RATIONAL G-SURFACES

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ABSTRACT. In this paper the author determines the structure of complete rational surfaces on which one can define a group action in such a way that for each element of the group there exists a nonzero linear equivalence divisor class with nonnegative self-intersection index which is invariant with respect to this element. If one excludes the case when this action factors through an algebraic action of a linear algebraic group, then all such surfaces are elliptic bundles, and the action of the group preserves the family of fibers.

Bibliography: 11 titles.

Introduction

Let \( V \) be a complete irreducible smooth surface defined over an algebraically closed field \( k \), and let \( G \) be an (abstract) group acting on \( V \). The action is defined by a homomorphism \( G \to \text{Aut}(V) \). A surface \( V \) with an action of a group \( G \) is called a \( G \)-surface. A surface \( V \) is called rational if its field of rational functions is isomorphic to a purely transcendental extension of transcendence degree two of the base field \( k \). Our goal is to give a classification of rational \( G \)-surfaces. Next we clarify our viewpoint with respect to this problem. If on a \( G \)-surface \( V \) there exists a nonempty \( G \)-invariant finite set of points, then, considering the blowing up \( \Sigma : U \to V \) of this set, we obtain a new \( G \)-surface \( U \); thus it makes sense to classify only \( G \)-minimal surfaces, i.e. \( G \)-surfaces \( V \) for which each birational \( G \)-morphism \( V \to W \) onto another \( G \)-surface \( W \) is an isomorphism. Furthermore, for the time being we put aside the almost completely investigated case when \( G \) is a linear algebraic group acting algebraically on \( V \) (i.e. we are given a morphism \( G \times V \to V \) satisfying the usual axioms); in particular, this includes the case of a finite group \( G \) considered by Cantor, Wiman, Manin and Iskovskih, and the case of a connected linear algebraic group \( G \) (Enriques). A formally more general situation arises when the action of \( G \) on \( V \) factors (by means of a homomorphism \( G \to H \)) through an algebraic action of a linear algebraic group \( H \) on \( V \). This situation is characterized by the existence of a positive ample divisor \( D \) on \( V \) such that the complete linear system \( |D| \) is \( G \)-invariant. In this case we shall say that \( V \) is a \( G \)-projective surface. If there does not exist a positive ample divisor \( D \) on \( V \) for which the system \( |D| \) is \( G \)-invariant, the \( G \)-surface \( V \) will be called \( G \)-antiprojective.

We are mainly interested in the classification of rational \( G \)-antiprojective surfaces.

Example. The first example of such a surface was given by I. R. Šafarevič in [1], Chapter VII, §1. In this example \( V \) is obtained from the projective plane \( \mathbb{P}^2 \) by blowing

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up the nine basic points of a pencil of cubic curves (we assume that no three of these points lie on a line and no six lie on a conic); let \( \Sigma : V \to \mathbb{P}_2 \) be the corresponding morphism. The pencil of cubic curves lifted on \( V \) gives rise to a fiber bundle \( \rho : V \to \mathbb{P}_1 \) whose fibers are irreducible curves of genus 1. The map \( \rho \) has many sections, including the exceptional curves \( L_1, \ldots, L_9 \) blown down to points under the morphism \( \Sigma \). Since the group of classes of divisors of degree zero acts on an elliptic curve, the divisors \( L_i - L_j \) define translations along the fibers of \( V \) which generate a free abelian subgroup \( G \) of rank 8 in \( \text{Aut}(V) \) (a formula for the rank of such subgroups is given in Proposition 7 of this paper). If \( V \) were \( G \)-projective, there would exist a \( G \)-invariant ample linear system, and the trace of this system on the generic fiber of \( \rho \) would be invariant with respect to all transformations from \( G \); however the group of translations of an abelian variety preserving the equivalence class of an ample divisor is finite (see [10], Chapter II, §6).

Although in this paper we prove some general results on antiprojective surfaces, we mainly discuss those antiprojective surfaces which are close to \( G \)-projective surfaces. We explain what we mean by this closeness. It turns out (see Theorem 1) that a rational \( G \)-surface \( V \) is \( G \)-projective if and only if there exists a divisor \( D \) on \( V \) with the following properties:

\[
(D^2) > 0, \quad (1)
\]

\[
gD \sim D \text{ for all } g \text{ from } G. \quad (2)
\]

If we keep condition (2), but replace (1) by the nonstrict inequality (assuming, of course, that \( D \) is not linearly equivalent to zero), then a \( G \)-surface satisfying the new requirements is not necessarily \( G \)-projective, as is shown by the above example in which \( D \) is equal to the fiber of \( \rho \).

**Definition.** A rational antiprojective \( G \)-surface \( V \) is called \( G \)-pseudoprojective if for each \( g \) from \( G \) there exists a divisor \( D_g \) on \( V \) such that

1) \( D_g \) is not linearly equivalent to zero,
2) \((D_g^2) > 0\), and
3) \(g(D_g) \sim D_g\).

We prove (Theorem 2) that a rational \( G \)-pseudoprojective and \( G \)-minimal surface \( V \) has the following properties:

a) \((K_V^2) = 0, \text{rk}(\text{Pic}(V)) = 10\).

b) For some natural \( m \), the system \([-mK_V]\) is one-dimensional, has no fixed components, and gives rise to a map \( \rho : V \to \mathbb{P}_1 \) whose fibers are curves of genus 1 on \( V \) which do not contain exceptional components. The pencil \([-mK_V]\) is the preimage under a birational morphism \( \Sigma : V \to \mathbb{P}_2 \) of a Halphen pencil, i.e. a pencil of curves of degree \( 3m \) with nine fundamental points of multiplicity \( m \) which are resolved by \( \Sigma \) (see [5] for further information of Halphen pencils).

c) If \( \{S_1, \ldots, S_r\} \) is the set of all reducible fibers of the map \( \rho \), and \( \mu_i = (\text{the number of components of the fiber } S_i) \), 1 \(< \ i \ < \ r \), then

\[
\sum_{i=1}^{r} (\mu_i - 1) < 8.
\]

If a surface \( V \) satisfies conditions a), b), and c), then it is \( \text{Aut}(V) \)-antiprojective.
In a sense, one may regard the group of automorphisms of an Aut-antiprojective surface as a discrete group of its transformations. From this point of view, for minimal Aut(V)-antiprojective surfaces V we have the following analogue of the Fedorov-Schoenflies-Bieberbach theorem from geometric crystallography: Aut(V) contains a subgroup of finite index generated by translations along the elliptic curves of the pencil \([-mK_V]\); the precise statement of this result is given in Theorem 2 and Proposition 7, and the description of the subgroup of translations is given in the proof of this proposition. Proposition 7 and Theorem 2 confirm a conjecture of Manin [11].

Finally we remark that there exist rational G-surfaces which are antiprojective but not pseudoprojective. We shall consider such surfaces in another paper; however we give an example of such a surface in Proposition 10; this example disproves the theorem in [11].

The paper is organized as follows. In §1 we prove some lemmas on linear systems; these lemmas are either known or easy to prove, but there are no convenient references. In the short §2 we describe four surfaces W_i, i = 1, 2, 3, 4, which are needed in order that in the statement of Proposition 7 we may assume that V does not coincide with W_i, although these W_i also present certain interest by themselves. In §2 we state all facts about W_i without proofs, because these proofs require a lot of space without involving significant new ideas. The reader may start with §3, where we state the main results. In addition to the results mentioned above, we would like to draw attention to Proposition 2 to the effect that a G-antiprojective surface is antiprojective with respect to a cyclic subgroup of G.

Notation and conventions. We assume that the base field k is algebraically closed and its characteristic is not equal to 2 or 3. We make this last assumption in order to avoid discussion of quasielliptic pencils.

In what follows V denotes a complete irreducible nonsingular surface. Beginning with §2, we assume that V is rational.

We denote by q(V) the irregularity of V and by K_V or simply K the canonical divisor class on V.

A divisor means a Cartier divisor.

By ~ (respectively, ≡) we denote the linear (numerical) equivalence of divisors; the group of classes of divisors with respect to linear (numerical) equivalence is denoted by Pic(V) (Num(V)). For a rational surface V we have Pic(V) = Num(V).

Nonnegative divisors on V are called curves and are considered as subschemes of V.

By D, possibly with indices or primes, we denote only divisors; similarly, by C we denote only curves.

By supp D we denote the support of D; for a divisor D on V, supp D is a reduced curve.

A component of a divisor is by definition an irreducible component of its support. Comp(D) denotes the set of components of D.

μ(D) denotes the number of elements in Comp(D).

A connected component of a nonzero divisor D is a nonzero divisor D' such that supp D' is connected and

$$\text{supp } D' \subset \text{supp } D,$$

$$\text{supp } D' \cap \text{supp } (D - D') = \emptyset.$$

(D \cdot D') denotes the intersection index of divisors on V, and (D^2) denotes the self-intersection index.
A positive divisor $D$ on $V$ is called *quasi-ample* if $(D \cdot C) > 0$ for each curve $C$ on $V$.

An element $d$ of an abelian group is called *primitive* if it cannot be written in the form $md'$ with $m > 2$; in particular, a divisor is called primitive if it is a primitive element of the group of divisors.

For a divisor $D$ on $V$ we denote by $|D|$ the set of all curves $C$ on $V$ such that $C \sim D$. It is clear that

$$\dim |D| = \dim \mathcal{H}^*(V, \mathcal{O}_V(D)) - 1.$$  

We say that a system $|C|$ has no fixed components if there is no positive $D$ such that $|C - D| + D = |C|$. A positive $D$ satisfying this equality is called the fixed component of the system $|C|$ if $|C - D|$ has no fixed components.

For a curve $C$ on $V$ and a divisor $D$ on $V$ such that

$$\text{Comp}(D) \cap \text{Comp}(C) = \emptyset$$  

we denote by $\text{Tr}_C(D)$ the divisor on $C$ induced by $D$ under the embedding of $C$ in $V$.

For a curve $C$ on $V$, let $\text{tr}_C : \text{Pic}(V) \rightarrow \text{Pic}(C)$ be the homomorphism associating to the class represented by a divisor $D$ satisfying condition (1) the class of the divisor $\text{Tr}_C(D)$; this class will also be denoted by $\text{tr}_C(D)$.

We define

$$\mathcal{P}(V) = \text{Num}(V) \otimes \mathbb{R}$$

($= \text{Pic}(V) \otimes \mathbb{R}$ for a rational $V$),

$A(V)$ is the subgroup of classes of quasi-ample divisors modulo numerical equivalence. $\mathcal{O}(V)$ is obtained by removing the zero vector from the closed convex cone in $\mathcal{P}(V)$ generated by $A(V)$. We recall that, in a real vector space, the closed convex cone generated by a set coincides with the closure of the set of all positive linear combinations of elements of this set.

A ray on a cone is a subset $\mathbb{R}^+ e$, where $e$ is a point of the cone, $e \neq 0$, and $\mathbb{R}^+$ is the set of positive real numbers.

The intersection index induces a bilinear form on $\mathcal{P}(V)$ which will be denoted by $(x \cdot y)$.

The set of those $x$ from $\mathcal{P}(V)$ for which $(x^2) > 0$ has two connected components.

The set of nonzero vectors belonging to the closure of the component containing the class of an ample divisor $V$ will be denoted by $I(V)$.

$I^0(V)$ is the set of those $x$ from $I(V)$ for which $(x^2) = 0$.

We note that

$$A(V) \subset \mathcal{O}(V) \subset I(V)$$

and $(x \cdot y) > 0$ for all $x$ and $y$ from $I(V)$; moreover, if $(x \cdot y) = 0$, then $x$ and $y$ are collinear and belonging to $I^0(V)$ (see Lemma 1 and its proof).

If $V$ is a $G$-surface, then $G$ acts naturally on $\mathcal{P}(V)$; moreover, Num($V$) is stable with respect to the action of $G$ (i.e. all transformations from $G$ are integral with respect to any basis of $\mathcal{P}(V)$ which is also a basis of Num($V$)). The subsets $A(V)$, $\mathcal{O}(V)$, $I(V)$, and $I^0(V)$ of the set $\mathcal{P}(V)$ are also stable with respect to $G$. 
The linear automorphism of \( \mathcal{G}(V) \) induced by an element \( g \) from \( G \) will be denoted by the same letter \( g \).

We shall often omit \( G \) in terms such as ‘\( G \)-projective’.

A reference to formula (n) is to the formula with this number which is nearest to the reference.

§1. Intersection indices and linear systems

**Lemma 1.** If \( D \) and \( D' \) are divisors on \( V \), \( (D^2) > 0 \), \( (D'^2) > 0 \) and \( (D \cdot D') = 0 \), then the classes of \( D \) and \( D' \) in \( \text{Num}(V) \) are proportional, i.e. there exists a pair \( (\alpha, \alpha') \) of nonzero integers such that

\[
\alpha D = \alpha' D'.
\]

Moreover, if \( D' \cong 0 \) (e.g. if \( D' > 0 \)), then \( (D^2) = 0 \) and the coefficient \( \alpha \) in (1) is not equal to zero.

**Proof.** The second assertion follows from the first. The first assertion follows from the theorem on intersection indices on surfaces (see [2], Lecture 18) to the effect that the quadratic form of self-intersections in \( \mathcal{G}(V) \) is nondegenerate and can be written (over \( \mathbb{R} \)) in the form \( x_0^2 - x_1^2 - \cdots - x_n^2 \) and that a linear subspace contained in the light cone of this quadratic form is at most one-dimensional. The lemma is proved.

