Curves of simple contact on algebraic surfaces.

By

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I. Introduction

Introduction.

The intersection of surfaces has been discussed chiefly with regard to postulation and equivalence*), i.e., the number of independent conditions imposed on a surface by making it contain a certain element, and the number of points of intersection of three surfaces absorbed by the common element. These numbers are important in connection with transformation-theory. In the plane, the common elements can only be points, and contact can always be regarded as the limit of intersection in distinct adjacent points; this is not true in three dimensions. Then there may also be common curves, and though a curve of contact does arise when two distinct curves of intersection move up and coincide, this is not the most general case of contact along the curve. If a surface contains a curve, it cannot be thought of as containing an adjacent curve, unless the surface is specialized in a way that involves a certain number of singular points of the surface lying on the curve.

This paper determines the postulation and equivalence of a curve of contact on a family of surfaces of sufficiently high degree. Noether's**)

lines are closely followed, but the results are also proved by other methods. The relation of the curve of contact to its residual is found; this information is useful in the theory of twisted curves. We first examine the cases which do arise from the coincidence of two adjacent curves; this leads to formulae which apply also in the general case, when the expression may be negative which was at first interpreted as the number of intersections of the two curves.

Approached in this way, the general case is that in which the family of surfaces, of degree \( \lambda \) say, are required to touch a given surface, of lower degree \( \kappa \), along the given curve of order \( m \), where \( \lambda \) exceeds a certain limit depending on \( \kappa, m \). The formulae do not apply to the case, more general from another point of view, in which the surfaces do not touch a surface of lower degree along the curve. The other chief restriction is that the only singularities considered are multiple points of the simplest type, i.e., on surfaces, conic nodes, and on curves, double points with distinct tangents.

The theory of moduli*) is not directly applicable to the question, because the given conditions do not determine the modulus, without considerations such as those of § IV below. In certain cases, another method would be to use a birational transformation in which the given curve corresponds to a single point in the second space, and to carry out the enumerations with regard to the transformed figure; but this process only transfers the difficulties of the question.

The following symbols and technical terms will be used:

\[
\binom{n}{3} = \frac{1}{6} n (n-1)(n-2),
\]

\[(P_i)_\lambda = \text{postulation of } C_\lambda \text{ as an } i\text{-fold curve on a surface of degree } \lambda \]
\[
= \frac{1}{2} i(i+1)(\lambda+2)m - \frac{1}{12} i(i+1)(2i+1)\varphi,
\]

\[(E_i)_\lambda = \text{equivalence of } C_\lambda \text{ as an } i\text{-fold curve on surfaces of degrees } \lambda, \mu, \nu,
= i^3(\lambda+\mu+\nu)m - i^3\varphi,
\]

where
\[
\varphi = m(m+1) - 2h = r + 2m + 2d;
\]

\( m \) is the order of the curve, \( d, h \) the numbers of its real and apparent double points, and \( r \) its rank (Noether's formulae).

Functions which equated to zero give a surface having \( C \) as a simple (or nodal) line are called simple (or nodal) terms.

A function is *linearly independent* of another function \( f \) if it is not the sum of a multiple of \( f \) and nodal terms.

The surface \( f = 0 \) is denoted by simply \( f \), which means both the surface and the function according to the context.

Points of intersection of curves and surfaces not lying on \( C \) are called *free*.

II. Intersection of adjacent curves.

Let \( C_m, C'_m \) be adjacent curves of the same character lying upon a surface \( f_x \) having \( D_x \) nodes at simple points of \( C \) and \( d_2 \) nodes at double points of \( C \). Let \( C, C' \) have \( N' \) intersections at simple points of \( C \), and let \( d_1 = d - d_2 \) be the number of double points of \( C \) at simple points of \( f \).

Now let \( C' \) move on \( f \) to coincide with \( C \). Then there are ultimately four intersections of \( C, C' \) in the neighbourhood of each of the \( d_1 \) points and two in the neighbourhood of each of the \( d_2 \) points; the postulation and equivalence of the system \( C, C' \), on any surface \( f_x \) of sufficiently high degree, are

\[
2P_1 - N, \quad 2E_1 - 2N,
\]

where

\[
N = N' + 4d_1 + 2d_2.
\]

Now project \( C, C' \) from any point \( A \) into two plane curves \( c, c' \) of class \( r \) having \( r^2 \) common tangents, which are the traces of the common tangent planes to \( C, C' \) which pass through \( A \). When \( C' \) moves up to and coincides with \( C \), the limiting positions of these \( r^2 \) common tangents fall into three classes.

