Degree Complexity of a Family of Birational Maps: II. Exceptional Cases

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Abstract We determine the degree complexity for all elements of a family $k_F$ of birational maps which was introduced and studied in Bedford et al. (Math Phys Anal Geom 11:53–71, 2008).

Keywords Birational maps · Degree complexity

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1 Introduction

Let $\mathbb{P}^2$ denote the complex projective space, and let $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be a rational map. We will consider its iterates $f^n = f \circ f \circ \cdots \circ f$. A basic invariant of iteration is the degree complexity, or the exponential rate of growth:

$$\delta(f) = \lim_{n \to \infty} \left( \frac{\deg(f^n)}{n} \right).$$

Here we consider the family of birational maps $k_F$ defined in Section 2 below for an arbitrary polynomial $F$. If we regard

$$F_a(w) = a_0 + a_1 w + \cdots + a_N w^N$$

as depending on the complex parameters $a = (a_0, \ldots, a_N) \in \mathbb{C}^{N+1}$, then the dependence $a \mapsto \delta(k_{F_a})$ is lower semi-continuous in the Zariski topology. This means that the set $\{a : \delta(k_{F_a}) \leq t\}$ is an algebraic variety for all $t$. In particular, the value of $\delta(k_{F_a})$ is equal to a constant value $\delta^*_N$ outside a proper subvariety of $\mathbb{C}^{N+1}$. 

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A parameter $a$ is said to be exceptional if $\delta(k_{F_a}) < \delta^*_N$. Exceptional maps are of special interest because the lower degree growth indicates the presence of internal symmetries and non-generic behaviors. Such symmetries often make $\delta$ more difficult to compute. For instance, there is a birational map $K$ on the projectivized space of $q \times q$ matrices (see [8], and [11]). The degree growth of the restriction of the map $K$ to the space of cyclic matrices was shown to be the largest root of the polynomial $x^2 - (q^2 - 4q + 2)x + 1$ (see [9]). However, the degree growth of the same map $K$, restricted to the smaller space of cyclic, symmetric matrices, depends in a much more complicated way on the number $q$ (for primes $q$ it was determined in [4], and for general $q$ it was determined in [6]).

In the case of the family $k_{F_a}$, the numbers $\delta^*_N$ were determined in [7]. Here we consider the map $a \rightarrow \delta(k_{F_a})$ for the full family; we determine the exceptional values as well as the associated rates of degree growth.

**Theorem 1** Suppose that $F_a$ is as above, and $N = \deg(F_a)$ is even. If $a_0 = 2/(m + 1)$ for some integer $m \geq 0$, then $\delta(k_{F_a})$ is the largest root of the polynomial $x^{2m+1}(x^2 - (N + 1)x - 1) + x^2 + N$. Otherwise, $\delta(k_{F_a}) = \delta^*_N$ is the largest root of $x^2 - (N + 1)x - 1$.

The behavior of this family is more complicated when the degree $N$ is odd. For instance, we have

**Theorem 2** Suppose that $F_a$ is as above, and $N = 3$. Then we have the following cases:

**Case 1:** $a_2 \neq a_3$.
- If $a_0 = 2/(1 + m)$ for some integer $m \geq 0$, then $\delta(k_{F_a})$ is the largest root of the polynomial $x^{2m+1}(x^3 - 3x^2 - 4x - 1) + x^3 + x^2 + 3x + 2$. Otherwise, $\delta(k_{F_a}) = \delta^*_3$ is the largest root of the polynomial $x^3 - 3x^2 - 4x - 1$.

**Case 2:** $a_2 = a_3$.
- 2a. If $a_0 = 2$, then $k_{F_a}$ is an automorphism, $\delta(k_{F_a}) = 1$. Moreover the degree growth is quadratic.
- 2b. If $a_0 = 2/(1 + m)$ for some integer $m \geq 1$, then $\delta(k_{F_a})$ is the largest root of the polynomial $x^{2m}(x^3 - 3x^2 - 2x - 1) + x^2 + x + 3$.
- 2c. If $a_0 = 2 + \frac{1}{2(l + 1)}$ for some integer $l \geq 1$, then $\delta(k_{F_a})$ is the largest root of the polynomial $x^{2l+2}(x^3 - 3x^2 - 2x - 1) + 3x^2 + x + 1$.
- 2d. Otherwise, $\delta(k_{F_a})$ is the largest root of the polynomial $x^3 - 3x^2 - 2x - 1$.

So we see that for case $N = 3$, there are (infinitely many) linear functions $L_1, L_2, L_3, \ldots$ depending on the variable $a = (a_0, a_1, a_2, a_3)$, and the different cases are determined by conditions of the form $L_s(a) = 0$ for certain values of $s$, and $L_t(a) \neq 0$ for certain values of $t$. Thus the sets of exceptional parameters are constructed by linear functions.
We will find in Section 5 that this is typical of the general case for $N$ odd. We also find that there are no automorphisms in the family $k_{F_0}$ other than the ones given in [7].

One difference between the cases when degree $N$ is even or odd is the following. When $N$ is even, the exceptional cases are characterized by a single condition whether $a_0 = 2/(1 + m)$ for some integer $m \geq 0$ or not. When $N$ is odd there is in addition other conditions for exceptional cases, the number of these exceptional conditions are $(N + 3)/2$. In the proofs of Theorems 1 and 3, following the general frame of Diller and Favre in [12] for working with birational maps of a surface, we will construct spaces $Z$ which is a composition of finite point-blowups of $\mathbb{P}^2$, whose induce map $k_Z$ is good (say, A.S. or 1-regular, see [13] for details). We mention here a special phenomena that happens when $N$ is odd: if $j$ exceptional conditions are satisfied, we need to construct spaces $Z_1, \ldots, Z_j$ where each $Z_l$ is a composition of two point-blowups of $Z_l$. In other words, if $N$ is odd, when a new exceptional condition occurs, we need to blowups two more points.

2 Properties of $k_F$

With $F$ as in (1.2), we define two involutions:

$$j_F(x, y) = (-x + F(y), y), \quad i(x, y) = \left(1 - x - \frac{x - 1}{y}, -y - 1 - \frac{y}{x - 1}\right).$$

and we set $k = k_F = j_F \circ i$.

We recall the following sets from [7]:

$$C_1 = \{x_0 = 0\}, \quad C_2 = \{x_0 = x_1\}, \quad C_3 = \{x_2 = 0\},$$

$$C_4 = \{-x_0^2 + x_0x_1 + x_1x_2 = 0\},$$

$$C'_1 = C_1, \quad C'_2 = \left\{1 + \frac{x_1}{x_0} - F\left(\frac{x_2}{x_0}\right) = 0\right\}, \quad C'_3 = C_3,$$

$$C'_4 = \left\{\frac{x_2}{x_0} - \left(1 + \frac{x_2}{x_0}\right) \left(1 + \frac{x_1}{x_0} - F\left(\frac{x_2}{x_0}\right)\right) = 0\right\}.$$

The exceptional hypersurfaces of $k_F$ are mapped as

$$k_F : C_4 \mapsto [1 : -1 + a_0 : 0] \in C_3, \quad C_1 \cup C_2 \cup C_3 \mapsto e_1.$$

The points of indeterminacy of $k_F$ are $e_1 = [0 : 1 : 0], e_2 = [0 : 0 : 1]$, and $e_{01} = [1 : 1 : 0]$. The exceptional curves for $k_F^{-1}$ are mapped as (Fig. 1)

$$k_F^{-1} : C'_1 \cup C'_3 \mapsto e_1, \quad C'_2 \mapsto e_2, \quad C'_4 \mapsto e_{01}.$$
3 Degree $N$ is Even

Let us start by recalling the space $X$ constructed in Section 3 of [7]. We define the complex manifold $\pi : X \rightarrow \mathbb{P}^2$ (see Figure 3.1 in [7]) by blowing up points $e_1, p_1, \ldots, p_{N-1}$ in the following order:

i) blowup $e_1 = [0 : 1 : 0]$ and let $E_1$ denote the exceptional fiber over $e_1$,

ii) blowup $q = E_1 \cap C_4$ and let $Q$ denote the exceptional fiber over $q$,

iii) blowup $p_1 = E_1 \cap C_1$ and let $P_1$ denote the exceptional fiber over $e_1$,

iv) blowup $p_j = P_{j-1} \cap E_1$ with exceptional fiber $P_j$ for $2 \leq j \leq N - 1$.

