THE REAL PLANE CREMONA GROUP IS A NON-TRIVIAL AMALGAM

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ABSTRACT. We show that the real Cremona group of the plane is a non-trivial amalgam of two groups amalgamated along their intersection.

1. INTRODUCTION

The plane Cremona group is the group Bir_k(\mathbb{P}^2) of birational transformations of \mathbb{P}^2 defined over a field k. [C2013, Appendix by Y. de Cornulier] shows that Bir_k(\mathbb{P}^2) is not isomorphic to a non-trivial amalgam of two groups if k is algebraically closed. The best one can obtain is that Bir_k(\mathbb{P}^2) is isomorphic to a non-trivial amalgam modulo one simple relation [B2012, L2010, I1984] or that is isomorphic to a generalised amalgamated product of three groups, amalgamated along all pairwise intersections [W1992]. The world looks different for k = \mathbb{R}. We show that Bir_R(\mathbb{P}^2) is indeed isomorphic to a non-trivial amalgamated product of two groups.

The group Bir_R(\mathbb{P}^2) contains two groups of de Jonquières transformations:

\[ \mathcal{J}_* = \{ f \in \text{Bir}_R(\mathbb{P}^2) \mid f \text{ preserves the pencil of lines through } [1 : 0 : 0] \} \]
\[ \mathcal{J}_o = \{ f \in \text{Bir}_R(\mathbb{P}^2) \mid f \text{ preserves the pencil of conics through } p_1, \bar{p}_1, p_2, \bar{p}_2 \} \]

where \( p_1, p_2 \in \mathbb{P}^2 \) are two non-real points such that \( p_1, \bar{p}_1, p_2, \bar{p}_2 \) are not collinear. The group Bir_R(\mathbb{P}^2) is generated by Aut_R(\mathbb{P}^2), \mathcal{J}_*, \mathcal{J}_o [BM2014], and the groups \( \mathcal{J}_*, \mathcal{J}_o \) are not conjugate in Bir_R(\mathbb{P}^2) [Z2015, Proposition 5.3]. Over the field \mathbb{C}, we can send the pencil of conics through four points, no three collinear, onto a pencil of lines through a point, so the complex versions of the two groups are conjugate in Bir_C(\mathbb{P}^2).

We write \( A := \text{Aut}_R(\mathbb{P}^2) \) and denote by \( G_\circ \subset \text{Bir}_R(\mathbb{P}^2) \) the subgroup generated by \( A \) and \( \mathcal{J}_o \), and by \( G_* \subset \text{Bir}_R(\mathbb{P}^2) \) the subgroup generated by \( A \) and \( \mathcal{J}_* \).

**Main Theorem.** The group Bir_R(\mathbb{P}^2) is isomorphic to the non-trivial amalgamated product of \( G_* \) and \( G_\circ \) along their intersection.

The intersection \( G_* \cap G_\circ \) contains all elements of \( \mathcal{J}_* \) with only one real base-point (Remark 3.2). Over \( \mathbb{C} \) the complex analogues of the groups \( G_\circ, G_* \) are equal, and they are equal to Bir_C(\mathbb{P}^2) [C1901], so over \( \mathbb{C} \) the analogous claim of the main theorem is true but trivial.

The claim that Bir_R(\mathbb{P}^2) is isomorphic to the amalgam is a corollary of the structure theorem [Z2015, Theorem 4.4], and the proof that it is non-trivial is a (much shorter) adaption of its proof. As a matter of course, much of this article is very similar to [Z2015, §4].

**Corollary 1.1.** The group Bir_R(\mathbb{P}^2) acts on a tree, and all its linear algebraic subgroups are conjugate to a subgroup of \( G_* \) or of \( G_\circ \).
Corollary 1.1 can be checked directly for the subgroups of odd order in [Y2016] and for the infinite algebraic subgroups in [RZ2016].

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2. THE REAL CREMONA GROUP IS AN AMALGAM

In this section we show that Bir$_{\mathbb{R}}(\mathbb{P}^2)$ is isomorphic to the amalgamated product of $G_*$ and $G_o$ along their intersection. Non-triviality will be proven in the next section.

Definition 2.1. A standard quintic transformation is an element of Bir$_{\mathbb{R}}(\mathbb{P}^2)$ of degree 5 whose base-points are three pairs of non-real conjugate base-points in $\mathbb{P}^2$. (See [BM2014, Example 3.1] for a definition via blow-ups.)

Remark 2.2. For any standard or special quintic transformation $f$ there exist $\alpha, \beta \in A$ such that $\beta f \alpha \in J_o$ [Z2015, Lemma 3.10]. In particular, all standard quintic transformations are contained in $G_o$.

We make use of the following structure theorem. Let $G$ be the free group $A \ast J_* \ast J_o$ amalgamated along all pairwise intersections, and let $w: A \cup J_* \cup J_o \to G$ be the canonical word map.