1. The traces of the \( n \) tangent planes to \( f_x \) at points of \( C \), which pass through \( A \), where \( n \) is the class of the developable formed by all these tangent planes. Considering the intersections of \( C \) with the first polar of \( A \) with regard to \( f_x \), we have

\[
(x-1)m = n + D_x + 2d_2.
\]

2. The \( N' + 2d \) tangents to \( c \) at the projections of the ultimate intersections of \( C, C' \).

3. The double tangents to \( c \) each counted twice. The number of these \( = \) the number of double points on the reciprocal of \( c \), whose degree is \( r \) and class \( m \),

\[
= \frac{1}{2} \{ r(r-1) - m \}.
\]

Therefore

\[
r^2 = (x-1)m - D_x - 2d_2 + N' + 2d + r(r-1) - m,
\]

\[
N = N' + 2d + 2d_1 = \rho + D_x - zm,
\]

or

\[
D_x = N + zm - \rho \geq zm - \rho.
\]
Therefore unless the surface \( f_x \) has at least \( zm - q \) nodes at simple points of \( C \), it cannot be thought of as containing a curve \( C' \) adjacent to \( C \).

Now when \( C, C' \) coincide, the postulation and equivalence remain unaltered, and hence for the cases of contact which arise in this way, their values are

\[
2P_1 - N, \quad 2E_1 - 2N,
\]

where

\[
N = q + D_x - zm.
\]

Now \( f_1 \) touches the same developable as \( f_x \) and has the same \( d_z \) nodes:

\[
\lambda m - D_z = n + m + 2d_z = zm - D_x,
\]

and is independent of \( \lambda \) or \( x \); also \( N = q + D_1 - \lambda m \), which has a meaning even when it is negative, and can no longer mean the number of intersections of adjacent curves. We shall show that these formulae still hold.

III. Postulation, Noether's method.

1. Straight line.

Let the equations of the straight line \( C \) be \( z = w = 0 \), and let the given surface \( f_x \) containing \( C \) be

\[
f_x = z\Phi_{x-1}(x, y) + w\Phi'_{x-1}(x, y) + \text{higher powers of } z, w,
\]

and let it have \( D_x \) nodes on \( C \).

Then the tangent plane to \( f_x \) at any point \((x, y, 0, 0)\) of \( C \) is given by the group of terms in \( f_x \) linear in \( z, w \), if in them \( x, y \) are regarded as constants. Now if \((x_1, y_1, 0, 0)\) is a node of \( f_x \), at this point the tangent plane is indeterminate and \( \Phi, \Phi' \) both vanish, and \( \Phi, \Phi' \) have a common factor \( xy_1 - yx_1 \), which does not affect the position of the tangent plane at other points and can be removed from their equations. Similarly for all the other nodes.

Therefore \( \Phi, \Phi' \) have a common factor of degree \( D_x \), whose linear factors vanish one at each node; let the remaining factors of \( \Phi, \Phi' \) be \( \Psi, \Psi' \); then these have no common factor.

Then the surface of degree \( x' = x - D_x \), viz.

\[
f_x' = z\Psi + w\Psi',
\]

has no node on \( C \); it has the same tangent plane as \( f_x \) except at the \( D_x \) points where \( f_x \) is singular, and at these points \( f_x' \) touches the limiting position of the tangent planes to \( f_x \) as we approach the node along \( C \).

