NOTES ON SCISSORS CONGRUENCE

DANNY CALEGARI

Abstract. These are notes on scissors congruence, based on a graduate course taught at U Chicago by Daniil Rudenko in Spring 2022

Contents

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1. Euclidean Scissors Congruence

Two finite Euclidean polyhedra P, Q in $\mathbb{E}^n$ are said to be *equidecomposable* if each polyhedron can be dissected (along finitely many flat planes) into subpolyhedra which can be reassembled (by orientation-preserving isometries of $\mathbb{E}^n$) to form the other polyhedron.

Equidecomposability is an equivalence relation on isometry classes of finite Euclidean polyhedra in each dimension. The goal of the theory of scissors congruence is to find computable algebraic invariants of isometry types of polyhedra, so that two polyhedra have equal invariants if and only if they are equidecomposable.

1.1. **Dimension 2.**

Theorem 1.1 (Bolyai–Gerwien)**.** *Two finite Euclidean planar polygons are equidecomposable if and only if they have the same area.*

Proof. We shall show that any triangle T is equidecomposable with a rectangle R with the same area where R has one side of length 1. This will prove the theorem as follows: if P is a planar polygon we decompose P into triangles $T_1, \cdots, T_n$. Each T_j is equidecomposable to a rectangle R_j with one side of length 1. Stack the R_j to make a rectangle R with one side of length 1 and the other side of length equal to area(P).

Let T be a triangle. First we show that T is equidecomposable with a parallelogram Q. This may be accomplished by first subdividing T into 4 congruent triangles all similar to T and then rotating the 'top' triangle to form Q.

Next we show that any two parallelograms with equal base and the same area (i.e. the parallelograms differ by a shear parallel to the base) are equidecomposable. If the parallelograms Q, Q' are close, their symmetric difference is two isometric triangles, one in $Q - Q'$ and one in $Q' - Q$. This proves the claim, since any shear-equivalent Q, Q' may be joined by a sequence of intermediate $Q \sim Q_1 \sim Q_2 \sim \cdots \sim Q_n \sim Q'$ where each Q_i is close enough to Q_{i+1} to apply the symmetric difference trick.

Now, any Q is shear equivalent to Q' with one edge of length 1. But then Q' is shear equivalent to the desired rectangle R. $\square$

1.2. **Zylev's Theorem.**

Equidecomposability classes of Euclidean polyhedra in each dimension n form an abelian semigroup, with disjoint union as the group operation (which we denote $+$). The *Euclidean scissors congruence group* $\mathcal{P}(\mathbb{E}^n)$ is the Grothendieck group of this semigroup. In other words, two polyhedra P, Q are equivalent in $\mathcal{P}(\mathbb{E}^n)$ if and only if there is a third polyhedron R so that $P + R \sim Q + R$.

Zylev's theorem says that the natural map from the semigroup to $\mathcal{P}(\mathbb{E}^n)$ is injective. Before we prove this theorem we prove a simple lemma.

Lemma 1.2. *Let P, Q be polyhedra in $\mathbb{E}^n$ with* $\mathrm{vol}(P) < \mathrm{vol}(Q)$*. Then P is equidecompsable with a subpolyhedron Q' of Q.*

Proof. Consider the lattice decomposition of $\mathbb{E}^n$ into cubes of edge length ϵ. Since $\mathrm{vol}(P) < \mathrm{vol}(Q)$ there is an ϵ and an integer N so that P is contained in the interior of N disjoint cubes, and the interior of Q contains N disjoint cubes. The lemma follows. □

Theorem 1.3 (Zylev)**.** *For any n, two Euclidean polyhedra P, Q in $\mathbb{E}^n$ satisfy $P \sim Q$ if and only if $P + R \sim Q + R$.*

Proof. We need to show $P + R \sim Q + R$ implies $P \sim Q$. We have a (discontinuous) piecewise isometry $\phi : P + R \to Q + R$, defined on a subset of full measure (i.e. away from a finite union of hyperplanes).