Theorem 2.3 ([Z2015, Theorem 4.4]). The group Bir$_{\mathbb{R}}(\mathbb{P}^2)$ is isomorphic to $G$ modulo the following relations:

(rel. 1) Let $\theta_1, \theta_2 \in J_o$ be standard quintic transformations and $\alpha_1, \alpha_2 \in A$.

$$w(\alpha_2)w(\theta_1)w(\alpha_1) = w(\theta_2) \quad \text{if} \quad \alpha_2 \theta_1 \alpha_1 = \theta_2.$$

(rel. 2) Let $\tau_1, \tau_2 \in J_* \cup J_o$ be both of degree 2 or of degree 3 and $\alpha_1, \alpha_2 \in A$.

$$w(\tau_1)w(\alpha_1) = w(\alpha_2)w(\tau_2) \quad \text{if} \quad \tau_1 \alpha_1 = \alpha_2 \tau_2.$$ 

(rel. 3) Let $\tau_1, \tau_2, \tau_3 \in J_*$ all be of degree 2, or $\tau_1, \tau_2$ of degree 2 and $\tau_3$ of degree 3, and $\alpha_1, \alpha_2, \alpha_3 \in A$.

$$w(\tau_2)w(\alpha_1)w(\tau_1) = w(\alpha_3)w(\tau_3)w(\alpha_2) \quad \text{if} \quad \tau_2 \alpha_1 \tau_1 = \alpha_3 \tau_3 \alpha_2.$$

Proposition 2.4. The group Bir$_{\mathbb{R}}(\mathbb{P}^2)$ is isomorphic to the amalgamated product of $G_*$ and $G_o$ along their intersection.

Proof. Let $G$ be the amalgamated product of $G_*$ and $G_o$ along their intersection. By the universal property of amalgamated products there exists a homomorphism $\psi: G \to \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, which is the identity map on $A, J_*$ and $J_o$. On the other hand, we have a canonical surjective homomorphism $A \ast J_* \ast J_o \to G \to G$. We claim that $G \to G$ factors through Bir$_{\mathbb{R}}(\mathbb{P}^2)$. By Theorem 2.3 it suffices to check that relations (rel. 1), (rel. 2) and (rel. 3) are satisfied in $G$.

Relation (rel. 1): standard quintic transformations are contained in $G_o$ (Remark 2.2). They are not contained $G_*$ because the kernel of the abelianisation of Bir$_{\mathbb{R}}(\mathbb{P}^2)$ contains $G_*$ but does not contain any of the standard quintic transformations [Z2015, Remark 3.19(3), Proposition 5.3].
Relation (rel. 2) and (rel. 3) are relations among linear, quadratic and cubic transformations. Any transformation of degree $\leq 3$ is contained in $G_*$, some of them in $G_* \cap G_{\circ}$, and relations (rel. 2) and (rel. 3) hold in $G_*$. 

So, the morphism $G \to G$ factors through $\text{Bir}_R(\mathbb{P}^2)$, and the induced morphism $\varphi: \text{Bir}_R(\mathbb{P}^2) \to G$ is the identity on $A, J_*, J_{\circ}$. Thus $\psi = \varphi^{-1}$. 

$\square$

3. THE GROUP $G_{\circ}$ IS A STRICT SUBGROUP OF $\text{Bir}_R(\mathbb{P}^2)$

This section is devoted to the proof that $G_{\circ}$ is a strict subgroup of $\text{Bir}_R(\mathbb{P}^2)$ (Corollary 3.8). The key idea is to show that it does not contain any element of degree 2 having three real base-points - for instance, it does not contain the standard quadratic transformation $[x : y : z] \mapsto [yz : xz : xy]$. To show this, we prove that any element in $G_{\circ}$ is the composition of elements in $A \cup J_{\circ} \cup H$ (see Definition 3.2 for the definition of $H_* \subset G_{\circ} \cap J_*$) such that the successive degree of the composition is non-decreasing (Proposition 3.7)

3.1. Properties of the groups $J_*$ and $J_{\circ}$. Before we run off to prove our claims, we quickly review the groups $J_*$ and $J_{\circ}$ and some of their properties.

Remark 3.1. The characteristic of an element $f \in \text{Bir}_R(\mathbb{P}^2)$ is the tuple $(\deg(f); m_1^{e_1}, \ldots, m_k^{e_k})$, where $m_1, \ldots, m_k$ are the different multiplicities of the base-points of $f$ and $e_i$ the number of base-points of multiplicity $m_i$.

An element of $J_{\circ}$ has characteristic

$$
\left( d; \left( d - 1 \right)^4, 2^{d-2} \right), \quad \text{if } \deg(f) \text{ is odd}
$$

$$
\left( d; \left( \frac{d}{2} \right)^2, \left( \frac{d - 2}{2} \right)^2, 2^{\frac{d-2}{2}}, 1 \right), \quad \text{if } \deg(f) \text{ is even}
$$

Moreover, if $d \geq 2$, $p_1, \tilde{p}_1, p_2, \tilde{p}_2$ are base-points of multiplicity $\frac{d-1}{2}$ or $\frac{d}{2}$ and $\frac{d-2}{2}$ and all other base-points are of multiplicity 2 or 1 [Z2015, Lemma 3.2].

In particular, $J_{\circ}$ contains only quadratic maps with one real and a pair of non-real conjugate base-points.

The group $J_*$ is isomorphic to $\text{PGL}_2(\mathbb{R}(x)) \rtimes \text{PGL}_2(\mathbb{R})$ because on the chart $z = 1$ all of its elements are of the form

$$
f: (x, y) \mapsto \left( \frac{\alpha(y)x + \beta(y)}{\gamma(y)x + \delta(y)}, \frac{ay + b}{cy + d} \right)
$$

with $a, b, d, c \in \mathbb{R}$, $ad - bc \neq 0$ and $\alpha, \beta, \delta, \gamma \in \mathbb{R}[y]$, $\alpha \delta - \beta \gamma \neq 0$.

