THE CREMONA TRANSFORMATIONS OF A CERTAIN PLANE
SEXTIC

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For references see: Sturm, Die Lehre von den geometrischen Verwandtschaften, Leipzig, 1909.

I. Degenerate Homaloids.

A plane Cremona transformation of degree \( n \) transforms the straight lines of the first plane into a homaloidal family in the second plane; that is, into a doubly infinite family of rational curves of degree \( n \), such that two general curves of the family meet in one and only one variable point of intersection. The remaining intersections are all accounted for by the base points common to the whole family. The straight lines of the second plane are transformed into a homaloidal family in the first plane, also with a set of base points.

To every point \( p \) in the first plane there corresponds a unique point \( P \) in the second plane: except that to a base point \( a \) there corresponds not a single point but a curve \( J \). The set of curves \( J \) corresponding to the whole set of base points, taken with proper multiplicities, constitute the Jacobian of the second homaloidal family.

To a straight line \( l \) passing through \( a \) there corresponds a homaloid which breaks up into \( J \), corresponding to the single point \( a \), and a rational curve \( \phi \) of degree less than \( n \), corresponding to all the rest of \( l \). Then \( J \) meets \( \phi \) in one point other than base points; \( J \) is exactly determined by its passages through base points, and \( \phi \) has one degree of freedom, corresponding to the one degree of freedom of the straight line \( l \) through \( a \). We have thus a singly infinite family \( (\phi) \) of rational curves, which are transformed into straight lines by a Cremona transformation of degree higher than that of \( \phi \).

If \( l \) joins the base points \( a_1, a_2 \), the corresponding homaloid breaks up into \( J_1, J_2, \phi \), where \( J_1, J_2 \) are Jacobian curves, and \( \phi \) corresponds properly to \( l \). Each of \( J_1, J_2 \) meets \( \phi \) in one point other than base points, and \( \phi \) is exactly determined by its passages through base points. If three or more of the base points \( a \) lie on a straight line \( l \), to this there corresponds...
properly a rational curve $\phi$, which is more than determined by its passages through base points, which means that the conditions presented to it, by the required multiplicities at the base points, are not independent: but are related to one another in the same way as are the conditions presented to the straight line $l$ by the collinear points $a$.

Now, if we are given the rational curve $\phi$, determined or over-determined by the conditions of having certain base points of assigned multiplicities, the question arises whether we can always regard it as the effective part of a degenerate homaloid of higher degree. The answer is negative, and in the third part of this paper an example is given of a rational curve which cannot be transformed into a straight line by a Cremona transformation of however high degree.

Assume that $\phi$ is the effective part of a degenerate homaloid of degree $n+\nu$, the remaining part consisting of an aggregate $J_\nu$ of degree $\nu$ of $k$ curves $J$ belonging to the Jacobian of the homaloidal family. Then $J_\nu$ has no multiple points except the base points of the family, and no two curves $J$ intersect elsewhere; each $J$ meets $\phi$ at one other point. Let $\Sigma$ extend to all the base points of the family, including all the assigned multiple points of $\phi$ and possibly other points. Let $\phi$, $J_\nu$ have $r$, $p$ branches respectively through a specimen base point; for the additional points introduced into $\Sigma$, the value of $r$ is 0 or 1. There are a series of geometrical facts expressed by equations between these numbers.

$\phi$ is rational: $\Sigma \frac{1}{2} r (r-1) = \frac{1}{2} (n-1)(n-2)$.

$\phi$ can be transformed into a straight line $l$ passing through the $k$ base points in the other plane which correspond to the $k$ curves $J$; if all the conditions, presented to $\phi$ by its assigned multiple points, were independent, it would have $2-k$ degrees of freedom. We shall say that the apparent freedom is $2-k$; if $k>2$, the conditions cannot be independent:

$\Sigma \frac{1}{2} r (r+1) = \frac{1}{2} n(n+3)-2+k$.

The multiplicity $r+p$ of $\phi$. $J_\nu$ at each base point is the same as that of a homaloidal family of degree $n+\nu$; this family is rational:

$\Sigma \frac{1}{2} (r+p)(r+p-1) = \frac{1}{2} (n+\nu-1)(n+\nu-2)$,

and has two degrees of freedom:

$\Sigma \frac{1}{2} (r+p)(r+p+1) = \frac{1}{2} (n+\nu)(n+\nu+3)-2$.

$J_\nu$ is an aggregate of $k$ rational curves, and is therefore of genus

\[
J_\nu
\]
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\( -(k-1) \); all its singularities occur in \( \Sigma \):

\[ \Sigma \frac{1}{2} \rho (\rho -1) = \frac{1}{2}(\nu-1)(\nu-2)-1+k; \]

and \( J \) is exactly determined:

\[ \Sigma \frac{1}{2} \rho (\rho +1) = \frac{3}{2} \nu(\nu +3). \]

All but \( k \) of the \( n \nu \) intersections of \( J, J \) fall at base points:

\[ \Sigma \rho = n\nu -k. \]

These are equivalent to five independent equations:

\[ \Sigma \rho^2 = n^2 -1+k, \quad \Sigma \rho = \nu^2 +k, \]
\[ \Sigma \varrho = 3n -3+k, \quad \Sigma \rho = 3\nu -k, \]
\[ \Sigma \rho = n\nu -k. \]

Now in the case studied below, of a sextic with ten double points,

\( n = 6, \quad r = 2, 1 \text{ or } 0, \quad \Sigma \rho \leq 2\Sigma \rho, \)
\[ 6\nu -k \leq 2(3\nu -k), \quad k \leq 0, \]

which is impossible if \( k \) is positive. It will be shewn that this curve actually exists; and it cannot be transformed into a straight line by any plane Cremona transformation whatever.