Here we use the notational convention that if $S$ is a curve at one stage of the construction, then $S$ will denote its strict transforms at subsequent stages (Fig. 2).
The coordinate projection at $P_j$ ($1 \leq j \leq N - 1$) is chosen as follows

$$\pi_j : X \ni (s, u) \mapsto [s^{j+1}u : 1 : s^j u] \in \mathbb{P}^2.$$  

In this coordinate $P_j = \{s = 0\}$. For convenience we will use the notations $u \in P_j$ or $[u]_{P_j}$ to indicate the point of $P_j$ which has coordinate $(0, u)$ in this coordinate projection.

Let $k_X := \pi_X^{-1} \circ k_F \circ \pi_X$ denote the induced birational map of $X$. The exceptional curves for $k_X$ are $C_1, C_2, C_4, P_1, \ldots, P_{N-2}$. The curves $C_1, C_2, P_1, \ldots, P_{N-2}$ are mapped to the same point $1/a_N \in P_{N-1}$, while $C_4$ is mapped to the point $[1 : -1 + a_0 : 0] \in C_3$. By Lemmas 3.2 and 3.3 in [7], the only way that an exceptional curve can be mapped to a point of indeterminacy is that $a_0 = 2/(m + 1)$ for some integer $m \geq 0$, and in this case we have $k_X^{2m+1}C_4 = [1 : 1 : 0]$.

If $a_0 = 2/(m + 1)$ we construct the new manifold $Z$ by blowing up the manifold $X$ at the points

$$r_0 = k_X(C_4) = [1 : -1 + a_0 : 0] \in C_3,$$

$$q_1 = k_X(r_0) \in Q, \quad r_1 = k_X(q_1) \in C_3,$$

$$\ldots$$

$$q_m = k_X(r_{m-1}) \in Q, \quad r_m = k_X(q_m) = [1 : 1 : 0] = e_{01} \in C_3.$$  

Call $R_0, Q_1, R_1, \ldots, Q_m, R_m$ the exceptional fibers of this blowup. Let $k_Z$ be the induced birational map of $Z$ (Fig. 3).

**Lemma 1** If $a_0 = 2/(m + 1)$ then the curves $C_4, R_0, Q_1, R_1, \ldots, Q_m, R_m$ are not exceptional for $k_Z$.

**Proof** It suffices to check that $C_4$ is not exceptional. We choose a local projection for $R_0$ as

$$Z \ni (s, u) \mapsto [1 : -1 + a_0 + su : s].$$

---

**Fig. 3** The space $Z$ when $a_0 = 2/(m + 1)$. New exceptional fibers are lying on $e_{01} = r_m$ and its pre-images.
In this coordinate chart $R_0 = \{ s = 0 \}$. If we rewrite $k[x_0 : x_1 : x_2]$ as

$$k[x_0 : x_1 : x_2] = \left[ 1 : -1 - \frac{x_0^2 - x_0x_1 - x_1x_2}{x_0x_1} + F \left( \frac{x_0^2 - x_0x_1 - x_1x_2}{x_0(x_1 - x_0)} \right) : \frac{x_0^2 - x_0x_1 - x_1x_2}{x_0(x_1 - x_0)} \right]$$

then it can be seen that

$$k_Z : C_4 \ni [x_0 : x_1 : x_2] \mapsto a_1 + \frac{x_1}{x_0} \in R_0.$$ 

Hence $C_4$ is not exceptional. \Box

The induced map $k_Z$ acts as follows

$$k_Z : E_1 \mapsto E_1, \quad P_{N-1} \mapsto P_{N-1}, \quad C_1, C_2, P_1, \ldots, P_{N-2} \mapsto \frac{1}{a_N} \in P_{N-1},$$

$$Q \mapsto C_3 \mapsto Q,$$

$$k_Z : C_4 \mapsto R_0 \mapsto Q_1 \mapsto R_1 \mapsto \ldots \mapsto Q_m \mapsto R_m \mapsto C_4,$$

$$k^{-1}_Z : C_1, P_1, \ldots, P_{N-1} \mapsto -\frac{1}{a_N} \in P_{N-1}.$$

If $S$ is a curve in $Z$, we will use the notation $S$ to denote its class in $\text{Pic}(Z)$. Let $H \in \text{Pic}(Z)$ denote the class of a generic line. Then $H, E_1, P_1, \ldots, P_{N-1}, Q, Q_1, \ldots, Q_m, R_0, \ldots, R_m$ form an ordered basis for the space $\text{Pic}(Z)$. The curves $C_1, C_2, C_3, C_4$ can be represented in this basis as

$$C_1 = H - E_1 - Q - \sum_{j=1}^{N-1} (j + 1) P_j - \sum_{j=1}^{m} Q_j,$$

$$C_2 = H - R_m,$$

$$C_3 = H - E_1 - Q - \sum_{j=1}^{N-1} j P_j - \sum_{j=1}^{m} Q_j - \sum_{j=0}^{m} R_j,$$

$$C_4 = 2H - E_1 - 2Q - \sum_{j=1}^{N-1} j P_j - 2 \sum_{j=1}^{m} Q_j - R_m.$$
From this, we see that $k^*_Z : Pic(Z) \to Pic(Z)$ is as follows

$$k^*_Z(H) = (2N + 1)H - NE_1 - (N + 1)Q - (N + 1) \sum_{j=1}^{N-1} jP_j$$
$$- (N + 1) \sum_{j=1}^m Q_j - (N + 1)R_m,$$

$k^*_Z(E_1) = E_1,$

$k^*_Z(Q) = C_3 = H - E_1 - Q - \sum_{j=1}^{N-1} jP_j - \sum_{j=1}^m Q_j - \sum_{j=0}^m R_j,$

$k^*_Z(P_j) = 0, \ 1 \leq j \leq N - 2,$

$k^*_Z(P_{N-1}) = C_1 + C_2 + \sum_{j=1}^{N-1} P_j = 2H - E_1 - Q - \sum_{j=1}^{N-1} jP_j - \sum_{j=1}^m Q_j - R_m,$

$k^*_Z(R_0) = C_4 = 2H - E_1 - 2Q - \sum_{j=1}^{N-1} jP_j - 2\sum_{j=1}^m Q_j - R_m,$

$k^*_Z(R_j) = Q_j, \ 1 \leq j \leq m, \ k^*_Z(Q_j) = R_{j-1}, \ 1 \leq j \leq m.$

**Proof of Theorem 1** If $a_0 \neq 2/(1 + m)$ for any integer $m \geq 0$, $\delta(k_F)$ was computed in [7]. It was shown in this case that $\delta(k_F)$ is the largest root of $x^2 - (N + 1)x - 1$.