The characteristic of any element of $J_*$ is $(d; d - 1, 1^{2d-2})$, where $[1 : 0 : 0]$ is the base-point of multiplicity $d - 1$.

Definition 3.2. We denote by $H \subset J_*$ the subgroup of elements having at most one real base-point.

If an element of $H$ is non-linear, all its base-points different from $[1 : 0 : 0]$ are pairs of non-real conjugate points. For any quadratic transformation $f \in \text{Bir}_R(\mathbb{P}^2)$ with one real and a pair of non-real conjugate base-points there exist $\alpha, \beta \in A$ such that $\beta f \alpha \in H$. In fact, the group $H$ is generated by its linear and quadratic elements and is hence contained in $G_{\circ}$ (we can follow the argument of [AC2002, Theorem 8.4.3]).
Definition 3.3. A special quintic transformation is an element of Bir$_R(\mathbb{P}^2)$ of degree 5 whose base-points are three pairs of non-real conjugate base-points, two of them points in $\mathbb{P}^2$ and one pair infinitely near one of the other two pairs. (See [BM2014, Example 3.2] for a definition via blow-ups.)

Remark 3.4. (1) A standard (resp. special) quintic transformation $f$ has multiplicity 2 in each of its base-points. In particular, there is no conic containing all its base-points. The inverse $f^{-1}$ is also a standard (resp. special) quintic transformation and $f$ sends the pencil of conics through two pairs of base-points onto a pencil of conics through two pairs of base-points of $f^{-1}$.

(3) Pick any three pairs of non-real conjugate points, not all six points contained in one conic. Suppose they are all points in $\mathbb{P}^2$ (resp. two pairs are points in $\mathbb{P}^2$ and one pair is in the first neighbourhood of one of the other two pairs). Then there is a standard (resp. special) quintic transformation with these points as its base-points $[\text{BM2014, Example 3.1, Example 3.2}], [\text{RV2005, §1}].$

(4) For any standard or special quintic transformation $f$ there exist $\alpha, \beta \in \mathcal{A}$ such that $\beta f \alpha \in \mathcal{J}_o [\text{Z2015, Lemma 3.10}].$ In particular, all standard and special quintic transformations are contained in $G_o.$

For $f \in \text{Bir}_R(\mathbb{P}^2)$ and a point $p,$ let $m_f(p)$ the multiplicity of $f$ in $p.$

Lemma 3.5 ([Z2015, Lemma 4.8]). Let $f \in \mathcal{J}_o$ be non-linear and $g \in \text{Bir}_R(\mathbb{P}^2).$ Suppose that $\deg(fg) \leq \deg(g)$ (resp. $\deg(fg) > \deg(g)$). Then there exists a base-point $q \notin \{p_1, \bar{p}_1, p_2, \bar{p}_2\}$ of $f$ of multiplicity 2 such that

\[
\sum_{i=1}^m \deg(p_i) + \deg(q) \geq \deg(g) \quad (\text{resp.} \quad \deg(q) > \deg(g))
\]

or $f$ has a simple base-point $r$ and there exists $i \in \{1, 2\}$ such that

\[
2 \deg(p_i) + \deg(q) \geq \deg(g), \quad \text{where} \quad m_f(p_i) = \deg(f)/2 \quad (\text{resp.} \quad \deg(g)).
\]

Lemma 3.6 ([Z2015, Lemma 4.14]). In Lemma 3.5 (ineq 1), suppose that $q$ is a real proper point of $\mathbb{P}^2.$ Then one of the following two assertions holds:

(1) there exists $h \in \mathcal{J}_o$ of degree 3 with base-points $p_1, \bar{p}_1, p_2, \bar{p}_2, q.$

(2) $\deg(f)$ is even and there exists $h \in \mathcal{J}_o$ of degree 2 with base-points $p_i, \bar{p}_i, q$ and $2 \deg(p_i) + \deg(q) \geq \deg(g), \quad \text{where} \quad m_f(p_i) = \frac{\deg(f)}{2}.$

3.2. The group $G_o$ is a strict subgroup of Bir$_R(\mathbb{P}^2).$ The following proposition is the key to proving that $G_o$ is a strict subgroup of Bir$_R(\mathbb{P}^2).$ The lemmas used in its proof are stated and proven in the next subsection, and to prove them we only use the properties of $\mathcal{J}_o, \mathcal{J}_1$ and $\mathcal{H}$ listed in the previous subsection.

Proposition 3.7. For any $f \in G_o$ there exist $g_1, \ldots, g_m \in \mathcal{A} \cup \mathcal{J}_o \cup \mathcal{H}$ such that $f = g_m \cdot \cdots \cdot g_1$ and such that $\deg(g_i) \leq \deg(g_{i+1}g_i \cdots g_1),$ for $i = 1, \ldots, m - 1.$

Proof. We write $f = f_N \cdots f_1$ for some $f_1, \ldots, f_N \in \mathcal{A} \cup \mathcal{J}_o \cup \mathcal{H}.$ Define $d_i := \deg(f_i \cdots f_1)$ and

\[
D := \max\{d_i \mid i = 1, \ldots, m\}, \quad n := \max\{i \mid d_i = D\}, \quad k = \sum_{i=1}^n (\deg(f_i) - 1).
\]
We claim that there exist \( h_1, \ldots, h_k \in \mathcal{A} \cup \mathcal{J}_0 \cup \mathcal{H} \) such that
\[
f_{n+1} f_n f_{n-1} = h_k \cdots h_1
\]
and the pair \((D', k')\) associated to the composition \( f_N \cdots f_{n+2} h_k \cdots h_1 f_{n-2} \cdots f_1\) is strictly smaller that \((D, k)\) with respect to the lexicographic order.

