Invariants and Covariants of the Cremona Cubic Surface

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Invariants and Covariants of the Cremona Cubic Surface.

BY C. P. SOITSLEY.

Introduction.

If a, b, . . . , f are properly chosen cubic curves on $P_3^6$, i.e., on six points in a plane, the cubic surface mapped on the plane by these curves may be given by the equation

$$a^3 + b^3 + \ldots + f^3 = 0,$$

where the variables are subject to the relations

$$a + b + \ldots + f = 0,$$

$$\bar{a}a + \bar{b}b + \ldots + \bar{f}f = 0.$$

This form is known as the hexahedral cubic surface of Cremona.*

As to the invariants and covariants of the Sylvester pentahedral form of the surface much is known and given in explicit form in Salmon's geometry of three dimensions. Nothing, however, is known as to the invariants and covariants of this Cremona form.

The purpose of this paper is to obtain some of these invariants and covariants and to outline the further steps necessary for the determination of the invariants and linear covariants.

The results are important in determining the lines on a general cubic surface. In particular for the Cremona form, the equations of the lines are known. If, then, one can find a typical representation for the general cubic surface in terms of the Cremona form and can determine the irrational invariants for the Cremona form, then the required lines of the general surface can be determined from the known lines of the Cremona cubic surface.

This requires the calculation of the invariants and linear covariants for the Cremona form and their identification with the corresponding invariants and covariants of the general cubic surface.

The results obtained are valuable also for any study that involves the behavior of the lines of a cubic surface with reference to the covariants of the surface.

Section 1.

Given in $S_3$ the cubic surface $(\alpha x)^3 = (\beta x)^3 = \ldots$ a known comitant $|\alpha \beta \gamma \xi| \ldots (\delta x) \ldots$ becomes $|\alpha \beta \gamma \eta \zeta \xi| \ldots (\delta x)$, when the surface is taken in $S_3$, and the variables are subject to the conditions $\Sigma \gamma_1 x_1 = 0$, $\Sigma \zeta_1 x_1 = 0$.

---

* Mathematische Annalen, Vol. XIII.
This process is known as the transference principle of Clebsch. The straightforward method for obtaining the corresponding comitant in \( S_3 \) would be to eliminate two variables, thus getting the surface in \( S_3 \), and for this form to calculate the comitant, but this would give an unsymmetrical result; whereas, by the Clebsch transference principle we obtain a symmetrical result.

The process of contravariant differentiation is not affected by going up to \( S_3 \) from \( S_3 \) in this way. This fact is noted immediately on performing corresponding operations in both dimensions.

Let us examine the degree of the invariants and covariants of the Cremona form. A covariant of a cubic surface of degree \( d \), weight \( u \), class \( c \), order \( \sigma \), has \( 3d \) symbols \( \alpha \); distributed in \( w-c \) symbols \( |\alpha \beta \gamma \delta| \), \( c \) symbols \( |\alpha \beta \gamma \xi| \) and \( \sigma \) symbols \( (ax) \), whence \( 3d = 4(w-c) + 3c + \sigma \), or \( 3d = 4w-c+\sigma \). These symbols become by Clebsch's principle of transference \( |\alpha \beta \gamma \delta \eta \zeta| \), \( |\alpha \beta \gamma \xi \eta \zeta'| \), \( (ax) \), where for the Cremona cubic surface the coefficients \( \alpha \) are now numerical and the coefficients \( \eta \) all are 1 and the coefficients \( \zeta \) are \( \bar{a}, \bar{b}, \ldots, \bar{f} \). Hence, if \( d' \) is the degree in \( \bar{a}, \bar{b}, \ldots, \bar{f} \) of the covariant for the Cremona form, we have

\[
d' = w = \frac{1}{3} (3d-\sigma-c) .
\]

Hence for this canonical form, the invariants have the degree \( d' = 6, 12, 18, 24, 30, 75 \); the linear covariants have the degree \( d' = 8, 14, 20, 32 \).

The invariants of the surface being invariants of \( P_6^2 \), must be expressible in terms of the rational and symmetric functions of the irrational system \( \bar{a}, \ldots, \bar{f} \) of \( P_6^2 \). The linear covariants of the surface can be expressed in terms of the functions

\[
K_i = \bar{a}^i a + \bar{b}^i b + \ldots + \bar{f}^i f, \quad (i = 2, \ldots, 5).
\]

Let us now calculate for \( P_6^2 \), referred to a special coordinate system, the irrational system \( \bar{a}, \ldots, \bar{f} \), and from these form the rational system \( a_1, \ldots, a_6 \),* which are the elementary symmetric functions of \( \bar{a}, \ldots, \bar{f} \).

Section 2.

If we take \( P_6^2 \) in the prepared form

\[
\begin{align*}
(1) & \quad 1 \quad 0 \quad 0, \\
(2) & \quad 0 \quad 1 \quad 0, \\
(3) & \quad 0 \quad 0 \quad 1,
\end{align*}
\]

\[
\begin{align*}
(4) & \quad 1 \quad 1 \quad 1, \\
(5) & \quad x \quad y \quad u, \\
(6) & \quad z \quad t \quad u,
\end{align*}
\]

we have (loc. cit., p. 170)

\[
\begin{align*}
3 \bar{a} & = \rho - 3(ux + ut), \\
3 \bar{b} & = \rho - 3(ux + yz), \\
3 \bar{c} & = \rho - 3(uy + ut), \\
3 \bar{d} & = \rho - 3(uy + uz), \\
3 \bar{e} & = \rho - 3(uy + xt), \\
3 \bar{f} & = \rho - 3(uz + xt), \\
\rho & = u(x + y + z + t) + xt + yz.
\end{align*}
\]

Let us introduce the following notation:

\[
\begin{align*}
\gamma_1 &= x + y + z + t, \\
\gamma_2 &= (x + t)(y + z), \quad \overline{\gamma}_2 = yz + xt, \\
\gamma_3 &= xyz + yzt + xyt + xzt, \\
\gamma_4 &= xyt.
\end{align*}
\]