Now if a surface \( f_1' \) of any degree is required to touch \( f_x \) along \( C \), it also touches \( f_x' \), and the surface \( f_{x'} \), formed from \( f_2 \) in the same way, must coincide with \( f_x' \).
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Therefore

\[ \lambda - D_\lambda = \gamma - D_\gamma = n + 1, \]

where \( n \) is the class of the common tangent developable to \( f_\lambda, f_\gamma \) along \( C \):

\[ f_\lambda \equiv \Phi_{D_\lambda} \cdot f_\gamma + \text{quadratic and higher powers of } x, w, \]

and \( f_\lambda \) is subject to \( 3\lambda - D_\lambda \) conditions, i.e. the postulation of contact with \( f_\gamma \) along \( C \) is

\[ P = 3\lambda - D_\lambda = 2P_1 - N, \]

where

\[ N = 2 + D_\lambda - \delta = 1 - n. \]

Now this straight line of contact can be thought of as a pair of adjacent straight lines in two cases only, when they do or do not intersect, \( N = 1 \) or \( 0, n = 0 \) or \( 1, D_\lambda = \lambda - 1 \) or \( \lambda - 2 \). But \( D_\lambda \) can have any value from 0 to \( \lambda - 1 \) inclusive (if \( D_\lambda > \lambda - 1 \), then \( C \) is a nodal line on \( f_\lambda \), and there are \( \lambda \) types of contact for surfaces of degree \( \lambda \) along a straight line.

2. Total intersection.

Let \( C_m \) be the total intersection of \( f_\gamma \) and another surface \( g_\alpha \). The general surface of degree \( \lambda \) containing \( C \) is

\[ f_\lambda \equiv f_\gamma \Phi_{-\nu} + g_\alpha \Phi_{-\alpha}. \]

If \( p_\nu, p_\alpha \) are the tangent planes to \( f_\gamma, g_\alpha \) at any point of \( C \), then the tangent plane to \( f_\lambda \) at the same point is

\[ p_\nu \Phi + p_\alpha \Phi', \]

where the coordinates of the point of contact are substituted in \( \Phi, \Phi' \).

Now since this plane coincides with \( p_\nu \), and in general \( p_\alpha \) is a different plane:

\[ \Phi' \equiv 0 \text{ at all points of } C', \text{ and } \Phi \text{ is a simple and } g\Phi' \text{ a nodal term.} \]

In considering the number of independent terms in \( f_\lambda \), we must admit

1. nodal and higher terms, in number \( \left( \frac{\lambda + 3}{3} \right) - (P_2)_3 \),

2. multiples of \( f_\gamma \), where in order not to count terms twice, we must exclude all simple and higher terms from the multiplier, leaving \( (P_1)_{-\nu} \) independent terms.

Thus the postulation is

\[ P = (P_2)_{-\nu} - (P_1)_{-\nu} = (2\lambda + \nu + 4)m - 2q \]

\[ = 2P_1 - (\rho - \nu m), \]

and since every node of \( f_\gamma \) on \( C \) is a double point of \( C \), we have \( D_\gamma = 0 \) and our previous formula (at the end of § II) is established for this case.

3. Any curve.

Now let \( C_m \) degenerate (the total number of apparent double points remaining unaltered), into two curves \( C_m', C_m'' \) intersecting one another
in \( I \) points, supposed simple points of each, of which \( I_1 \) are simple points of \( f_x \) and \( I_2 \) are nodes of \( f_x \), where \( I = I_1 + I_2 \):

\[
m = m' + m'', \quad \varphi = \varphi' + \varphi'' + 2I,
\]

\[
P = \{(2\lambda + x + 4)m' - 2\varphi'\} + \{(2\lambda + x + 4)m'' - 2\varphi''\} - 4I.
\]

But

\[
P = P' + P'' - \sigma_1 I_1 - \sigma_2 I_2,
\]

where \( P', P'' \) are the postulations of contact with \( f_x \) along \( C', C'' \) separately, and \( \sigma_1, \sigma_2 \) are the reductions in total postulation due to the two kinds of intersection respectively. Then

\[
P' = (2\lambda + x + 4)m' - 2\varphi' - \frac{1}{2} (4 - \sigma_1) I_1 - \frac{1}{2} (4 - \sigma_2) I_2.
\]

Now \( \sigma_1, \sigma_2 \) depend only on the behaviour of the curves and surfaces in the immediate neighbourhood of the intersection, and not on their general characteristics (degree, genus etc). They are therefore absolute constants and can be determined from the simplest possible cases, viz. when \( C', C'' \) are straight lines, and \( f_x \) is a plane or a pair of planes. This gives \( \sigma_1 = 4, \sigma_2 = 2 \), and

\[
P' = (2\lambda + x + 4)m' - 2\varphi' - I_2.
\]

Now \( C' \) is any curve on \( f_x \), and \( I_2 \) is the number of nodes of \( f_x \) at simple points of \( C' \).