If the claim holds, the proof finishes as follows. The procedure stops at the pair \((D'', k'')\) where \( D'' = \deg(f) \) and \( f = f'_n \cdots f'_1\). We define
\[
I := \max \{ i \mid \deg(f'_i) > \deg(f'_i + \cdots + f'_1) \}, \quad f' := f'_{i+1} \cdots f'_1
\]
By construction, we have \( \deg(f') < \deg(f) \). We repeat the process for \( f' \). After finitely many steps we will not find such an \( I \) anymore, and the composition we have obtained is the one in the statement.

Let us prove the claim. If \( D = 1 \), all \( f_i \) are linear maps, so \( f \) is linear, and the claim holds with \( m = 1 \).

Suppose that \( D \geq 2 \). Any consecutive letters \( f_i, f_{i+1} \) that are both contained in \( \mathcal{A} \) or in \( \mathcal{J}_0 \) we replace by their product. This does not increase the pair \((D, k)\). We can therefore assume that no two consecutive letters are both contained in \( \mathcal{A} \) or in \( \mathcal{J}_0 \). By definition of \( n \), we have \( d_n > d_{n-1} \) and in particular,
\[
f_{n+1} \in \mathcal{J}_0 \setminus \mathcal{A}, \quad f_n \in \mathcal{A} \setminus \mathcal{J}_0, \quad f_{n-1} \in \mathcal{J}_0 \setminus \mathcal{A} \quad \text{and} \quad d_{n-2} \leq d_{n-1} = d_n = D.
\]

(1) If \( f_{n-1}, f_{n+1} \in \mathcal{H} \), we apply Lemma 3.11 to \( f_{n-1}, f_n, f_{n+1} \). There exist \( h_1 \in \mathcal{H}, h_2 \in \mathcal{A}, h_3, \ldots, h_k \in \mathcal{A} \cup \mathcal{H} \) such that \( f_{n+1} f_n f_{n-1} = h_k \cdots h_1 \) and
\[
\deg(h_1) = \deg(f_{n-1}) - 1,
\]
\[
\deg(h_2 h_1 f_{n-2} \cdots f_1) = \deg(h_1 f_{n-2} \cdots f_1) \leq D,
\]
\[
\deg(h_i h_2 h_1 f_{n-1} \cdots f_1) < D, \quad i = 3, \ldots, k.
\]
Then \( D' \leq D \) and if \( D' = D \) then \( n' \leq n \) and
\[
k' = \sum_{i=1}^{n-2} (\deg(f_i) - 1) + (\deg(h_1) - 1) < \sum_{i=1}^{n-1} (\deg(f_i) - 1) = k.
\]

(2) Else, we proceed as follows. Let \( m_i(t) \) be the multiplicity of \((f_i \cdots f_1)^{-1}\) in the point \( t \). If \( f_{n-1} \in \mathcal{J}_0 \), then by Lemma 3.5 there exists a base-point \( q \notin \{ p_1, \bar{p}_1, p_2, \bar{p}_2 \} \) of \( f_{n-1}^{-1} \) of multiplicity 2 such that
\[
m_{n-1}(p_1) + m_{n-1}(p_2) + m_{n-1}(q) \geq D
\]
or \( \deg(f_{n-1}) \) is even and there exists a simple base-point \( r \) of \( f_{n-1}^{-1} \) such that
\[
2m_{n-1}(p_1) + m_{n-1}(r) \geq D, \quad \text{where} \quad m_{f_{n-1}^{-1}}(p_i) = \frac{\deg(f_{n-1})}{2}.
\]

- If \( q \) is a non-real point, we can assume that it is a proper point of \( \mathbb{P}^2 \) or in the first neighbourhood of one of \( p_1, \bar{p}_1, p_2, \bar{p}_2 \). Since \( m_{f_{n-1}^{-1}}(q) = 2 \), the points \( p_1, \ldots, \bar{p}_2, q, \bar{q} \) are not on one conic (Remark 3.1). By Remark 3.4 there exists a standard or special quintic transformation \( \theta_{n-1} \in \mathcal{J}_0 \) with these 6 points as base-points.
- If \( q \) is a real point, we can assume that it is a proper point of \( \mathbb{P}^2 \) (it cannot be infinitely near one of \( p_1, \ldots, \bar{p}_2 \)). By Lemma 3.6 there exists \( \theta_{n-1} \in \mathcal{J}_0 \) of degree 3 with base-points \( p_1, \ldots, \bar{p}_2, q \) or there exists \( \theta_{n-1} \in \mathcal{J}_0 \) of degree 2 with base-points \( p_i, \bar{p}_i, q \) where \( m_{\theta_{n-1}^{-1}}(p_i) = \frac{\deg(f_{n-1})}{2} \) and \( 2m_{n-1}(p_i) + m_{n-1}(q) \geq D \).
- If there is no such \( q \), we have \( r \) as in (1). If \( r \) is not a proper point of \( \mathbb{P}^2 \) we replace it by the (real) base-point of \( f_{n-1}^{-1} \) to which it is infinitely near. This does not change Inequality (1). Remark 3.1 implies that \( p_i, \bar{p}_i, r \) are not collinear, so there exists \( \theta_{n-1} \in \mathcal{J}_o \) of degree 2 with base-points \( p_i, \bar{p}_i, r \) [Z2015, Lemma 3.4].