Let us suppose now that $a_0 = 2/(1 + m)$ for some integer $m \geq 0$. Then by Lemma 1, we see that for every exceptional curve $\Gamma$, the images $k^j_Z(\Gamma), \ j \geq 1$, are disjoint from the determinacy locus. It follows that $(k^m_Z)^* = (k^*_Z)^m$ for all integer $n \geq 1$. It follows that $\delta(k_F)$ is the spectral radius of $k^*_Z$. Thus it is the largest root of the characteristic polynomial of $k^*_Z$, which is

$$P(x) = -x[x^{2m+1}(x^2 - (N + 1)x - 1) + x^2 + N].$$

\[\square\]

**4 Degree $N = 3$**

In this section we will prove Theorem 2. First we consider the more general case where $N \geq 3$ is an odd number. First we recall the construction of spaces $Y$ and $Y_1$ constructed in [7]. We start from the space $X$ constructed in the
previous section. Then the line $C_1$ and all blowup fibers $P_j$ ($1 \leq j \leq N - 2$) are all exceptional for both $k_X$ and $k_X^{-1}$. Moreover $C_2$ is exceptional for $k_X$:

$$k_X : C_1, C_2, P_1, \ldots, P_{N-2} \mapsto \frac{1}{a_N} \in P_{N-1},$$

$$k_X^{-1} : C_1, P_1, \ldots, P_{N-2} \mapsto \frac{1}{a_N} \in P_{N-1}.$$ 

Hence when $N$ is odd the image of all exceptional curves of $k_X$ coincide with a point of indeterminacy $\zeta_0 = \frac{1}{a_N} \in P_{N-1}$. Let $\pi_Y : Y \to \mathbb{P}^2$ be the blowup of $X$ at the point $\zeta_0 \in P_{N-1}$, and let $P_N$ be the exceptional fiber. At $P_N$ we use the coordinate projection (Fig. 4)

$$\pi_N : (u, s) \in Y \mapsto \left[ s^N (su + \zeta_0) : 1 : s^{N-1} (su + \zeta_0) \right] \in \mathbb{P}^2.$$

We set $Y_1 = Y$ if $a_0 \neq \frac{2}{m+1}$ for every integer $m \geq 0$. Otherwise, as in the previous section, define $\pi_1 : Y_1 \to \mathbb{P}^2$ to be the blowup of $Y$ at the points

$$r_0 = [1 : -1 + a_0 : 0] \in C_3,$$

$$q_1 = k_X(r_0) \in Q, r_1 = k_X(q_1) \in C_3,$$

$$\ldots$$

$$q_m = k_X(r_{m-1}) \in Q, r_m = k_X(q_m) = [1 : 1 : 0] \in C_3,$$

and call $R_0, Q_1, R_1, \ldots, Q_m, R_m$ the exceptional fibers of this blowup (Fig. 5).

**Lemma 2** $k_Y$ maps $P_N \leftrightarrow P_{N-2}$ by the following formulas

$$P_N \ni u \mapsto \frac{1}{-a_N^2 u - (N - 1)a_N + a_{N-1}} \in P_{N-2},$$

$$P_{N-2} \ni u \mapsto \frac{1 + a_{N-1} u}{a_N^2 u} \in P_N.$$ 

**Fig. 4** The space $Y$ which is the blowup of $X$ at a point on $P_{N-1}$
Proof (Sketch) First, write

\[ k\left[ s^N (su + \zeta_0) : 1 : s^{N-1} (su + \zeta_0) \right] = [z_0(s, u) : z_1(s, u) : z_2(s, u)], \]

then if \( u \in P_N \) its image \( w \in P_{N-2} \) under \( k_{Y_1} \) can be computed as

\[ w = \lim_{s \to 0} \frac{z_2^{N-1}}{z_0^{N-2} z_1}. \]

Now to compute the image of \( w \in P_{N-2} \) we do as follows: If \( u \in P_N \) then applying the above argument to the inverse map \( k^{-1} \) will give its image \( w = g(u) \in P_{N-2} \) under the map \( k_{Y_1}^{-1} \). Now the inverse \( u = g^{-1}(w) \) is the image of \( w \in P_{N-2} \) under the map \( k_{Y_1} \).

\[ \square \]

In a similar way, we also have

**Lemma 3** If we set

\[ \xi_1 = -\frac{a_{N-1}}{a_N^2}, \]
\[ \xi_1 = \frac{-(N-1)a_N + a_{N-1}}{a_N^2}, \]

then

\[ k_{Y_1} : C_1, C_2, P_1, P_2, \ldots, P_{N-3} \mapsto \xi_1 \in P_N, \]
\[ k_{Y_1}^{-1} : C_1, P_1, P_2, \ldots, P_{N-3} \mapsto \xi_1 \in P_N. \]

Hence the map \( k_{Y_1}^2 : P_N \to P_N \) is

\[ P_N \ni u \mapsto u + \frac{(N-1)a_N - 2a_{N-1}}{a_N^2} = u + \xi_1 - \xi_1 \in P_N. \] (4.1)
From (4.1), we see that the orbit of $\zeta_1$ (hence also the orbit of all exceptional curves of $kY_1$) is

\[ k_{Y_1}^{2m}(\zeta_1) = \zeta_1 + m(\zeta_1 - \xi_1) \in P_N. \tag{4.2} \]

Hence the orbit of all exceptional curves of $kY_1$ will contain a point of indeterminacy iff that indeterminacy point is $\xi_1$, that is iff $\zeta_1 + m(\zeta_1 - \xi_1) = \xi_1$. The last condition is satisfied iff $\xi_1 = \zeta_1$, that is iff the coefficients $a_N$ and $a_{N-1}$ of the polynomial $F_N(z)$ satisfy the linear equation

\[-a_{N-1} = -(N-1)a_N + a_{N-1}.\]

Hence if $a_{N-1} \neq \frac{(N-1)a_N}{2}$ then the map $kY_1$ satisfies the condition $(k_{Y_1}^n)^* = (k_{Y_1}^*)^n$ for all integer $n \geq 1$, while if $a_{N-1} = \frac{N-1)a_N}{2}$ then the image of all exceptional curves of $kY_1$ is the point of indeterminacy $\zeta_1 = \xi_1$.

**Proof of Theorem 2** Let $Y_1$ be as above. Since $N = 3$ we have

\[
\zeta_1 = -\frac{a_2}{a_3}, \quad \xi_1 = -2a_3 + a_2 = \frac{a_2}{a_3}.
\]

Then (4.2) becomes

\[
k_{Y_1}^{2m}\left(-\frac{a_2}{a_3}\right) = -\frac{a_2}{a_3} - m \frac{2a_2 - 2a_3}{a_3}. \tag{4.3}
\]

\[\square\]

In Case 1: $a_2 \neq a_3$, it follows that the orbit of exceptional curves of $kY_1$ does not contain a point of indeterminacy. Thus $(k_{Y_1}^n)^* = (k_{Y_1}^*)^n$ for all integer $n \geq 1$, and so $\delta(k_F)$ is the spectral radius of $k_{Y_1}$, which is the largest root of the polynomial given in the statement of Theorem 2.