If we drop the dashes and write \( D_x \) for \( I_2 \), we have

\[
P = (2\lambda + x + 4)m - 2\varphi - D_x
\]

\[
= 2P_1 - (\varphi + D_x - \lambda m);
\]

and this formula is general, provided only that \( \lambda \) is great enough for Noether's formulae for \( (P_2)_{\lambda}, (P_1)_{\lambda-x} \) to be applicable.

IV. Postulation, second method.

We shall now give two direct algebraic proofs that if \( f_x \) has a node at a simple point \( A \) of \( C \), we can construct a surface \( f' \) touching \( f \) along \( C \), but with a simple point at \( A \) and therefore linearly independent of \( f \). Then in the general equation of the family of degree \( \lambda \) touching \( f \) along \( C \), we admit not only nodal terms and multiples of \( f \), but also multiples of \( f' \); and we shall prove that the latter give only one independent term.

Also all other nodes of \( f \) on \( C \) are nodes of \( f' \), so if we form another surface \( f'' \) corresponding to another node \( B \) of \( f \), then \( f'' \) has a node at \( B \), and \( f'' \) has not; \( f'' \) is independent of \( f' \) as well as of \( f \), and the reduction in postulation is exactly \( D_x \).
Now at a node of $f$ at a double point of $C$, $f$ touches two different planes along the two branches of $C$, and it is geometrically impossible to construct $f'$ touching $f$ along $C$, and having a simple point at this node of $f$, so that such nodes do not reduce the postulation.

This gives a second proof of the formula

$$P = (P_x)_k - (P_y)_l x - D_z,$$

which is equivalent to the one obtained in the last paragraph.

We shall first treat as an example the simplest curve which is not a total intersection, viz. the twisted cubic.

1. Twisted cubic.

Let the curve $C_s$ be given parametrically by

$$x:y:z:w = l:1:8:8,$$

and also as the common intersection of the three quadrics

$$g = xz - y^2, \quad g' = xw - yz, \quad g'' = yw - z^2.$$

Let $p, p', p''$ be the tangent planes to $g, g', g''$ at the point of $C$ whose parameter is $\theta$.

Now let $f_x$ be a surface containing $C$ and having a node at $(1,0,0,0)$ i. e. the point $\theta = 0$. Then we may assume

$$f_x \equiv \sum_{i=1}^{x-2} x^{x-1-2}(y, z, w^i),$$

in which the general term can also be written

$$x^{x-1-2}(y, z, w^i, y, z, w^i),$$

The tangent plane to $f_x$ at the point $\theta$ is

$$P_x \equiv \sum(\theta, \theta^2, \theta^3, \theta^4, \theta^5, \theta^6)$$

$$= \theta \sum(1, \theta, \theta^2, \theta^3, \theta^4, \theta^5, \theta^6)$$

$$= \theta K \text{ say}.$$

Now $K$ can be interpreted as the tangent plane at the point $\theta$ to

$$f_x' \equiv \sum x^{x-1-2}(x, y, z, y, z, w^{i-1}),$$

which has a simple point at $\theta = 0$. Since $f, f'$ have identical tangent planes at all points of $C$ except $\theta = 0$ and $\theta = \infty$, where they are respectively singular, they are in contact along $C$.

Now

$$xf - yf'' = \sum x^{x-1-2}(0, g, y, g', y, w^{i-1}),$$
which is nodal, and similarly $zf'$, $wf'$ only differ from multiples of $f$ by nodal terms, but $zf''$ is linearly independent of $f$.

Therefore, if $f_{k-\infty}$ is subject to the single condition of passing through the node of $f_k$, then $f_{k-\infty}f_k$ is not independent; i.e., the multiples of $f_k$ give only one independent term in $f_k$.