For purposes of calculation it is more convenient to use the sums of the powers of \( \overline{a}, \ldots, \overline{f} \) for the rational invariants of \( P^3 \), rather than their symmetric functions, i.e., if \( p_2 = \Sigma \overline{a}^2 \), \( p_3 = \Sigma \overline{a}^3 \), \ldots, \( p_5 = \Sigma \overline{a}^5 \), and if we substitute the values of \( \overline{a}, \ldots, \overline{f} \), given by (4), we find

\[
9p_2 = 6 \left[ (2y_1^2 - 3y_2 - 6y_3) u^2 - (2y_1 \overline{y}_2 - 3y_2) u + 2(\overline{y}_2^2 - 3y_4) \right],
\]

\[
27p_3 = 3 \left[ (9y_1 \overline{y}_2 + 18y_1y_2y_3 - 4y_3^2 - 27y_3) u^3 + 3 (2y_1 \overline{y}_2^2 - 6y_2^2 - 3y_1y_2y_3 + 36y_4) u^2 + 3 (2y_1 \overline{y}_2 - 3y_1y_4 - 3\overline{y}_2y_3) u \\
+ 2\overline{y}_2(9y_4 - 2y_3^2) \right],
\]

\[
81p_4 = 18 \left[ (2y_1^4 - 12y_1y_2y_3 - 6y_1^2y_2 + 18y_1^2y_3 + 18y_2^2y_3 - 18y_4^3) u^4 \right. \\
+ 2 (3y_1 \overline{y}_2^2 - 2y_1 \overline{y}_2^2 - 6y_1 \overline{y}_2y_3 + 9y_2^2y_3 + 9y_1y_2y_3 + 3y_3^2y_3 + 9y_1y_4) u^3 \\\n+ 3 (2y_1 \overline{y}_2 - 2y_1y_3 - 6y_1 \overline{y}_2y_3 + 6\overline{y}_2y_4 + 7y_3^2 - 10y_2y_4) u^2 \\\n+ 2 (3\overline{y}_2y_3 - 2y_1 \overline{y}_2y_4 + 6y_1 \overline{y}_2y_4 - 9y_3y_4) u + 2 (\overline{y}_2^2 - 6\overline{y}_2y_4 + 9y_4^2) \right],
\]

\[
243p_5 = 5 \left[ (-4y_1^4 + 30y_1^2y_2y_3 + 15y_1^3y_3 - 27y_1^2y_3 - 5y_1y_2y_3 + 27y_2^2y_3 + 27y_1y_2y_4 + 15y_1y_3y_4 + 27y_2y_3y_4 - 54y_4y_3) u^5 \\
- 27y_1 \overline{y}_2^2 - 54y_1y_2y_3 - 27y_1y_4 - 81y_2y_3y_4 + 81y_2y_3y_4) u^4 \\
+ 3 (10y_1 \overline{y}_2y_3 - 45y_1^2y_2y_3 - 36y_3^2y_3 + 54y_3y_3 + 270y_2y_3 - 15y_1y_2y_3) u^3 \\\n+ 108y_1y_2y_3 + 108y_2y_3y_4 + 54y_1y_2y_4 + 27 \overline{y}_2y_3y_4 + 216y_2y_3y_4 - 108y_3y_4) u^2 \\\n+ 6 (9y_1 \overline{y}_2y_3 - 27y_3^2y_3 + 9y_1 \overline{y}_2y_4 - 108y_1y_2y_3 - 3y_3y_4 - 2y_1 \overline{y}_2y_3) u \\\n+ 243y_2y_3 - 18y_1 \overline{y}_2y_4 + 15y_2y_3y_4 - 9y_1y_2y_4) u^3 \\\n+ 6 (3y_2^2y_4 - 6y_2^2 - 2y_1 \overline{y}_2y_4 + 6y_1 \overline{y}_2y_3 + 108y_2y_4) u \\\n- 21 \overline{y}_2y_3y_4 + 27y_1y_2y_4 - 15y_2y_3y_4 - 324y_2y_4) u^2 \\\n+ 3 (10y_1 \overline{y}_2y_3 - 45y_1 \overline{y}_2y_3 - 27y_1y_2y_4 - 15y_1y_3y_4 + 54y_2y_3y_4) u \\
+ 2 (4y_2^2 - 6y_2^2 - 81y_2y_4) \right].
\]

\[
729a_6 = (4y_1^6 - 36y_1^4y_2 - 18y_1^3y_3 + 54y_2^2y_3 - 162y_1y_2y_3 + 81y_1y_2y_3^2) u^6 \\
+ 81y_1 \overline{y}_2y_3y_4 + 243y_1y_2y_3y_4 + 729y_2y_3y_4) u^6 \\
+ 3 (6y_1 \overline{y}_2y_3 - 4y_1y_3y_4 + 24y_4 \overline{y}_2y_3 + 21y_1y_3y_4 + 90y_3y_4 - 135y_2y_2y_3 - 27y_1y_2y_4 - 27y_1y_3y_4 \\
- 27y_1y_2y_4 - 108y_1y_2y_3 - 351y_1y_2y_4 + 162y_2y_3y_4) u^5 \\
+ 3 (-y_1y_3y_4 - 3y_1y_4 - 6y_2y_2y_3 - 9y_2y_3y_3 - 9y_1y_2y_4 - 27y_1y_2y_4 + 9y_2y_2y_3 + 9y_2y_2y_3 - 27y_1y_2y_4 \\
+ 9y_2y_2y_3 + 81y_2y_3y_4 - 27y_1y_2y_3 - 108y_2y_3y_4 + 27y_2y_3y_4 \\
+ 54y_2y_3y_4 + 27y_2y_3y_4 + 27y_2y_3y_4) u^5 \\
+ (26y_3 \overline{y}_2y_3 - 117y_3y_2y_3 - 72y_1 \overline{y}_4 + 162y_1 \overline{y}_2y_4 + 9y_1y_2y_4 + 225y_2y_3y_4 - 81y_2y_3y_4 - 162y_3y_4) u^5 \\
+ 3 (3y_2y_3y_4 - y_2y_4 + 3y_2y_4 + 3y_1y_2y_4 + 3y_1y_2y_4 - 30y_1y_2y_4 - 6y_3y_4 - 45y_2y_3y_4 + 12y_3y_4) u^4 \\
+ 3 (24y_2y_2y_4 - 4y_2y_4 - 27y_1y_2y_4 + 6y_3y_3 - 27y_2y_3y_4) u \\
+ (4y_3^2 - 36y_2y_4 + 81y_2y_4).}
\]
We can change to symmetric functions at any time by means of the relations
\[
\begin{align*}
p_2 &= -2a_3, \\
p_3 &= -3a_3, \\
p_4 &= -4a_4 + 2a_3^2, \\
p_5 &= -5a_3a_4 + 5a_6, \\
p_6 &= -6a_6 + 6a_2a_4 + 3a_5^2 - 2a_3^2.
\end{align*}
\]
(7)