In Case 2: $a_2 = a_3$, we have $\zeta_1 = -\xi_1 = -\frac{1}{a_3}$. Hence $\zeta_1$ and $\xi_1$ are both the image of exceptional curves $C_1, C_2$ of $kY_1$, and the image of the exceptional curve $C_1$ of $k_{Y_1}^{-1}$. We define a complex manifold $\pi_{Y_2}: Y_2 \to \mathbb{P}^2$ by blowing up $Y_1$ at the point $-\frac{1}{a_3} \in P_3$, and call $P_4$ the exceptional fiber of this blowup. We use a local coordinate projection at $P_4$ as follows:

\[
\pi_4: Y_2 \ni (s, u) \mapsto \left[ s^3 \left( s^2 u - \frac{1}{a_3} s + \frac{1}{a_3} \right) : 1 : s^2 \left( s^2 u - \frac{1}{a_3} s + \frac{1}{a_3} \right) \right] \in \mathbb{P}^2.
\]
The induced map $k_{Y_2}$ is as follows:

\[
k_{Y_2} : P_4 \ni u \mapsto \left[ 0 : 1 : \frac{1}{\text{-}a_3 + a_1 + a_2^2 u} \right] \in C_1,
\]

\[
k_{Y_2} : C_1 \ni [0 : 1 : u] \mapsto \frac{1 + (a_3 - a_1)u}{a_3 u} \in P_4,
\]

\[
k_{Y_2} : C_2 \mapsto \left[ \frac{a_3 - a_1}{a_3^2} \right] \mapsto [0 : 0 : 1] = e_2.
\]

Thus the orbit of the exceptional curve $C_2$ encounters an indeterminacy point.

Let $\pi_{Y_3} : Y_3 \to \mathbb{P}^2$ be the complex manifold obtained by blowing up $Y_2$ at two points $e_2 = [0 : 0 : 1]$ and $\frac{a_3 - a_1}{a_3^2} \in P_4$, and let $E_2$ and $P_5$ be the exceptional fibers of this blowup. We use a local coordinate projection at $P_5$ as

\[
Y_3 \ni (s, u) \mapsto \left[ s^3 \left( s^3 u + s^2 \frac{a_3 - a_1}{a_3^2} - s \frac{1}{a_3} + \frac{1}{a_3} \right) : 1 : s^2 \left( s^3 u + s^2 \frac{a_3 - a_1}{a_3^2} - s \frac{1}{a_3} + \frac{1}{a_3} \right) \right],
\]

and use a local coordinate projection at $E_2$ as (Fig. 6)

\[
E_2 \ni (s, u) \mapsto [s : su : 1] \in \mathbb{P}^2.
\]

Then the induced map $k_{Y_3}$ is as follows:

\[
k_{Y_3} : P_5 \ni u \mapsto -a_3^2 u - a_3 + 2a_1 + a_0 - 4 \in E_2,
\]

\[
k_{Y_3} : E_2 \ni u \mapsto -u - a_3 + 2a_1 - a_0 + \frac{1}{a_3^2} \in P_5,
\]

\[
k_{Y_3} : E_2 \ni u \mapsto u + 2a_0 - 5 \in E_2,
\]

\[
k_{Y_3} : C_2 \mapsto \left[ -\frac{a_3 - 2a_1 + a_0}{a_3^2} \right]_{P_5} \mapsto 2a_0 - 4 \in E_2.
\]

**Fig. 6** The space $Y_3$ in case $N = 3$ and $a_2 = a_3$
The point $0 \in E_2$ is a point of indeterminacy for $k_{Y_3}$.

2a. If $a_0 \neq 2 + \frac{l}{2(l+1)}$ for $l \geq 0$, then from (4.4), the orbit of the exceptional curve $C_2$ of $k_{Y_3}$ does not contain the point of indeterminacy $0 \in E_2$. It follows $(k_{Y_3}^n)^* = (k_{Y_3}^*)^n$ for all integer $n \geq 1$. Then a computation of $k_{Y_3}^*$ on $H^{1,1}(Y_3)$ similar to that of Section 3 completes the proof of Theorem 4 for this case.

2b. If $a_0 = 2 + \frac{l}{2(l+1)}$ for an integer $l \geq 0$, then from (4.4) it follows that the orbit of $C_2$ contains the point of indeterminacy $0 \in E_2$. We define a complex manifold $\pi : Z \to \mathbb{P}^2$ by blowing up $Y_3$ at the points

$$ p_6 = \left[-\frac{a_3 - 2a_1 + a_0}{a_3^2}, \right], \quad s_0 = k_{Y_3}(s_0) = [2a_0 - 4]_{E_2}, $$

$$ s_1 = k_{Y_3}^2(s_0), \ldots, s_{2l} = k_{Y_3}^{2l+1}(s_0) = [0]_{E_2}, $$

and let $P_6, S_0, S_1, \ldots, S_{2l}$ the exceptional fibers of this blowup. Then, as in the proof of Lemma 1, it can be shown that the curves $C_2, P_6, S_0, \ldots, S_{2l}$ are not exceptional for $k_Z$. It follows $(k_{Z}^n)^* = (k_{Z}^*)^n$ for all integer $n \geq 1$. Then a computation of $k_{Z}^*$ on $H^{1,1}(Z)$ similar to that of Section 3 completes the proof of Theorem 2 for this case (Fig. 7).

5 Degree $N$ is Odd

In this section we will describe the degree complexities of all elements of the family $k_F$ having odd degrees.

For fixed $N$, define for $0 \leq j \leq N$

$$ L_j(a_0, a_1, \ldots, a_N) = (a_{N-j} + a_{N-j+1}) - \sum_{l=0}^{j} (-1)^l a_{N-l} \binom{N-l}{j-l}, $$

(5.1)
where \( \binom{n}{j} \) is the binomial coefficient.

These linear functions will determine all exceptional parameters of the family \( k_F \) when \( \deg(F) = N \) is odd.

**Theorem 3** Suppose that \( N = \deg(F) \geq 3 \) is odd. Define \( h \) as the largest integer in \([0, N - 2]\) for which

\[
L_j(a_0, a_1, \ldots, a_n) = 0
\]

for all \( 0 \leq j \leq h \). Then exactly one of the following occurs:

**Case 1:** \( h < N - 2 \).

If \( a_0 = 2/(1 + m) \) for some integer \( m \geq 0 \), then \( \delta(k_F) \) is the largest real root of the polynomial

\[
(1 + x^{2m+1})(x^3 - Nx^2 - (N - h + 1)x - 1) + (N + 1)x^2 + (2N - h + 1)x + N - h.
\]

Otherwise \( \delta(k_F) \) is the largest real root of the polynomial \( x^3 - Nx^2 - (N + 1 - h)x - 1 \).

**Case 2:** \( h = N - 2 \).

2a. If \( a_0 = 2/(1 + m) \) for some integer \( m \geq 0 \), and \( a_0 = \frac{N+1}{2} + \frac{l}{2(1+l)} \) for some integer \( l \geq 0 \), then \( N = 3 \), \( a_0 = 2 \), and the map \( k_F \) is an automorphism with \( \delta(k_F) = 1 \). Moreover the degree growth is quadratic.

In the remaining cases, we assume that \( N \geq 5 \).

2b. If \( a_0 = 2/(1 + m) \) for some integer \( m \geq 0 \), then \( \delta(k_F) \) is the largest real root of the polynomial \( x^{2m}(x^3 -Nx^2 - 2x - 1) + x^2 + x + N \).