By writing $P + R = P + R_1 + R_2$ where $2\,\mathrm{vol}(R_2) < \mathrm{vol}(P)$ and induction, we may reduce to the case where $2\,\mathrm{vol}(R) < \mathrm{vol}(P) = \mathrm{vol}(Q)$. Let $S = \phi^{-1}(Q) \cap P$. Then $\mathrm{vol}(S) > \mathrm{vol}(R)$ so by Lemma 1.2 we can find a subpolyhedron $A \subset S$ and a piecewise isometry $\alpha : A \to R$. Let $B = \phi(A) \subset Q$.

Let $P' = (P - A) + R$. This is an honest polyhedron because A is a subpolyhedron of P. Evidently $P \sim P'$ by the map which is the identity on $P - A$ and α on A, so it will suffice to prove that $P' \sim Q$.

But $\phi(P') = (Q - B) + R$ and $(Q - B) + R \sim Q$ by the map which is the identity on $Q - B$ and $\phi \circ \alpha^{-1}$ on R. So $P \sim P' \sim Q$ and we are done. □

Remark 1.4. Lemma 1.2 (and hence also Zylev's theorem) is true but harder to prove in spherical or hyperbolic space.

1.3. The Dehn invariant.

Let P be a polyhedron in $\mathbb{E}^3$. The *Dehn invariant* is defined by

$$D(P) := \sum_e \ell(e) \otimes \alpha(e) \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z}$$

where the sum is taken over edges e of P, and where $\ell(e)$ is the length of e and $\alpha(e)$ is the dihedral angle. This map is evidently additive on disjoint union and invariant under isometry.

Theorem 1.5 (Dehn)**.** *The function D is constant on scissors congruence classes. It therefore extends to a homomorphism $D : \mathcal{P}(\mathbb{E}^3) \to \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z}$.*

Proof. It suffices to show $D(P) = D(A) + D(B)$ when A, B may be obtained from P by a single straight cut. The cut creates new edges in A and B when it slices through a face of P, but these appear in pairs with dihedral angles summing to π. Likewise, a cut that slices through an edge of P will produce two new edges whose dihedral angles sum to a corresponding dihedral angle in P. The proof follows. □

As a consequence, one may exhibit (following Dehn) pairs of 3d Euclidean polyhedra with the same volume which are not equidecomposable. For example, take P the unit cube and Q the unit volume regular tetrahedron. Evidently $D(P) = 0$ whereas $D(Q) = 6\ell \otimes \alpha$ for some real ℓ and for $\alpha = \cos^{-1}(1/3)$.

One may show that α is not a rational multiple of π and therefore $6\ell \otimes \alpha \neq 0$.

1.4. Prisms.

A *prism* in $\mathbb{E}^n$ is the Minkowski sum of a polyhedron P in $\mathbb{E}^{n-1}$ with an interval. Any 3-dimensional prism is equidecomposable to a cube, by mirroring the proof of the Bolyai–Gerwien Theorem (first we show P is scissors congruent to a parallelepiped by taking the Minkowski sum of an equidecomposition of its base with the same interval; then we argue, analogously to the 2d case, that shear equivalent parallelepipeds are equidecomposable).

Sum with an orthogonal unit interval therefore determines an injection $p : \mathcal{P}(\mathbb{E}^2) \to \mathcal{P}(\mathbb{E}^3)$ whose image is the span of prisms. The image is in the kernel of D, and volume gives an isomorphism to $\mathbb{R}$. Let $\mathcal{P}^1(\mathbb{E}^3)$ denote the quotient group $\mathcal{P}(\mathbb{E}^3)/p\mathcal{P}(\mathbb{E}^2)$; this is the group of "polyhedra modulo prisms".