In all three cases we define \( h_1 := \theta_{n-1} f_{n-1} \in \mathcal{J}_o \). Then

\[
\deg(h_1 f_{n-2} \cdots f_1) = \deg(\theta_{n-1} f_{n-2} \cdots f_1) \leq D \quad \text{and} \quad \deg(\theta_{n-1} f_{n-2} \cdots f_1) < \deg(f_{n-1}).
\]

We proceed analogously if \( f_{n+1} \in \mathcal{J}_o \), and find \( \theta_{n+1} \in \mathcal{J}_o \) satisfying the analogous inequalities with “<”, and define \( h_k := f_{n+1} \theta_{n+1}^{-1} \in \mathcal{J}_o \). Analogously, we have

\[
\deg(\theta_{n+1} f_n \cdots f_1) < D, \quad \deg(h_k) < \deg(f_{n+1}).
\]

Now, we use Lemma 3.9 and Lemma 3.13. The constructions are summarised in the commutative diagrams below.

If \( f_{n-1}, f_{n+1} \in \mathcal{J}_o \setminus \mathcal{H} \), we apply Lemma 3.9 to \( \theta_{n-1}^{-1}, f_n, \theta_{n+1} \). There exist \( h_2 \in \mathcal{A}, h_3, \ldots, h_{k-1} \in \mathcal{A} \cup \mathcal{J}_o \) such that \( \theta_{n+1} f_{n+1} \theta_{n-1}^{-1} = h_{k-1} \cdots h_2 \) and

\[
\deg(h_i \cdots h_3 h_2 h_1 f_{n-2} \cdots f_1) < D, \quad i = 3, \ldots, k-1.
\]

If \( f_{n-1} \in \mathcal{J}_o \setminus \mathcal{H} \) and \( f_{n+1} \in \mathcal{H} \), we apply the last part of Lemma 3.13 to \( \theta_{n-1}, f_n, f_{n+1} \). There exist \( h_2 \in \mathcal{A}, h_3, \ldots, h_k \in \mathcal{A} \cup \mathcal{J}_o \cup \mathcal{H} \) such that \( f_{n+1} f_n \theta_{n-1} = h_k \cdots h_2 \) and

\[
\deg(h_i \cdots h_3 h_2 h_1 f_{n-2} \cdots f_1) < D, \quad i = 3, \ldots, k.
\]

If \( f_{n-1} \in \mathcal{H} \) and \( f_{n+1} \in \mathcal{J}_o \setminus \mathcal{H} \), we apply Lemma 3.13 to \( f_{n-1}, f_n, \theta_{n+1} \). There exist \( h_1 \in \mathcal{J}_o, h_2 \in \mathcal{A}, h_3, \ldots, h_{k-1} \in \mathcal{A} \cup \mathcal{J}_o \) such that \( \theta_{n+1} f_{n+1} \theta_{n-1} = h_{k-1} \cdots h_1 \) and

\[
\deg(h_1) = \deg(f_{n-1}) - 1, \quad \deg(h_1 f_{n-2} \cdots f_1) = \deg(h_2 h_1 f_{n-2} \cdots f_1) \leq D,
\]

\[
\deg(h_i \cdots h_3 h_2 h_1 f_{n-2} \cdots f_1) < D, \quad i = 3, \ldots, k-1.
\]

We claim that in each case, the pair \((D', k')\) is strictly smaller than \((D, k)\). Indeed, the above conditions imply \( D' \leq D \). If \( D' = D \) then \( n' \leq n \) and

\[
k' = \sum_{i=1}^{n-2} (\deg(f_i) - 1) + (\deg(h_1) - 1) < \sum_{i=1}^{n-1} (\deg(f_i) - 1) = k.
\]

\[ \square \]

**Corollary 3.8.** The group \( G_0 \) contains no maps of degree 2 having more than one real base-point. In particular, \( G_0 \subseteq \text{Bir}_\mathbb{R}(\mathbb{P}^2) \) is a strict subgroup.
Proof. Suppose that $G_0$ contains an element $\tau$ of degree $\deg(\tau) = 2$ with three real base-points. By Proposition 3.7, there exist $g_1, \ldots, g_n \in A \cup J_0 \cup H$ such that $\tau = g_n \cdots g_1$ and

$$\deg(g_i \cdots g_1) \leq \deg(g_{i+1} \cdots g_1), \quad i = 1, \ldots, n - 1.$$ 

We may suppose that $\deg(g_1) \geq 2$, and then $\deg(g_i \cdots g_1) = 2$ for $i = 1, \ldots, n$. This is only possible if

- $\deg(g_1) = 2$,
- $\deg(g_i) = 1, 2, 3, 4$ for $i = 2, \ldots, n$,
- if $\deg(g_i) = 2, 3, 4$, then $g_i$ and $(g_{i-1} \cdots g_1)^{-1}$ share 2, 3, 3 base-points of multiplicity respectively;
- if $\deg(g_i) = 3$, these are a double and two simple base-points.