**Lemma 2.** Let \( S \) be a divisor on \( V \), \( S = a_0 S_0 + \cdots + a_r S_r \), where \( r > 0 \) and \( a_i S_i \) is a connected component of \( S \) for \( 0 < i < r \), \( a_i > 0 \), \( a_i \in \mathbb{Z} \), and \( S_i \) is a primitive divisor. Let

\[
\text{Comp}(S) = \{E_1, \ldots, E_n\}.
\]

Suppose that a) \( S_i \cong 0 \) for \( 0 < i < r \), and b) \( (S \cdot E) = 0 \) for all \( E \in \text{Comp}(S) \). Then the following assertions are true:

1) The quadratic form

\[
Q = (x_0 E_1 + \cdots + x_n E_n)^2
\]

on \( \bigoplus^r_{i=1} \mathbb{Z} E_i \) is nonpositive.

2) For each \( i, 0 < i < r \), the classes of all components of \( S_i \) form a linearly independent subset in \( \text{Num}(V) \).

3) The maximal completely isotropic submodule for the form \( Q \) is generated by \( \{S_0, \ldots, S_r\} \), i.e. for each divisor \( D \) such that

\[
\text{supp } D \subseteq \text{supp } S, \quad (D^2) = 0
\]

there exist integers \( \mu_0, \ldots, \mu_r \) such that \( D = \mu_0 S_0 + \cdots + \mu_r S_r \).

**Proof.** We observe that from b) it follows that

\[
(S_i^2) = (S_i \cdot E_j) = 0 \quad \text{for} \quad 0 \leq i \leq r, \quad 1 \leq j \leq n.
\]

1) For a divisor \( D = x_1 E_1 + \cdots + x_n E_n \), from the equality \( (S_0^2) = (D \cdot S_0) = 0 \), assumption a) and Lemma 1 it follows that \( (D^2) < 0 \).

2) Let \( i = 0 \), \( \{E_1, \ldots, E_i\} \subseteq \text{Comp}(S_0) \), \( y_1 E_1 + \cdots + y_i E_i \equiv 0 \), \( y_1 > 0 \), \( y_i \in \mathbb{Z} \) and \( y_1 \neq 0 \), \( 1 < r < i \). Since all the \( y_j \) cannot be positive, transferring the summands with negative coefficients to the right, we may rewrite the dependence between \( E_1, \ldots, E_i \) in

\[
(S_i^2) = (S_i \cdot E_j) = 0 \quad \text{for} \quad 0 \leq i \leq r, \quad 1 \leq j \leq n.
\]
the form \( H_0 \equiv H_1 \), where \( H_0 \) and \( H_1 \) are positive divisors and

\[
\text{supp } H_i \subset \text{supp } S_0, \quad \text{Comp } (H_i) \cap \text{Comp } (H_i) = \emptyset.
\]

Let \( E \) be a component of \( S_0 \) which intersects \( H_0 \) but does not lie in \( H_0 \) (the existence of \( E \) follows from the connectedness of \( S_0 \)). From Lemma 1 and the relations

\[
(H_0^2) = (H_0 \cdot H_1) \geq 0, \quad (H_0 \cdot S_0) = (S_0^2) = 0
\]

it follows that there exist integers \( \alpha \) and \( \beta \) such that \( \alpha > 0 \) and \( \alpha H_0 \equiv \beta S_0 \), so that \( 0 < \alpha (H_0 \cdot E) = \beta (S_0 \cdot E) = 0 \). This contradiction proves 2).

3) Let \( D \) be a divisor satisfying (1), \( D \neq 0 \). Changing, if necessary, the order of the \( S_i \), we may write

\[
D = b_0 D_0 + \cdots + b_r D_r, \quad \text{supp } D_j \subset \text{supp } S_j,
\]

where \( b_j \in \mathbb{Z}, b_j 
eq 0, \) and \( D_j \) is primitive, \( 0 < j < s < r \). Since \( (D^2) = (D_j \cdot D_j) = 0 \) for \( 0 < l < j < s \) and since \( (D_j^2) < 0 \) by 1), we infer that \( (D^2) = 0 \) for \( 0 < j < s \).

Lemma 1 and the equality \( (D_j \cdot S_j) = 0 \) show that there exists a pair of integers \( \alpha_j, \beta_j \) such that \( \alpha_j > 0 \) and \( \alpha_j D_j \equiv \beta_j S_j \). However, this last numerical equivalence can actually be replaced by the equality \( \alpha_j D_j = \beta_j S_j \), since otherwise we would obtain a relation of linear dependence between the classes of components of \( S_j \) in Num(V) contrary to assertion 2) above. Since \( D_j \) and \( S_j \) are proportional primitive divisors, they are equal to each other modulo a sign. The lemma is proved.

**Lemma 3.** Let \( S = s_1 E_1 + \cdots + s_r E_r \) be a positive connected divisor on \( V \), and let \( \{E_1, \ldots, E_r\} = \text{Comp}(S) \) and \( (S \cdot E_i) = 0 \) for \( 1 < i < r \). Then for each divisor \( D \) on \( V \) such that \( (D \cdot S) = 0 \) there exists a natural \( n \) and a divisor \( D_0 \) such that

\[
\text{supp } D_0 \subset S, \quad ((D-D_0) \cdot E_i) = 0 \quad \text{for } 1 \leq i \leq r.
\]

The least such \( n \) depends only on \( S \) and on the \( r \) integers \( (D \cdot E_1), \ldots, (D \cdot E_r) \).

**Proof.** We denote by \( N \) the \( r \)-dimensional vector space \( \mathbb{Q}^r \) over the field \( \mathbb{Q} \) (the space of divisors with rational coefficients whose support is contained in \( S \)). The space \( N \) has a distinguished basis \( E_1, \ldots, E_r \). We may regard the intersection matrix \( \{(E_i \cdot E_j)\} \) as the matrix of a linear endomorphism \( A \) of \( N \). By assertion 3) of Lemma 2 we have \( \text{rk}(A) = r - 1 \), i.e. \( \dim A(N) = r - 1 \). More precisely, \( A(N) \) coincides with the kernel of the linear form \( \lambda \) on \( N \) which associates to an element

\[
x = x_1 E_1 + \cdots + x_r E_r \in N
\]

the number

\[
\lambda(x) = s_1 x_1 + \cdots + s_r x_r.
\]

For an element

\[
\nu = (D \cdot E_1) E_1 + \cdots + (D \cdot E_r) E_r \in N
\]

we have

\[
\lambda(\nu) = (D \cdot S) = 0;
\]

the number

\[
\lambda(\nu) = s_1 \nu_1 + \cdots + s_r \nu_r.
\]
hence there exists an element \( w = w_1 E_1 + \cdots + w_r E_r \in \mathbb{N} \) for which \( A(w) = \nu \). If \( n \) is a common denominator of the numbers \( w_1, \ldots, w_r \) and we set
\[
D_n = (nw_1)E_1 + \cdots + (nw_r)E_r,
\]
the above equality can be written in the form
\[
(D_n \cdot E_i) = n(D \cdot E_i), \quad 1 \leq i \leq r.
\]

The lemma is proved.

**Lemma 4.** Let \( S \) be a positive primitive connected divisor on \( V \) such that \((S \cdot E) = 0\) for \( E \in \text{Comp}(S)\). Suppose that \( \dim |IS| > 1 \) for some natural \( l \), and let \( m \) denote the least natural \( l \) with this property (i.e. \( m > 0 \), \( \dim |mS| > 1 \) and \( |IS| = \{IS\} \) for \( 1 < l < m \)). Then 1) the system \( |mS| \) has no fixed components, and 2) \( \dim |mS| = 1 \).

**Proof.** 1) Let \( D_0 \) be a fixed component of the system \( |mS| \), \( D_0 > 0 \), and set \( D = mS - D_0 \). Then \( \dim |D| > 1 \) and \( |D| \) has no fixed components; hence
\[
(D^3) \geq 0, \quad (D \cdot D_3) \geq 0,
\]
and since
\[
\text{supp } D_n \subseteq S, \quad (D_n \cdot S) = 0, \quad ((mS)^2) = 0,
\]
we infer that
\[
(D_n^3) = (D^n) = (D \cdot D_n) = 0.
\]
From assertion 3) of Lemma 2 it follows that there exist natural \( \beta \) and \( \beta_0 \) such that \( D = \beta S \) and \( D_0 = \beta_0 S \); hence
\[
\begin{align*}
mS & = D + D_0 = (\beta + \beta_0)S, \quad m = \beta + \beta_0, \\
1 \leq \beta < m, \quad \dim |\beta S| = \dim |D| \geq 1,
\end{align*}
\]
which contradicts the definition of \( m \).

2) Suppose that \( \dim |mS| > 2 \). We pick two points on \( V \), one of which lies on a component \( E \) of the curve \( S \) and the other does not belong to \( S \). In the system \( |mS| \) there exists a divisor \( D \) containing this pair of points. Then \( D = D_0 + D_1 \), where
\[
D_0 \geq 0, \quad D_1 > 0, \quad \text{supp } D_0 \subseteq S, \quad \text{Comp } (D_1) \cap \text{Comp } (S) = \emptyset.
\]
Furthermore, \( D_0 \) is positive since otherwise we would have \((D \cdot D_1) > 0\), which contradicts the equalities \((E \cdot D) = (E \cdot mS) = 0\).

Furthermore, if \( D_1 \cap S \neq \emptyset \), then
\[
\begin{align*}
0 < (S \cdot D_1) = (S \cdot D) = 0,
\end{align*}
\]
and hence \( D_1 \cap S = \emptyset \), from which it follows that \( D_1 \cap D_0 = \emptyset \). From the equalities \((D_1 \cdot D_0) = (D \cdot D_0) = 0\) it follows that \((D_0^3) = 0\); hence, by Lemma 2, \( D_0 = \mu S \) for some natural \( \mu \) and
\[
D_1 \sim (m-\mu)S, \quad 0 < m-\mu < m,
\]
\[
\dim |(m-\mu)S| \geq 1,
\]
which contradicts the definition of \( m \). The lemma is proved.
LEMMA 5. Let $S$ be a divisor on a surface $V$ with $q(V) = 0$ satisfying the conditions of Lemma 4, let $m$ be the integer defined in that lemma, let $\rho : V \to \mathbb{P}_1$ be the map defined by the linear system $|mS|$, and let $C^*$ be the generic fiber of $\rho$ (i.e. $C^* = \rho^{-1}(\xi)$, where $\xi$ is the generic point of $\mathbb{P}_1$, so that $k(\xi) = k(\mathbb{P}_1)$ and $C^*$ is a curve over $k(\mathbb{P}_1)$). Then the following assertions are true:

1) The curve $C^*$ is absolutely irreducible, and in particular all members of the system $|mS|$ are connected.

2) The homomorphism $\text{tr}_{C^*} : \text{Pic}(V) \to \text{Pic}(C^*)$ maps the group $\text{Pic}(V)$ onto the group $\text{Pic}(C^*/k(\mathbb{P}_1))$ of classes of divisors on $C^*$ defined over the field $k(\mathbb{P}_1)$.

3) The group $\text{Ker}(\text{tr}_{C^*})$ is generated by the set consisting of: a) the class of $S$, b) the classes of the supports of the nonprimitive irreducible members of $|mS|$, and c) the classes of the components of the reducible members of $|mS|$.

4) If there are no nonprimitive members in $|mS|$ except possibly $mS$, then the module of relations between the generators of the abelian group $\text{Ker}(\text{tr}_{C^*})$ indicated in 3) is generated by the relations

$$\alpha_i E_i + \cdots + \alpha_r E_r - mS \sim 0,$$

where $\alpha_i E_i + \cdots + \alpha_r E_r$ is a reducible member of $|mS|$ and $E_i, \ldots, E_r$ are its components.

5) If $S_1, \ldots, S_r$ is the set of all reducible fibers of the map $\rho$ whose fibers, with the possible exception of $mS$, are primitive and $\mu_i = \mu(S_i)$ (the number of components of the curve $S_i$), then

$$\text{rk}(\text{Ker}(\text{tr}_{C^*})) = \sum_{i=1}^{r} (\mu_i - 1) + 1.$$

PROOF. 1) If $C^*$ were not absolutely irreducible, then the subfield $k(\mathbb{P}_1)$ (or, more precisely, $\rho^* k(\mathbb{P}_1)$) would not be algebraically closed in $k(V)$. The closure of $k(\mathbb{P}_1)$ in $k(V)$ is the field of functions $k(B)$ on a nonsingular curve $B$, the inclusion $k(\mathbb{P}_1) \subset k(B)$ gives rise to a morphism $\pi : B \to \mathbb{P}_1$, and the inclusion $k(B) \subset k(V)$ gives rise to a rational map $f$ of the surface $V$ onto the curve $B$, which is actually a morphism since $\pi \circ f = \rho$ and the linear system

$$f^{-1}(|\pi^{-1}(x)|) = \rho^{-1}(|x|) = |mS|$$

(here $x \in \mathbb{P}_1$) has no fixed points. From the equality $q(V) = 0$ it follows that $B \simeq \mathbb{P}_1$; hence the fibers of $f$ form a linear system. Let $D = f^{-1}(f(s))$, where $s \in S$, be a member of this system. We have $(D^2) = 0, 0 < D < mS$, and, by assertion 3) of Lemma 2, $D = \mu S, 0 < \mu < m$. From the definition of $m$ and the fact that $|\mu S|$ is one-dimensional it follows that $\mu = m$ and $D = mS$; hence $f$ is defined by the system $|mS|$ and $\pi$ is an isomorphism.

Assertions 2) and 3) are essentially proved in [1], Chapter IV, §7 (Russian p. 60, English pp. 66–67).