There is a certain amount of arbitrariness in the way of putting $(y, z, w)'$ into the form $(y, z, w)' y, z, w) = \frac{1}{2}$, but the various forms of $f'$ that arise differ only by nodal terms.


Let the curve $C_m$ be given parametrically by

$$x : y : z : w = X\Phi_{m-1}(x, y, z) : Y\Phi_{m-1}(x, y, z) : Z\Phi_{m-1}(x, y, z),$$

where

$$\Phi_m(x, y, z) = 0;$$

the last equation is that of a plane projection of $C$. Let $C$ be given also as the common intersection of as many surfaces $g_a$ as may be necessary.

We assume

$$f_m = \sum f_{m-\alpha} g_a,$$

and its tangent plane at any point of $C$ is

$$P_m = \sum \Phi_{m-\alpha} p_a,$$

where $\Phi$ is what $f$ becomes when $xyzw$ are replaced by their values in terms of the parameters of the point, and $p$ is the tangent plane to $g$ at the point. Since these planes $p$ all contain the tangent line to $C$, they are connected by a certain number of linear identities.

Now let $f$ have a node at the point $X = Y = Z$; then at this point the plane $P_m$ is indeterminate, and by using if necessary the identities between the $p$, we can put $P_m$ into the form

$$P_m = \sum \{(X - Y)\Phi_{m-\alpha} + (Y - Z)\Psi_{m-\alpha}\} p_a.$$

Now since $X = Y = Z$ is a point of $C$, it lies on $\Phi_m$; and

$$\Phi_m = (X - Y)\Phi_{m-1} + (Y - Z)\Psi_{m-1}. $$

We suppose that $\Phi_{m-1}$ does not vanish at the point; then

$$\Phi_{m-1}P_m = (Y - Z)\sum \Phi_{m-\alpha} p_{m-\alpha} = (Y - Z)K,$$

say, where

$$\Phi_{m-\alpha} p_{m-\alpha} = \Phi_{m-1} \Psi_{m-\alpha} - \Psi_{m-1} \Phi_{m-\alpha}.$$

Now $K$ can be interpreted as the tangent plane to

$$f_m = \sum \Phi_{m-\alpha} (x, y, z) g_a,$$

of degree

$$x' = (x - a) + m + \alpha - 2,$$

which touches $f$ along $C$ and has a simple point at the node of $f$, and
is therefore linearly independent of \( f \); but if \( f_{\lambda-x'} \) passes through the node, its parametric expression has the form

\[(X - Y)\Phi + (Y - Z)\Psi,\]

and the tangent plane to \( f_{\lambda-x'} \cdot f'_{x}\) is

\[\Phi_{i-1}^{x'-a} \cdot (\Phi_{m-1}^{y} - \Psi_{m-1}^{y} \Phi)^{P_{x}},\]

so that \( f_{\lambda-x'} \cdot f'_{x} \) is the sum of a multiple of \( f \) and nodal terms.


If \( f'_{x} \) touches \( f_{x} \) along \( C \), there must exist an identity

\[(1) \quad f'_{x} \varphi_{a-x'} - f_{x} \varphi^{a-x} = \text{nodal terms},\]

where \( \varphi \) does not contain \( C \).

Now suppose that \( \varphi \) is quite arbitrary; then (1) imposes \((P_{2})_{x} \) conditions upon the \((x^{'}+3) + \frac{(\alpha-x^{'}+3)}{3} \) coefficients of \( f'_{x}, \varphi' \). But since

\[\left(\frac{x^{'}+3}{3}\right) + \left(\frac{\alpha-x^{'}+3}{3}\right) - (P_{2})_{x} = \left(\left(\frac{x^{'}+3}{3}\right) - (P_{2})_{x^{'}}\right) + \left(\frac{\alpha-x^{'}+3}{3}\right) - (P_{2})_{a-x} + (P_{2})_{x^{'}-x},\]

the only solutions are the obvious ones in which \( f', \varphi' \) are the sums of nodal terms and the same multiple of \( f, \varphi \) respectively.

But if \( \varphi \) is specialized, some of the \((P_{2})_{x} \) conditions may fall upon the coefficients of \( f, \varphi \) and not on those of \( f', \varphi' \); in this case there are solutions for \( f' \) other than those just enumerated.