2c. If \( a_0 = \frac{N+1}{2} + \frac{l}{2(1+l)} \) for some integer \( l \geq 0 \), then \( \delta(k_F) \) is the largest real root of the polynomial \( x^{2l+2}(x^3 - Nx^2 - 2x - 1) + Nx^2 + x + 1 \).

2d. Otherwise, \( \delta(k_F) \) is the largest real root of the polynomial \( x^3 - Nx^2 - 2x - 1 \).

The proof of this theorem will be given in Appendix 2, but here we discuss how the linear functions \( L_j \) are derived.

Since \( F_N(z) \) is a polynomial of degree \( N \), the function

\[
s^N F \left( -1 - \frac{1}{s} \right) + (1 + s)s^N F \left( \frac{1}{s} \right)
\]

is a polynomial of degree \( N + 1 \), and we have

\[
s^N F \left( -1 - \frac{1}{s} \right) + (1 + s)s^N F \left( \frac{1}{s} \right) = a_0 s^{N+1} + \sum_{j=0}^{N} L_j(a_0, \ldots, a_N)s^j.
\]
The numbers $\zeta_1$ and $\xi_1$ in the previous section can be constructed as follows:

\[
\frac{1}{a_N^3 s} \left[ \frac{1 + s}{\xi_0 + su} - (1 + s)s^N F \left( \frac{1}{s} \right) \right] = \zeta_1 - u + O(s),
\]

\[
\frac{1}{a_N^2 s^2} \left[ \frac{1 + s}{\xi_0 + \xi_1 s + s^2 u} - (1 + s)s^N F \left( \frac{1}{s} \right) \right] = \xi_1 - u + O(s).
\]

Then

\[
\zeta_1 - \xi_1 = -\frac{1}{a_N^2 s} \left[ s^N F \left( -1 - \frac{1}{s} \right) + (1 + s)s^N F \left( \frac{1}{s} \right) \right] + O(s).
\]

Hence $\zeta_1 - \xi_1 = -L_1(a_0, \ldots, a_N)/a_N^2$, so the vanishing of $L_1$ corresponds to the case $\zeta_1 = \xi_1$.

If $\zeta_1 = \xi_1$ and $N \geq 5$, define complex numbers $\zeta_2$ and $\xi_2$ as follows

\[
\frac{1}{a_N^3 s^2} \left[ \frac{1 + s}{\xi_0 + \xi_1 s + s^2 u} - (1 + s)s^N F \left( \frac{1}{s} \right) \right] = \zeta_2 - u + O(s),
\]

\[
\frac{1}{a_N^2 s^2} \left[ \frac{1 + s}{\xi_0 + \xi_1 s + s^2 u} + s^N F \left( -1 - \frac{1}{s} \right) \right] = \xi_2 - u + O(s).
\]

Then $\zeta_2 - \xi_2 = -L_2(a_0, \ldots, a_N)/a_N^2$ ($\zeta_2$ and $\xi_2$ will play the similar roles to that of $\zeta_1$ and $\xi_1$). However $L_2 = nL_1/2 = 0$ hence $\zeta_2 = \xi_2$. Then, if we define $\zeta_3$ and $\xi_3$ by

\[
\frac{1}{a_N^3 s^3} \left[ \frac{1 + s}{\xi_0 + \xi_1 s + \xi_2 s^2 + s^3 u} - (1 + s)s^N F \left( \frac{1}{s} \right) \right] = \zeta_3 - u + O(s),
\]

\[
\frac{1}{a_N^2 s^3} \left[ \frac{1 + s}{\xi_0 + \xi_1 s + \xi_2 s^2 + s^3 u} + s^N F \left( -1 - \frac{1}{s} \right) \right] = \xi_3 - u + O(s).
\]

we have $\zeta_3 - \xi_3 = -L_3(a_0, \ldots, a_N)/a_N^3$. Note that now $L_3$ is not a linear combination of $L_1$ and $L_2$; hence in general $\zeta_3 \neq \xi_3$.

Continuing, we assume that $\zeta_1 = \xi_1, \zeta_2 = \xi_2, \zeta_3 = \xi_3$ and $N \geq 7$. Then we can define $\zeta_4$ and $\xi_4$ in the same manner, and $\zeta_4 - \xi_4 = -L_4(a_0, \ldots, a_N)/a_N^3$. Note in this case that $L_4$ is a linear combination of $L_1, L_2$ and $L_3$. Hence $\zeta_4 = \xi_4$. We can continue defining $\zeta_5$ and $\xi_5$, and have that $\zeta_5 - \xi_5 = -L_5(a_0, \ldots, a_N)/a_N^3$. Moreover $L_5$ is not a linear combination of $L_1, L_2, L_3$ and $L_4$.

In Appendix 1 we will show that $L_2j$ is a linear combination of $L_1, \ldots, L_{2j-1}$, while $L_{2j+1}$ is not a linear combination of $L_1, \ldots, L_{2j}$ for all $j$.

As a consequence of Theorem 3, we show that there are no automorphisms in the family $k_F$ other than the ones given in [7]. The following lemma, which is used in the proof of Theorem 4, and also the proof of Theorem 4, are suggested to us by the referee.

**Lemma 4**

a) Let $f : \mathcal{Z} \to \mathcal{Z}$ be an automorphism of a surface $\mathcal{Z}$. Let $\delta(f)$ be the spectral radius of the linear map (which is also the complexity degree of $f$) $f^*$:
Let $f: \mathcal{Z} \to \mathcal{Z}$. Then either $\delta(f) = 1$, or the minimal polynomial $p(x)$ of $\delta(f)$ is symmetric. That is, if $d$ is the degree of $p(x)$ we have

$$x^d p(1/x) = p(x).$$

b) Let $f: \mathcal{Z} \to \mathcal{Z}$ be a birational map of a surface $\mathcal{Z}$. Assume that $f$ is 1-regular and that $f$ is birational equivalent to an automorphism. Let $\chi(f)(x)$ be the characteristic polynomial of the linear map $f^*: H^{1,1}(\mathcal{Z}) \to H^{1,1}(\mathcal{Z})$. Then roots of $\chi(f)(x)$ are $\delta(f)$, $1/\delta(f)$, and/or 0, and/or algebraic numbers of complex modulus 1.

In particular, let $g(x)$ be a factor of $\chi(f)(x)$ which is a monic polynomial and whose coefficients are integers and such that $g(0) \neq 0$. Then $g$ is either symmetric or anti-symmetric. That is if $d$ is the degree of $g(x)$ we have:

either

$$x^d g(1/x) = g(x),$$

or

$$x^d g(1/x) = -g(x).$$

Moreover if $g$ is anti-symmetric then $g(1) = 0$.

Proof We remark first that if $f: \mathcal{Z} \to \mathcal{Z}$ is a birational map of a surface $\mathcal{Z}$, then $\chi(f)(x)$ is a monic polynomial (that is it is a polynomial with integer coefficients, and its leading coefficient is 1). Hence if $\lambda \in \mathbb{Q}$ is a root of $\chi(f)(x)$ then either $\lambda \in \mathbb{N}$ or 0.

a) Since $f: \mathcal{Z} \to \mathcal{Z}$ is an automorphism, both $f$ and $f^{-1}$ are 1-regular. Moreover $(f^{-1})^* = (f^*)^{-1}$ and $\delta(f) = \delta(f^{-1})$. By $(f^{-1})^* = (f^*)^{-1}$ we have that if $\lambda$ is any root of $\chi(f)$ then $1/\lambda$ is a root of $\chi(f^{-1})$, and vice versa. In particular, $1/\delta(f)$ is also a root of $\chi(f)$.