We turn to the proof of 4). Suppose that $D \sim 0$, where $D$ is a divisor whose components lie either in $S$ or in reducible members of $|mS|$. Such $D$ can be written in the form $D_0 + \cdots + D_t$, where $\text{supp } D_0 \subset S, D_i \neq 0$ for $1 < i < t$ and $\text{supp } D_i$ belongs to a reducible member $S_i$ of the system $|mS|$. Since $(D^2) = (D \cdot D) = \langle D^2 \rangle = 0$ for
0 < i < t, assertion 3) of Lemma 2 shows that we can write
\[ D = \mu_0 S + \mu_1 S_1 + \cdots + \mu_t S_t, \]
where \( \mu_i \in \mathbb{Z}, \) 0 < i < t. Since
\[ S_i - mS \sim 0, \quad 1 \leq i \leq t, \quad (2) \]
we have
\[ 0 \sim D \sim \mu_0 S + m \left( \sum_{i=1}^{t} \mu_i \right) S = \left( \mu_0 + m \sum_{i=1}^{t} \mu_i \right) S; \]
hence
\[ \mu + m \sum_{i=1}^{t} \mu_i = 0, \quad D = \sum_{i=1}^{t} \mu_i \left( S_i - mS \right) \]
and so \( D \sim 0 \) is a consequence of relations of type (1) emerging under the development of equivalences of type (2).

Assertion 5) follows from 4). The lemma is proved.

**Lemma 6.** Suppose that \( \text{Pic}(V) = \text{Num}(V) \). Let \( C \) be a curve on \( V \) such that \( l \text{tr}_c(C) \neq 0 \) for an arbitrary natural \( l \). Then the restriction of the quadratic form of self-intersections to the subgroup \( \text{Ker}(\text{tr}_c) \) of \( \text{Pic}(V) \) is negative definite.

**Proof.** Let \( D \) be a divisor whose class in \( \text{Pic}(V) \) is a nonzero element of \( \text{Ker}(\text{tr}_c) \). We claim that \( (D^2) < 0 \). From Lemma 1 it follows that \( (D^2) < 0 \). If \( (D^2) = 0 \), then by the same lemma there exist integers \( \alpha \) and \( \beta \) such that \( \alpha \neq 0 \) and \( \alpha D \sim \beta C \). Since the group \( \text{Pic}(V) \) does not have torsion, \( \beta \neq 0 \). It follows that the class of \( \beta C \) belongs to \( \text{Ker}(\text{tr}_c) \), which is impossible by our assumption. The lemma is proved.

**Lemma 7.** Let \( V \) be a rational surface, and let \( C \in | -K_V | \). Then the following assertions are true:

1) \( C > 0 \).

2) \[ H^1(C, \mathcal{O}_C) = 0. \]

3) For a divisor \( D \) with \( 0 < D < C \)
\[ H^1(D, \mathcal{O}_D) = \{ 0 \}. \]

4) The curve \( C \) is connected.

5) If \( (K^2_V) < 0 \), then \( C \) is primitive.

**Proof.** 1) If \( C = 0 \), then
\[ H^0(V, \mathcal{O}_V(nK)) = H^0(V, \mathcal{O}_V) = k, \]
which contradicts the rationality of \( V \). We also note that from the rationality of \( V \) it follows that
\[ H^1(V, \mathcal{O}_V) = \{ 0 \}. \]

2) To the exact sequence of sheaves
\[ 0 \rightarrow \mathcal{O}_V(-C) \rightarrow \mathcal{O}_V \rightarrow \mathcal{O}_C \rightarrow 0. \]
there corresponds the exact sequence of cohomology groups
\[ H^1(\mathcal{O}_V) \to H^1(\mathcal{O}_V^C) \to H^2(\mathcal{O}_V(-C)) \to H^3(\mathcal{O}_V). \] (5)

By Serre duality,
\[
H^2(\mathcal{O}_V(-C)) \cong H^0(\mathcal{O}_V(K+C)) = H^0(\mathcal{O}_V) = k,
\]
\[
H^0(\mathcal{O}_V) \cong H^0(\mathcal{O}_V(K)) = H^0(\mathcal{O}_V(-C)) = \{0\},
\]
and (1) follows from (3) and (5).

3) Since \( K + D \sim -C + D < 0 \), we have
\[
0 = H^0(\mathcal{O}_V(K+D)) \cong H^2(\mathcal{O}_V(-D)),
\]
and using (3) and the analogue of the exact sequence (5) in which \( C \) is replaced by \( D \) we obtain (2).

4) If \( C = D + E, D > 0, E > 0 \) and \( D \cap E = \emptyset \), then
\[
k \cong H^1(\mathcal{O}_C) \cong H^1(\mathcal{O}_D) \oplus H^1(\mathcal{O}_E);
\]
however by 3) the summands in the right are equal to zero.

5) Suppose that \( C = mD, m \geq 2 \) and \( D > 0 \). From assertion 3), the inequality \( 0 \geq (K^2) = m^2(D^2) \), and the formula
\[
(1/2) (D^3 + (D \cdot K)) = \dim H^1(\mathcal{O}_D) - \dim H^0(\mathcal{O}_D)
\]
it follows that
\[
\dim H^0(\mathcal{O}_D) = \frac{m-1}{2} (D^3) \leq 0,
\]
which contradicts the inclusion \( k \subset H^0(\mathcal{O}_D) \). The lemma is proved.

**Lemma 8.** Let \( V \) be a rational surface, let \( C \) be a connected primitive curve on \( V \) such that
\[
(C \cdot E) = (K_V \cdot E) = 0
\]
for each \( E \) from \( \text{Comp}(C) \), and suppose that there exists a natural \( l \) for which \( \text{tr}_{\mathcal{C}}(lC) = 0 \), and let \( m \) be the minimal \( l \) with this property. Then the following assertions are true:

1) \( \dim |lC| = 0 \) for \( 0 < l < m \).
2) \( H^1(V, \mathcal{O}_V(lC)) = \{0\} \) for \( 0 < l < m \).
3) \( \dim |lC| = 1 \).

(Thus the integer \( m \) introduced here for \( C \) would coincide with the integer \( m \) defined in Lemma 4 if in the statement of that lemma we replace \( S \) by \( C \) and suppose that \( V \) is rational.)

4) The system \( |mC| \) has no fixed components.
5) If \( C \in |K_V| \), then in the system \( |mC| \) there is at most one nonprimitive divisor; more precisely, if \( m = 1 \), there are no such divisors, and if \( m > 1 \), then \( mC \) is the only nonprimitive member of \( |mC| \).

**Proof.** 1)–2) We denote by \( \mathcal{T}_{\mathcal{C}}(lC) \) the invertible sheaf \( \mathcal{T}_{\mathcal{C}}(lC) \) on \( C \). Since for \( 1 < l < m \) this sheaf is nontrivial and since all powers of the restrictions of this sheaf to
the components of the curve $C$ are trivial, we infer (see [7], §2, the lemma and Corollary 1) that

$$H^i(C, O_C(I_C)) = \{0\} \quad \text{for} \quad i = 0, \, 1; \, 1 \leq l < m. \quad (6)$$

Exact sequence (4) multiplied by $\theta_\nu(I_C)$ gives rise to the following exact cohomology sequence:

$$0 \to H^0(\mathcal{O}_V((l - 1)C)) \to H^0(\mathcal{O}_V(lC)) \to H^0(\mathcal{O}_C(lC)) \to H^1(\mathcal{O}_V((l - 1)C)) \to H^1(\mathcal{O}_V(lC)) \to H^1(\mathcal{O}_C(lC)). \quad (7)$$

We shall prove 1) and 2) by induction on $l$. For $l = 0, \, 1$ is obvious and 2) is (3).

Suppose that 1) and 2) are proved for $l - 1$. Then 1) follows from equalities 1) and 2) for $l - 1$, equality (6) for $i = 0$, and the exact sequence (7), and 2) follows from (6) for $i = 1, \, 2$) for $l - 1$, and (7).

3) We need to show that

$$\dim H^0(V, \mathcal{O}_V(mC)) = 2;$$

but this follows from 1), equality 2) for $l = m - 1$, exact sequence (7) for $l = m$, and the equalities

$$H^0(C, \mathcal{O}_C(mC)) = H^0(C, \mathcal{O}_C) = k$$

(the last equality follows from the result in [7] cited above).

4) This follows from Lemma 4.

5) For $m = 1$ this follows from assertion 5) of Lemma 7. Let $m > 1$ and $n > 2$, let $S$ be a curve, and let $nS$ be a nonprimitive member of $|mC|$ which does not coincide with $mc$. Comparing the intersection indices of the left and right sides of the equivalence $mc \sim nS$ with the exceptional curve $L$ on $V$, we infer that $m = nl$, where $l = (S \cdot L)$, from which it follows that $0 < l < m$ and $nlC \sim nS$, so that $lC \sim S$, $S \in |lC|$ and $\dim |lC| > 1$, which contradicts 1). The lemma is proved.

**Lemma 9.** Let $V$ be a rational surface with $(K^2) = 0$. Then the following assertions are true:

1) $|K_V| \neq \emptyset$.

2) If $C \in |K_V|$, $(C \cdot E) = 0$ for all $E \in \text{Comp}(C)$ and $\Sigma : V \to W$ is a birational morphism of $V$ onto a surface without exceptional curves, then $W = \mathbb{P}_2$, or $W = F_0$, or $W = F_2$.

3) If $C$ satisfies the conditions indicated in 2), then there exists a birational morphism $\Sigma : V \to \mathbb{P}_2$.

**Proof** (cf. pp. 1074–1078 in [5]). 1) The Riemann-Roch inequality for $V$ yields

$$\dim H^0(\mathcal{O}_V(-K)) + \dim H^0(\mathcal{O}_V(2K)) \geq (K^2) + 1 = 1;$$

moreover, the second summand on the left side of this inequality vanishes.

2) This assertion would follow from the fact that on $V$ there are no nonsingular rational curves $H$ with $(H^2) < -3$. But for such a curve we would have $(H \cdot K) > 1$; hence $H \notin \text{Comp}(C)$, so that $(H \cdot C) > 0$, which is impossible since $C \sim -K$.

3) Let $\Sigma : V \to W$ be a birational morphism of $V$ onto a minimal model $W$. Suppose that $W \neq \mathbb{P}_2$. Then $\text{rk}(\text{Pic}(W)) = 2$ and $\text{rk}(\text{Pic}(V)) = 10 - (K^2) = 10$, so that $\Sigma$ can be
represented as a composition \( \Sigma = \sigma_8 \circ \sigma_7 \circ \cdots \circ \sigma_1 \), where \( \sigma_i \) is the blowing down of an exceptional curve \( L_i \), \( 1 < i < 8 \). Set \( P = \sigma_8(L_8) \) and \( T = \sigma_\gamma \circ \sigma_6 \circ \cdots \circ \sigma_1 \).

If \( W = F_0 \) and \( E_1 \) and \( E_2 \) are the generatrices of \( F_0 \) passing through \( P \), then the morphism \( \tau \circ T \), where \( \tau \) is the blowing down of the curve \( \sigma_\gamma(E_1 \cup E_2) \), maps \( V \) onto \( P_2 \).

If \( W = F_2 \), then, for the reason indicated in the proof of 2), \( P \) does not lie on a curve \( B \) for which \( (B^2) = -2 \). We denote by \( E \) the generatrix of \( F_2 \) passing through \( P \). The image of the morphism \( \tau \circ T \), where \( \tau \) is the blowing down of the curve \( \sigma_\gamma(E \cup B) \), coincides with \( P_2 \). The lemma is proved.

2. Rational elliptic surfaces with two degenerate fibers

In this section we introduce three surfaces \( W_2, W_3, W_4 \) and a family of surfaces \( W_i(\gamma) \) depending on parameter \( \gamma, \gamma \in k, \gamma \neq 0, \gamma \neq 1, W_i(\beta) = W_i(\gamma) \) if and only if \( I(\beta) = I(\gamma) \), where

\[
I(x) = (x^2 - x + 1)^2(x - 1)^{-2}.
\]

In what follows we shall write \( W_i \) in place of \( W_i(\gamma) \).