The characteristics in Remark 3.1 and Remark 3.2, and all $g_i$ being contained in $J_0 \cup H$ of degree $\leq 4$ implies that the quadratic maps $g_i \cdots g_1$ have two non-real base-points for $i = 1, \ldots, n$. This contradicts $\tau = g_n \cdots g_1$ having three real base-points. \hfill \Box

Proof of the Main Theorem. By Proposition 2.4 is isomorphic to the amalgamated product of $G_*$ and $G_0$ along their intersection. Furthermore, $G_*$ is a strict subgroup of Bir$_R(\mathbb{P}^2)$ by [Z2015, Proposition 5.3], and $G_0$ is a strict subgroup by Corollary 3.8. \hfill \Box

Proof of Corollary 1.1. By the Main Theorem, Bir$_R(\mathbb{P}^2)$ acts on the Bass-Serre tree $T$ of the amalgam $G_* \ast_{G_0} G_0$. Any linear algebraic subgroup $H$ of Bir$_R(\mathbb{P}^2)$ has bounded degree [BF2013, Remark 2.20], and therefore fixes a vertex of $T$. It is thus conjugate to a subgroup of $G_*$ or of $G_0$. \hfill \Box

3.3. Lowering the degree. In this section we prove the lemmas used in the proof of Proposition 3.7. Let $\varphi \in$ Bir$_R(\mathbb{P}^2)$, $f_1, f_3 \in J_0 \cup H$ and $f_2 \in A$.

Lemma 3.9. Suppose that $f_1, f_3 \in J_0$ be standard or special quintic transformations and that

$$\deg(g) \leq \deg(f_1 g) \quad \text{and} \quad \deg(f_3 f_2 f_1 g) < \deg(f_2 f_1 g).$$

Then there exists $h_1 \in A, h_2, \ldots, h_m \in A \cup J_0$ such that $f_3 f_2 f_1 = h_m \cdots h_1$ and

$$\deg(h_i \cdots h_2 h_1 g) < \deg(f_1 g), \quad i = 2, \ldots, m.$$ 

Proof. Follows from [Z2015, Lemma 4.10] by taking $\Lambda$ to be the linear system of $(f_1 g)^{-1}$. \hfill \Box

Lemma 3.10. Let $f_1, f_3 \in H$ be of degree 2. Suppose that

$$\deg(g) \leq \deg(f_1 g) \quad \text{and} \quad \deg(f_3 f_2 f_1 g) < \deg(f_2 f_1 g).$$

Then there exist $h_1 \in A, h_2, \ldots, h_m \in A \cup H$ such that $f_3 f_2 f_1 = h_m \cdots h_1$ and

$$\deg(h_i \cdots h_2 h_1 g) < \deg(f_1 g), \quad i = 2, \ldots, m.$$ 

Proof. If $\deg(f_3 f_2 f_1) = 1$, we put $h_m = h_1 = f_3 f_2 f_1$.

If $\deg(f_3 f_2 f_1) = 2$, then $f_1^{-1}$ and $f_3 f_2$ have exactly two common base-points, namely their pairs of non-real conjugate base-points. It follows that the quadratic map $f_3 f_2 f_1$ has two non-real base-points and a real base-point. So there exist $\alpha, \beta \in A$ such that $\beta f_3 f_2 f_1 \alpha \in H$.

The claim follows with $h_1 = \alpha^{-1}, h_2 = \beta f_3 f_2 f_1 \alpha$ and $h_m = h_3 = \beta^{-1}$.

If $\deg(f_3 f_2 f_1) = 3$, then the maps $f_1^{-1}$ and $f_3 f_2$ have exactly one common base-point, namely $[1 : 0 : 0]$. Hence $f_3 f_2 f_1 \in J_*$. Since moreover $f_1, f_3$ have only one real base-point,
$f_3f_2f_1$ has only one real base-point. Thus $f_3f_2f_1 \in \mathcal{H}$ and the claim follows with $h_1 = h_m = f_3f_2f_1$.

If $\deg(f_3f_2f_1) = 4$, then the maps $f_1^{-1}$ and $f_3f_2$ have no common base-points. The following construction is visualised in the diagram below. Let $[1 : 0 : 0], p, \tilde{p}$ be the base-points of $f_1^{-1}$ and $r, q, \tilde{q}$ the base-points of $f_3f_2$, and define $D := \deg(f_1g)$. By $m(t)$ we denote the multiplicity of $(f_1g)^{-1}$ in the point $t$. The assumptions $\deg(g) \leq D$ and $\deg(f_3f_2f_1g) < D$ imply that

\[(2) \quad m([1 : 0 : 0]) + 2m(p_i) \geq D, \quad m(r) + 2m(q) > D.\]

If $m([1 : 0 : 0]) \geq m(r)$, then Inequality (2) implies that

\[m([1 : 0 : 0]) + 2m(q) \geq m(r) + 2m(q) \geq D,\]

which means that $[1 : 0 : 0], q, \tilde{q}$ are not collinear. In particular, there exists $\alpha \in \mathcal{A}$ and $\theta \in \mathcal{H}$ of degree 2 such that $\theta\alpha$ has base-points $[1 : 0 : 0], q, \tilde{q}$. The above inequality implies

\[\deg(\theta\alpha f_1g) = 2D - m([1 : 0 : 0]) - 2m(q) < D.\]

The maps $f_1^{-1}$ and $\theta\alpha$ have one common base-point, and the maps $\theta\alpha$ and $f_3f_2$ have two common base-points. We proceed as in (3) and (2).