It can, of course, be mapped on a straight line by a Riemann transformation; for example, a quartic curve of the singly infinite family, determined by the ten double points and three fixed simple points of the sextic, meets the sextic in one variable point \( P \), whose coordinates are therefore rational functions of the parameter of the family, which may be taken as the coordinate, on a straight line, of a variable point \( P' \) corresponding to \( P \).

The sextic can also be transformed into a curve of lower degree by a Cremona space transformation; the sextic is the projection of a twisted sextic with one actual double point, which is the residual intersection of a sextic cone with ten double edges, standing on the plane curve, and a cubic monoid, with the same vertex as the cone, passing through nine of these edges. The twisted sextic is projected from its double point into a plane quartic. This succession of projections can be brought about by a single Cremona space transformation, which does not, however, transform the rest of the plane of the sextic into the plane of the quartic.
II. Extension of Noether's Theorem.

Noether* has proved that every Cremona transformation is the result of compounding a series of quadratic transformations; because the sum of the three highest multiplicities of a homaloidal family at base points is always greater than the degree of the family, and a quadratic transformation, with these three points as fundamental triad, lowers the degree of the family. So instead of seeking a Cremona transformation of higher degree, which transforms \(f\) into a straight line, we may seek a quadratic transformation which lowers the degree of the curve.

Noether's proof rests on a manipulation of the equations which express that a homaloidal family is rational:

\[
\Sigma \frac{1}{2}r(r-1) = \frac{1}{2}(n-1)(n-2),
\]
and that it has two degrees of freedom:

\[
\Sigma \frac{1}{2}r(r+1) = \frac{1}{2}n(n+3) - 2,
\]
or the equivalent pair

\[
\Sigma r^2 = n^2 - 1, \quad \Sigma r = 3n - 3,
\]
where \(n\) is the degree of the family, \(\Sigma\) extends to all its base points, and \(r\) is the multiplicity of one of them.

Assume that there are at least three multiple base points; the other cases are trivial, and require separate treatment. Let \(x, y, z\) be the three highest values of \(r\), and let \(\Sigma'\) denote a summation from which these three points are excluded. Let

\[
x = z + \lambda, \quad y = z + \mu;
\]

then

\[
\lambda \geq \mu \geq 0, \quad z \geq r, \quad z \geq 1,
\]

\[
\Sigma' r^2 = n^2 - 1 - x^2 - y^2 - z^2,
\]

\[
\Sigma' r = 3n - 3 - x - y - z.
\]

Multiplying the second of these by \(z\) and subtracting the first from it,

\[
\Sigma' r(z-r) = z(3n-3-x-y-z) - (n^2 - 1 - x^2 - y^2 - z^2),
\]
which can be put in the form, obtained by Prof. Baker in a more general

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* Math. Annalen, Bd. 3, p. 167; Bd. 5, p. 635.
case,

\[(x + y + z - n)(n + \lambda + \mu) = \Sigma' r(z - r) + 2z(\lambda + \mu) + 2\lambda \mu + 3(z - 1) + 2.\]

Now \(n + \lambda + \mu, z - 1, 2\) are positive; \(r, z - r, \lambda, \mu\) are positive or zero; the right-hand side of the last equation is positive, and

\[x + y + z > n.\]

We generalize this as follows: for any curve \(\phi\) of degree \(n\) having assigned multiplicities \(r\) at the points included in \(\Sigma\), let

\[\Sigma \frac{1}{2} r(r - 1) = \frac{1}{2}(n - 1)(n - 2) - p,\]

\[\Sigma \frac{1}{2} r(r + 1) = \frac{1}{2}n(n + 3) - f,\]

whence

\[\Sigma r^2 = n^2 + 1 - p - f = n^2 - i \text{ say},\]

\[\Sigma r = 3n - 1 + p - f.\]

Then \(p\) is the genus of \(\phi\); it cannot be negative if \(\phi\) is a proper curve; \(f\) is the apparent freedom of the family having the same multiple points as \(\phi\); if the conditions which these points present are not independent, \(f\) can be negative. If there are at least two curves of the family, \(i = f + p - 1\) is the number of their intersections which do not fall at base points; it can be negative if there is only one curve \(\phi\) having the assigned multiple points.