If P is a polyhedron and $\mu \in \mathbb{R}^+$ let $\mu \cdot P$ denote the dilation of P by a factor μ. This operation definitely does *not* make $\mathcal{P}(\mathbb{E}^3)$ into a real vector space; it multiplies volumes by μ^3. However, it does make $\mathcal{P}^1(\mathbb{E}^3)$ into a real vector space:

Lemma 1.6. *For $\mu, \lambda \in \mathbb{R}^+$ and a polyhedron P we have $\mu \cdot P + \lambda \cdot P = (\mu + \lambda) \cdot P$ in $\mathcal{P}^1(\mathbb{E}^3)$.*

Proof. It suffices to prove it for tetrahedra. Let P be a tetrahedron and let e be an edge with vertices x, y. We disjointly embed $\mu \cdot P$ in $(\mu + \lambda) \cdot P$ so that vertices corresponding to x agree, and embed $\lambda \cdot P$ in $(\mu + \lambda) \cdot P$ so that vertices corresponding to y agree. Then $(\mu + \lambda) \cdot P - \mu \cdot P - \lambda \cdot P$ is a prism. $\square$

With respect to the real vector space structure on $\mathcal{P}^1(\mathbb{E}^3)$ the map $D : \mathcal{P}^1(\mathbb{E}^3) \to \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z}$ is real linear.

1.5. **Statement of the Dehn–Sydler–Jessen Theorem.** Sydler's Theorem is the statement that $D : \mathcal{P}^1(\mathbb{E}^3) \to \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z}$ is injective. Jessen computed its cokernel. The combination of these results is the Dehn–Sydler–Jessen Theorem:

Theorem 1.7 (Dehn–Sydler–Jessen)**.** *The following sequence is exact:*

$$0 \to \mathcal{P}(\mathbb{E}^2) \xrightarrow{p} \mathcal{P}(\mathbb{E}^3) \xrightarrow{D} \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z} \xrightarrow{J} \Omega^1_{\mathbb{R}/\mathbb{Z}} \to 0$$

where we shall define $\Omega^1_{\mathbb{R}/\mathbb{Z}}$ and the map J shortly. We have already shown p is injective and $Dp = 0$, so the theorem reduces to showing

$$0 \to \mathcal{P}^1(\mathbb{E}^3) \xrightarrow{D} \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z} \xrightarrow{J} \Omega^1_{\mathbb{R}/\mathbb{Z}} \to 0$$

is exact. We refer to this reduced sequence as the *DSJ complex* in the sequel. The actual proof of the theorem is rather involved and will take the next several sections; in this section we define the terms and the maps, prove that it really is a chain complex, and prove the surjectivity of J.

The group $\Omega^1_{\mathbb{R}/\mathbb{Z}}$ is the group of Kähler differentials of $\mathbb{R}$ over $\mathbb{Z}$. It is generated (as a real vector space) by expressions of the form dr for $r \in \mathbb{R}$ modulo relations

(1) $d(rs) = rd(s) + sd(r)$ for $r, s \in \mathbb{R}$;
(2) $d(r + s) = dr + ds$ for $r, s \in \mathbb{R}$; and
(3) $dr = 0$ for $r \in \mathbb{Q}$.

The map J is defined on generators by

$$J(\ell \otimes \alpha) = \ell \frac{d \sin \alpha}{\cos \alpha}$$

whenever α is not a rational multiple of π. One verifies that $J(n\ell \otimes \alpha) = J(\ell \otimes n\alpha)$ for any integer n.

Remark 1.8. Morally, J wants to be the map $J(\ell \otimes \alpha) = \ell\, d\alpha$.

Lemma 1.9. *J is surjective and $JD = 0$*

Proof. First we show $JD = 0$. Schläffli's differential formula says that for any smooth variation of a Euclidean polyhedron one has the formula $\sum_e \ell(e) d\alpha(e) = 0$. Once one knows this theorem is true, one may deduce that there is an algebraic proof of it via trigonometry; and this algebraic proof together with the definition of J and the defining properties of Kähler differentials implies that $JD = 0$.