To determine the cubic curves \(a, \ldots, f\) on the prepared set (3), solve the five relations (loc. cit., p. 168) with \(\Sigma a = 0\), thus getting the following values, where we write in order the coefficients of \(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\):
\[
\begin{align*}
a &= \frac{u^3 - ux - uz}{2} \quad \text{or} \quad \frac{-u^3 + uy + ut, xz + uz - ux, xt - xz - yz, ut - uy - ty, yz - tx + ty, 2(ux - ut)}{2}
\end{align*}
\]
(8)

Combining these with the values of \(\bar{a}, \ldots, \bar{f}\), given by (4), we find immediately the functions \(K_2, \ldots, K_5\).

From the values of \(p_2, \ldots, p_5\), given by (6), we find that \(p_2^2 - 4p_4\) or \(a_3^2 - 4a_4\), and 12\(p_5 - 5p_2p_3\) or \(a_9a_9 - 2a_6\) contain \(u^4\) as a factor. If, then,
\[
q_4 = p_2^2 - 4p_4, \quad q_5 = 12p_5 - 5p_2p_3,
\]
we find
\[
\begin{align*}
q_4 &= u^2\left[(\gamma_2^3 - 4\gamma_4)u^2 + 2(2\gamma_4 - \gamma_3^2)u + \gamma_3^2 - 4\gamma_2\gamma_4\right], \\
q_5 &= 2430u^2\left[(3\gamma_2\gamma_3 - 2\gamma_1\gamma_4 - \gamma_1^2\gamma_2^3)u^3 \\
&\quad + (2\gamma_3^3 - 8\gamma_2\gamma_4 - \gamma_3\gamma_2\gamma_3 + 5\gamma_1^2\gamma_4 - 12\gamma_2\gamma_4)u^2 \\
&\quad + (3\gamma_2\gamma_4 - \gamma_3^2\gamma_2 - 7\gamma_1\gamma_2\gamma_4 - 2\gamma_1\gamma_2\gamma_4 - \gamma_1\gamma_2^2 + 18\gamma_2\gamma_4)u \\
&\quad + (6\gamma_2^3\gamma_4 - 3\gamma_2\gamma_4^2 - 2\gamma_2^2\gamma_4 - 3\gamma_1\gamma_2\gamma_4 - 24\gamma_4^3)\right].
\end{align*}
\]
(9)
Calculating the terms free of \( u \) in the functions \( K_2, \ldots, K_5 \), I find

\[
\begin{align*}
K_2 &= 36y_4(xt-yz)(x_1x_2^2-x_0x_2^2), \\
K_3 &= 0, \\
K_4 &= 108y_4(\gamma_2^2-3\gamma_4)(xt-yz)(x_1x_2^2-x_0x_2^2), \\
K_5 &= 36y_4(-2\gamma_2^2-81\gamma_4)(xt-yz)(x_1x_2^2-x_0x_2^2);
\end{align*}
\]

hence, the combinations \( K_4 = 9p_2K_2 - 4K_5 \) contain \( u \) as a factor. It follows from the value of \( K_5 \) that it could not be used to advantage in expressing the linear covariants.

An interesting particular case occurs when the last two points of (3) are apolar to the pencil of conics on the first four, \( x_2x_3 = x_3x_1 = x_1x_2 \); i.e., when two of the points are a corresponding pair in the quadratic Cremona involution with fixed points at the other four. This requires that \( xt+yz=u(y+t) \), whence from (4) \( \alpha + \delta + \eta = \beta + \varepsilon = c + \varepsilon = \lambda \). Since \( \Sigma \alpha = 0 \), \( \lambda = 0 \) and, therefore, \( p_3 = \Sigma \alpha^3 = 0 \) and \( p_5 = \Sigma \alpha^5 = 0 \). If, conversely, \( p_3 = p_5 = 0 \) and if five points of \( P_3 \) be fixed there are fifteen possible positions for the sixth. These all are accounted for by choosing any four of the five as fixed points or any three of the five as fixed points. Hence,

The necessary and sufficient condition that the invariants of \( P_3 \) of odd degree (other than the skew invariant \( d_2\sqrt{\alpha} \)) vanish is that two of the points be apolar to the pencil of conics on the other four.

Section 3.

Let the cubic surface be given by

\[
U = (ax)^8 = (bx)^8 = \ldots = (dx)^8, \quad \text{or} \quad x_1^8 + x_2^8 + \ldots + x_6^8 = 0,
\]

for the Cremona form, where the variables are subject to the relations

\[
\begin{align*}
x_1 + x_2 + \ldots + x_6 &= 0, \\
\bar{a}x_1 + \bar{b}x_2 + \ldots + \bar{\iota}x_6 &= 0.
\end{align*}
\]

The Hessian is the locus of points whose polar quadrics are cones and, therefore, obtained by writing the discriminant of a polar quadric.

