(4.18)

for $i \geq 0$. A divisor $E \in M^0_{\leq D}$ is in $Z_D(i)$ if its strict transform $f^e(E)$ is of degree $\leq D$, and $f^e(f^e(E))$ is also of degree $\leq D$, up to $(f^e)^i(E)$ which is also of degree at most $D$. The subsets $Z_D(i)$ form a decreasing sequence of Zariski closed subsets (in the disjoint union $M^0_{\leq D}$ of the $M^0_D$, $D' \leq D$). The strict transform $f^e$ maps $Z_D(i + 1)$ in $Z_D(i)$. There exists a minimal integer $\ell(D) \geq 0$ such that

$$Z_D(\ell(D)) = \bigcap_{i \geq 0} Z_D(i);$$

(4.19)

we denote this subset by $Z_D(\infty) = Z_D(\ell(D))$. By construction, $Z_D(\infty)$ is stable under the operator $f^e$; more precisely, $f^e(Z_D(\infty)) = Z_D(\infty) = (f^e)^{-1}(Z_D(\infty))$.

Let $\tau: \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a lower bound for the inverse function of $\ell$:

$$\ell(\tau(n)) \leq n \quad (\forall n \geq 0).$$

(4.20)

Assume that $\max\{\deg(f^m) \mid 0 \leq m \leq n_0\} \leq \tau(n_0)$ for some $n_0 \geq 1$. Then $\deg((f^e)^i(\eta)) \leq \tau(n_0)$ for every integer $i$ between 0 and $n_0$: this implies that $\eta$ is in the set $Z_{\tau(n_0)}(\ell(\tau(n_0))) = Z_{\tau(n_0)}(\infty)$, so that the degree of $(f^e)^m(\eta)$ is bounded from above by $\tau(n_0)$ for all $m \geq 0$. From Equation (4.17) we deduce that the sequence $(\deg(f^m))_{m \geq 0}$ is bounded. This proves the following lemma.

**Lemma 4.6.** Let $\tau$ be a lower bound for the inverse function of $\ell$. If

$$\max\{\deg(f^m) \mid 0 \leq m \leq n_0\} \leq \tau(n_0)$$

for some $n_0 \geq 1$, then the sequence of degrees $(\deg(f^m))_{n \geq 0}$ is bounded.

So, to conclude, we need to compare $\ell: \mathbb{Z}_{\geq 0} \to \mathbb{Z}^+$ to the function $S: \mathbb{Z}_{\geq 0} \to \mathbb{Z}^+$ of paragraph 4.1 (recall that $S$ depends on the parameters $k = \dim(\mathbb{P}^k_1)$ and $d = \deg(f)$ and that $\ell$ depends on $f$).

Let us describe $Z_D(i + 1)$ more precisely. For each $i$, and each $E \in G$ of degree $\deg(E) \leq dD$ consider the subset $T_E(\overline{\tau(Z_D(i))}) \cap M_{dD}$; this is a subset of $M_{dD}$ which is made of divisors $W$ such that $\pi_5(W)$ is contained in $Z_D(i)$, and the union of all these subsets when $E$ varies is exactly the set of points $W$ in $M_{dD}$ with a projection $\pi_5(W)$ in $Z_D(i)$. Thus, we define

$$(f^*)^{-1}(T_E(\overline{\tau(Z_D(i))})) = \{ V \in M^0_{\leq D} \mid f^e(\overline{\tau(V)}) \in T_E(\overline{\tau(Z_D(i))}) \}.\quad (4.21)$$
These sets are closed subsets of $M_{\leq D}$, and
\[ Z_D(i + 1) = Z_D(i) \bigcap \bigcup_{E \in G, \text{deg}(E) \leq dD} \pi_\circ \left( (f^*)^{-1}(T_E(\chi(Z_D(i)))) \right). \] (4.22)

Now, write $Z_D'(i) = Z_D(i) \setminus Z_D(\infty)$, and note that it is a decreasing sequence of open subsets with $Z_D'(j) = \emptyset$ for all $j \geq \ell(D)$.

We shall say that a closed subset $L$ of $M^0_{\leq D} \setminus Z_D(\infty)$ for the Zariski topology is **piecewise linear** if all its irreducible components are equal to the intersection of $M^0_{\leq D} \setminus Z_D(\infty)$ with a linear projective subspace of some $M_{D'}$, $D' \leq D$. Let Lin$(a, b, c)$ be the family of closed piecewise linear subsets of $M^0_{\leq D} \setminus Z_D(\infty)$ of dimension $a$, with at most $c$ irreducible components, and at most $b$ irreducible components of maximal dimension $a$. Then:

1. $Z_D'(i + 1) = \{ E \in Z_D'(i) \mid f^\circ(E) \in Z_D'(i) \} = \pi_\circ (f^* Z_D'(i)) \bigcap \bigcup_{E \in G} T_E(Z_D'(i))$, where $E$ runs over the elements of $G$ of degree $\text{deg}(E) \leq dD$.
2. In this union, every irreducible component of $T_E(Z_D'(i))$ is piecewise linear.

Recall that $q = (dD)^m(D + 1)$ was introduced in Section 4.1. If $Z$ is any closed piecewise linear subset of $M^0_{\leq D} \setminus Z_D(\infty)$ that contains exactly $c$ irreducible components, the set
\[ \pi_\circ (f^* Z) \bigcap \bigcup_{E \in G, \text{deg}(E) \leq dD} T_E(E) \] (4.23)
has at most $qc^2 = (dD)^m(D + 1)c^2$ irreducible components (this is just a crude estimate : the factor $(D + 1)$ comes from the number of irreducible components of $M_{\leq D}$, and the factor $(dD)^m$ from the fact that $G$ contains at most $(dD)^m$ elements of degree $\leq dD$). Let us now use that the sequence $Z_D'(i)$ decreases strictly as $i$ varies from 0 to $\ell(D)$, with $Z_D'(\ell(D)) = \emptyset$. If $0 \leq i \leq \ell(D) - 1$, and if $Z_D'(i)$ is contained in Lin$(a, b, c)$, we obtain

1. if $b \geq 2$, then $Z_D'(i + 1)$ is contained in Lin$(a, b - 1, qc^2)$;
2. if $b = 1$, then $Z_D'(i + 1)$ is contained in Lin$(a - 1, qc^2, qc^2)$.

This shows that
\[ \ell(D) \leq S\left( \binom{k + D}{k} - 2 \right) + 1 \] (4.24)

where $S$ is the function introduced in the Equation (4.7) of Section 4.1. Since $\chi_{d,k}$ satisfies $\ell(\chi_{d,k}(n)) \leq n$ for every $n \geq 1$, the conclusion follows.
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