To show that J is surjective: note that we may surject onto any element of the form sdr when $|r| < 1$ is irrational by choosing $r = \sin(\alpha)$ and suitable ℓ. When $|r| > 1$ we use the relation $d(r \cdot r^{-1}) = r^{-1}dr + rdr^{-1}$ so that $dr^{-1} = -r^{-2}dr$. □

1.6. **Orthoschemes.** Let $x := x_1, x_2, \cdots, x_n$ be a vector of positive real numbers. Associated to this sequence is an n-simplex in $\mathbb{E}^n$ whose 0th vertex is at the origin, and whose ith vertex has coordinates $x_1, x_2, \cdots, x_i, 0, 0, \cdots, 0$. The isometry type of this simplex is called the *orthoscheme* associated to x.

By abuse of notation we let x^2 denote the vector $x_1^2, x_2^2, \cdots, x_n^2$. Note that the distance between vertex i and vertex j of $\Delta(x)$ is $\sum_{k=i+1}^j x_k^2$, and this set of pairwise distances (evidently) characterizes the orthoscheme associated to x up to isometry.

It follows that the ith face of this orthoscheme is itself an orthoscheme, associated to the vector y for which

$$y^2 = (x_1^2, x_2^2, \cdots, x_{i-1}^2, x_i^2 + x_{i+1}^2, \hat{x}_{i+1}^2, x_{i+2}^2, \cdots, x_n^2)$$

Let $O(a, b, c)$ be the 3d orthoscheme associated to the vector a, b, c. Then three of the dihedral angles of O are $\pi/2$ (those associated to the edges 02, 12, 13) and the other three edges 01, 03, 23 have acute dihedral angles α, β, γ respectively.

Let e_i be the positive unit normal to face i. The Gram matrix G has ij entry the dot product $e_i \cdot e_j$ which is equal to 1 if $i = j$ and $-\cos(\text{angle}(ij))$ otherwise. Thus:

$$G = \begin{pmatrix} 1 & -\cos(\alpha) & 0 & -\cos(\beta) \\ -\cos(\alpha) & 1 & 0 & 0 \\ 0 & 0 & 1 & -\cos(\gamma) \\ -\cos(\beta) & 0 & -\cos(\gamma) & 1 \end{pmatrix}$$

which has determinant $\sin^2(\alpha)\sin^2(\gamma) - \cos^2(\beta)$. On the other hand, since the e_i are linearly dependent, this determinant must be zero. Hence we obtain the identity

$$\sin^2(\alpha)\sin^2(\gamma) = \cos^2(\beta) = \sin^2(\beta^\vee)$$

where $\beta^\vee := \pi/2 - \beta$.

Evidently $\tan(\alpha) = c/b$ and $\tan(\gamma) = a/b$. Scaling the orthoscheme does not affect the dihedral angles. We call an orthoscheme *normalized* if it is scaled so that $a = \cot(\alpha)$. For a normalized orthoscheme, $c = \cot(\gamma)$ and $b = ac$. The edge 03 satisfies

$$\text{length}^2(03) = a^2 + b^2 + c^2 = (1 + \cot^2(\alpha))(1 + \cot^2(\gamma)) - 1 = \cot^2(\beta^\vee)$$

Hence the Dehn invariant of the normalized orthoscheme $O(\cot(\alpha), \cot(\alpha)\cot(\gamma), \cot(\gamma))$ is

$$D(O(\cot(\alpha), \cot(\alpha)\cot(\gamma), \cot(\gamma))) = \cot(\alpha) \otimes \alpha - \cot(\beta^\vee) \otimes \beta^\vee + \cot(\gamma) \otimes \gamma$$

In the remainder of this section we state three orthoscheme identities and prove two of them.