According to the principle of transference of Clebsch it is given by

\[
H = \frac{1}{24} |\alpha \beta \gamma \delta \eta \zeta|^2(ax)(\beta x)(\gamma x)(\delta x) \quad \text{or} \quad \Sigma\left\{ x_1x_2x_3x_4\bar{\xi}\bar{\zeta}\right\}^{15}
\]

for the Cremona form, where \( \bar{ij} = \left| \begin{array}{cc} \eta_i & \eta_j \\ \zeta_i & \zeta_j \end{array} \right| \), and since \( \eta_1, \ldots, \eta_6 = 1, \ldots, 1, \zeta_1, \ldots, \zeta_6 = \bar{a}, \ldots, \bar{\iota} = \bar{b}_1, \ldots, \bar{b}_6 \), we have \( \bar{ij} = b_i - b_j \). The super- and subscript of \( \Sigma \) refer to the fifteen terms obtained by choosing the pair 56 from 1, \ldots, 6.
The contravariant $S$, as given by the Clebsch transference principle, is

$$S = \frac{1}{6} \left| \alpha \beta \gamma \eta \zeta \xi \right| \left| \alpha \beta \delta \eta \zeta \xi \right| \left| \alpha \gamma \delta \eta \zeta \xi \right| \left| \beta \gamma \delta \eta \zeta \xi \right| \text{ or } = 4 \sum_{(55)}^{15} \left[ 156 \ 256 \ 356 \ 456 \right]$$

(14)

for the Cremona form, where

$$\sum_{ijk} = \begin{vmatrix} \xi_i & \xi_j & \xi_k \\ \eta_i & \eta_j & \eta_k \\ \zeta_i & \zeta_j & \zeta_k \end{vmatrix}.$$

By expanding and collecting the coefficients of $\xi$ we have

$$\frac{1}{4} S = \sum_{(1)}^{6} S_1 + \sum_{(2)}^{5} S_{1,2} \xi_1^2 + \sum_{(12)}^{15} S_{1,2} \xi_1^2 \xi_2^2 + \sum_{(23)}^{10} S_{1,23} \xi_1^2 \xi_2^2 \xi_3^2 + \sum_{(1234)}^{15} S_{1234} \xi_1 \xi_2 \xi_3 \xi_4.$$

Let us indicate the coefficients of $S$ as follows:

$$\frac{1}{4} S = \sum_{(1)}^{6} S_1 + \sum_{(2)}^{5} S_{1,2} + \sum_{(12)}^{15} S_{1,2} + \sum_{(23)}^{10} S_{1,2} + \sum_{(1234)}^{15} S_{1234}.$$

We find that these coefficients expressed in terms of $a_2, \ldots, a_6$, the elementary symmetric functions of $b_1, \ldots, b_6$ are:

$$S_1 = -15a_4 + 4a_2^2 + 9a_2b_1 - 6a_2b_2^2 - 9b_1^2,$$

$$S_{1,2} = 12a_4 - 4a_2^2 + 3a_2|1| - 2a_2|1,2| - 6|0,1,0| - 6b_2[a_2 + 2b_1 b_2^2],$$

$$S_{1,2} = 6a_4 - 9a_2|1| + 2a_2|5, -2| + 10|1, -2, 1|,$$

$$S_{1,2} = -8a_4 + 4a_2^2 - a_2[10b_1 + |1]| + 2a_2[b_1 |4| + |3, -2|]},$$

$$S_{1,2} = 2[2b_1 |4|, -9| + 2|, -3, 0|],$$

$$S_{1,2} = 8a_2^2 + 18a_2|1| + 12a_2|1, -2| - 4|2, -7, -4|.$$

Here $\{ \ldots \}$ refers to the coefficients of polynomials in $p_1, p_2$ the symmetric combinations of the two isolated letters.

Identities among these which are sometimes useful are:

$$\{1, -4, 3\} + a_2|1, -2| - a_2|1, -1| + a_2|1| - a_5 = 0,$$

$$[b_1 + b_1^2|1| + b_1^2|1|, -1| + b_1|1, -2| + |1, -3, 1|,,$$

$$+ a_3[b_1^2 - b_1|1| + |1, -1|] - a_3[b_1 + |1|] + a_4 = 0.$$

(16)

The coefficients of $S$ are connected by the relations

$$4 S_1 \eta_1 + \sum_{(2)}^{5} S_{1,2} \eta_2 = 0,$$

$$3 S_{1,2} \eta_1 + 2S_{1,2} \eta_2 + \sum_{(3)}^{4} S_{1,23} \eta_3 = 0,$$

$$2(S_{1,2} \eta_1 + S_{2,31} \eta_2 + S_{2,32} \eta_3) + S_{1234} \eta_4 + S_{1235} \eta_5 + S_{1236} \eta_6 = 0.$$
Hence, \( \gamma_1, \ldots, \gamma_6 = 1, \ldots, 1 \), or \( b_1, \ldots, b_6 \), also \( \sum_{2}^{5} S_{2,1} = \sum_{2}^{5} S_{1,2} = -4 S_1 \).

A numerical check is furnished by taking \[ \{ b_1 b_2 b_3 b_4 b_5 b_6 \} \text{ with } a_2 = -6, a_3 = -4, a_4 = 9, a_5 = 12, a_6 = 4, \] while \( S_5 = S_6 = 81, S_{56} = 18 \cdot 27, S_{5,6} = S_{6,5} = -18 \cdot 18, \) and all the other coefficients \( S \) vanish.

By operating with \( S \) on \( H \) (in operating with a form of order \( n \) on a form of class \( m > n \) we remove the factor \( m(m-1) \ldots (m-n+1) \) after differentiation) we get the first invariant

\[
I_1 = 24 [a_2^3 - 3a_3^2 - 16a_2a_4 + 12a_6].
\]

By operating with \( S \) on \( U^2 \) we get the first covariant quadric, i.e., a \( C_{6,2,0} \). Denoting this quadric by \( Q \), we find

\[
5 Q = \sum_{1}^{6} (20 S_1 + \sum_{2}^{5} S_{2,1}) x_1^2 + 4 \sum_{12}^{15} S_{12} x_1 x_2 = \sum_{1}^{6} Q_{11} x_1^2 + \sum_{12}^{15} Q_{12} x_1 x_2,
\]

where

\[
Q_{11} = 12 S_1, \quad Q_{12} = 4 S_{12},
\]

\( (S_1 \text{ and } S_{12} \text{ are given by (15)).} \)