We fix coordinates \((x_0 : x_1 : x_2)\) on \( P_2 \); the curve defined by an equation \( f(x_0, x_1, x_2) = 0 \) will be denoted by \([f(x_0, x_1, x_2)]\). We consider the following four pencils of cubic curves on \( P_2 \):

\[
\Delta_1 = \{(\lambda x_2 (x_2 - x_0)(x_2 - \gamma x_0) + \mu x_0 x_2^2)\},
\]

\[
\Delta_2 = \{(\lambda \Phi + \mu x_2^2)\},
\]

\[
\Delta_3 = \{(\lambda \Phi + \mu x_0 x_2)\},
\]

\[
\Delta_4 = \{(\lambda \Phi + \mu x_0 x_1)\},
\]

where \( \Phi = x_0 x_1^2 + x_2^3 \), and \((\lambda : \mu)\) are homogeneous coordinates. Let \( 1 < i < 4 \), and let \( \Sigma_i : W_i \rightarrow P_2 \) be the birational morphism resolving the fundamental points of the pencil \( \Delta_i \). The surface \( W_i \) has the structure of a minimal elliptic surface; we denote by \( \rho_i : W_i \rightarrow P_1 \) the corresponding morphism; the pencil \( \Delta_i \) consists of the images of fibers of \( \rho_i \) under the morphism \( \Sigma_i \). The morphism \( \rho_i \) is defined by the one-dimensional linear system \( |{-K_{W_i}}| \) which does not have fixed components; we shall say that a member of this system has the same coordinates \((\lambda : \mu)\) as the corresponding member of the pencil \( \Delta_i \). We denote by \( C_i \) (\( C'_i \)) the member of \( |{-K_{W_i}}| \) with coordinates \( \lambda = 1, \mu = 0 \) \((\lambda = 0, \mu = 1)\). Using Kodaira’s symbols (cf. [6]) to denote the type of a degenerate fiber of an elliptic bundle, one can show that the types of the curves \( C_i, C'_i \) are given by the following table:

<table>
<thead>
<tr>
<th>( C_i )</th>
<th>( C'_i )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>( C'_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma_0 )</td>
<td>( \Gamma_0' )</td>
<td>( \Pi )</td>
<td>( \Pi^* )</td>
<td>( \Pi )</td>
<td>( \Pi^* )</td>
</tr>
</tbody>
</table>

Table 1

\( C_i \) and \( C'_i \) are the only singular curves in the system \( |{-K_{W_i}}| \) (we recall that the characteristic of the base field is greater than three).
Let \( p(D) \) denote the number of those components of a divisor \( D \) that enter it with multiplicity one. Then \( p(C) = p(C') = \nu_i \), and the numbers \( \nu_i \) are given by the following table:

<table>
<thead>
<tr>
<th>( \nu_i )</th>
<th>( \nu_1 )</th>
<th>( \nu_2 )</th>
<th>( \nu_3 )</th>
<th>( \nu_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

**Table 2**

The number of exceptional curves on \( W_i \) is equal to \( \nu_i \), and one can index these curves \( L_1, \ldots, L_\nu (\nu = \nu_i) \) and the components \( E_1, \ldots, E_\nu \) (respectively, \( E_1', \ldots, E_\nu' \) \( (\nu = \nu_i) \)) entering \( C_i (C_i') \) with multiplicity 1 in such a way that

\[
(E_\alpha \cdot L_\beta) = (E_\alpha' \cdot L_\beta') = -(L_\alpha \cdot L_\beta) = \delta_{\alpha\beta},
\]

where \( \delta_{\alpha\beta} \) is the Kronecker symbol and \( 1 < \alpha, \beta < \nu, \nu = \nu_i \).

We consider the following two actions of the group \( \mathbb{G}_m \) on \( \mathbb{P}_2 \):

\[
a((x_0 : x_1 : x_2)) = (x_0 : ax_1 : x_2),
\]

(1)

\[
a((x_0 : x_1 : x_2)) = (x_0 : a^2x_1 : a^3x_2),
\]

(2)

where \( a \in k^* \). The action (1) preserves the pencil \( \Delta_i \); hence, using \( \Sigma_i \), we can lift this action on \( W_i \), where it defines a one-dimensional subgroup \( H_i \) in \( \text{Aut}(W_i) \), \( i = 2, 3, 4 \).
For $1 < i < 4$, $H_i$ is a normal subgroup of finite index in $\text{Aut}(W_i)$; the quotient of the group of automorphisms by this subgroup is described in Table 3.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Aut}(W_i)/H_i$</td>
<td>${1}$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>$\mathbb{S}_3$</td>
<td></td>
</tr>
</tbody>
</table>

$\nabla$ $\nu$ $0$ $9/4$ $\neq 0$, $\neq 9/4$

Aut $(W_1(\nu))/H_1$ $A_4 \times (\mathbb{Z}/2\mathbb{Z})$ $G_8 \times (\mathbb{Z}/2\mathbb{Z})$ $(\mathbb{Z}/2\mathbb{Z})^3$

(Here $\mathbb{S}_4$ is the symmetric group of order $8$, $A_4$ is the alternating subgroup in $\mathbb{S}_4$, and $G_8$ is the subgroup of order 8 in $\mathbb{S}_4$ generated by the cyclic permutation $(1, 2, 3, 4)$ and the transposition $(1, 3)$.)

**Proposition A.** Let $V$ be a rational surface with $(K_V^2) = 0$, and suppose that there exists a curve $C$ in $|-K_V|$ such that $(C \cdot E) = 0$ for $E \in \text{Comp}(C)$. Denote by $\text{Aut}_0(V)$ the subgroup in $\text{Aut}(V)$ consisting of all elements acting identically on $\text{Pic}(V)$. Then the dimension of the algebraic group $\text{Aut}_0(V)$ is positive if and only if $V$ is isomorphic to one of the surfaces $W_i$, $i = 1, 2, 3, 4$.

**Proposition B.** Let $V$ be a rational surface with $(K_V^2) = 0$ such that the linear system $|\mathcal{F}^m\mathcal{K}_V|$ is one-dimensional and does not have fixed components, and let $\rho : V \to \mathbb{P}_1$ be the morphism defined by this system. Then $\rho$ has at least two degenerate fibers, and if it has exactly two such fibers, then $m = 1$ and $V$ is isomorphic to one of the surfaces $W_i$.

Another description of the surfaces $W_i$ has been given by Dolgachev. Let $X_i = \mathbb{P}_1 \times E_i$, where $1 < i < 4$, $E_i$ for $i \neq 4$ is the member of the pencil $\Delta_i$ with coordinates $\lambda = 1$, $\mu = 1$, and $E_4 = E_2^3$. Consider the cyclic groups $G_i = \mathbb{Z}/(i+1)\mathbb{Z}$, $i = 1, 2, 3$, and $G_4 = \mathbb{Z}/6\mathbb{Z}$. Let $g_i$ be an arbitrary generator of $G_i$, and let $\xi_i$ be a primitive root of unity of degree equal to the order of $G_i$. We define an action of $G_i$ on $X_i$ by the following formula (where $1 < i < 4$):

$$g_i ((t_0 : t_1) \times (x_0 : x_1 : x_2)) = (t_0 : \xi_i t_1) \times (x_0 : \xi_i^3 x_1 : \xi_i^6 x_2).$$

Then $W_i$ is obtained as a result of an economical resolution of singularities of the quotient surface $X_i/G_i$. The action of $G_m$ on $W_i$ arises from the action of $G_m$ on $\mathbb{P}_1$ defined by the formula

$$a((t_0 : t_1)) = (t_0 : a t_1), \quad a \in k^*.$$

§3. G-projective and G-antiprojective rational surfaces

A variety $X$ with an action of a group (i.e. a homomorphism $G \to \text{Aut}(X)$) is called a $G$-variety. A complete $G$-variety is called $G$-projective if there exist an action of $G$ on a projective space $\mathbb{P}_n$ and a $G$-embedding $X \to \mathbb{P}_n$. The $G$-projectivity of $X$ is equivalent to
each of the following two properties:
(A) There exists a positive ample divisor \( D \) on \( X \) such that \( gD \sim D \) for all \( g \in G \); in other words, the linear system \( |D| \) is \( G \)-invariant.
(B) The action of \( G \) on \( X \) factors through an algebraic action of a linear algebraic group \( H \) on \( X \).

For a rational \( G \)-surface \( V \) condition (A) can be somewhat weakened; more precisely, we have

**Theorem 1.** A rational \( G \)-surface \( V \) is \( G \)-projective if and only if there exists a divisor \( D \) on \( V \) such that \( (D^2) > 0 \) and \( gD \sim D \) for all \( g \in G \).

**Proof.** Suppose that \( D \) is such a divisor, and let \( H \) be a very ample divisor on \( V \). The group \( G \) acts on the \( \mathbb{Z} \)-module \( \text{Pic}(V) \) and preserves the quadratic form induced by the self-intersection index, so we obtain a homomorphism

\[
\alpha : G \to \text{Aut}(\text{Pic}(V)).
\]

The submodule

\[
D^\perp = \{ x \in \text{Pic}(V) \mid (x \cdot D) = 0 \}
\]

of \( \text{Pic}(V) \) is stable under the action of \( G \), and the self-intersection form is negative definite on \( D^\perp \); hence the group \( \alpha(G) \) whose elements preserve \( D^\perp \) and the equivalence class of \( D \) is finite. Let

\[
\alpha(G) = \{ f_1, \ldots, f_n \}, \quad g_i \in \alpha^{-1}(f_i), \quad 1 \leq i \leq n.
\]

The divisor

\[
H_0 = \sum_{i=1}^n g_i H
\]

is very ample and is contained in the \( G \)-invariant system \( |H_0| \). The theorem is proved.

In what follows we shall consider only rational surfaces; we shall sometimes omit \( G \) in such terms as \( G \)-projective and \( G \)-antiprojective.

**Proposition 1.** Let \( V \) and \( W \) be two \( G \)-surfaces, and let \( \Sigma : V \to W \) be a birational \( G \)-morphism; then either both these surfaces are projective or both are antiprojective.

**Proof.** 1) Suppose that \( W \) is projective, and let \( D \) be a divisor on \( W \) belonging to the \( G \)-invariant system \( |D| \) such that \( (D^2) > 0 \). Then \( \Sigma^*(D) \) has the same properties, and therefore \( V \) is projective.

2) Suppose that \( V \) is projective. Assuming that \( \Sigma \) is not an isomorphism, we denote by \( C \) the reduced curve consisting of all irreducible curves mapped to points under the morphism \( \Sigma \). It is clear that \( C \) is \( G \)-invariant and the intersection matrix of its components is unimodular and negative definite (more precisely, the quadratic form defined by this matrix is integrally equivalent to the form \( -x_1^2 - x_2^2 - \cdots - x_r^2 \)). We shall use some constructions pertaining to the theory of blowing down exceptional curves (see the proof of Theorem 2.3 in [8]). Let \( D \) be a very ample positive divisor on \( V \) for which the system \( |D| \) is \( G \)-invariant. Replacing, if necessary, \( D \) by some multiple of it, we may assume that

\[
H^1(V, \mathcal{O}_V(D)) = H^2(V, \mathcal{O}_V(D)) = \{0\}.
\]
Since the intersection matrix of the components of $C$ is unimodular, there exists a divisor $Z$ such that

$$\text{supp } Z \subseteq C, \quad ((D + Z) \cdot E) = 0$$

for $E \in \text{Comp}(C)$. Since for each $g$ from $G$ and each $E$ from $\text{Comp}(C)$

$$((D + gZ) \cdot E) = ((D + Z) \cdot g^{-1}(E)) = 0$$

it follows that $Z$ is a $G$-invariant divisor. Furthermore, in [8] it is shown that $Z > 0$ and there exists a divisor $D_0$ in $|D + Z|$ such that $D_0 \cap C = \emptyset$. Now it is clear that $\Sigma(D_0)$ is ample and $|\Sigma(D_0)|$ is a $G$-invariant system; hence $W$ is projective. The proposition is proved.

We remark that if $H$ is a subgroup of $G$, then a $G$-surface $V$ has a natural structure of $H$-surface. If $H$ is the cyclic subgroup of $G$ generated by an element $g$ from $G$, then instead of saying that $V$ is $H$-(anti)projective we shall say that $V$ is $g$-(anti)projective.

**Proposition 2.** A $G$-surface $V$ is antiprojective if and only if $V$ is $g$-antiprojective for some element $g$ from $G$.

**Proof.** The sufficiency is clear: if $V$ is $g$-antiprojective for some $g$ from $G$, then it is $G$-antiprojective. Suppose that $V$ is $G$-antiprojective but $g$-projective for each $g$ from $G$, so that for each $g$ there exists a positive ample divisor $D_g$ such that $g(D_g) = D_g$. Let $L$ be an ample divisor on $V$, and let $l$ be its class in $\text{Pic}(V)$. We have a natural homomorphism

$$\alpha : G \to \text{Aut}(\text{Pic}(V)).$$

If we know that the image of $\alpha$ is finite, then, taking a positive divisor from the class $\Sigma_{g \in \alpha(G)} s(l)$, we would obtain an ample divisor $D$ contained in the $G$-invariant system $|D|$.

We claim that to show that the group $\alpha(G)$ is finite it suffices to show that it is periodic. In fact, if for each $g$ from $G$ there exists a natural number $m$ for which $\alpha(g)^m = 1$, then the boundedness of orders of all elements $\alpha(g)$ follows from the Jordan theorem on the finiteness of the set of conjugacy classes of finite subgroups in $GL(n, \mathbb{Z})$ (or from the following elementary argument: the characteristic polynomial for $\alpha(g)$ has integral coefficients and factors into a product of cyclotomic polynomials whose degrees are bounded by the number $\text{rk}(\text{Pic}(V))$; however the degree $\varphi(q)$ of the cyclotomic polynomial vanishing at a primitive root of unity of degree $q$ is equal to the value of the Euler function at $q$ and tends to infinity as $q \to \infty$). Now that we know that the orders of $\alpha(g)$ are bounded, the finiteness of $\alpha(G)$ follows from Burnside's theorem to the effect that every periodic subgroup with bounded exponent in $GL(n, \mathbb{C})$ is finite (see [9], §36).

It remains to show that $\alpha(G)$ is periodic. Each element $\alpha(g)$ preserves the class of the divisor $D_g$ in $\text{Pic}(V)$ and the orthogonal complement $D_g^\perp$ of this class in $\text{Pic}(V)$. By the index theorem, the quadratic form defined by the self-intersection index is negative definite on $D_g^\perp$; hence the transformation of $D_g^\perp$ induced by $\alpha(g)$ has finite order, from which it follows that $\alpha(g)$ also has finite order. The proposition is proved.

**Lemma 10.** A rational $G$-surface $V$ is antiprojective if and only if $V$ is $H$-antiprojective for each subgroup $H$ of finite index in $G$. 
PROOF. If

$$G = \bigcup_{i=1}^r g_i H,$$

and \( L \) is a positive ample divisor for which \(|L|\) is \( H \)-invariant, then the divisor \( M = \sum_i g_i L \) belonging to the \( G \)-invariant system \(|M|\) is ample. The lemma is proved.

**Lemma 11.** Let \( E \) and \( D \) be two divisors on a \( G \)-surface \( V \) such that \( 0 < E < D \) and the system \(|D|\) is \( G \)-invariant. Denote by \( G_E \) the subgroup of those elements \( h \) from \( G \) for which \( hE \sim E \). Then \(|G : G_E| < \infty\).