As before, let \(x, y, z\) be the three highest values of \(r\), and let these three points be excluded from \(\Sigma'\). We find the relation

\[(x + y + z - n)(n + \lambda + \mu) = \Sigma' r(z - r) + 2z(\lambda + \mu) + 2\lambda \mu + (f - p + 1)(z - 1) + 2 - 2p. \quad (I)\]

The conclusion \(x + y + z > n\) follows in a great variety of cases besides that of the homaloidal family, \(p = 0, f = 2\), just considered. It applies to:

- any family of curves of genus 1, \(p = 1, f \geq 1\);
- any family of rational curves, \(p = 0, f = 1 \text{ or } 2\);
- any rational curve exactly determined, \(p = 0, f = 0\);
- any rational curve for which one of the conditions is a consequence of the others, so that the apparent freedom is \(-1\), \(p = 0, f = -1\).
It also applies, but not so obviously, when \( p = 0, f = -2 \). The right-hand side of (I) is then

\[
\Sigma' r(z-r) + z(2\lambda + 2\mu - 1) + 2\lambda \mu + 3,
\]

and is certainly positive unless \( \lambda = \mu = 0, z = x = y \). Now \( \Sigma' \) extends to multiple points for which \( z > r > 2 \). The terms for which \( r = z \) are each zero; let \( \beta \) be the number of effective terms, for which \( r < z \); then \( \beta \geq 0 \), and since

\[
r(z-r) = (r-1)(z-r-1)+(z-1),
\]

therefore

\[
\Sigma' r(z-r) \geq \beta(z-1),
\]

and the right-hand side of (I)

\[
> (\beta-1)(z-1)+2,
\]

and is positive unless \( \beta = 0, z \geq 3 \). Then \( r = z = x = y \), and all the multiple points are of the same order \( z \); let their number be \( a \); the original equations give

\[
az^2 = n^2+3, \quad az = 3n+1;
\]

therefore

\[
z = \frac{n^2+3}{3n+1}, \quad z-3 = \frac{n(n-9)}{3n+1}.
\]

Since \( n \) and \( 3n+1 \) have no common factor, \( 3n+1 \) is a factor of \( |n-9| \), which could only be if either

(i) \( n = 1 \) or \( 2, \quad z = 1 \),

which is impossible since the base points are multiple, or

(ii) \( n = 9, \quad z = 3 \),

but then \( a \) is not integral.

If \( p = 0, f = -3 \), the right-hand side of (I) is

\[
\Sigma' r(z-r) + 2z(\lambda + \mu - 1) + 2\lambda \mu + 4;
\]

if this is not positive, \( \lambda = \mu = 0 \), and it becomes

\[
\Sigma' r(z-r) - 2z + 4 \geq (\beta-2)(z-1)+2,
\]

which is positive unless either

(i) \( \beta = 0, \quad x = y = z = r \geq 2 \),

or

(ii) \( \beta = 1, \quad x = y = z \quad \geq 3 \).
(i) If $\beta = 0$, as before,

$$Z = \frac{n^2 + 4}{3n + 2}, \quad Z = \frac{n(n - 6)}{3n + 2},$$

and since $n$, $3n + 2$ can only have the factor 2 in common, $3n + 2$ is a factor of $2 | n - 6 |$. This can only be if $n = 1, 2$ (which do not apply) or 6; hence $n = 6, z = 2, a = 10$, and we are led again to the sextic with ten double points.

(ii) If $\beta = 1$, there are $a - 1 (> 3)$ base points of multiplicity $z (> 3)$, and one of lower multiplicity $r (> 2)$, then

$$(a - 1)z^2 + r^2 = n^2 + 4 \geq 31, \quad n \geq 6, \quad r^2 \leq n^2 - 23,$$

therefore

$$z = \frac{n^2 + 4 - r^2}{3n + 2 - r}.$$

If $n = 6, r = 2$ or 3

7, 2  5
8, 2  6
9, 2  7
10, 2  8

and the only case in which $z$ is an integer above 2 is

$n = 9, r = 2, z = 3, a = 10$.

If $f = -3$, then whatever the genus,

$$\Sigma \frac{1}{r}(r + 1) = \frac{1}{2}n(n + 3) + 3$$

$$= 3a_2 + 6a_3 + 10a_4 + 15a_5 + 21a_6 + 28a_7 + \ldots,$$

where $a_r$ is the number of base points of multiplicity $r$. If, as before,

$x + y + z \leq n$, then $n \geq 6$, since $z \geq 2$.

If $n = 6, x = y = z = 2, a_2 = 10$, the case already referred to.

If $n = 7$ or 8, $r \leq 7$, and casting out the threes, $a_1 + a_7 \equiv 2 \pmod{3}$; there are at least two fourfold or higher points, and $x + y + z > 8$.

Thus the rational sextic with ten double points is the simplest curve, as regards genus, degree and freedom, which cannot be transformed into a curve of lower degree by any plane Cremona transformation. The next curve is of degree 9 at least.

III. The Sextic with Ten Double Points.

The total intersection of a quadric and a cubic surface is a twisted sextic curve with six apparent double points. If the two surfaces touch
at four points the sextic has, in addition, four actual double points, and its projection from any point of space on to any plane is a plane sextic with ten double points. This way of generating the curve was suggested by Mr. Richmond.