By operating with \( Q \) on \( S \) we get the contravariant quadric \( C_{10,0,2} \). Denoting this quadric by \( q \) we find

\[
5 \cdot 3 q = \sum_{1}^{6} q_{11} \xi_1^2 + \sum_{12}^{15} q_{12} \xi_1 \xi_2,
\]

where if \( q_4 = a_2^3 - 4a_4 \) and \( q_5 = a_2 a_6 - 2a_5 \),

\[
\begin{align*}
\frac{1}{72} q_{11} &= -3 \cdot 32 q_4 b_1^4 + 3 \cdot 8 q_5 b_1^3 - 3 \cdot 2 [I_4 + 8a_2 q_4] b_1^2 \\
&+ 3 \cdot 2 [5a_2 q_5 + 17a_5 q_4] b_1 + [5a_2 I_4 + 15a_3 q_5 + 27q_4^2 - 15 a_5^2 q_4], \\
\frac{1}{32} q_{12} &= -32 a_2 a_6 - 135 a_3 a_5 - 216 a_4^2 + 198 a_5^2 a_4 - 36 a_6^2 \\
&+ 153 a_5 a_6 \{1 \} - 108 a_3 a_4 \{1 \} + 27 a_2 a_3 \{1 \} + 12 a_5 \} - 48, 33 \} \\
&+ 2a_2 a_4 \{1, 144, -47 \} + 81 a_5 \{0, 1 \} - 36 a_5 \{2, 1 \} + 6 a_5 \{45, -4 \} \\
&- 248 a_2 a_3 \{0, 1 \} + 36 a_5 \{6, -9 \} - 14 \{2 a_2 \} - 27, -11, 173 \} \\
&+ 6 a_3 \{0, -27, 61 \} + 2a_2 \{0, 16, 93 \} - 83 \{12, 0, 0, 10, 13, -6 \}. 
\end{align*}
\]
By operating with $q$ on $U$ we get the first linear covariant $C_{11,1,0}$. Denoting this covariant by $L_1$ we find

$$L_1 = -16q_4K_4 + 4q_5K_5 + [-I_1 = -8a_2q_4]K_2. \quad (22)$$

By operating with $q$ on $Q$ we get the second invariant $I_2$, where

$$5^2 \cdot 6I_2 = 2 \sum_{(1)} q_{11}Q_{11} + \sum_{(12)} q_{12}Q_{12}. \quad (23)$$

By another method I have found a second invariant to be

$$I'_2 = 2^8 q_3^2 q_2 + 3 \cdot 5 \cdot 2^4 q_5 q_4 q_3 - 2 \cdot 3 \cdot 5^3 q_4^3 - 2 \cdot 3 \cdot 5^2 q_2^2 q_4^2 + 3 \cdot 5^2 I_1 q_4 q_2; \quad (24)$$

hence, we must find $I_2 = c_0I'_2 + c_1I'_2$, where $c_0$ and $c_1$ are numerical constants.

If we denote by $M$ the mixed concomitant obtained by writing the plane equation of the polar quadric of any point with respect to the cubic surface, we have, according to the principle of transference of Clebsch,

$$M = \left\{a_{y} \gamma \varepsilon \xi \right\} \left\{(ax)(\beta x)(\gamma x), \right\} = \sum_{(1)} x_1 x_2 x_3 \overline{4565}, \quad (25)$$

for the Cremona form.

By operating on this with $\frac{5}{4}Q = \sum_{(1)} S_{11}x_1^2 + \sum_{(12)} S_{12}x_1x_2$, we get a $C_{0,0,0}$. Denoting this by $N$ we have

$$N = \sum_{(123)} x_1 x_2 x_3 \left[3 \sum_{(4)} (S_{4565} - S_{5046}) \right] = \sum_{(123)} C_{123} x_1 x_2 x_3, \quad (26)$$

where we find

$$C_{123} = 24q_4(\sigma_1^2 - 3\sigma_2) + 36a_4(\sigma_1\sigma_2 - 9\sigma_3) + 72a_2(3\sigma_1\sigma_2 - \sigma_3^2) + 3[19(4\sigma_3^2 - \sigma_1^2\sigma_2^2) + 28\sigma_2^2 - 150\sigma_1\sigma_2\sigma_3 + 81\sigma_3^3]. \quad (27)$$

Here $\sigma_1, \sigma_2, \sigma_3$ are the elementary symmetric functions of $b_4, b_5, b_6$.

The result of operating with $N$ on $S$ is a $C_{13,0,1}$. Denoting this by $C$, we have

$$C = \sum_{(1)} \xi_1 \left[ \sum_{(5)} (S_{1234}C_{234} + 2S_{11,56}C_{156}) \right] = C_1\xi_1 + \ldots + C_6\xi_6. \quad (28)$$