Now (1) shows that \( f' \) must contain \( C \), and that the nodes of \( f \) on \( C \) are nodes on \( f' \varphi \), and, if \( \varphi \) is general, are nodes on \( f' \); i.e., some of the \((P_{2})_{x} \) necessary conditions express that \( f' \) has nodes at these points.

Now subject \( \varphi \) to the single condition of passing through a node \( A \) of \( f \) on \( C \); then one of the necessary conditions is satisfied by virtue of this specialization and does not fall on the coefficients of \( f', \varphi' \), and in this case there is just one independent solution for \( f' \).

We arrive at the same solution \( f' \), whatever be the surface \( \varphi \) through \( A \). For, the same problem, with the same \( f, \varphi \) but with \( x^{'} + \varepsilon, \alpha + \varepsilon \) instead of \( x^{'} , \alpha \), is satisfied by \( f_{x} \cdot f' \) where \( f_{x} \) is arbitrary. But we have seen that there is only one such solution; so \( f_{x} \cdot f' \) ceases to be independent if \( f_{x} \) is subject to a certain single condition, which is clearly that it pass through \( A \), i.e. \( f' \) is the solution if we start with \( f_{x} \) instead of \( \varphi \).

Therefore the existence of the node of \( f_{x} \) at \( A \) reduces by 1 the postulation of contact with \( f_{x} \) along \( C \), whatever be the degree of the family considered.
V. Equivalence.

1. Cayley's method.*)

Let \( f_\lambda, f_\mu, f_\nu \) be three surfaces touching one another along \( C_m \). Let them have \( D_\lambda, D_\mu, D_\nu \) distinct nodes at simple points of \( C \) respectively, and \( d_\lambda \) common nodes at double points of \( C \). Let the residual intersection \( C'_{\lambda,\mu-2m} \) of \( f_\lambda, f_\mu \) meet \( C \) in \( I \) points. These are in general distinct from the nodes; for one of the \( D_\lambda \) points is simple on \( f_\mu \), and through it pass two branches of the total intersection, viz. \( C \) counted twice; and one of the \( d_\lambda \) points is a common node, and through it pass four branches of the intersection, viz. the two branches of \( C \) each counted twice. At each of the \( I \) points, \( C' \) touches \( f_\nu \).

Therefore \( C' \) meets \( f_\nu \) in \( \nu(\lambda \mu - 2m) - 2I \) points not lying on \( C \), which are the free intersections of \( f_\lambda, f_\mu, f_\nu \); the remaining intersections are absorbed by \( C \); the equivalence is

\[
E = \lambda \mu \nu - \{ \nu(\lambda \mu - 2m) - 2I \} = 2\nu m + 2I.
\]

To find \( I \), consider an auxiliary surface \( f_\omega \) containing \( C \) but not touching \( f_\lambda \), and having no singularities on \( C \). It meets \( C' \) in the \( I \) points on \( C \) and in \( \omega(\lambda \mu - 2m) - I \) other points, which are the free intersections of \( f_\lambda, f_\mu, f_\omega \).

Now the residual intersection \( C'_{\lambda,\omega-2m} \) of \( f_\lambda, f_\omega \) passes through the \( D_\lambda \) nodes of \( f_\omega \), but not through the \( d_\lambda \) nodes; let it also meet \( C \) in \( I' \) simple points, at which it touches \( f_\mu \).

Therefore \( C'' \) meets \( f_\mu \) in \( \mu(\lambda \omega - m) - D_\lambda - 2I' \) other points which are again the free intersections of \( f_\lambda, f_\mu, f_\omega \). Equating these two values, we have

\[
I = (\mu - 2\omega)m + D_\lambda + 2I'.
\]