Take homogeneous coordinates $x_1x_2x_3x_4$, the four points of contact $A_1A_2A_3A_4$ being the corners of the tetrahedron of reference. The general quadric through these points is

$$Q \equiv a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 + a_4'x_1x_4 + a_5'x_2x_4 + a_6'x_3x_4 = 0,$$

or, say, $$2Q \equiv x_1u_1 + x_2u_2 + x_3u_3 + x_4u_4,$$

where $$u_1 \equiv a_3x_2 + a_2x_3 + a_1'x_4, \ldots, u_4 = a_1'x_1 + a_2'x_2 + a_3'x_3,$$

and $u_1 \ldots u_4$ are the tangent planes to $Q$ at $A_1 \ldots A_4$.

The general cubic surface touching the same planes at the same points is

$$C' \equiv c_1x_1^2u_1 + c_2x_2^2u_2 + c_3x_3^2u_3 + c_4x_4^2u_4 + b_1x_2x_3x_4 + b_2x_1x_3x_4 + b_3x_1x_2x_4 + b_4x_1x_2x_3 = 0,$$

and the total intersection is a twisted sextic $S$ with four actual double points $A_1 \ldots A_4$.

(i) $S$ is in general a proper sextic. It also lies on the surface

$$C' - Q(c_1x_1 + \ldots + c_4x_4) \equiv b_1x_2x_3x_4 + \ldots + b_4x_1x_2x_3 = 0,$$

where $b_1' \ldots b_4'$ are other constants, and its projection from $A_4'$ on the plane $A_1A_2A_3$ is given by eliminating $x_4$ between (1) and (3). We find

$$(b_1'x_2x_3 + b_2'x_3x_1 + b_3'x_1x_2)(a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2) - b_4'x_1x_2x_3u_4 = 0,$$

which is a proper plane quartic if $a_1 \ldots b_4'$ have general values. Hence, if $S$ broke up, it could only be into one or two straight lines through $A_4'$ and a proper residual of degree 5 or 4; but by the same argument, the straight lines would have to pass through each of $A_1 \ldots A_4$, which is impossible; therefore $S$ does not break up.

(ii) $S$ has six apparent double points. Take any two points $Y(y_1, y_2, y_3, y_4), Z$, of space, and express the condition that the straight line $YZ$ shall meet $S$ in $X$; then $X$ lies upon $Q$ and $C'$. We may put $x = \lambda y + \mu z$ for all suffixes; let the result of substitution be

$$\lambda^3Q_{11} + \lambda\mu Q_{12} + \mu^2Q_{22} = 0,$$

$$\lambda^3C_{11} + \lambda^2\mu C_{12} + \lambda\mu^2C_{22} + \mu^3C_{22} = 0.$$
Eliminate \( \lambda : \mu \); this gives the equation

\[
F = 0, \quad 0, \quad Q_{11}, \quad Q_{12}, \quad Q_{22} = 0,
\]

of degree 6 in \( y \), 6 in \( z \). If \( Y \) is fixed, this equation in \( z \) represents the sextic cone projecting \( S \) from \( Y \) and vice versa.

If \( Y \) is a point of \( S \), then \( Q_{11} = C_{111} = 0 \): we can divide by \( \mu \) before eliminating, and obtain

\[
K' = 0, \quad Q_{12}, \quad Q_{22} = 0,
\]

of degree 2 in \( y \), 5 in \( z \). If \( Y \) is fixed, this equation in \( z \) represents the quintic cone projecting \( S \) from \( Y \). If \( Z \) is fixed, this equation in \( y \) represents a quadric, meeting \( S \) in twelve points \( Y \), which are such that each of the cones projecting \( S \) from these points passes through \( Z \); in other words, such that each of the straight lines \( YZ \) meets \( S \) again, in another of the twelve points of intersection of the quadric \( K'(y) \) with \( S \). These twelve points therefore fall into six pairs, each collinear with \( Z \); as viewed from the arbitrary point \( Z \), the curve \( S \) has six apparent double points.

(iii) The six apparent double points of \( S \) are in addition to its four actual double points. If the coordinates \((1, 0, 0, 0)\) of \( A_1 \) are substituted for \( y \),

\[
Q_{12} \text{ becomes } a_2 x_3 + a_3 z_3 + a_4 z_4 \equiv u_z \text{ say,}
\]

\[
C'_{112} \equiv c_1 u_z,
\]

\[
C'_{122} \equiv 2c_1 z_1 u_z + c_3 a_3 x_3^2 + c_3 a_4 x_3^2 + b_3 z_3 z_4 + b_4 z_2 z_4 + b_4 z_2 z_3,
\]

and \( K' \) breaks up into \( u_z \) and another factor, but does not vanish. Hence \( K'(y) \), which contains the ends of the six chords of \( S \) through \( Z \), does not contain the four actual double points, which are therefore in addition to the apparent double points.
(iv) The six chords of $S$ through any point lie on a quadric cone. If we put $y = z$, $Q_{12}$ becomes $2Q_{22}$, and $C_{112}$, $C_{122}$ each become $3C_{222}$; then

$$K' = Q_{22}^2 C_{222} \begin{bmatrix} 0, & 2, & 1 \\ 2, & 1, & 0 \\ 3, & 3, & 1 \end{bmatrix} = -Q_{22}^2 C_{222},$$

which does not vanish; $K'(y)$ does not in general pass through $Z$, and the six chords of $S$ are not generators of $K'(y)$, unless $Z$ lies either on $Q$ or on $C'$.