**Proof.** Let \( L \) be a very ample divisor on \( V \). The set of numbers \((gE \cdot L)\), where \( g \) belongs to \( G \), is bounded since \((gE \cdot L) < (gD \cdot L) = (D \cdot L)\); hence the set of numerical equivalence classes of curves \( gE \) is finite (cf. [2], Lecture 16), and since \( V \) is rational, the number of linear equivalence classes of curves \( gE \) is also finite. The group \( G \) permutes these classes, and \( G_E \) is the stabilizer of the class of the curve \( E \). The lemma is proved.

**Proposition 3.** Let \( V \) be an antiprojective rational \( G \)-surface, and let \( D \) be a divisor on \( V \) such that \((D^2) = 0\) and \( gD \sim D \) for all \( g \in G \). Then the following assertions are true:

1) \((D \cdot K_F) = 0\), and in particular the arithmetic genus \( p_a(D) \) of the divisor \( D \) is equal to one.
2) If \( D \) is quasi-ample (i.e. \((D \cdot C) > 0 \ \forall C\)), then \(|D| \neq \emptyset\).
3) If \( D > 0 \), then \((E \cdot D) = 0\) for all \( E \in \text{Comp}(D)\).
   Furthermore, suppose that \( V \) is \( G \)-minimal and \( D > 0 \). Then, in addition,
4) \( D \) does not have exceptional components.
5) \((E \cdot K_F) = 0\) for all \( E \in \text{Comp}(D)\).
6) If \( D \) is connected and reducible and \( E \in \text{Comp}(D) \), then \( E = P_1 \) and \((E^2) = -2\).

**Proof.** 1) Let \( D_n = n(D \cdot K)\) \( + K \). Then \( gD_n \sim D_n \) for all \( g \in G \). If \((D \cdot K) \neq 0\), then \((D^2) > 0\) for large \( n \), and from Theorem 1 it follows that \( V \) is projective.

2) We prove this by induction on the rank \( r(V) \) of the group \( \text{Pic}(V) \). If \( r(V) = 1 \) or \( 2 \), assertion 2) is true for the trivial reason that \( V \) cannot be \( G \)-antiprojective. By 1) and the Riemann-Roch theorem we have

$$\dim H^0(\mathcal{O}_V(D)) + \dim H^0(\mathcal{O}_V(K-D)) > (1/2)(D \cdot (D-K)) + 1 = 1,$$

from which it follows that either \(|D| \neq \emptyset\) and 2) holds, or \(|K-D| \neq \emptyset\). Suppose that we have the last possibility and let \( L \) be an exceptional curve on \( V \) and \( D_1 \) a divisor from \(|K-D|\) so that \( D_1 > 0 \) and \( D_1 \sim K-D \). From the quasi-amenenseness of \( D \) it follows that \((L \cdot D) > 0\) and

\[
(L \cdot D_1) = (L \cdot K) - (L \cdot D) = -1 - (L \cdot D) \leq -1.
\]

Since \( D_1 > 0 \) and \((L \cdot D_1) < -1\), we infer that \( L \in \text{Comp}(D_1) \). From Lemma 11 applied to \( L \) and \( D_1 \) it follows that there exists a subgroup \( G_L \) of finite index in \( G \) such that \( hL \sim L \) for all \( h \in G_L \). Two exceptional curves are equivalent if and only if they coincide with each other; hence \( G_L \) is the stabilizer of \( L \) in \( G \). Let \( \sigma : V \to \overline{V} \) be the blowing down of the curve \( L \); \( \overline{V} \) has a natural structure of a \( G_L \)-surface, and we set \( \overline{D} \equiv \sigma(D) \). By Proposition 1, \( \overline{V} \) is \( G_L \)-antiprojective; for \( h \in G_L \) we have \( h\overline{L} \sim \overline{L} \), and
hence \((\overline{D}^2) < 0\). On the other hand,
\[
(\overline{D}^2) = (D^2) + (D \cdot L)^2 = (D \cdot L)^2 \geq 0,
\]
and hence
\[
(\overline{D}^2) = 0, \quad (D \cdot L) = 0, \quad \sigma^*(\overline{D}) = D.
\]
From the induction hypothesis and the equality \(r(\overline{V}) = r(V) - 1\) it follows that \(|\overline{D}| \neq \emptyset\). Then for an arbitrary \(D_0\) from \(|\overline{D}|\) we have
\[
\sigma^*(D_0) \geq 0, \quad \sigma^*(D_0) \sim \sigma^*(\overline{D}) = D;
\]
hence \(|D| \neq \emptyset\).

3) Let \(E \in \text{Comp}(D)\), and let \(G_E\) be the subgroup of \(G\) defined in Lemma 11. The divisor
\[
E = n(D \cdot E)D + E
\]
belongs to a \(G_E\)-invariant equivalence class, and if \((D \cdot E) \neq 0\), then \((E^2) > 0\) for large \(n\), from which it would follow that \(V\) is \(G_E\)-projective, which is impossible by Lemmas 10 and 11.

4) Let \(L\) be an exceptional component in \(D\), and let \(G_L\) be the subgroup of finite index in \(G\) defined by this component. In the proof of 2) we saw that \(G_L\) coincides with the stabilizer of \(L\) in \(G\). The equality \(G_L = G\) is impossible, since otherwise the action of \(G\) on \(V\) would originate from an action of \(G\) on \(W\), where \(W\) is obtained as a result of the blowing down \(\sigma : V \rightarrow W\) of \(L\), which contradicts the \(G\)-minimality of \(V\). Consider the decomposition
\[
G = \bigcup_{i=1}^{r} g_iG_L, \quad r = [G : G_L], \quad g_i = e,
\]
and set \(L_i = g_iL, 1 < i < r\); we observe that \(L_1 = L\) and \(L_i \neq L_j\) for \(i \neq j\).

It is clear that \(G\) permutes the components of the curve \(\bigcup_i L_i\). These components cannot be mutually nonintersecting, since otherwise the blowing down \(\Sigma : V \rightarrow U\) of \(\bigcup_i L_i\) would be a \(G\)-morphism (for a natural action of \(G\) on \(U\)), which is impossible since \(V\) is \(G\)-minimal. Thus we may assume that \((L_1 \cdot L_2) > 0\). Then \(L_2 \cap D \neq \emptyset\) and
\[
(L_2 \cdot D) = (g_2L_1 \cdot g_2D) = (L_1 \cdot D) = 0,
\]
so that \(L_2 \subset D\), the curve \(C = L_1 \cup L_2\) is contained in \(D\) and its self-intersection index is nonnegative:
\[
(C^2) = (L_1^2) + (L_2^2) + 2(L_1 \cdot L_2) = 2((L_1 \cdot L_2) - 1) \geq 0.
\]
Since \(V\) is \(G\)-antiprojective we have \((C^2) = 0\) and, by 1), \((C \cdot K) = 0\); but on the other hand
\[
(C \cdot K) \equiv (L_1 + L_2 \cdot K) = -2.
\]

5)–6) If \(E \in \text{Comp}(D)\), then \(V\) is \(G_E\)-antiprojective and from Theorem 1 it follows that \((E^2) < 0\); moreover, if \((E^2) = 0\), then \(E\) is a connected component of \(\text{supp} D\) (otherwise \(nE + D\) would have a positive self-intersection index for large \(n\)), and for such \(E\) we have \((E \cdot K) = 0\) by 1). In the proof of 5) and 6) we shall assume that \(D\) does
not have components with self-intersection index zero. A curve $E$ cannot simultaneously satisfy the inequalities $(E^2) < 0$ and $(E \cdot K) < 0$, since otherwise from the nonnegativity of $p_a(E)$ it would follow that $E$ is exceptional; hence $(E \cdot K) > 0$ for all $E \in \text{Comp}(D)$.

If for some $E_0 \in \text{Comp}(D)$ we had $(E_0 \cdot K) > 0$, then from the equality $(D \cdot K) = 0$ it would follow that there exists a curve $E_1 \in \text{Comp}(D)$ with $(E_1 \cdot K) < 0$, and the above argument shows that this is impossible. Assertion 5) is proved.

If $D$ is connected and reducible and $E \in \text{Comp}(D)$, then from the relations

\[(E^2) < 0, \quad (E \cdot K) = 0, \quad p_a(E) \geq 0,\]

it follows that

\[(E^2) = -2, \quad p_a(E) = 0, \quad E \cong \mathbb{P}^1.\]

The proposition is proved.

In what follows we shall use the objects $\mathcal{P}(V)$, $A(V)$, $\mathcal{A}(V)$, $I(V)$, and $I^0(V)$ described in the list of notation.

**Proposition 4.** Let $V$ be a G-surface, and let $g \in G$.

1) There exists a $g$-invariant ray in $\mathcal{A}(V)$, i.e. there exists a vector $e$ from $\mathcal{A}(V)$ such that

\[g(e) = \lambda e, \quad \lambda > 0.\]  

Suppose moreover that $V$ is $g$-antiprojective.

2) Each $g$-invariant ray from $I(V)$ lies in $I^0(V)$, i.e. it is isotropic.

If $\lambda$ in (1) is rational, then:

3) $\lambda = 1$,

4) $\mathbb{R}^+e$ is the only $g$-invariant ray in $I(V)$, and

5) the ray $\mathbb{R}^+e$ contains a vector from Pic($V$) which can be represented by a positive connected primitive quasi-ample divisor.

**Proof.** 1) The set of rays in $\mathcal{A}(V)$ (i.e. $\mathcal{A}(V)/\mathbb{R}^+$) in its natural topology is homeomorphic to the $(r - 1)$-dimensional disk, where $r = \text{rk}(\text{Pic}(V))$. Hence 1) is a consequence of the fixed point theorem.

2) From $e \in I(V)$ it follows that $(e^2) > 0$. If $(e^2) > 0$, then $(e^2) = (g(e)^2) = \lambda^2(e^2)$, so that $\lambda = 1$ and $e$ is $g$-invariant. Since the subset Pic($V$) in $\mathcal{P}(V)$ is stable with respect to $g$, the linear subspace $\mathcal{P}(V)^g$ consisting of all $g$-invariant vectors has a basis contained in Pic($V$). The inequality $(e^2) > 0$ implies that the intersection $\mathcal{P}(V)^g \cap \text{Int} I(V)$ is nonempty; hence this intersection contains a vector $d$ from Pic($V$). For a divisor $D$ representing $d$ we have $(D^2) > 0$ and $gD \sim D$, which by Theorem 1 contradicts the $g$-antiprojectivity of $V$.

3) Clearly $\lambda$ is a root of the characteristic polynomial of the linear transformation $g$ of the space $\mathcal{P}(V)$. Since both $g$ and $g^{-1}$ are integral transformations, $\lambda$ and $\lambda^{-1}$ are integral algebraic numbers, i.e. $\lambda$ is a unit of the ring of integral algebraic numbers. From this it follows that if $\lambda \in \mathbb{Q}$ and $\lambda > 0$, then $\lambda = 1$.

4) Let $\mathbb{R}^+e_0$ be a $g$-invariant ray in $I(V)$ not coinciding with $\mathbb{R}^+e$, so that

\[e_0 \in I(V), \quad g(e_0) = \lambda e_0, \quad \lambda > 0.\]

Then, by 2), $(e_0^2) = 0$. Since the intersection index of arbitrary noncollinear vectors from
\( I(V) \) is positive, we have
\[
0 < (e \cdot e_\pi) = (ge \cdot ge_\pi) = \lambda_\pi (e \cdot e_\pi), \quad \lambda_\pi = 1,
\]
\[
e_\pi \in \mathcal{P}(V)^\pi, \quad e_\pi + e \in \mathcal{P}(V)^\pi, \quad ((e_\pi + e)^2) > 0,
\]
which contradicts 2).

5) From 4) it follows that
\[
\mathcal{P}(V)^\pi \cap I(V) = R^+ e_\pi, \quad \mathcal{P}(V)^\pi \cap \{x \in \mathcal{P}(V), \quad (x^2) = 0\} = \text{Re}.
\]
The subspace \( \mathcal{P}(V)^\pi \) has a basis lying in \( \text{Pic}(V) \), and if the intersection of the corresponding rational subspace with the light cone is one-dimensional, this intersection is also rational; hence in \( R^+ e \) there is a nonzero vector \( d \) lying in \( \text{Pic}(V) \). Using assertion 2) of Proposition 3, we can choose a positive divisor \( D \) belonging to the class \( d \). Let \( mD_0 \)
be a connected component of \( D \), where \( m > 1 \) and \( D_0 \) is a primitive divisor. Since by assertion 3) of Proposition 3 \( (E \cdot D) = 0 \) for all \( E \) from \( \text{Comp}(D) \), we have
\[
(D_0 \cdot D) = (D_0^2) = (D_0^2) = 0.
\]
Applying Lemma 1, we choose a nonzero pair of integers \( \alpha_0, \alpha \) such that \( \alpha_0 D_0 \sim \alpha D \).

From the positivity of \( D \) and \( D_0 \) it follows that \( \alpha_0 > 0 \), so that the class of \( D_0 \) belongs to the ray \( R^+ e \) and \( D_0 \) is the required divisor. The proposition is proved.

**PROPOSITION 5.** Let \( V \) be a rational G-surface, and suppose that for each element \( \xi \) from \( G \) there exists a \( g \)-invariant vector in the cone \( I(V) \). Then there exists a \( G \)-invariant vector in \( \mathcal{P}(V) \).