Suppose that $m([1 : 0 : 0]) < m(r)$. Inequalities (2) imply that

\[m(r) + 2m(p) > m([1 : 0 : 0]) + 2m(p) \geq 2D,\]

so $r, p, \tilde{p}$ are not collinear. There exists $\alpha \in \mathcal{A}$ and $\theta \in \mathcal{H}$ of degree 2 such that $\theta\alpha$ has these points as base-points. The claim follows as above. \qed

**Lemma 3.11.** Let $f_1, f_3 \in \mathcal{H}$. Suppose that

\[\deg(g) \leq \deg(f_1g) \quad \text{and} \quad \deg(f_3f_2f_1g) < \deg(f_2f_1g)\]

Then there exist $h_1 \in \mathcal{H}, h_2 \in \mathcal{A}, h_3, \ldots, h_m \in \mathcal{A} \cup \mathcal{H}$ such that $f_3f_2f_1 = h_m \cdots h_1$ and $\deg(h_1) = \deg(f_1) - 1$.

\[\deg(h_2h_1g) = \deg(h_1g) \leq \deg(f_1g),\]

\[\deg(h_i \cdots h_3h_2h_1g) < \deg(f_1g), \quad i = 3, \ldots, m.\]

If the inequality in the assumption is strict, all inequalities are strict.

**Proof.** By Remark 3.2, $f_1$ and $f_3$ both have exactly one real base-point, and the rest of their base-points are pairs of non-real conjugate points. Let $m(q)$ be the multiplicity of $(f_1g)^{-1}$ in the point $q$, and $D := \deg(f_1g) = \deg(f_2f_1g)$. Both $f_1$ and $f_3f_2$ have characteristic $(d; d - 1, 1^{2d-2})$, hence there exist non-real base-points $q_1, q_2$ of $f_1^{-1}$ and $f_3$, respectively, such that

\[(3) \quad m([1 : 0 : 0]) + 2m(q_1) \geq D, \quad m(f_2^{-1}([1 : 0 : 0])) + 2m(f_2^{-1}(q_2)) > D.\]
We can assume that \( q_1, q_2 \) are points in \( \mathbb{P}^2 \) or on the exceptional divisor of \([1 : 0 : 0]\) or \( f_2^{-1}([1 : 0 : 0]) \), respectively. Furthermore, the points \([1 : 0 : 0], q_1, \bar{q}_1 \) and the points \( f_2^{-1}([1 : 0 : 0]), q_2, \bar{q}_2 \) are not collinear because of Bézout theorem.

If \( q_1, q_2 \) are both in \( \mathbb{P}^2 \), there exist \( \tau_1, \tau_2 \in \mathcal{H} \) of degree 2 with base-points \([1 : 0 : 0], q_1, \bar{q}_1 \) and \([1 : 0 : 0], q_2, \bar{q}_2 \) respectively. We define \( h_1 := \tau_1 f_1 \in \mathcal{H} \) and \( h_m := f_3 \tau_2^{-1} \in \mathcal{H} \). By construction we have

\[
\deg(h_1) = \deg(f_1) - 1, \quad \deg(h_m) = \deg(f_3) - 1
\]

and by Inequalities (3) that

\[
\deg(h_1 g) = \deg(\tau_1 f_1 g) = 2D - m([1 : 0 : 0]) - 2m(q_1) \leq D
\]
\[
\deg(\tau_2 f_2 f_1 g) = 2D - m(f_2^{-1}([1 : 0 : 0])) - 2m(f_2^{-1}(q_2)) < D.
\]

We apply Lemma 3.10 to \( f_1, \ldots, f_{n-2}, h_1, \tau_1, f_2, \tau_2 \) to get the claim.

Let us consider what to do if \( q_1 \) or \( q_2 \) is on the exceptional divisors of \([1 : 0 : 0]\).

If they are both on the exceptional divisors, respectively, then \( f_1 \) and \( f_3 \) are both of degree at least 3 and

\[
2m(q_1) \leq m([1 : 0 : 0]), \quad 2m(f_2^{-1}(q_2)) \leq m(f_2^{-1}([1 : 0 : 0])).
\]

Now, Inequalities (3) imply that

\[
2m([1 : 0 : 0]) \geq D, \quad 2m(f_2^{-1}([1 : 0 : 0])) > D
\]

Since \( f_1 \) and \( f_3 \) are of degree at least three, this is only possible if \( f_2^{-1}([1 : 0 : 0]) = [1 : 0 : 0] \).

In particular \( f_2 \in \mathcal{H} \). We put \( h_1 = h_m = f_3 f_2 f_1 \in \mathcal{H} \).

Let us assume that \( f_2 \notin \mathcal{H} \). Then either \( q_1 \) or \( q_2 \) are points in \( \mathbb{P}^2 \). Suppose that \( q_1 \) is in \( \mathbb{P}^2 \) and \( q_2 \) is on the exceptional divisor of \( f_2^{-1}([1 : 0 : 0]) \). Then, because of the above reasoning, we have

\[
2m(f_2^{-1}([1 : 0 : 0])) > D \geq 2m([1 : 0 : 0]).
\]

Inequalities (3) get us

\[
m(f_2^{-1}([1 : 0 : 0])) + 2m(q_1) > m([1 : 0 : 0]) + 2m(q_1) \geq D.
\]

Then \([1 : 0 : 0], f_2(q_1), f_2(\bar{q}_1)\) are not collinear and all of them are points of \( \mathbb{P}^2 \). There exists \( \tau'_2 \in \mathcal{H} \) of degree 2 with base-points \([1 : 0 : 0], f_2(q_1), f_2(\bar{q}_1)\). We proceed as in the first case with \( \tau'_2 \) instead of \( \tau_2 \). If \( q_2 \) is a point of \( \mathbb{P}^2 \) and \( q_1 \) is on the exceptional divisor of \([1 : 0 : 0]\), we proceed analogously.