This investigation holds if instead of $C'$ we take $C = C' + Q \cdot P$, where $P$ is any plane, for $C$ is another cubic surface through $S$, and touches the same planes as $C$ at $A_1 \ldots A_4$. We have thus a whole family of quadrics ($K$) such as $K'$ through the same twelve points of $S$, which is, in fact, the family $K' + \lambda Q$. In particular, if $P$ is chosen so that $C$ passes through $Z$, we have $C_{222} = 0$, and $K$ vanishes when we put $y = z$. This quadric $K$ passes through $Z$, meets each of the six chords of $S$ through $Z$ in three points, and therefore contains them altogether. Hence $K$ is a quadric cone on which the six chords lie.

Since now $C_{222} = 0$, we have identically

$$K = Q_{22} Q_{12} \begin{bmatrix} Q_{22}, & Q_{22} \\ C_{112}, & C_{122} \end{bmatrix} = Q_{22} K_2, \text{ say},$$

$$F = Q_{22} \begin{bmatrix} 0, & Q_{11}, & Q_{12}, & Q_{22} \\ Q_{11}, & Q_{12}, & Q_{22}, & 0 \\ 0, & C_{111}, & C_{112}, & C_{122} \\ C_{111}, & C_{112}, & C_{122}, & 0 \end{bmatrix} = Q_{22} F_6, \text{ say},$$

where

$$F_6 = K_2 (Q_{12} C_{111} - Q_{11} C_{112}) + (Q_{11} C_{122} - Q_{22} C_{111})^3 = K_2 K_4 + K_3^2, \text{ say},$$

the suffixes shewing the degrees in $y$. Here $F_6$, $K_2$ are cones containing the six chords of $S$ through $Z$, which are double edges on $F_6$; this identity shews that they also lie on the cubic surface $K_3$. They are therefore double on $K_2^2$ as well as on $F_6$; therefore, by the same identity, they are double on $K_2 \cdot K_4$. Since they are simple but not double edges on $K_2$, they lie on $K_4$.

The four actual double points $A_1 \ldots A_4$ of $S$ are double points on $F_6$; they lie on $Q$, $C$, and their coordinates make $Q_{11}, C_{111}$ vanish; they therefore lie on $K_3, K_4$, but not on $K_2$. They are double on $F_6 - K_3^2$, and therefore double on $K_2 \cdot K_4$; since they do not lie on $K_2$, they are double on $K_4$. 
The point $Z$ lies on $C$ but not on $Q$. It is six-fold on $F_6$, and double on $K_2$; the substitution of $z$ for $y$ make $C_{111}$, $C_{112}$, $C_{222}$ all vanish, so $Z$ lies on $K_3$ and $K_4$. It is therefore triple at least on $F_6-K_2, K_4$, and therefore triple on $K_3^2$; this can only be if it is double on $K_3$. It is then fourfold on $F_6-K_3^2$, and therefore fourfold on $K_3.K_4$; since it is double on $K_2$, it is double on $K_4$.

These multiplicities can be set out in a table:

<table>
<thead>
<tr>
<th>Surface</th>
<th>The Six Chords</th>
<th>$A_1 ... A_4$</th>
<th>$Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_6$</td>
<td>2</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>$K_2$</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$K_3$</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$K_4$</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Take the sections of these four surfaces by the plane $A_1A_2A_3$, by putting $y_4 = 0$. Let the six chords meet the plane in $B_1 ... B_6$, and let $A_4 Z$ meet it in $A_4$. The four plane curves are:

- a sextic $f_6$ with ten double points at $A_1, A_2, A_3; A_4; B_1 ... B_6$; this will be indicated by $f_6(A_1 ... A_4 B_1 ... B_6)^3$;
- a conic $k_2(B_1 ... B_6)^1$;
- a cubic $k_3(A_1A_2A_3B_1 ... B_6)^1$;
- a quartic $k_4(A_1A_2A_3)^2 (B_1 ... B_6)^1$;

with the relation $f_6 \equiv k_2 + k_3^2$.

The existence of $f_6$ presents three conditions to the ten points $A_1 ... B_6$; $k_2$ gives a single condition, among $B_1 ... B_6$: we shall call it a condition curve; $k_3$ gives no condition, it is exactly determined by its nine simple points; $k_4$ is another condition curve, for it is exactly determined by its three double points $A_1A_2A_3$ and five simple points $B_1 ... B_5$: there is therefore one condition that $B_6$ should lie on this quartic.

There exists another cubic $k_3'$ determined by passing through the nine points $A_2A_3A_4B_1 ... B_6$. Then $f_6+\lambda k_2^3$ is a family of sextics $k_6(A_2A_3A_4B_1 ... B_6)^3$, and each meets $k_2$ in $B_1 ... B_6$ counted twice and no other point. If we use $\lambda$ to make $k_6$ pass through another point of $k_2$, it must break up into $k_2$, which passes through $B_1 ... B_6$ once and not through $A_2A_3A_4$, and a quartic $k_4'(A_2A_3A_4)^3 (B_1 ... B_6)^1$. Similarly there exist two other condition quartics $k_4''(A_1A_2A_3)^3 (B_1 ... B_6)^1$ and $k_4'''(A_1A_2A_3)^3 (B_1 ... B_6)^1$: the point $A_4$ is on exactly the same footing as $A_1A_2A_3$. 
By employing other exactly determined curves instead of these cubics, we can prove that there exist an unlimited number of condition curves of higher degree determined by the ten double points of the sextic: see below. Each of these curves, whose existence gives a single condition, is of apparent freedom — 1.