**PROOF.** In the case when \( V \) is projective the existence of such a vector follows from the definition of projectivity, so in what follows we shall assume that \( V \) is antiprojective. By Proposition 2, there exists an element \( g \) in \( G \) such that \( V \) is \( g \)-antiprojective. By our assumption, in \( I(V) \) there exists a nonzero \( g \)-invariant vector \( \xi \). Using Proposition 4, we may assume that \( \xi \) is a primitive element of Pic \( (V) \).

Our goal is to show that \( \xi \) is \( g_1 \)-invariant for an arbitrary element \( g_1 \) from \( G \). Setting \( \eta = g_1 \xi \), we shall obtain a contradiction from the hypothesis \( \eta \neq \xi \).

The vectors \( \xi \) and \( \eta \) are noncollinear, since otherwise \( g_1 \xi = \lambda \xi \) and from the rationality of \( \xi \) it would follow that \( \lambda \in \mathbb{Q} \), so that by Proposition 4, \( \lambda = 1 \) and \( \eta = \xi \). Since \( \xi \) and \( \eta \) are noncollinear, \( (\xi \cdot \eta) > 0 \).

**LEMMA 12.** Let \( E \) be a quadratic Z-module, let \( \mathcal{E} = E \otimes \mathbb{R} \), and let \( (x \cdot y) \) be the inner product in \( E \) and \( \mathcal{E} \); suppose that the quadratic form \( (x^2) \) is nondegenerate and is equivalent to the form \( x_1^2 - x_2^2 - \cdots - x_n^2 \) over \( \mathbb{R} \). Let \( I \) be the set of nonzero vectors of the closure in \( \mathcal{E} \) of one of the two connected components of the set of those \( x \in \mathcal{E} \) for which \( (x^2) > 0 \), and let
\[
I^0 = I \cap \{x \in \mathcal{E}, \quad (x^2) = 0\}.
\]

Let \( G \) be the group of those automorphisms of the quadratic Z-module \( E \) which preserve \( I \cap E \), let \( \mathcal{G} \) be the group of linear automorphisms of \( \mathcal{E} \) preserving the scalar product and the cone \( I \), and let \( \xi \) and \( \eta \) be two fixed noncollinear vectors from \( E \cap I^0 \).

Let \( \xi^\perp \) be the orthogonal complement of \( \xi \) in \( \mathcal{E} \).
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Let \( \pi : \mathcal{S} \rightarrow \xi^+ \cap \eta^+ \) be the orthogonal projection, so that for \( x \in \mathcal{S} \)

\[
\pi(x) = x - \frac{(x \cdot \eta)}{(\xi \cdot \eta)} \xi - \frac{(x \cdot \xi)}{(\xi \cdot \eta)} \eta.
\]

Let \( T_\xi \) be the group of transformations from \( \mathcal{S} \) preserving \( \xi \) and inducing the identity automorphism of the quotient space \( \xi^+ / R \xi \); in a similar way we define \( T_\eta \).

Then the following assertions are true:

1) If \( \xi \) is an invariant vector for a transformation \( g \) from \( G \), then the automorphism of the quotient group \( (\xi^+ \cap E) / Z \xi \) induced by \( g \) has finite order, i.e., \( g^m \) lies in \( T_\xi \) for some natural \( m \).

2) The map \( \psi : T_\xi \rightarrow \xi^+ \cap \eta^+ \) associating to each \( t \) from \( T_\xi \) the vector \( \psi(t) = \pi(t(\eta)) \) is an isomorphism between the group \( T_\xi \) and the additive group of vectors of the space \( \xi^+ \cap \eta^+ \). Similarly, the following map is an isomorphism of groups:

\[
\psi : T_\eta \rightarrow \xi^+ \cap \eta^+, \quad \psi(s) = \pi(s(\xi)).
\]

3) If \( t \) and \( s \) are nonidentical transformations, \( t \in T_\xi \), \( s \in T_\eta \) and there exists a vector \( y \) from \( I \) such that \( t(y) = s(y) \), then the vectors \( \psi(t) \) and \( \psi(s) \) are collinear and similarly directed, i.e., \( \psi(t) \in R \psi(s) \).

4) If \( t \) and \( s \) satisfy the conditions indicated in 3), then \( t \) and \( s^{-1} \) do not satisfy these conditions.

Deduction of Proposition 5 from Lemma 12. Set \( t = g^m \), where \( m \) is as indicated in 1), and let \( s = g_1 g_2^{-1} \). Clearly \( t \in T_\xi \) and \( s \in T_\eta \). By our assumption, in \( I(V) \) there are a \( t^{-1} \)-invariant vector \( y_1 \) (i.e., \( t(y_1) = s(y_1) \)) and an \( s \)-invariant vector \( y_2 \) (i.e., \( t(y_2) = s^{-1}(y_2) \); but this contradicts 4).

Proof of Lemma 12. 1) The map \( (x + Z \xi) \mapsto (x^2) \) defines a negative definite quadratic form on \( (\xi^+ \cap E) / Z \xi \) which is invariant with respect to the automorphism of this quotient group induced by \( g \).

2) An arbitrary vector \( x \) from \( \mathcal{S} \) can be written in the form

\[
x = a \xi + b \eta + \zeta,
\]

where

\[
\zeta = \pi(x), \quad a = (x \cdot \eta) / (\xi \cdot \eta), \quad b = (x \cdot \xi) / (\xi \cdot \eta),
\]

and comparing the scalar squares of the left- and right-hand sides of (1) we see that

\[
ab = ((x^2) - (\xi^2)) / 2 (\xi \cdot \eta).
\]

If we write \( t(\eta) \) in the form (1), then \( (\xi \cdot t(\eta)) = (\xi \cdot \eta) \) and by (2) \( b = 1 \), so that

\[
t(\eta) = u \xi + \eta + \zeta,
\]

moreover, \( \xi = \phi(t) \) and by (3)

\[
u = - (\xi^2) / 2 \xi \cdot \eta.
\]

From the triviality of the action of \( t \) on \( \xi^+ / R \xi \) it follows that for an arbitrary vector \( \theta \) from \( \xi^+ \cap \eta^+ \)

\[
f(\theta) = \theta + \lambda_\theta \xi, \quad \lambda_\theta \in R,
\]
whence

\[ t^{-1}(\theta) = 0 - \lambda_{e} \xi; \]

taking into account (4) and the equality \( (\eta \cdot t^{-1}(\theta)) = (t(\eta) \cdot \theta) \), we infer that

\[ \lambda_{\varphi} = - (\theta / \xi \cdot \eta). \]  

(7)

From (4) and (6) it follows that \( \varphi \) is a homomorphism, from (4)–(7) it follows that the kernel of \( \varphi \) is trivial, and since for each \( \xi \) from \( \xi^{-1} \cap \eta^{+} \) equalities (4)–(7) and \( s(\xi) = \xi \) define a transformation \( t \) from \( T_{\xi} \), \( \varphi \) is epimorphic.

3) Let \( t \in T_{\xi} \) and \( s \in T_{\eta} \), where \( t \) and \( s \) are nonidentical, \( t \) is defined by (4)–(7), and \( s \) is given by

\[ s(\xi) = \xi + \nu \eta + \rho, \quad s(\eta) = \eta, \quad s(\theta) = \theta + \mu_{e} \eta, \]

where

\[ \nu = (\rho \cdot \xi) / (\xi \cdot \eta), \quad \mu_{e} = - (\rho \cdot \theta) / (\xi \cdot \eta). \]

Let

\[ y = a_{\xi} + b_{\eta} + \theta, \quad \theta = \xi^{-1} \cap \eta^{+}. \]

Since \( y \in I \), \( y \not\in \mathbb{R} \xi \) and \( y \not\in \mathbb{R} \eta \), we have

\[ (y \cdot \eta) > 0, \quad (y \cdot \xi) > 0, \quad a > 0, \quad b > 0. \]

From the equalities

\[ t(y) = s(y), \quad \pi(t(y)) = \pi(s(y)), \]

\[ t(y) = (a + \lambda_{e} + \lambda_{e}) \xi + b_{\eta} + (b_{\xi} + \theta), \]

\[ s(y) = a_{\xi} + (b_{\xi} \lambda_{e} + \lambda_{e}) \eta + (a_{\eta} \lambda_{e} + \theta) \]

it follows that \( b_{\xi} = a_{\eta} \), i.e. \( b_{\xi} = a_{\eta} \).

4) Since the nonzero vectors \( \psi(t), \psi(s) \) and \( -\psi(s) \) cannot be collinear and have one and the same direction, this assertion follows from the equality \( \psi(s^{-1}) = -\psi(s) \).

Lemma 12 and Proposition 5 are proved.

**Proposition 6.** Let \( C \) be a curve on a G-surface \( V \) such that

a) \( C \) is connected and primitive,

b) the system \( |C| \) is G-invariant,

c) \( (E \cdot C) = (E \cdot K_{G}) = 0 \) for all \( E \in \text{Comp}(C) \), and

d) \( \dim |C| = 0 \) for all natural \( l \).

Then \( V \) is G-projective.

**Proof.** From our assumptions and Lemma 8 it follows that

\[ l \text{ tr}_{G}(C) \neq 0 \quad \text{for} \quad l \geq 1. \]  

(1)

For simplicity we shall write \( \text{tr} \) instead of \( \text{tr}_{G} \). To prove the proposition it suffices to find a subgroup of finite index in \( G \) with respect to which \( V \) is projective.

Since \( G \) preserves both \( \text{Ker}(\text{tr}) \) and the quadratic form on \( \text{Ker}(\text{tr}) \) defined by the self-intersection index, and since by Lemma 6 this quadratic form is negative definite, there exists a subgroup \( G_{0} \) of finite index in \( G \) acting trivially on \( \text{Ker}(\text{tr}) \). Since the system \( |C| = \{ C \} \) is G-invariant, we may assume that \( G_{0} \) preserves all components of \( C \).
We observe that each irreducible curve which does not have common points with \( C \) is \( G_0 \)-invariant; in fact, the class of such a curve \( L \) in \( \text{Pic}(V) \) lies in \( \text{Ker}(\text{tr}) \); hence \((L^2) < 0, gL \sim L \) for \( g \in G_0 \) and if \( gL \neq L \), then \( 0 < (gL \cdot L) = (L^2) < 0 \).

Consider the group \( \text{Pic}(C) \). Its subgroup \( \text{Pic}_0(C) \) whose elements correspond to invertible sheaves on \( C \) whose restrictions to an arbitrary component of \( C \) have degree zero has the structure of one-dimensional connected algebraic group. An equivalence class of \( \text{Pic}(C) \) modulo \( \text{Pic}_0(C) \) is defined by a collection of degrees of the restrictions of an invertible sheaf on the components of the curve \( C \). These equivalence classes can be interpreted as connected components of the locally algebraic group \( \text{Pic}(C) \). The group \( G \) acts on \( \text{Pic}(C) \), and its subgroup \( G_0 \) preserves each connected component of \( \text{Pic}(C) \).

We shall show that for an arbitrary connected component \( X \) of \( \text{Pic}(C) \) such that \( X \) contains an element of the form \( \text{tr}(D) \), where \( D \in \text{Div}(V) \) and \( (D \cdot C) = 0 \), there exists a subgroup \( G_X \) of finite index in \( G_0 \) whose action on \( X \) is trivial.

Suppose first that \( X = \text{Pic}_0(C) \).

If \( X \) is a one-dimensional abelian variety, the existence of \( G_X \) follows from the finiteness of the group of automorphisms.

If \( X \) is the additive group \( G_a \), then \( G_0 \) preserves the point 0 and a nonzero element of \( \text{tr}(C) \) in \( X \) (this element lies in \( \text{Pic}_0(C) \) since \( (C \cdot E) = 0 \) for all \( E \) from \( \text{Comp}(C) \)); hence \( G_0 \) acts trivially on \( X \) and \( G_X = G_0 \).

If \( X \) is the multiplicative group \( G_a \), then from the fact that all transformations from \( G_0 \) preserved the element \( \text{tr}(C) \in X \) it follows that we may assume that \( G_X \) coincides either with \( G_0 \) or with its subgroup of index two.

For \( X = \text{Pic}_0(C) \) the subgroup \( G_X \) of \( G_0 \) will be denoted by \( G^0 \).

Let \( D \) be a divisor on \( V \) such that \( (D \cdot C) = 0 \), and let \( X \) be the connected component of \( \text{Pic}(C) \) containing \( \text{tr}(D) \). We shall show that there exists a subgroup \( G_X \) in \( G^0 \) with \([G : G_X] < \infty \).

For an element \( g \) from \( G^0 \) we define an element \( \gamma_g \) from \( \text{Pic}_0(C) \) by the equality

\[
\gamma_g = g(d) - d,
\]

where \( d \) is an arbitrary element of \( X \). The element \( \gamma_g \) does not depend on the choice of \( d \), since if \( d' \) is another element of \( X \), then

\[
d - d' \in \text{Pic}_0(C), \quad g(d) - g(d') = d - d', \quad g(d) - d = g(d') - d'.
\]

The map \( g \mapsto \gamma_g \) defines a homomorphism \( \gamma : G^0 \to \text{Pic}_0(C) \), since for \( g, h \in G \) we have

\[
\gamma_{gh} = gh(d) - d = (gh(d) - h(d)) + (h(d) - d) = \gamma_g + \gamma_h.
\]

To establish the existence of \( G_X \) it suffices to show that \( \gamma(G^0) \) is finite (then we can set \( G_X = \text{Ker} \gamma \)). By Lemma 3, there exist a natural number \( n \) and a divisor \( D_0 \) on \( V \) such that

\[
\text{supp} D_0 \subset C, \quad \text{tr}(nD - D_0) \equiv \text{Pic}_0(C),
\]

so that if we set \( d = \text{tr}(D) \) and \( d_0 = \text{tr}(D_0) \), then

\[
nD - D_0 \equiv \text{Pic}_0(C).
\]

Since \( G^0 \) preserves the components of \( C \), for \( g \in G^0 \) we have

\[
g(D_0) = D_0, \quad g(d_0) = d_0, \quad g(nD - D_0) = nD - d_0;
\]
hence

\[ n(gd - d) = 0, \quad n_Y = 0, \]

i.e. \( \gamma(G^2) \) lies in the kernel of multiplication by \( n \) in \( \text{Pic}_0(C) \). In the case when \( \text{Pic}_0(C) = G \) and the base field has positive characteristic this kernel is not necessarily finite, but if this is so, then (1) does not hold. The existence of \( G_X \) is proved.