Multiplying $S_{1,56}$ and $C_{156}$ together, summing for $\Sigma$ and indicating by $(ijkl)$ the symmetric sum for five things of $\Sigma b_i b_j b_k b^l b^m$, we get
\[ \sum_{S_{1,56}} C_{156} = 6 \{ 24 q_4 (4 a_2^2 - 8 a_4 - 10 a_3 b_1) [2(2) - (11)] \\
+ \{ 24 q_4 [-a_3 + 8 a_3 b_1] + 36 a_3 (4 a_2^2 - 8 a_4 - 10 a_3 b_1) \} [(21) - 2(111)] \\
+ 48 a_3 [-a_3 + 8 a_3 b_1] [(211) - 6(1111)] \\
+ 24 a_3 [4 a_2^2 - 8 a_4 - 10 a_3 b_1] [(211) - 3(22)] \\
+ 32 a_4 q_4 [9(22) - (211) - 12(1111)] \\
+ \{ 4 a_2^2 - 8 a_4 - 10 a_3 b_1 \} [38(411) - 57(42)] \\
+ 10(321) - 30(33) - 30(222) \} \\
+ 32 q_4 b_1 [12(32) - 8(311) + 6(221) - 9(2111)] \\
+ 144 a_2 a_3 [2(221) - (2111) - 40(11111)] \\
+ 24 a_6 [-a_3 + 8 a_3 b_1] [3(2111) - 2(221)] \\
+ 96 a_2^2 [(2211) - 9(222) + 12(21111)] \\
+ 16 \cdot 12 a_3 b_1 [2(321) - 6(3111) + 3(2211) - 18(21111)] \\
+ 16 q_4 [6(42) - 4(411) + 5(321) - 15(3111)] \\
+ 18(222) - 6(2211)] \\
+ 48 a_3 [2(421) - 6(4111) + 5(3211) - 60(31111)] \\
+ 9(2221) - 18(22111) \} \\
+ 96 a_2 b_1 [4(3211) - 8(322) + 18(22111) - 9(2221)] \\
+ 2 \{ -a_3 + 8 a_3 b_1 \} [19 \cdot 12 (31111) - 19(421)] \\
+ 20(3211) - 20(331) - 30(2221] \\
+ 48 a_2 [2(4211) - 4(422) + 15(32111) - 5(3221)] \\
+ 18(22211) - 36(2222)] \\
+ 4 a_5 [19 \cdot (4211) - 6(422) + 24(41111)] + 10 \{ 3(3221) - 3(332) \\
- 18(2222) + 6(32111) - 2(3311) - 6(22211) \} \\
+ 38 [8(4411) - 8(442) + 30(43111) - 5(4321) \\
+ 12(42211) - 18(4222)] \\
+ 20 [2(4321) - 4(433) - 6(4222) + 20(33211) - 15(3331) \\
+ 3(3221) - 6(3322) - 180(22222) \} \}.

In order to get \( \Sigma S_{1234} C_{234} \), we write

\[ S_{1234} = 2 \{ 4 a_2^2 + 9 a_3 (b_3 + b_6) + 6 a_2 (b_3^2 + b_6^2) - 4 (b_3^2 + b_6^3) - 2 b_6 b_6 (b_3^2 + b_6^2) + 12 b_3^2 b_6 \}, \]

\[ C_{234} = 3 \{ 8 q_4 b_3^2 (19 a_3 b_1 + 12 a_3 b_1^2) (b_3 + b_6) + (8 q_4 + 12 a_3 b_1 - 24 a_2 b_1^2 \\
- 19 b_1^2) (b_3^2 + b_6^2) + (-8 q_4 - 72 a_3 b_1 + 24 a_2 b_1^2 + 38 b_1^2) b_6 - 10 b_1^2 (b_3^2 + b_6^2) \\
+ (12 a_3 + 24 a_2 b_1 + 10 b_1^2) (b_3 + b_6) b_6 - 19 b_1^2 (b_3^2 + b_6^2) + 10 b_1^2 b_6 b_6 (b_3^2 + b_6^2) \\
+ (-24 a_2 - 30 b_1^2) b_3^2 b_6^2 + 38 b_1 b_6 b_6 (b_3^2 + b_6^2) + 10 b_1 b_5 b_6 (b_3 + b_6) \\
- 19 b_2^2 b_5 (b_3^2 + b_6^2) - 10 b_5^2 b_5 \}. \]
In the Cremona Cubic Surface.

\[ \Sigma_{1234} C_{234} = 6 \left| 320 a_3^2 q_1 b_1^2 + [72 a_3 q_1 b_1^2 + 48 a_3 a_1^2 b_1^2 - 32 a_3^2 q_1 b_1] \right| 4 (1) \]

\[ + \left[ -76 a_3^2 b_1^2 + 12 ( -8 a_3^2 + 9 a_3^2 + 4 a_3 q_1) b_1 + 24 (2 a_3^2 a_3 - 3 a_3 q_1) b_1 \right. \]

\[ + 32 a_3^2 q_1 \right] 4 (2) \]

\[ + [8 \cdot 19 a_3^2 b_1^2 + 24 (9 a_3^2 + 4 a_3^2) b_1 - 144 (a_3 q_1 + 2 a_3^2 a_3) b_1 - 32 a_3^2 q_1] \right] 4 (11) \]

\[ + \left[ -9 \cdot 19 a_3 b_1^2 - 40 a_3 a_1^2 b_1 + 144 a_3 a_3 b_1^2 + 12 (9 a_3^2 - 4 a_3 q_1) b_1 \right. \]

\[ + 72 a_3 q_1 \right] 4 (3) \]

\[ + [9 \cdot 19 a_3 b_1^2 + 40 a_3 a_1^2 b_1 + 72 a_3 a_3 b_1^2 + 12 (8 a_3^2 + 9 a_3^2 - 4 a_3 q_1) b_1 \]

\[ + 48 a_3^2 a_3] \right] 4 (21) \]

\[ + [ -6 \cdot 19 a_3 b_1^2 - 90 a_3 b_1^2 - 4 (55 a_3^2 + 8 q_1) b_1 + 72 a_3 a_3 b_1 + 48 a_2 q_1] \right] 4 (4) \]

\[ + [12 \cdot 19 a_3 b_1^2 + 8 (23 a_3^2 - 2 q_1) b_1^2 - 24 \cdot 9 a_3 a_3 b_1 + 12 (9 a_3^2 - 4 a_2 q_1)] \right] 4 (31) \]

\[ + [-12 \cdot 19 a_3 b_1^2 + 180 a_3 b_1^2 + 24 ( -17 a_3^2 + 4 q_1) b_1^2 + 24 \cdot 24 a_3 a_3 b_1 \]

\[ + 24 (9 a_3^2 - 4 a_3 q_1) b_1 + 72 a_3 q_1 \right] 4 (22) \]

\[ + [ -60 a_3 b_1^2 - 3 \cdot 73 a_3 b_1^2 + 32 a_3 b_1] \right] 4 (5) \]

\[ + [60 a_3 b_1^2 - 9 \cdot 17 a_3 b_1^2 + (8 \cdot 37 a_3^2 + 48 q_1) b_1 + 72 a_3 a_3] \right] 4 (21) \]

\[ + [ -60 a_3 b_1^2 - 90 a_3 b_1^2 - 4 (55 a_3^2 + 8 q_1) b_1 + 72 a_3 a_3 b_1 + 48 a_2 q_1] \right] 4 (32) \]