We claim that roots of $\chi(f)$ are $\delta(f)$, $1/\delta(f)$, and/or algebraic numbers of complex norm 1. We have two cases:

–Case 1: $\delta(f) = 1$. Then $\chi(f)$ can not have roots $\lambda$ with $|\lambda| < 1$. Because otherwise then $1/\lambda$ is a root of $\chi(f^{-1})$ with $|1/\lambda| > 1$, which is a contradiction to $\delta(f^{-1}) = \delta(f) = 1$.

–Case 2: $\delta(f) > 1$. Then from Theorem 5.1 in [12], $\delta(f)$ is a root of multiplicity 1 and it is the only root $\lambda$ of $\chi(f)$ with $|\lambda| > 1$. The same is true for $\chi(f^{-1})$. Hence using the observation above about roots of $\chi(f)$ and $\chi(f^{-1})$ we conclude that $\chi(f)$ has no other roots of complex norm not equal to 1 other than $\delta(f)$ and $1/\delta(f)$.

Now we complete the proof of a). Assume that $\delta(f) > 1$. Then by the remark from the beginning of the proof, $\delta(f) \notin \mathbb{Q}$ (other wise $1 = 1/\delta(f)$ is also a rational root of $\chi(f)(x)$ which is contradict to that remark). Let $p(x)$ be the minimal polynomial of $\delta(f)$. Then it is a root of $\chi(f)$, and
all of its coefficients are integers. Now use Case 2 above we now show that $p(x)$ is symmetric. We have

$$\prod_{\alpha: p(\alpha) = 0} |\alpha| = |p(0)| \geq 1,$$

here one of the $\alpha$’s is $\delta(f)$, another of the $\alpha$’s is $1/\delta(f)$, and the others are algebraic numbers of complex norm 1. This proves a).

b) The proof of b) use the proof of a) and is similar to that of a).

**Theorem 4** Suppose $N = \deg(F) \geq 2$. Assume that the map $k_F$ is birationally conjugate to an automorphism. Then $N = 3$, and the map $F$ is that described in Case 2a) of Theorem 3.

**Proof** In Theorems 1 and 3, we constructed spaces $Z$ for which the induced map $k_Z$ is 1-regular, and introduced polynomials $h(x)$ such that $h(x) = (x - 1)^2 g(x)$ where $g(x)$ factors of the corresponding characteristic polynomials of $k_Z$ which has $\delta(k_F)$ as a root and $g(0) \neq 0$. Hence we can apply Lemma 4 b) to rule out cases for which $k_F$ can not be birationally conjugate to an automorphism.

-Case 1: $N \geq 2$ is even. In this case we have two subcases, which for convenience we list in the same order to that of the statement of Theorem 1:

Subcase 1a) In this case $g(x) = x^{2m+1}(x^2 - (N + 1)x - 1) + x^2 + N$. Then $g(x)$ is neither symmetric nor anti-symmetric. Hence by Lemma 4, $k_F$ does not conjugate with an automorphism.

Subcase 1b): In this case $g(x) = x^2 - (N + 1)x - 1$. Although in this case $g(x)$ is anti-symmetric, we see that $g(x)$ is irreducible and has two roots $\lambda$ and $-1/\lambda$. If $k_F$ was to be conjugate to an automorphism, by Lemma 4, the two roots of $g(x)$ should be $\lambda$ and $1/\lambda$. Hence in this case $k_F$ does not conjugate to an automorphism.

-Case 2: $N \geq 3$ is odd. We have several subcases, which for convenience we list in the same order that of the statement of Theorem 3:

Subcase 2.1 a) In this case $g(x) = (1 + x^{2m+1})[x^3 - Nx^2 - (N - h + 1) x - 1] + (N + 1)x^2 + (2N - h + 1)x + N - h$. Since $N - h \geq 2$, $g(x)$ is neither symmetrical or anti-symmetrical. Hence as in Case 1a), $k_F$ does not conjugate to an automorphism.

Subcase 2.1 b) In this case $g(x) = x^3 - N x^2 - (N + 1 - h)x - 1$. In this case $g(x)$ can not be symmetric. Although $g(x)$ can be anti-symmetric but we always have $g(1) = -N - (N + 1 - h) < 0$. Hence from Lemma 4, it follows that $k_F$ does not conjugate to an automorphism.
Subcase 2.2 a) It is proved in [7] that in this case $k_F$ does conjugate to an automorphism.

For the other subcases 2.2 b, c, d, it can be easily seen that $g(x)$ is neither symmetric nor anti-symmetric. Hence in these cases, $k_F$ does not conjugate to an automorphism. □

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Appendix 1: A System of Linear Equations

In this section we explore the system of linear equations defined in in (5.1). Functions $L_j = L_j(a_0, \ldots, a_n)$ for some first values of $j$ are:

- $L_0 = a_n + [-a_n] = 0$,
- $L_1 = (a_n + a_{n-1}) + [-na_n + a_{n-1}] = -(n-1)a_n + 2a_{n-1}$,
- $L_2 = (a_{n-1} + a_{n-2}) + [-a_n(n) + a_{n-1}(n-1) - a_{n-2}(n-2)] = \frac{n}{2}L_1$.

We will explore the properties of systems of linear equations of the form

$$L_j(a_0, a_1, \ldots, a_n) = 0$$ (6.1)

for all $j = 0, 1, 2, \ldots, m$, where $0 \leq m < n$ is a constant integer. It will be convenient to write equations (6.1) as

$$-(a_{n-j} + a_{n-j+1}) = -a_n\binom{n}{j} + a_{n-1}\binom{n-1}{j-1} + \ldots + (-1)^{j+1}a_{n-j}\binom{n-j}{0}$$ (6.2)

Changing the order of indexes ($b_j := a_{n-j}$), the equations (6.2) can be written in a more convenient form

$$-(b_j + b_{j-1}) = -b_0\binom{n}{j} + b_1\binom{n-1}{j-1} + \ldots + (-1)^{j+1}b_j\binom{n-j}{0}.$$ (6.3)

Lemma 5 If $0 \leq m < n$, and $m$ is odd, and if $b_0, b_1, \ldots, b_n$ satisfy the equations (6.3) for all $j = 1, 3, 5, \ldots, m$ then $b_0, b_1, \ldots, b_n$ also satisfy (6.3) for all $j = 0, 2, 4, \ldots, m + 1$.