\[ \square \]

**Lemma 3.12.** Let \( f_1 \in \mathcal{H} \) be of degree 2 and \( f_3 \in \mathcal{J}_o \) a standard or special quintic transformation. Suppose that

\[
\deg(g) \leq \deg(f_1 g), \quad \text{and} \quad \deg(f_3 f_2 f_1 g) < \deg(f_2 f_1 g)
\]

Then there exist \( h_1 \in \mathcal{A}, h_2, \ldots, h_m \in \mathcal{A} \cup \mathcal{J}_o \cup \mathcal{H} \) such that \( f_3 f_2 f_1 = h_m \cdots h_1 \) and

\[
\deg(h_1 g) \leq \deg(f_1 g),
\]
\[
\deg(h_i \cdots h_3 h_2 h_1 g) < \deg(f_1 g) \quad i = 2, \ldots, n.
\]

If we have strict inequality in the assumption, all inequalities are strict.

The same claim holds if \( f_1 \in \mathcal{J}_o \) is a standard or special quintic transformation and \( f_3 \in \mathcal{H} \) is of degree 2.
Lemma 3.13. Let $f_1 \in \mathcal{H}$ and $f_3 \in \mathcal{J}_o$ a standard or special quintic transformation. Suppose that

$$\deg(g) \leq \deg(f_1 g), \quad \text{and} \quad \deg(f_3 f_2 f_1 g) < \deg(f_2 f_1 g)$$
Then there exist \( h_1 \in \mathcal{H}, h_2 \in \mathcal{A}, h_3, \ldots, h_m \in \mathcal{A} \cup \mathcal{J} \cup \mathcal{H} \) such that \( f_3 f_2 f_1 = h_m \cdots h_1 \) and \( \deg(h_1) = \deg(f_1) - 1, \)
\[
\deg(h_1 g) \leq \deg(f_1 g),
\]
\[
\deg(h_i \cdots h_3 h_2 h_1 g) < \deg(f_1 g) \quad i = 3, \ldots, n.
\]

If we have strict inequality in the assumption, all inequalities are strict.

If \( f_1 \in \mathcal{J} \) is a standard or special quintic transformation and \( f_3 \in \mathcal{H} \), we have
\[
\deg(h_i \cdots h_2 h_1 g) < \deg(f_1 g) \quad i = 2, \ldots, m.
\]

Proof. Let \( m(s) \) be the multiplicity of \( (f_1 g)^{-1} \) in the point \( s \) and define \( D := \deg(f_1 g) \). Since \( f_1 \) has characteristic \((d; d - 1, 1^{2d-2})\) and is contained in \( \mathcal{H} \), there exist a non-real base-point \( q \) of \( f_1^{-1} \) such that
\[
m([1 : 0 : 0]) + 2m(q) \geq D.
\]
We can assume that they are points in \( \mathbb{P}^2 \) or on the exceptional divisor of \([1 : 0 : 0]\). Furthermore, the points \([1 : 0 : 0], q, \bar{q}\) are not collinear by Bézout theorem.

(1) Suppose that \( q, \bar{q} \) are points in \( \mathbb{P}^2 \). Then there exists \( \tau \in \mathcal{H} \) of degree 2 such that \( \tau \) has base-points \([1 : 0 : 0], q, \bar{q}\). We put \( h_1 := \tau f_1 \in \mathcal{H} \). Note that
\[
\deg(h_1) = \deg(f_1) - 1,
\]
\[
\deg(h_1 g) = \deg(\tau f_1 g) = 2D - m([1 : 0 : 0]) - 2m(q) \leq D.
\]
We apply Lemma 3.12 to \( g' := h_1 g \) and \( \tau^{-1}, f_2, f_3 \) to get \( h_2, \ldots, h_m \).

(2) Suppose that \( q, \bar{q} \) are on the exceptional divisor of \([1 : 0 : 0]\). Then \( m([1 : 0 : 0]) \geq 2m(q) \). Call \( r_1, \bar{r}_1, r_2, \bar{r}_2, r_3, \bar{r}_3 \) the base-points of \( f_3 f_2 \), and suppose that
\[
m(r_1) \geq m(r_2) \geq m(r_3)
\]
and \( r_1 \) is a point in \( \mathbb{P}^2 \). Lemma 3.5 implies that
\[
m(r_1) + m(r_2) + m(r_3) > D.
\]
We get \( m([1 : 0 : 0]) \geq \frac{2D}{3} \) and \( m(r_1) > \frac{2D}{3} \), and so \( m([1 : 0 : 0]) + 2m(r_1) > D \). In particular, the points \([1 : 0 : 0], r_1, \bar{r}_1 \) are not collinear, and hence there exists \( \tau \in \mathcal{H} \) of degree 2 with these three points its base-points. As above we put \( h_1 := \tau f_1 \in \mathcal{H} \) and note that
\[
\deg(h_1) = \deg(f_1) - 1, \quad \deg(h_1 g) = \deg(\tau f_1 g) < D,
\]
and apply Lemma 3.12 to \( g' := h_1 g \) and \( \tau^{-1}, f_2, f_3 \).

If \( f_1 \) is a standard or special quintic transformation and \( f_3 \in \mathcal{H} \), we construct \( h_1, \ldots, h_m \) analogously. \( \square \)

References


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