To obtain the actual equation of the plane sextic, put \( y_4 = 0 \) in the determinant \( F \). In order to simplify the expressions, take the coordinates of \( Z \) to be \((1, 1, 1, 1)\); there is no loss of generality, as \( Q \), \( C \) have no metrical properties. Also when we replace \( C' \) by \( C \equiv C' + Q \cdot P \), we can use the four coefficients of \( P \) to make the new cubic surface not only pass through \( Z \), but also have double points at \( A_1 \ldots A_3 \). The effect of this is the same as if in \( C' \) we assume

\[
c_1 = c_2 = c_3 = 0, \quad c_4 (a_1' + a_2' + a_3') + b_1 + b_2 + b_3 + b_4 = 0,
\]

and the plane curve is

\[
f_6 = c_1 c_2 c_3 c_4 (a_1' + a_2' + a_3') + b_1 + b_2 + b_3 + b_4 = 0,
\]

where

\[
q_{11} = a_1 y_2 y_3 + a_2 y_3 y_1 + a_3 y_1 y_2,
q_{12} = a_1 y_1 (a_3 + a_2 + a_1') + y_2 (a_3 + a_1 + a_2') + y_3 (a_1 + a_2 + a_3'),
q_{22} = a_1 + a_2 + a_3 + a_1' + a_2' + a_3',
c_{111} = y_2 y_3 (b_1 + b_2) + y_3 y_1 (b_2 + b_3) + y_1 y_2 (b_3 + b_4),
c_{112} = y_2 (c_1 a_1 + b_2 + b_3 + b_4) + y_3 (c_2 a_2 + b_3 + b_1 + b_4) + y_1 (c_3 a_3 + b_1 + b_2 + b_3).
\]

The equation of \( f_6 \) contains the eleven constants \( a_1 a_2 a_3 a_1' a_2' a_3' b_1 b_2 b_3 b_4 c_4 \), subject to one condition; as the first ten enter homogeneously, there are nine independent parameters. In the plane, we have fixed the coordinate system by taking \( A_1 A_2 A_3 \) as the triangle of reference and assigning the coordinates \((1, 1, 1)\) to \( A_4 \). The twelve coordinates of the remaining six double points \( B_1 \ldots B_6 \) are functions of the nine parameters, and are therefore connected by three independent relations. It is probable that the existence of three condition curves, for example \( k_2 \), \( k_4 \), \( k_4' \), ensures the existence of \( f_6 \); but I have no proof that this is so.
IV. Transformations of the Sextic.

A quadratic transformation, with any three of the ten double points \( A, B \) as fundamental triad, transforms \( f_6 \) into another plane sextic with ten double points, but these may be differently grouped. If the fundamental triad is \( B_4B_5B_6 \),

\[
\begin{align*}
    f_6(A_1 \ldots A_4 B_1 \ldots B_6)^2 & \quad \text{becomes} \quad h_6(A_1 \ldots A_4 B_1 \ldots B_6)^2, \\
    k_2(B_1 \ldots B_6)^3 & \quad \text{becomes} \quad h_1(B_1 B_2 B_3)^3, \\
    k_3(A_1 A_2 A_3 B_1 \ldots B_6)^3 & \quad \text{becomes} \quad h_3(A_1 A_2 A_3 B_1 \ldots B_6)^3, \\
    k_4(A_1 A_2 A_3)^2(B_1 \ldots B_6)^3 & \quad \text{becomes} \quad h_5(A_1 A_2 A_3 B_1 B_2 B_3 B_6)^3(B_1 B_2 B_3)^3,
\end{align*}
\]

and for these new curves there is the identity \( h_6 \equiv h_1 h_5 + h_3^2 \). Apparent freedom is not altered by the transformation; \( h_3 \) is exactly determined, and \( h_1, h_5 \) are condition curves. The ten double points of \( h_6 \) are grouped into three collinear points \( B_1B_2B_3 \), and seven others, instead of into six points on a conic, and four others.