To find \( I' \), consider the locus \( f_\lambda + \rho - 2 \) of points whose polar planes with regard to \( f_\lambda, f_\omega \) meet in a straight line which intersects an arbitrary straight line. This meets \( C \) in \( (\lambda + \rho - 2)m \) points, consisting of the \( I' \) intersections of \( C, C'' \), \( } \) being points of contact of \( f_\lambda, f_\omega \), the \( d_\lambda \) double points of \( C \), the \( D_\lambda + d_\lambda \) nodes of \( f_\lambda \) and the \( \rho \) points of \( C \) at which the tangent line to \( C \) meets the arbitrary line. Therefore:

\[
I' = (\lambda + \rho - 2)m - \rho - D_\lambda - 2d_\lambda - 2d_\omega,
I = (2\lambda + \mu)m - 2\rho - D_\lambda,
E = 2(\lambda + \mu + \nu)m - 4\rho + 2(\lambda m - D_\lambda)
= 2E_1 - 2N,
\]

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where

\[ N = \varphi + D_2 - \lambda m. \]

Since \( E \) must be symmetrical in \( \lambda, \mu, \nu \), this proves again that \( D_2 - \lambda m \) is independent of \( \lambda \).

2. Salmon’s method.*)

The value of \( I \) can also be found as follows.

Let \( C, C' \) have \( H \) apparent intersections:

\[ I + H = (\lambda \mu - 2m)m. \]

To find \( H \), consider the equation \( F' = 0 \) obtained by eliminating \( \varepsilon \) between

\[
\begin{align*}
\varepsilon \mu (x + x', \varepsilon y + y', \varepsilon z + z', \varepsilon w + w') &= f_\lambda (x', y', z', w'), \\
\varepsilon \mu (x + x', \varepsilon y + y', \varepsilon z + z', \varepsilon w + w') &= f_\mu (x', y', z', w').
\end{align*}
\]

It is of degree \( \lambda \mu - 1 \) in \( xyzw \), and of degree \( (\lambda - 1) (\mu - 1) \) in \( x'y'z'w' \).

Let these be the coordinates of two points \( A, A' \). Now if \( A' \) is a common point of \( f_\lambda, f_\mu \), then \( F \), regarded as the locus of \( A \), is the cone vertex \( A' \) standing on the total intersection of \( f_\lambda, f_\mu \); i.e., if \( A' \) is on \( C' \), then \( F \) consists of:

the cone of degree \( m \) standing on \( C \) counted twice and
the cone of degree \( \lambda \mu - 2m - 1 \) standing on \( C' \).

But if \( A' \) is on \( C \), then \( F \) consists of:

the cone of degree \( m - 1 \) standing on \( C \) counted twice,
the cone of degree \( \lambda \mu - 2m \) standing on \( C' \) and
the common tangent plane at \( A' \) to \( f_\lambda, f_\mu \).

Now let \( A \) be fixed, and regard \( F' \) as the locus of \( A' \). It meets \( C \) in \( (\lambda - 1) (\mu - 1)m \) points consisting of:

the \( H \) feet of straight lines through \( A \) meeting \( C \) and \( C' \),
the \( 2h \) feet of straight lines through \( A \) meeting \( C \) twice,
the \( d_2 \) common nodes of \( f_\lambda, f_\mu \) and
the \( n \) points of \( C \) at which the common tangent plane to \( f_\lambda, f_\mu \) passes through \( A \), where as before

\[ n = (\lambda - 1)m - D_2 - 2d_2. \]

Also, since when \( A' \) is on \( C \), \( F \) has a squared factor which vanishes at the \( 2h \) points, these count twice in the intersection of \( F, C \).

Therefore

\[
\begin{align*}
H &= (\lambda - 1) (\mu - 1)m - 4h - 2d_2 - n, \\
I &= (\lambda + \mu)m - 2\varphi - D_2 + \lambda m,
\end{align*}
\]

as before.

VI. Examples and observations.

1. Common nodes.

The family \((f_i)\) has in general no common nodes other than the \(d_2\) nodes at double points of \(C\), but three particular members of the family may have common nodes, which will absorb some of the free intersections and increase the equivalence. If \(D\) of the nodes are common, Cayley's method gives \(2D\) as the additional equivalence.

E.g., let \(C\) be the twisted cubic, and \(f_\infty\) the quadric cone \(g\); then \(d_2 = 0, D_\infty = 1\). Now \(g\) is touched along \(C\) by the cubic family

\[ f_3 = wg'' - zg'' + f_1 g, \]

where \(f_1\) is arbitrary.