To show that a subgroup \( H \) of finite index in \( G \) acts trivially on \( \text{Pic}(V) \) it suffices to show that \( H \) acts trivially on a subgroup \( M \) of \( \text{Pic}(V) \) having finite index in \( \text{Pic}(V) \).

We proceed with the construction of \( M \). Let \( E_1, \ldots, E_r \) be a finite collection of irreducible curves on \( V \), whose classes generate \( \text{Pic}(V) \), and let \( n_i = (E_i \cdot C) \). Suppose that \( n_j > 0 \) for \( 1 < i < s \), and \( n_j = 0 \) for \( s + 1 < j < r \). We note that each \( E_j \) with \( s + 1 < j < r \) is \( G \)-invariant, since such an \( E_j \) is either contained in \( C \) or does not intersect with it. We set

\[ m = n_1 n_2 \ldots n_r, \quad m_i = m/n_i, \]

\[ L_i = m_i E_i \] \( (1 \leq i \leq s) \).

Let \( M \) be the subgroup generated by the classes of the curves \( L_1, \ldots, L_s, E_{s+1}, \ldots, E_r \).

It is clear that \([\text{Pic}(v) : M] < \infty \) and

\[ (L_1 \cdot C) = m, \quad (L_i \cdot C) = m_i, \quad (L_{i-1} L_i \cdot C) = 0. \]

(2)

Let \( X_i \) be the connected component of \( \text{Pic}(C) \) containing \( \text{tr}(L_i - L_j) \), let \( G \) be the subgroup in \( G_0 \) consisting of all elements acting trivially on \( X_i \) \( (1 < i, j < s) \), and let \( H = \cap \_i G_i \). Since the intersection of a finite number of subgroups of finite index in \( G \) has finite index, we see that \([G : H] < \infty \).

For an arbitrary element \( h \) from \( H \) we shall show that

\[ hL_1 \sim L_1 \] \( (1 \leq i \leq s) \).

(3)

In fact, \( \text{tr}(L_i - L_j) \) is stable with respect to the action of \( h \); hence

\[ \text{tr}(hL_i - L_i) = \text{tr}(hL_j - L_j), \quad (1 \leq i, j \leq s), \]

and setting

\[ D = hL_i - L_i, \]

we can write

\[ hL_i - L_i = D \cdot D, \]

where \( D_i \in \text{Ker}(\text{tr}) \), \( D_i = 0 \). We shall show that the class of \( D_i \) does not depend on \( i \), and therefore all the \( D_i \) are equivalent to zero. In fact, if \( A \in \text{Ker}(\text{tr}) \), then \( hA = A \) and

\[ (D_i \cdot A) = \left( (L_i - L_i - D) \cdot A \right) \]

\[ = (L_i - A) - (D \cdot A) = -(D \cdot A) \]

i.e. \( (D_i \cdot A) \) does not depend on \( i \), and since the self-intersection quadratic form on \( \text{Ker}(\text{tr}) \) is nondegenerate (see Lemma 6), the class of \( D_i \) is uniquely determined by the values \( (D_i \cdot A) \), where \( A \) runs through \( \text{Ker}(\text{tr}) \).
RATIONAL $G$-SURFACES

Thus $D_i \sim 0$ for $1 < i < s$, and (5) can be rewritten in the form

$$hL_i \sim L_i + D.$$  \hspace{1cm} (6)

Comparing the self-intersection indices of the left and right sides of (6), we come to the equality

$$2(L_i \cdot D) + (D^3) = 0;$$  \hspace{1cm} (7)

hence the number

$$(L_i \cdot D) = \lambda$$  \hspace{1cm} (8)

does not depend on $i$, and

$$(D^3) = -2\lambda.$$  \hspace{1cm} (9)

To show that the divisor $mD - \lambda C$ is (numerically and hence linearly) equivalent to zero it suffices to verify that it has zero intersection with the generators of the group $M$. By (2)--(4) for $s + 1 < j < r$ we have

$$((mD - \lambda C) \cdot E_j) = m(D \cdot E_j) = m((hL_i - L_i) \cdot E_j) = 0,$$

and by (2) and (8) for $1 < i < s$

$$((mD - \lambda C) \cdot L_i) = 0.$$

Thus

$$mD \sim \lambda C,$$  \hspace{1cm} (10)

from which it follows that $m^2(D^3) = \lambda^2(C^3) = 0$; by (9) $\lambda = 0$, and then from (10) it follows that $D \sim 0$, which together with (6) implies (3). Thus we have shown that $H$ acts trivially on $\text{Pic}(V)$, and in particular $H$ preserves the class of an ample divisor; hence $V$ is $H$-projective and $G$-projective. The proposition is proved.

**Definition.** A $G$-surface $V$ is called $G$-pseudoprojective if it is antiprojective and for each $g$ from $G$ there exists a $g$-invariant vector in $I(V)$.

This definition of pseudoprojectivity is equivalent to the definition given in the Introduction, since if the class of the divisor $D_\alpha$ indicated in the Introduction does not lie in $I(V)$, it suffices to replace $D_\alpha$ by $(-D_\alpha)$.

**Theorem 2.** Let $V$ be a rational $G$-pseudoprojective and $G$-minimal surface. Then the following assertions are true:

1) $(K_\alpha^2) = 0$ and $\text{rk}(\text{Pic}(V)) = 10$.

2) There exists a natural number $m$ such that the system $|-mK_\alpha|$ is one-dimensional, has no fixed components, and all fibers of the map $\rho : V \to \mathbb{P}_1$ defined by this system are curves of genus 1 which do not contain exceptional components.

3) If $(S_1, \ldots, S_r)$ is the set of all reducible fibers of the map $\rho$, then

$$\sum_{i=1}^{r} (\mu(S_i) - 1) < 8.$$  \hspace{1cm} (1)
Assertion 3) of Theorem 2 can be made more precise in the following way:

**Proposition 7.** If $V$ is a rational surface satisfying conditions 1) and 2) of Theorem 2 and not coinciding with any of the surfaces $W_i$ ($i = 1, 2, 3, 4$; see §2), and $(S_1, \ldots, S_g)$ is the set of reducible fibers of $\rho$, then there exists an abelian subgroup $A$ in $\text{Aut}(V)$ satisfying the following conditions:

0) All transformations from $A$ induce translations on the generic fiber of $\rho$.

1) $[\text{Aut}(V): A] < \infty$.

2) $\text{rk}(A) = 8 - \sum_i (\mu(S_i) - 1)$.

3) For each nonperiodic element $g$ of the group $A$, the surface $V$ is $g$-antiprojective.

In particular, $V$ is $\text{Aut}(V)$-antiprojective if and only if inequality (1) holds.

**Proof of Theorem 2.** From the definition of pseudoprojectivity and Propositions 4 and 5 it follows that in the cone $\mathcal{C}(V)$ there exists a $G$-invariant vector $e$. By Propositions 3 and 4, in the ray $R^+e$ there exists a class from $\text{Pic}(V)$ containing a divisor $C$ which satisfies conditions a), b) and c) of Proposition 6.

Condition d) of this proposition does not hold for $C$, since otherwise $V$ would be $G$-projective. Hence there exists a natural number $l$ such that $\text{dim} |C| > 1$. Let $l_0$ be the smallest such $l$. By Lemma 4, $|l_0C|$ is a one-dimensional system without fixed components; it is easy to see that this system consists of curves of genus 1 and by assertion 4) of Proposition 3 there are no exceptional curves among its members. Thus $|l_0C|$ gives rise to a morphism $\rho : V \to \mathbb{P}_1$ defining on $V$ the structure of a minimal elliptic surface. From [1], Chapter VII, it follows that $(K_V^2) = 0$ for such $V$, and then from the Noether formula for rational surfaces it follows that

$$\text{rk}(\text{Pic}(V)) = 10 - (K_V^2) = 10,$$

which proves 1). Furthermore, the system $|l_0C|$ is nonempty (see Lemma 9) and $G$-invariant; hence from Lemma 7 and Proposition 3 it follows that we can choose a curve $C$ satisfying the conditions indicated in the beginning of the proof from the system $|l_0C|$. The number $l_0$ will be henceforth denoted by $m$. Thus the system $|mK_V|$ is one-dimensional and has no fixed components, the map $\rho$ defined by it has the required properties (which proves 2), and all its members with a possible exception of $mC$ are primitive (see Lemma 8).

Since for each surface $W_i$ from §2 $\text{Aut}(W_i)$ is a linear algebraic group acting algebraically on $W_i$, and since for the curves $C_i$, $C_i'$ on $W_i$

$$\mu(C_i) + \mu(C_i') = 10,$$

to prove 3) it suffices to carry out the

**Proof of Proposition 7.** We denote by $C^*$ the generic fiber of $\rho$ over the field $k(\mathbb{P}_1)$, by $K^*$ the kernel of the homomorphism $\text{Pic}(V) \to Z$ associating to the class of a divisor $D$ the number $(D \cdot K)$, and by $\text{tr}$ the homomorphism

$$\text{tr}_{C^*} : \text{Pic}(V) \to \text{Pic}(C^*).$$

We shall consider two sequences of homomorphisms of groups:

$$0 \to \text{Aut}(C^*/k(\mathbb{P}_1)) \xrightarrow{\delta} \text{Aut}(V) \xrightarrow{\mu} \text{Aut}(\mathbb{P}_1), \quad (1)$$

$$0 \to \text{Ker}(\text{tr}) \xrightarrow{\lambda} K^* \xrightarrow{\delta} \text{Aut}(C^*/k(\mathbb{P}_1)). \quad (2)$$
DEFINITION OF \( \iota \): By Theorem 1 from [1], Chapter VII, §1, an automorphism of the generic fiber of \( \rho \) extends to an automorphism of \( V \).

DEFINITION OF \( \kappa \): The group \( \text{Aut}(V) \) preserves the system \( \{-mK_V\} \) defining \( \rho \). Hence for a point \( x \) of \( P_1 \) we may put \( \kappa(g)(x) = \rho(g(y)) \), where \( y \in \rho^{-1}(x) \).

DEFINITION OF \( \lambda \): We denote by \( \lambda \) the natural inclusion \( \text{Ker}(\text{tr}) \subset K^\perp \).

DEFINITION OF \( \theta \): If the class of a divisor \( D \) lies in \( K^\perp \), then \( \deg(\text{tr}(D)) = 0 \). Hence the divisor \( \text{tr}(D) \) defines a translation \( \theta_D \) on the curve \( C^* / k(P_1) \) by means of the formula

\[
\theta_D(P) = \text{tr}(D) + P,
\]

where \( P \) is a geometric point of \( C^* \).

**Lemma 13.** Sequences (1) and (2) are exact.

**Proof.** The exactness of (1) is obvious. Furthermore, \( \lambda \) is an inclusion, and the exactness of (2) at \( K^\perp \) follows from the fact that a divisor of degree zero on \( C^* \) gives rise to the identical translation if and only if it is equivalent to zero. The lemma is proved.

**Lemma 14.** \([\text{Aut}(C^*/k(P_1)): \theta(K^\perp)] < \infty\).

**Proof.** The subgroup of \( \text{Aut}(C^*/k(P_1)) \) consisting of translations corresponding to classes of degree zero from \( \text{Pic}(C^*/k(P_1)) \) is a subgroup of finite index, and by Lemma 5

\[
\text{Pic}(C^*/k(P_1)) = \text{tr}(\text{Pic}(V)).
\]

**Lemma 15.** If \( V \) does not coincide with any of the surfaces \( W_i \) defined in §2 \((1 < i < 4)\), then

\[
[\text{Aut}(V) : \iota(\text{Aut}(C^*/k(P_1)))] < \infty,
\]

i.e. the image of the homomorphism \( \kappa \) is finite.

**Proof.** If the image of \( \kappa \) were infinite, then the map \( \rho \) would have at most two degenerate fibers, and by Proposition (B) from §2 \( V \) would coincide with one of the surfaces \( W_i \).

Consider the homomorphism

\[
T : K^\perp / \text{Ker}(\text{tr}) \rightarrow \text{Aut}(V),
\]

induced by the composition \( \iota \circ \theta \). From Lemma 13 it follows that \( T \) is a monomorphism, and from Lemmas 14 and 15 that the group \( A = T(K^\perp / \text{Ker}(\text{tr})) \) is a subgroup of finite index in \( \text{Aut}(V) \), which proves 0) and 1).

Next we compute the rank of \( A \). Since \( T \) is a monomorphism, we have

\[
\text{rk}(A) = \text{rk}(K^\perp) - \text{rk}(\text{Ker}(\text{tr}));
\]

but \( \text{rk}(K^\perp) = 9 \) and by Lemma 5

\[
\text{rk}(\text{Ker}(\text{tr})) = \sum_{i=1}^{r}(\mu(S_i) - 1) + 1,
\]

which yields 2).