Proof Fixed $0 \leq m < n$, where $m$ is odd. Let $b_0, b_1, \ldots, b_n$ satisfy the equations (6.3) for all $j = 1, 3, 5, \ldots, m$. To prove Lemma 5 it suffices to prove the following claim:

Claim 1: $b_0, b_1, \ldots, b_n$ also satisfy (6.3) for $j = m + 1$. 

\[Springer\]
The proof is divided in several steps.

i) Reduction 1: In equations (6.3) with \( j = 1, 3, \ldots, m \), pushing all \( b_i \) with \( i \) odd to the left hand-sided and pushing all \( b_i \) with \( i \) even to the right hand-sided we can rewrite them as

\[
2b_1 = b_0 \binom{n-1}{1},
\]
\[
b_1 \binom{n-1}{2} + 2b_3 = b_0 \binom{n}{3} + b_2 \binom{n-3}{1},
\]
\[
b_1 \binom{n-1}{4} + b_3 \binom{n-3}{2} + 2b_5 = b_0 \binom{n}{5} + b_2 \binom{n-2}{3} + b_4 \binom{n-5}{1},
\]
\[
: \quad b_1 \binom{n-1}{m-1} + b_3 \binom{n-3}{m-3} + \ldots + b_{m-2} \binom{n-m+2}{2} + 2b_m
\]
\[
= b_0 \binom{n}{m} + b_2 \binom{n-2}{m-2} + \ldots + b_{m-3} \binom{n-m+3}{3} + b_{m-1} \binom{n-m}{1}.
\]

The equation (6.3) for \( j = m + 1 \) which we want to prove in Claim 1 can be written as

\[
b_1 \binom{n-1}{m} + b_3 \binom{n-3}{m-2} + \ldots + b_{m-2} \binom{n-m+2}{3} + b_m \binom{n-m+1}{1}
\]
\[
= b_0 \binom{n}{m+1} + b_2 \binom{n-2}{m-1} + \ldots + b_{m-1} \binom{n-m+1}{2}.
\]

ii) Reduction 2: For any value of \( b_0, b_2, b_4, \ldots, b_{m-1} \) there exists a unique solution \( b_1, b_3, \ldots, b_m \) to the system (6.3) for \( j = 1, 3, \ldots, m \). For a proof of this claim we can use the rewritten system in Reduction 1.

iii) Reduction 3: Claim 1 is true in general case if we can show that it is true for the special case \( b_0 = 1, b_2 = b_4, \ldots = 0 \). For a proof use the special structure of the rewritten system in Reduction 1.

From now on in this proof we will assume that \( b_0 = 1, b_2 = b_4 = \ldots = 0 \). We rewrite Reduction 1 as
iv) Reduction 4: In equations (6.3) with $j = 1, 3, \ldots, m$, pushing all $b_i$ with $i$ odd to the left hand-sided and pushing all $b_i$ with $i$ even to the right hand-sided we can rewrite them as

\[
2b_1 = \binom{n-1}{1},
\]
\[
b_1\binom{n-1}{2} + 2b_3 = \binom{n}{3},
\]
\[
b_1\binom{n-1}{4} + b_3\binom{n-3}{2} + 2b_5 = \binom{n}{5},
\]
\[\vdots\]
\[
b_1\binom{n-1}{m-1} + b_3\binom{n-3}{m-3} + \ldots + b_{m-2}\binom{n-m+2}{2} + 2b_m = \binom{n}{m}.
\]

The equation (6.3) for $j = m + 1$ which we want to prove in Claim 1 can be written as

\[
b_1\binom{n-1}{m} + b_3\binom{n-3}{m-2} + \ldots + b_{m-2}\binom{n-m+2}{3} + b_m\binom{n-m+1}{1} = \binom{n}{m+1}.
\]

v) Reduction 5: Define

\[
\beta_1 = \frac{b_1}{n},
\]
\[
\beta_3 = \frac{b_3}{n(n-1)(n-2)},
\]
\[
\beta_5 = \frac{b_5}{n(n-1)(n-2)(n-3)(n-4)},
\]
\[\vdots\]

then $\beta_1, \beta_3, \beta_5, \ldots$ satisfy the following system of equations

\[
2\beta_1 = 1 - \frac{1}{n},
\]
\[
\frac{\beta_1}{2!} + 2\beta_3 = \frac{1}{3!},
\]
\[
\frac{\beta_1}{4!} + \frac{\beta_3}{2!} + 2\beta_5 = \frac{1}{5!},
\]
\[\vdots\]
\[
\frac{\beta_1}{(m-1)!} + \frac{\beta_3}{(m-3)!} + \ldots + \frac{\beta_{m-2}}{2!} + 2\beta_m = \frac{1}{m!}.
\]
What we want to prove in Claim 1 can be written as
\[ \frac{\beta_1}{m!} + \frac{\beta_3}{(m - 2)!} + \ldots + \frac{\beta_{m-2}}{3!} + \beta_{m} \left(1 + \frac{1}{n-m}\right) = \frac{1}{(m+1)!} \]

vi) Reduction 6: A universal system of linear equations
Let \( \theta_1, \theta_3, \theta_5, \ldots \) be the unique sequence satisfying the following system of infinitely many linear equations
\[ 2\theta_1 = 1, \]
\[ \frac{\theta_1}{2!} + 2\theta_3 = 0, \]
\[ \frac{\theta_1}{4!} + \frac{\theta_3}{2!} + 2\theta_5 = 0, \]
\[ \ldots \]
Then, for any sequence \( c_1, c_3, c_5, \ldots \), the unique solution to
\[ 2z_1 = c_1, \]
\[ \frac{z_1}{2!} + 2z_3 = c_3, \]
\[ \frac{z_1}{4!} + \frac{z_3}{2!} + 2z_5 = c_5, \]
\[ \ldots \]
is
\[ z_1 = c_1\theta_1, \]
\[ z_3 = c_3\theta_1 + c_1\theta_3, \]
\[ z_5 = c_5\theta_1 + c_3\theta_3 + c_5\theta_1, \]
\[ \ldots \]

vii) Reduction 7: Let \( \alpha_1, \alpha_3, \ldots \) be the unique sequence satisfying the following system
\[ 2\alpha_1 = \frac{1}{1!}, \]
\[ \frac{\alpha_1}{2!} + 2\alpha_3 = \frac{1}{3!}, \]
\[ \frac{\alpha_1}{4!} + \frac{\alpha_3}{2!} + 2\alpha_5 = \frac{1}{5!}, \]
\[ \ldots \]
Then it is easy to see that for $\beta_j$ in Reduction 4:

$$\beta_j = \alpha_j - \frac{1}{n} \theta_j,$$

for all $j = 1, 3, \ldots, m$, and what we wanted to prove in Claim 1 becomes

$$-\frac{1}{n} \left( \frac{\theta_1}{m!} + \frac{\theta_3}{(m-2)!} + \ldots + \frac{\theta_{m-2}}{3!} + \frac{\theta_m}{1!} \right) + \frac{1}{n-m} \left( \frac{\alpha_m - \theta_m}{m} \right) = 0.$$

Hence Claim 1 is proved if we can prove the following claim

Claim 2: For any $m \in \mathbb{N}$, $m$ odd then the following conclusions are true

$$\frac{\theta_1}{m!} + \frac{\theta_3}{(m-2)!} + \ldots + \frac{\theta_{m-2}}{3!} + \frac{\theta_m}{1!} - \frac{\theta_m}{m} = 0, \quad (6.4)$$

and

$$\alpha_m - \frac{\theta_m}{m} = 0. \quad (6.5)$$

viii) Proof of Claim 2:

Define a formal series

$$\theta(t) = \theta_1 - t^2 \theta_3 + t^4 \theta_5 - t^6 \theta_7 + \ldots$$

From the Reduction 6:

$$1 = \theta(t). \left( 2 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} \ldots \right) = \theta(t).(1 + \cos t).$$

Hence

$$\theta(t) = \frac{1}{1 + \cos t}.$$ 

Similarly, if we define

$$\alpha(t) = t \alpha_1 - t^3 \alpha_3 + t^5 \alpha_5 \ldots$$

then from Reduction 7

$$\alpha(t) = \frac{\sin t}{1 + \cos t}.$$ 

It follows that

$$\frac{d\alpha}{dt} = \theta(t),$$

which proves (6.5).