There are seven cubics such as \( h_3 \), each exactly determined by passing through the three points of the first group and six out of the seven points of the second group; four of these cubics arise by transformation from the four curves such as \( k_3 \); the other three, such as \( k_3'(A_1 \ldots A_4 B_1 \ldots B_6)^3 \), arise from the exactly determined quartics such as

\[
k_4(B_4 B_5)^3(A_1 \ldots A_4 B_1 B_2 B_3 B_6)^3.
\]

As above, a discussion of the family \( h_6 + \lambda h_3^2 \) proves that one of its members breaks up into \( h_1 \) and \( h_5 \); and so we can prove the existence of the four condition curves such as \( h_3 \); in just the same way, a discussion of \( h_6 + \lambda h_3^2 \) proves the existence of three condition curves \( h_5 \), passing once through \( B_1B_2B_3 \), and twice through the remaining sets of six points out of the seven of the second group. \( h_5'(A_1 \ldots A_4 B_4 B_5)^3(B_1 B_2 B_3)^3 \) arises from \( k_6'(B_4 B_5)^3(A_1 \ldots A_4)^3(B_1 B_2 B_3 B_6)^1 \), which is a condition curve connected with the original sextic, and whose existence could have been proved independently. Since \( k_6' \) divides the six \( B \)'s into two sets of two and four points, there are fifteen such curves; the remaining twelve give by transformation other condition curves, connected with the second type of sextic which has three collinear double points. These two principles, of transformation and of symmetry, lead to an infinite number of condition curves connected with either of the two types of sextic with ten double points.

Instead of the triad \( BBB \) in the auxiliary quadratic transformation, we can use a triad \( ABB, AAB, \) or \( AAA \), and we are led to other types of
sextics, each with ten double points grouped in some way, and a series of condition curves connected with each. In particular, a transformation using $A_1A_2B_1$, followed by another using $A_3A_4B_1$ (equivalent to a single cubic transformation) transforms the original $k_3(B_1 \ldots B_9)^1$ into a $k_4(B_1 \ldots A_4B_9)^1$, and transforms the original $k_5$'s into other quartics with triple points; there is now no grouping of the ten double points, which are all on the same footing for this type of sextic.

If we apply quadratic transformations systematically, using every possible variety of triads, to every type of sextic that arises, we obtain five different types and no more, given by the following table.

<table>
<thead>
<tr>
<th>Reference Number</th>
<th>Grouping of Double Points</th>
<th>Simplest Condition Curves</th>
<th>Some other Condition Curves</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A_1A_2; B_1 \ldots B_9$</td>
<td>$k_6(A_1A_2)^1$</td>
<td>$k_6((A_1 A_2)(B_1 \ldots B_7)^3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$k_6(B_1B_2)^3(B_3 \ldots B_9)^2$</td>
</tr>
<tr>
<td>2</td>
<td>$A_1A_2A_3; B_1 \ldots B_7$</td>
<td>$k_1(A_1A_2A_3)^1$</td>
<td>$k_6(B_1 \ldots B_9)^2(A_1A_2A_3)^1$</td>
</tr>
<tr>
<td>3</td>
<td>$A_1 \ldots A_4; B_1 \ldots B_6$</td>
<td>$k_2(B_1 \ldots B_9)^1$</td>
<td>$k_6(A_1A_2A_3)^3(B_1 \ldots B_9)^1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$k_6(B_1B_2)^3(A_1 \ldots A_4)^3(B_3 \ldots B_9)^1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$k_6(A_1)^4(B_1 \ldots B_9)^3(A_2A_3A_4)^2(B_1B_9)^1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$k_{10}(B_1)^2(A_1A_2)^4(B_2 \ldots B_9)^3(A_3A_4)^3(B_9)^1$</td>
</tr>
<tr>
<td>4</td>
<td>$A_1A_2A_3; B_1 \ldots B_7$</td>
<td>$k_3(A_1)^3(B_1 \ldots B_7)^1$</td>
<td>$k_6(B_1)^3(A_1A_2A_3)^2(B_2 \ldots B_9)^1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(three such)</td>
</tr>
<tr>
<td>5</td>
<td>$A_1 \ldots A_{10}$</td>
<td>$k_4(A_1)^4(A_7 \ldots A_{10})^1$</td>
<td>$k_6(B_1)^3(A_1A_2A_3)^2(B_2 \ldots B_9)^1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(ten such)</td>
</tr>
</tbody>
</table>

In the first type, what is for the sake of uniformity called $k_9$, a curve of zero degree, is a single point. Two of the double points, $A_1A_2$, coincide, and the sextic has a tacnode there; we may think of $A_2$ as the point adjacent to $A_1$ along the tacnodal tangent. Each of the condition curves $k_6$, $k_8$ has three branches through $A_1$, one of which passes through $A_2$, that is, touches the tacnodal tangent.

The existence of any of the condition curves in the last column can be proved, if we assume the sextic and any other condition curve of the same row, by considering a suitable curve exactly determined, corresponding to $k_9$ in the original discussion, and obtained from it by some series of transformations. For example, in the original case (3), to prove the existence of $k_6$ we assume $k_3$ and consider the exactly determined quartic $k_4(B_1B_2)^3(A_1 \ldots A_4B_9 \ldots B_9)^1$. Then $f_6\cdot[k_1(B_1B_2)^1]^3 + \lambda k_4^2$ is a singly
infinite family of octavics $k_8(B_1B_2)^4(A_1 \ldots A_4B_3 \ldots B_9)^2$, each meeting $k_3$ in the equivalent of sixteen points at $B_1 \ldots B_9$, and in no other point. One member of the family therefore breaks up into $k_9$ and another condition curve $k_6(B_1B_2)^3(A_1 \ldots A_4)^2(B_3 \ldots B_9)^1$. All the properties of any one type follow from the existence of any one of its condition curves; if there exists a condition quartic with three double points, the type is (3), and so on.