3) Suppose that \( V \) is \( g \)-projective, where \( g \in A, A = \text{Im} T \), and let \( L \) be an ample divisor on \( V \) whose class in \( \text{Pic}(V) \) is \( g \)-invariant. Then the class of \( \text{tr}(L) \) is \( \iota^{-1}(g) \)-invariant, and by Proposition 1 of [10], Chapter II, §6, \( \iota^{-1}(g) \) is an element of finite order; but then \( g \) is periodic.

The proposition is proved.
Theorem 2 and Proposition 7 confirm a conjecture in [11], a slightly more precise statement of which is as follows: a suitable power of an automorphism $g$ inducing a transformation of infinite order on $\text{Pic}(V)$, where $V$ is a rational surface with $\text{rk}(\text{Pic}(V)) = 10$, is elliptic (i.e. $V$ satisfies condition 2) of Theorem 2 and a power of $g$ lies in the group $A$ indicated in Proposition 7).

In [11] it is required that the order of $g$ itself be infinite, but taking an element $g$ of infinite order from $\text{Aut}_g(W) = G_n$, we would obtain nonelliptic automorphisms $g^n$, $n \neq 0$. To prove the above conjecture it suffices to observe that $V$ is $g$-antiprojective and $g$-pseudoprojective (since $g$ preserves $|-K_V| = |C|$), and so we can apply Theorem 2 and Proposition 7.

**Proposition 8.** Suppose that a surface $V$ is obtained by blowing up nine points in general position in $\mathbb{P}^2$, and let $\Sigma : V \to \mathbb{P}^2$ be the corresponding morphism. Then $\text{Aut}(V) = \{1\}$.

**Remark.** Nine points are in general position if the following conditions hold:
1) These points lie on a unique nonsingular cubic curve $C$ in $\mathbb{P}^2$.
2) The curve $C$ is neither harmonic nor equianharmonic in the sense of [4], Chapter VI, §9.
3) If we introduce on $C$ a commutative group law taking the inflection point as neutral element, then our nine points generate a free abelian group of rank nine.

**Proof.** Let $L = \Sigma^*$ (line in $\mathbb{P}^2$), let $L_1, \ldots, L_9$ be the exceptional curves on $V$ blown down to points under the map $\Sigma$, and let $C = \Sigma(C)$. We observe that $|-K_V| = \{C\}$, 

$$(C^2) = 0,$$

the classes of $L_1, L_2, \ldots, L_9$ generate $\text{Pic}(V)$, and their images under the map

$$\text{tr} = \text{tr}_C : \text{Pic}(V) \to \text{Pic}(C)$$

generate a free abelian group of rank 10 in $\text{Pic}(C)$; hence $\text{Ker}(\text{tr}) = \{0\}$. Since $\text{Aut}(V)$ preserves $|-K_V|$, we obtain a homomorphism

$$\varphi : \text{Aut}(V) \to \text{Aut}(C)$$

and for $g \in \text{Aut}(V)$ $\varphi(g)$ is either a translation or an automorphism of order two (an involution).

Suppose first that $\varphi(g)$ is the translation defined by a divisor $D$ of degree zero on $C$. Then

$$\text{tr}(g(L_1) - L_i) \sim d, \quad \text{tr}(g(L) - L_i) \sim 3d, \quad 1 \leq i \leq 9.$$ 

Since the class of $D$ corresponds to the image on $C$ of a divisor $D$ on $V$ (e.g. $D = g(L_1) - L_1$) and $\text{tr}$ is a monomorphism, we have

$$g(L_1) \sim L_i + D, \quad g(L) \sim L + 3D, \quad 1 \leq i \leq 9. \quad (1)$$

From $(D \cdot K) = 0$ it follows that

$$(D^3) = -2\lambda, \quad \lambda \in \mathbb{Z}, \quad (2)$$

and comparing the self-intersection indices of the two sides of equivalences (1) we see that $(L_7 \cdot D) = \lambda$ and $(L \cdot D) = 3\lambda$; hence the divisors $D$ and $\lambda C$ have the same intersections with the generators of $\text{Pic}(V)$; therefore

$$D \sim \lambda C, \quad (D^3) = \lambda^2 (C^3) = 0,$$

and by (2) $\lambda = 0$ and $D \sim 0$. But then by (1) $g$ acts trivially on $\text{Pic}(V)$; hence this action is generated by an action of $g$ on $\mathbb{P}^2$ preserving the nine points. Therefore $g = 1$. 


Next we show that $\varphi(g)$ cannot be an involution. An involution on $C$ is defined by means of the formula

$$\varphi(g)(P) + P \sim d,$$

where $P \in C$ and $d$ is a divisor of degree 2. In particular,

$$\text{tr}(gL_i + L_i) = d, \quad \text{tr}(gL + L) = 3d.$$

Taking a divisor $D$ on $V$ with $\text{tr}(D) = d$ (e.g. $D = gL_i + L_i$), we see that

$$(D \cdot K) = -2,$$

$$(gL_i \sim -L_i + D, \quad gL \sim -L + 3D). \quad (3)$$

From (3) we infer that

$$(D^*) = 2\lambda, \quad \lambda \in \mathbb{Z},$$

and (4) yields

$$(L_i \cdot D) = \lambda, \quad (L \cdot D) = 3\lambda;$$

hence

$$D \sim \lambda C, \quad \lambda = 0, \quad D \sim 0,$$

which contradicts (3). The proposition is proved.

Remarks. Let $V$ be a rational surface with $(K_Y^2) = 0$, and let $C \subseteq |-K_Y|$. In order that $V \in \text{Aut}(V)$-antiprojective it is necessary that the intersection index of the curve $C$ with an arbitrary component $E$ of $C$ be equal to zero (see Proposition 3). If $C$ is such a curve, then by Lemma 9 there exists a birational morphism $\Sigma : V \to \mathbb{P}_2$, where $\Sigma$ is a composition of nine successive blowings up of points on (or over) $\mathbb{P}_2$. If $V \in \text{Aut}(V)$-antiprojective, these nine points cannot be arbitrary (see Proposition 8)—they must be such that, for some natural $m$, the curve $mC$ would belong to the one-dimensional linear system $|mC|$ without fixed components (see Theorem 2), and in that case the images of the curves of this system form an Halphen pencil on $\mathbb{P}_2$ (see [5]), namely the pencil of curves of degree $3m$ for which the nine points that are going to be blown up are points of multiplicity $m$. Proposition 7 shows that in order that $V \in \text{Aut}(V)$-antiprojective it is necessary that this pencil not be too degenerate, i.e. that the reducible members of $|-mK_Y|$ not have too many components.

Proposition 9 below describes the action on $\text{Pic}(V)$ of transformations from the abelian group $A \subseteq \text{Aut}(V)$ constructed in Proposition 7 in the case when the pencil $|-mK_Y|$ does not have reducible members.

**PROPOSITION 9.** Suppose that a surface $V$ satisfies conditions 1) and 2) of Theorem 2 and that all members of the pencil $|-mK_Y|$ are irreducible. Then the following assertions are true:

1) The sequence

$$0 \to \mathbb{Z} \to K \to K_\perp \to \text{Aut}(V),$$

where $\iota = \iota \circ \theta$ (and $\theta$ are the same as in sequences (1) and (2) in the proof of Proposition 7), is exact.
2) If \( t_D \) denotes the image of the class of a divisor \( D \) under the map \( t \), then \( t_D \) acts on \( \text{Pic}(V) \) according to the formula

\[
t_D(E) \sim E - m(E \cdot K)D + (m(E \cdot D) + \frac{m^2}{2}(E \cdot K)(D^2))K.
\]

3) The transformation of the quotient group \( K^\perp / ZK \) induced by \( t_D \) is identical.

**Proof.** Let \( C^* \) be the generic member of the pencil \( |-mK_P| \) (so that \( C^* \) is a curve over \( k(P_i) \)), and let

\[
\text{tr} = \text{tr}_{C^*} : \text{Pic}(V) \rightarrow \text{Pic}(C^*).
\]

From Lemma 5 it follows that

\[
\text{Ker} \left( \text{tr} \right) = ZK,
\]

and therefore assertion 1) follows from Lemma 13.

In the proof of 2) we may assume that \( (E \cdot K) \neq 0 \), since the two sides of our formula are additive with respect to \( E \) and \( \text{Pic}(V) \) is generated by divisor classes having nonzero intersections with \( K \). For a geometric point \( P \) of the curve \( C^* \) we have

\[
\theta_D(P) \sim P + \text{tr}(D),
\]

where \( \theta_D \) is the image of \( D \) under the map \( \theta \), whence

\[
\text{tr}(t_D(E)) \sim E + (\text{deg}(\text{tr} E)) \text{tr}(D);
\]

applying (1) and the equality \( \text{deg}(\text{tr} E) = -m(E \cdot K) \), we see that

\[
t_D(E) \sim E - m(E \cdot K)D + \lambda K,
\]

where \( \lambda \in Z \). Comparing the self-intersection indices of the two sides of the last formula, we find \( \lambda \) and thus complete the proof of 2).

Assertion 3) is an immediate consequence of 2). The proposition is proved.

Next we give an example of an \( \text{Aut}(V) \)-antiprojective surface \( V \) which is not pseudo-projective. In this example we make use of an irreducible rational plane curve of degree 6 with ten double points. Such curves and the Cremona transformations associated with them were studied by Coble [3]. We shall need a sextic with ten double points such that

a) no three of these ten points are collinear,
b) no six of these points lie on a conic, and
c) no nine of these points form the fundamental set of a pencil of cubic curves.

We construct this curve in the following way. We take eight points \( P_0, \ldots, P_8 \) in general position in \( P_2 \). These points define a pencil of cubic curves. We pick a generic curve \( C \) belonging to this pencil; in a standard way we introduce a group law on \( C \) and pick a point \( P_0 \) on \( C \) which is not a base point of the pencil of cubics in such a way that the doubled sum of the nine points \( P_0, \ldots, P_8 \) is equal to zero. These nine points form the fundamental set for a pencil of Halphen sextics; all members of this pencil are irreducible, and among them there are twelve rational curves which have a tenth double point in addition to the base points. An arbitrary sextic from this dozen will do for our purposes.
PROPOSITION 10. Suppose that the ten double points of a plane sextic satisfy conditions a), b) and c), and let \( \Sigma : V \to P_2 \) be the blowing up of these ten points. Then the surface \( V \) is Aut(\( V \))-anti-projective, but not Aut(\( V \))-pseudo-projective.

PROOF. We choose any nine of our ten points, blow them up, and denote by \( \Sigma_1 : W \to P_2 \) the corresponding morphism. Then \( \Sigma = \Sigma_1 \circ \sigma_1 \), where \( \sigma_1 \) is the blowing up of the point whose image under \( \Sigma_1 \) is the double point of the sextic which does not belong to the set of nine distinguished points. Let \( D \) be a divisor on \( W \) whose class lies in \( K_W^+ \) but not in \( ZK_W \). In Proposition 9 we defined a transformation \( t_D \) from Aut(\( W \)) which preserves the pencil \(-2K_W\), its degenerate members, and their double points; hence this transformation lifts to a transformation

\[ t = \sigma_1^{-1} t_D \sigma_1 \in \text{Aut}(V). \]

We denote by \( \xi \) the class \( \sigma_1^{-1}(-K_W) \) in Pic(\( V \)) and by \( \xi^\perp \) the orthogonal complement of \( \xi \) in \( \mathcal{P}(V) = \text{Pic}(V) \otimes \mathbb{R} \). We observe that \( \xi \in I(\mathcal{P}) \), and by assertion 3) of Proposition 9 \( t \) induces the identity transformation on \( \xi^\perp / R \xi \).

Next we choose nine of our ten points not coinciding with the previously chosen nine points and perform the above constructions for these new points:

\[ \Sigma_2 : U \to P_2, \quad \Sigma = \Sigma_2 \circ \sigma_2 \]

we take a divisor \( E \) on \( U \) whose class lies in \( K_U^+ - ZK_U \) and introduce \( t_E, s = \sigma_2^{-1} t_E \sigma_2 \in \text{Aut}(V) \) and \( \eta = \sigma_2^{-1}(-K_U) \in I(\mathcal{P}) \); \( s \) induces the identity transformation on \( \eta^\perp / R \eta \). The existence of \( s \) and \( t \) makes \( V \) Aut(\( V \))-anti-projective. If \( V \) were Aut-pseudo-projective, in \( I(\mathcal{P}) \) there would exist an \( s \)-invariant vector and a \( t^\perp \)-invariant vector, which is impossible in view of Proposition 5 and Lemma 12 (see the deduction of the proposition from the lemma). If \( g \) denotes that one of the transformations \( st, t^\perp \) for which there are no \( g \)-invariant vectors in \( I(\mathcal{P}) \), then \( g \) does not preserve any pencil of curves on \( V \). The existence of such \( g \) refutes the theorem in [11]. In fact, we shall show that for an arbitrary natural number \( n \) there does not exist a \( g^n \)-invariant vector in \( I(\mathcal{P}) \). By Proposition 4, in \( \mathcal{P}(V) \) there exists a \( g \)-invariant ray \( R^+ e \), where \( g(e) = \lambda e \), \( \lambda \) being a positive irrational number which is a unit of the ring of integral algebraic numbers. Then \( \lambda^n \neq 1 \), \( g^n(e) = \lambda^n e \), and from assertion 4) of Proposition 4 it follows that a \( g^n \)-invariant vector and the \( g^n \)-invariant ray \( R^+ e \) cannot coexist in \( R^+ e \). The proposition is proved.

Incidentally, the argument formulated in the last three sentences of the above proof also proves the following fact.

PROPOSITION 11. If \( V \) is a rational G-surface, \( H \) is a subgroup of finite index in \( G \) and \( V \) is \( H \)-pseudo-projective, then \( V \) is \( G \)-pseudo-projective.

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