This family has four nodes on \(C\) all varying with \(f_1\); and \(D_3 = 4, D = 0\).

Now consider the three quartic surfaces touching \(g\) along \(C\) viz.

\[ @_1 g, @_2 g, (g', g'') \]

where \(r @_1\) are arbitrary.

Here \(D_4 = 7, D = 0, E = 42\), and we verify that the remaining 22 free intersections lie 2 on \(g, \Phi_1\); 8 on \(\Phi_2, \Phi_1\); and 12 on \(\Phi_3, \Phi_3\). But if \(\Phi_1\) passes through the node of \(g\), the three surfaces have a common node at this point, which absorbs the first two of the free intersections; \(D = 1, E = 44\).

2. Maxima and minima of \(P, E\).

If we only know that the surfaces \((f_i)\) touch one another along a given curve \(C_m\), without knowing what fixed surface they touch, then there are a whole series of values possible for \(P, E\) depending on the values of \(D_\infty = zm\), where \(x\) is the degree of any surface \(f_\infty\) containing \(C\), and \(D_\infty\) is the number of nodes of \(f_\infty\) at simple points of \(C\). When \(1\) is given, the maximum of \(P\) is \((P_1)_{1-1}\), when there is only one independent surface \(f_1\) for which \(C\) is not a nodal line; then \(f_\infty \equiv f_1\); and can be the most general surface of degree \(1\) through \(C\), which, if \(1\) is great enough, has no nodes at either simple or double points of \(C\).

The formula for \(P\) does not hold, since \(1 - x = 0\) and \((P_1)_{1-1}\) has no meaning; but instead

\[ P = P_2 - 1 = 3(1 + 2)m - \frac{5}{2} \phi - 1. \]

To find the equivalence in this case we may suppose that \(C\) is nodal on \(f_\infty, f_i\); then their residual intersection meets \(C\) in \(2(\mu + \nu)m - 4\phi\) points, and since now \(1 = \mu = \nu\), we have

\[ E = 8 \lambda m - 4\phi. \]

The minima of \(P, E\) when \(1\) is given are much more complicated, and depend on the nature of the surfaces of lowest degree through \(C\). Since \(zm - D_\infty = m + n + 2d_2\), the absolute minima of \(P, E\) for a given
value of $m$ occur when $n = d = 0$ and $f_x$ is a plane; then $C$ is a plane curve and $g = m(m + 1)$, giving

$$P = 2P_1 - m^2, \quad E = 2E_1 - 2m^2,$$

as for two distinct coplanar curves intersecting in $m^2$ points.

3. *Gaps in the series of values of $P$.*

When $\lambda$, $C_m$ are given, then $\lambda m - D_x$ varies between $\lambda m$ and a fixed minimum, but the intermediate values may not all be possible. E. g., consider the $C_5$ which with a straight line makes up the total intersection of a non-singular quadric $g_5$ and a cubic surface $g_8$ having one node on $C$. No other quadric passes through $C$, and if $x = 2$ then $D_x = 0$; but the general cubic surface through $C$ is

$$l_x = g_8 + f_1g_2,$$

for which $D_x = 0$ or 1 only.

Thus $\lambda m - D_x$ cannot lie between 10 and 14.

This is connected with the fact that if two distinct adjacent curves of this nature lie on the same quadric, they meet in 12 points; if they lie on the same cubic surface, they are residual to the same straight line $C_1'$, and meet in 8 or 7 points according as the surface has or has not a node on $C_1'$; but the number of their intersections cannot lie between 12 and 8.

4. *Application to twisted curves.*

Concerning the residual $C'_m - 2m$, we know that it meets $C$ in $I$ points, and the number $h'$ of its apparent double points is given by Salmon's formula (l. c., p. 311), provided we regard $C$ as a pair of curves meeting in $N$ points, i. e., as a curve of order $2m$ with $2h + m^2 - N$ apparent double points, and this holds even when $N$ is negative. E. g., let two cubic surfaces touch along a straight line;

$$I = 5, \quad N = -1, \quad h' = 12,$$

and the residual is a septic of genus 3 with one 5-secant.)*

Cambridge, January 1912.