In order to prove that a certain quadratic transformation changes a certain type into a certain other type, we only have to prove that it changes the simplest condition curve of the first type into some condition curve of the second.

Applied to type (1), a quadratic transformation using $B_1B_2B_3$ does not alter the type; using $A_1B_1B_2$, it gives type (2), from which (1) was originally derived. A specialized transformation, with two fundamental points coinciding at $A_1A_2$, does not alter the type.

Whatever the triad, the $k_1$ and $k_2$ of types (2) and (3), and suitable ones of the $k_3$'s of type (4), and of the $k_4$'s of type (5), are all transformed into rational curves of degree $\leqslant 4$. If the degree is 0, 1, 2, or 3, the type obtained is (1), (2), (3), or (4); while the two kinds of rational quartics, with three double points or one triple point, give types (3) and (5) respectively.

All these condition curves are rational; the only other curves of degree 6 or less, of any genus determined or over-determined by ten or fewer base points, are such that if we identified the base points with the double points of $f_6$, the number of intersections with $f_6$ would exceed the maximum, and $f_6$ would either break up or coincide with the condition curve; as, for example, $k_3(10)^1$ or $k_6(9)^2(1)^1$.

The curve $k_6(1)^3(7)^1(1)^1$ only appears in the particular case when the simple point is adjacent to the triple point, type (1). This can be proved directly: assume the existence of $f_6(AB_1 \ldots B_7CD)^3$ and $k_6(A)^3(B_1 \ldots B_7)^3(C)^1$. Then $f_6+\lambda[k_3(AB_1 \ldots B_7C)^1]^2$ is a family of sextics $k_6(AB_1 \ldots B_7C)^2$, each meeting $k$ in 36 points. If we use $\lambda$ to make $k'$ contain any other point of $k$, it coincides with it entirely. For this value of $\lambda$ therefore,

$$k \equiv f_6 + \lambda k_3^2 \equiv k'(AB_1 \ldots B_7C)^3,$$

shewing that $k$ has a double point at $C$ and not merely a simple point as assumed. But $k$ is already rational, and cannot have an additional double point; therefore $C$ coincides with one of the other singularities, either $A$ or $B$, where $f_6$ has a tacnode, two distinct branches touching the same tangent line $l$, and $k_3$ touches $l$ also; while all the family $k'$, including $k$, have two branches touching $l$ at $C$. If $C$ coincided with $B$, then $k$ would
have a tacnode, which reduces the genus by two, so that \( k \) would break up; hence \( C \) coincides with \( A \). Then \( k \) has a triple point with one branch touching \( l \); the other two branches form a curve having a double point at \( A \), which curve must be considered as touching any straight line through \( A \), and therefore touching \( l \); and \( k \) must be considered as having two branches through \( A \) touching \( l \). But the triple point is not specialized, and does not reduce the genus by more than before.

The equation of the singly infinite family \( k' \) having a tacnode at \( A(1, 0, 0) \), both branches touching \( x_2 = 0 \), has the form:

\[
(a + b\lambda) x_1^4 x_2^2 + x_1^2 x_2 \left((c_1 + d_1\lambda)x_2^2 + (c_2 + d_2\lambda)x_2x_3 + (c_3 + d_3\lambda)x_3^2\right)
\]

terms of lower degree in \( x_1 = 0 \),

and \( k \) is the member of the family for which \( \lambda = -a/b \), the first group of terms is absent, and \( k \) has an ordinary triple point, one branch touching \( x_2 \).

There probably exist other types of the sextic, which are such that no combination of the three conditions among the ten double points expresses the existence of a curve. But we can shew, generally, that if a condition curve exists, of any degree, genus or negative freedom, then the sextic is of one of the five types enumerated.

Let a proper curve \( k_v \) of degree \( v \) be over-determined by passing \( p_k \) times through the point \( A_k \) \((k = 1 \ldots 10, \rho \geq 0)\); let its genus be \( p \) and its freedom \( f \):

\[
\Sigma \frac{1}{2} \rho (\rho - 1) = \frac{1}{2}(v - 1)(v - 2) - p \quad (p \geq 0),
\]

\[
\Sigma \frac{1}{2} \rho (\rho + 1) = \frac{1}{2}v(v + 3) - f \quad (f \leq -1).
\]

Double and subtract:

\[
\Sigma 2\rho = 6v + 2(p - f - 1).
\]

If the \( A \)'s are the double points of \( f_6(10)^2 \), then \( \Sigma 2\rho \) is the number of intersections of \( k_v, f_6 \) which fall at \( A \)'s; if the curves do not break up nor coincide,

\[
\Sigma 2\rho \leq 6v, \quad p - f - 1 \leq 0,
\]

which can only be so if

\[
p = 0, \quad f = -1,
\]

and \( k_v \) is a rational curve of freedom \(-1\); and as we saw in Part I, it can be transformed into a straight line \( k_1 (3)^1 \) by a series of quadratic transformations, using triads of its multiple points, which changes \( k_6 \) into another sextic with ten double points, which is of type \((2)\) because of the existence of \( k_1 \). It was therefore formerly of one of the types which arise from \((2)\) by quadratic transformations, which are the five types enumerated above.