From Reductions 6 and 7 we have

$$\alpha_m = \frac{\theta_1}{m!} + \frac{\theta_3}{(m-2)!} + \ldots + \frac{\theta_{m-2}}{3!} + \frac{\theta_m}{1!}.$$ 

This equality and (6.5) imply (6.4). Hence we completed the proof of Lemma 5. \[\square\]
Lemma 6 Let $n \geq 3$ be an odd integer. Let $a_0, \ldots, a_n$ be a solution of the system of linear equations

$$L_j(a_0, a_1, \ldots, a_n) = 0$$

for all $j = 0, 1, 2, \ldots, n - 1$. Then

$$\sum_{j=2}^{n} (-1)^ja_j = 0.$$ 

Proof To prove the equality we need only to take the difference between the sum of odd-th equations and the sum of even-th equations. \hfill \square

Appendix 2: Proof of Theorem 3

Proof The proof is divided into some steps.

Step 1: If $h < N - 2$, we construct a sequence $Y_1, Y_2, \ldots, Y_{h+1}$ where $Y_{j+1} \rightarrow Y_j$ is a blowup of $Y_j$ at a point $\xi_j \in P_{N-1+j}$, where $P_{N-1+j}$ is the exceptional fiber of the blowup $Y_j \rightarrow Y_{j-1}$. Here $\xi_j$’s are constructed inductively in the same way as $\xi_1, \xi_2, \xi_3$ in Section 5. We use the coordinate projection at $P_{N-1+j}$ as follows

$$(u, s) \in Y_j \mapsto [s^N(\xi_0 + s\xi_1 + \ldots + s^{j-1}\xi_{j-1} + s^ju) \in \mathbb{P}^2].$$

The induced map $k_{Y_{h+1}}$ is as follows (see Lemma 2):

$$k_{Y_{h+1}} : C_1, P_{N-1-(h+1)} \mapsto \xi_{h+1} \in P_{N-1+h+1},$$

$$k^{-1}_{Y_{h+1}} : C_1, P_{N-1-(h+1)} \mapsto \xi_{h+1} \in P_{N-1+h},$$

where $\xi_{h+1}$ and $\xi_{h+1}$ are constructed in the same way as $\xi_1, \xi_2, \xi_3$. Moreover $k_{Y_{h+1}} : P_{N-1+(h+1)} \mapsto P_{N-1-(h+1)}$ is

$$P_{N-1+(h+1)} \ni u \mapsto \frac{(-1)^{N-(h+1)}}{-a_N^2u + a_N^2\xi_{h+1}} \in P_{N-1-(h+1)},$$

$$P_{N-1-(h+1)} \ni u \mapsto \frac{(-1)^{N-(h+1)}}{-a_N^2u} + \xi_{h+1} \in P_{N-1+(h+1)}.$$ 

Step 2: The case when $h = N - 2$ can be treated as the case when $a_2 = a_3$ in Theorem 2. We construct a sequence $Y_1, Y_2, \ldots, Y_{N-1}$ as in Step 1. Then the induced map $k_{Y_{N-1}}$ is as follows

$$k_{Y_{N-1}} : P_{N-1+N-1} \mapsto C_1,$$

$$k_{Y_{N-1}} : C_2 \mapsto \xi_{N-1} \mapsto e_2 = [0 : 0 : 1].$$
where $\zeta_{N-1} \in P_{N-1+N-1}$ is constructed as $\zeta_1, \zeta_2, \zeta_3$. Hence we see that the orbit of the exceptional curve $C_2$ contains the indeterminacy point $e_2$.

Let $Y_N \rightarrow Y_{N-1}$ be the blowup of two points $\zeta_{N-1} \in P_{N-1+N-1}$ and $e_2$, and call $P_{N-1+N}$ and $E_2$ the corresponding exceptional fibers of this blowup. We choose the coordinate projection at $P_{N-1+N}$ as

$$\{(u, s) \in Y_N | s^N (\zeta_0 + s \zeta_1 + \ldots + s^{N-1} \zeta_{N-1} + s^N u) : 1$$

$$: s^{N-1} (\zeta_0 + s \zeta_1 + \ldots + s^{N-1} \zeta_{N-1} + s^N u) \} \in \mathbb{P}^2,$$

and the coordinate projection at $E_2$ as

$$\{(u, s) \in Y_N | s : su : 1 \} \in \mathbb{P}^2.$$

Using computations as in Lemma 2 we can show that in case $h = N - 2$ then the induced map $k_{Y_N} : P_{2N-1} \leftarrow E_2$ is

$$k_{Y_N} : P_{2N-1} \ni u \mapsto -a^2_N u + a^2_N \xi_N - (N + 1) \in E_2,$$

$$k_{Y_N} : E_2 \ni u \mapsto \frac{-u + a^2_N \xi_N + 1}{a^2_N} \in P_{2N-1}.$$

Here $\zeta_N$ and $\xi_N$ is constructed in similar manner to that of $\zeta_1, \zeta_1, \zeta_2, \zeta_2$.

That the point $0 \in E_2$ is the unique indeterminacy point of $k_{Y_N}$ lying on $E_2$ is not hard to see. It also easy to see that $C_2$ is an exceptional curve for $k_{Y_N}$.

We have $C_2 \cap E_2 = 1 \in E_2$, which is a regular point of the map $k_{Y_N}$. Hence

$$k_{Y_N} (C_2) = k_{Y_N}([1]_{E_2}) = \zeta_N.$$

The map $k_{Y_N}^2 : E_2 \rightarrow E_2$ is $u \mapsto u + a^2_N (\xi_N - \zeta_N) - (N + 2)$, and $k_{Y_N} (\zeta_N) = a^2_N (\xi_N - \zeta_N) - (N + 1)$.

When $h = N - 2$, Lemma 5 implies $L_j = 0$ for all $j = 1, \ldots, N - 1$. From the formulas for $\xi_N$ and $\zeta_N$, it follows that

$$a^2_N (\xi_N - \zeta_N) = 2a_0 + \sum_{j=2}^{N} (-1)^j a_j.$$

Lemma 6 implies $a^2_N (\xi_N - \zeta_N) = 2a_0$. Hence the orbit of $C_2$ is

$$k_{Y_N}^{2l+2} : C_2 \mapsto 2a_0 (l + 1) - (N + 1) (l + 1) - l \in E_2.$$

Hence this orbit contains a point of indeterminacy point of $k_{Y_N}$ iff that point is $0 \in E_2$, that is iff there exists an integer $l \geq 0$ for which $2a_0 (l + 1) - (N + 1) (l + 1) - l = 0$. The latter condition is exactly the cases 5 and 6 of Theorem 3. If this is the case, then we construct a space $Z$ as the blowup of $Y_N$ at the points $\zeta_N \in P_{N-1+N}, k_{Y_N} (\zeta_N) \in E_2, k_{Y_N}^2 (\zeta_N) \in P_{N-1+N}, \ldots, k_{Y_N}^{2l+1} (\zeta_N) = 0 \in E_2$ as in the proof of of Theorem 2. Then the induced map $k_{Z}$ is good, that is it satisfies $(k_{Z}^*)^n = (k_{Z}^*)^*$ for all integer $n \geq 0$. Hence the spectral radius of $k_{Z}^*$ is $\delta(k_{Z})$. 

\[\Box\]
References