\[ + [4 \cdot 19 b_1^2 + 8 (23 a_3^2 - 10 q_1) b_1 - 144 a_3 a_3] \right] 4 (6) \]

\[ + [ -6 \cdot 19 b_1^2 + 12 a_3 b_1^2 - 32 b_1^2 + 606 a_3 b_1 + 16 q_1] \right] 4 (51) \]

\[ + [-12 \cdot 19 b_1^2 - 6 \cdot 89 a_3 b_1^2 + 24 \cdot 28 a_3 b_1 + 20 (4 q_1 - 11 a_3^2)] \right] 4 (42) \]

\[ + [28 \cdot 19 b_1^2 + 12 \cdot 42 a_3 b_1^2 - 6 \cdot 122 a_3 b_1 - 8 (16 q_1 + 5 a_3^2)] \right] 4 (33) \]

\[ + [40 b_1^4] 4 (7) \right] + [ -2 b_1^4 + 12 \cdot 11 a_3 b_1 - 48 a_3] \right] 4 (61) \]

\[ + [-180 b_1^4 - 84 a_3 b_1 - 9 \cdot 27 a_3] \right] 4 (52) \]

\[ + [160 b_1^4 + 48 \cdot 11 a_3 b_1 - 3 \cdot 47 a_3] \right] 4 (43) \]

\[ + [4 \cdot 19 b_1^2] 4 (8) \right] + [-2 b_1^2] 4 (71) \right] + [-16 \cdot 8 b_1^2 - 18 a_3] \right] 4 (62) \]

\[ + [2 \cdot 89 b_1^2 - 12 a_3] \right] 4 (53) \right] + [-4 \cdot 62 b_1^2 - 129 a_3] \right] 4 (44) \]

\[ + [-8 \cdot 19 a_3 b_1^2 + 4 \cdot 29 b_1^3 + 12 \cdot 33 b_1 (63) - 4 \cdot 32 b_1 (54) \]

\[ + 4 \cdot 19 (82) + 78 (73) - 4 \cdot 33 (64) - 4 \cdot 11 (55) \]

Using brackets for symmetric functions of six things, we have

\[ (i) = [i] - b_1^4, \]

\[ (ii) = [ii] - b_1^4 + [i] + b_1^4, \]

\[ (ij) = [ij] - b_1^4 + [j] - b_1^4 + 2 b_1^4, \]

thus, we get \( C_1 \) in final form by making these substitutions in (28) and (32) and collecting.

The polar form of (18) is \( \frac{1}{2} \left[ \sum_{(1)} x_1 (2 Q_{11} y_1 + \sum_{(2)} Q_{12} y_2) \right] \), and by operating with this on (20) we get a collineation \( C_{10,1,1} \). Denoting this by \( K \), we have

\[ 2 K = \sum_{(1)} x_1 (4 q_{11} Q_{11} + \sum_{(2)} q_{12} Q_{12}) + \sum_{(2)} x_2 (2 q_{12} Q_{22} + 2 q_{11} Q_{12} + \sum_{(3)} q_{13} Q_{23}) \].
If this collineation be written as

\[ 2^7 K = p_{11} \xi_1 + p_{32} \xi_2 + \ldots + p_{66} \xi_6, \]  

we find

\[
p_{11} = x_1 [d_1 + b_1 d_2 + b_1^2 d_2 + b_1^3 d_4 + b_1^4 d_5 + b_1^5 d_6] \\
+ K_2 [f_{2,0} + b_1 f_{2,1} + b_1^2 f_{2,2} + b_1^3 f_{2,3} + b_1^4 f_{2,4} + b_1^5 f_{2,5}] \\
+ K_4 [f_{4,0} + b_1 f_{4,1} + b_1^2 f_{4,2} + b_1^3 f_{4,3} + b_1^4 f_{4,4} + b_1^5 f_{4,5}] \\
+ K_5 [f_{5,0} + b_1 f_{5,1} + b_1^2 f_{5,2} + b_1^3 f_{5,3} + b_1^4 f_{5,4} + b_1^5 f_{5,5}],
\]

where

\[
d_1 = 2^4 \cdot 3^2 \cdot 53 I_1^2 + 2^4 \cdot 3^3 \cdot 53 a_3^2 - 5 \cdot 10181 a_3 q_4 I_1 + 2^4 \cdot 3^4 \cdot 5 q_4^2 + 2^6 \cdot 13 \cdot 1609 a_3^2 q_4 - 3 \cdot 7 \cdot 2911 I_1 + 3 \cdot 5^2 a_3^2 a_3 - 2^2 \cdot 5^3 a_2 a_4 q_4,
\]

\[
d_2 = 6 (2^4 \cdot 3 \cdot 359 q_5 I_1 - 2^4 \cdot 3^2 \cdot 53 a_3 I_1 - 3 \cdot 5^2 \cdot 257 a_4 q_5 + 2^7 \cdot 3^2 \cdot 5^2 q_5^2 + 5^2 a_3^2 q_5 + 2^4 \cdot 3^4 \cdot 5 a_3 q_4 + 3 \cdot 5 \cdot 143 q_4 a_3 q_5 - 5^4 a_2 a_4),
\]

\[
d_3 = 3 (-2^4 \cdot 3^2 \cdot 53 a_3 I_1 + 2^4 \cdot 461 a_3^2 I_1 + 2^5 \cdot 3^2 \cdot 5^2 q_5 - 2^4 \cdot 3 \cdot 2281 a_2 a_3 q_5 + 14153 a_2 a_3 q_5 + 2^4 \cdot 3^2 \cdot 5 a_3^2 q_4 - 2 \cdot 12097 a_3^2 q_4 + 2^4 \cdot 3^2 \cdot 7 a_3^2 q_5 + 5 \cdot 101 a_3^2),
\]

\[
d_4 = 12 (-2^4 \cdot 3^3 \cdot 53 a_3 I_1 - 2^4 \cdot 3 \cdot 7 \cdot 37 q_4 q_5 + 2^2 \cdot 3 \cdot 1049 a_3 q_5 + 5 \cdot 3473 a_3 a_4 q_5),
\]

\[
d_5 = 12 (2^3 \cdot 3^2 \cdot 83 a_3 I_1 - 2^4 \cdot 3^3 \cdot 359 a_3 q_5 - 2^4 \cdot 3 \cdot 41 q_4^2 - 3 \cdot 5539 a_3^2 q_4 + 25 a_4^2),
\]

\[
d_6 = 288 (1621 a_3 q_6 + 2^3 \cdot 3^2 a_4 q_6),
\]

\[
f_{2,0} = 6 (17 q_4 I_1 - 563 a_3 I_1 - 3^2 \cdot 5 q_5 - 2^3 \cdot 3^2 \cdot 5 a_3 a_5 + 2^3 \cdot 59 a_2 q_4 + 3 \cdot 487 a_3^2 q_4 + 2^6 \cdot 5 \cdot 89 a_3^2 q_4 - 2^2 \cdot 3 \cdot 143 a_3^2 a_3 - 2^4 \cdot 3 \cdot 5 a_4^2),
\]

\[
f_{3,0} = 18 (421 a_3 I_1 + 2 \cdot 3 \cdot 23 a_3^2 q_5 - 2 \cdot 5 \cdot 127 a_2 a_3 q_5 + 2^3 \cdot 3 \cdot 53 a_3^2),
\]

\[
f_{3,1} = 6 (-3 \cdot 7^2 - 3 \cdot 19 q_4 I_5 - 3 \cdot 7 \cdot 167 + 2 \cdot 11 \cdot 251 - 2^3 \cdot 3^3 \cdot 53 + 2^8 \cdot 3 \cdot 143 a_3^2 a_3),
\]

\[
f_{4,0} = 12 (-2^4 \cdot 11 \cdot 13 a_2 I_1 + 2 \cdot 3^2 \cdot 3 a_2 q_5 + 2^8 \cdot 3^2 \cdot 5 q_4^2 + 2^8 \cdot 5 \cdot 59 a_3^2 q_4 - 2^2 \cdot 3 \cdot 143 a_2 a_3^2),
\]

\[
f_{5,1} = 12 (2^2 \cdot 157 + 2 \cdot 3^2 \cdot 11 a_2 q_4 + 2^8 \cdot 3^2 \cdot 53 a_3^2),
\]

\[
f_{5,2} = 12 (619 - 3^2 \cdot 829 - 2^4 \cdot 3 \cdot 2 \cdot 667 + 2^2 \cdot 3 \cdot 461 - 2 \cdot 3 \cdot 143 a_3^2),
\]

\[
f_{5,3} = 36 (3 \cdot 3 \cdot 5 \cdot a_3 q_5 + 3 \cdot 7 \cdot 11 a_3 q_4),
\]

\[
f_{5,4} = 36 (-1093 + 3 \cdot 47 + 2 \cdot 143 a_3^2 q_3),
\]

\[
f_{5,5} = 36 (7 \cdot 389 - 3 \cdot 11 \cdot 17 - 2^2 \cdot 3 \cdot 53),
\]

\[
f_{5,6} = 36 (5 \cdot 101 - 3^2 \cdot 79 - 2 \cdot 3 \cdot 53),
\]

\[
f_{5,1} = 12 (2 \cdot 3 \cdot 197 I_2 - 2^3 \cdot 7^2 \cdot 11 a_2 q_4 + 2^8 \cdot 3^2 \cdot 53 a_3^2),
\]

\[
f_{6,1} = 12 (2 \cdot 3 \cdot 197 - 2 \cdot 5 \cdot 137 + 2^3 \cdot 3^3 \cdot 53 - 2 \cdot 3 \cdot 143 a_2^2),
\]

\[
f_{6,2} = 12 (2^3 \cdot 5 \cdot 3 \cdot 53 - 2^2 \cdot 19 \cdot 31 - 2^3 \cdot 3^3 \cdot 53 + 2 \cdot 3 \cdot 143),
\]

\[
f_{6,3} = 12 (2 \cdot 3^2 \cdot 5^2 \cdot 3^3 \cdot 3 \cdot 53 - 2 \cdot 3 \cdot 143),
\]

\[
f_{6,4} = 2^4 \cdot 3^3 \cdot 317 q_5 - 2^6 \cdot 3^2 \cdot 151 a_2 a_3,
\]

\[
f_{6,5} = 2^4 \cdot 3^3 \cdot 5 \cdot 7 q_5 - 2^9 \cdot 3^2 a_2 a_3,
\]

\[
f_{6,6} = 2^6 \cdot 3^2 \cdot 11 q_4 + 2^5 \cdot 3^2 \cdot 143 a_2^2,
\]

\[
f_{6,7} = 2^5 \cdot 3^2 \cdot 7 q_4 + 2^5 \cdot 3^2 \cdot 143 a_2^2,
\]

\[
f_{6,8} = 2^6 \cdot 3^2 \cdot 143 a_2^2.
\]
Thus the coefficient of $\xi_1$ in the collineation can be expressed in terms of $x_1, K_2, K_3, K_4, K_5$ linearly with coefficients which are functions of $a_2, a_3, q_4, q_5, I_1$ and $b_1, b_1^2, \ldots, b_1^5$.

As for the second invariant the sum $K_{1,1} + \ldots + K_{6,6}$ if not identically zero would furnish the self-apolarity invariant of the collineation which we could define to be $I_2$.

The result of operating with $C$ on $Q$ is a $C_{10,0,1}$, i.e., the second linear covariant $L_2$. Hence,

$$\frac{5}{2} L_2 = \sum_{(1)} (6 C_1 S_1 + \sum_{(2)} C_2 S_{12}) x_1.$$  (35)

The collineation $K$ sends $L_1$ into the third linear covariant $L_3$. This likewise is sent into the fourth linear covariant $L_4$ by the same collineation.

The three remaining invariants $I_3, I_4, I_5$ are obtained by operating with $C$ on $L_1, L_2, L_3$, respectively.