1.6.1. *Sydler's Lemma.* For $x \in (0,1)$ define $\alpha(x)$ to be the unique acute angle with $\sin^2(\alpha(x)) = x$, and define $t_1(x) := \cot(\alpha(x)) \otimes \alpha(x)$. For $x, y \in (0,1)$ let $\Delta(x,y)$ denote the normalized orthoscheme with $\alpha = \alpha(x)$ and $\gamma = \alpha(y)$. By the calculation above, we have the formula

$$D(\Delta(x,y)) = t_1(x) - t_1(xy) + t_1(y)$$

In other words, $D(\Delta(\cdot,\cdot))$ is a 2-coboundary on the multiplicative semigroup $(0,1)$. Since the coboundary of a coboundary is zero, it follows that for any $x, y, z \in (0,1)$ we have an identity

$$D(\Delta(y,z)) - D(\Delta(xy,z)) + D(\Delta(x,yz)) - D(\Delta(x,y)) = 0$$

Theorem 1.7 says that the Dehn invariant is injective on polyhedra modulo prisms; thus an identity between the Dehn invariants of four orthoschemes should imply an identity of the associated scissors congruence classes modulo prisms. This is Sydler's Lemma:

Theorem 1.10 (Sydler's Lemma)**.** *For any $x, y, z \in (0,1)$ we have an equality*

$$\Delta(y,z) + \Delta(x,yz) \sim \Delta(xy,z) + \Delta(x,y) \text{ mod prisms}$$

We defer the proof for now.

1.6.2. *Schläffli Identity.* For positive x, y, z we let $S(x,y,z)$ denote the "Schläffli simplex" in $\mathbb{R}^3$ (*not* an orthoscheme) which is the convex hull of the origin 0 and the vectors $p := (\sqrt{yz}, 0, 0)$, $q := (0, \sqrt{xz}, 0)$, $r := (0, 0, \sqrt{xy})$. If we let s denote the foot of the perpendicular from r to the edge pq then $ps0r$ and $qs0r$ are (unnormalized) orthoschemes

$$qs0r = O\left(\frac{x\sqrt{z}}{\sqrt{x+y}}, \frac{\sqrt{xyz}}{\sqrt{x+y}}, \sqrt{xy}\right), \quad ps0r = O\left(\frac{y\sqrt{z}}{\sqrt{x+y}}, \frac{\sqrt{xyz}}{\sqrt{x+y}}, \sqrt{xy}\right)$$

An orthoscheme $O(a,b,c)$ is normalized when $b = ac$; thus these orthoschemes have been scaled from normalization by factors of $\lambda_1 := x$ and $\lambda_2 := y$. Suppose the dihedral angles of $qs0r$ along qs and $0r$ are α and γ. Then

$$\cot(\alpha) := \frac{\sqrt{z}}{\sqrt{x+y}} \quad \text{so } \sin^2(\alpha) = \frac{x+y}{x+y+z} \text{ and } \cot(\gamma) := \frac{\sqrt{y}}{\sqrt{x}} \quad \text{so } \sin^2(\gamma) = \frac{x}{x+y}$$

We may therefore write these two orthoschemes as

$$qs0r = x\Delta\left(\frac{x+y}{x+y+z}, \frac{x}{x+y}\right), \quad ps0r = y\Delta\left(\frac{x+y}{x+y+z}, \frac{y}{x+y}\right)$$

Interchanging the roles of y and z gives another decomposition of $S(x,y,z)$ into orthoschemes, which is the Schläffli Identity:

Lemma 1.11 (Schläffli Identity)**.**

$$x\Delta\left(\frac{x+y}{x+y+z}, \frac{x}{x+y}\right) + y\Delta\left(\frac{x+y}{x+y+z}, \frac{y}{x+y}\right) \sim x\Delta\left(\frac{x+z}{x+y+z}, \frac{x}{x+z}\right) + z\Delta\left(\frac{x+z}{x+y+z}, \frac{z}{x+z}\right)$$

1.6.3. *Brick Identity.* A *brick* is a right-angled parallelepiped. Any brick may be decomposed along the long diagonal into six orthoschemes, coming in three isometric pairs. If the side lengths of the brick are a, b, c, the orthoschemes are $O(a, b, c)$ and its cyclic permutations. Now,

$$O(a,b,c) = \frac{ac}{b} O\left(\frac{b}{c}, \frac{b^2}{ac}, \frac{b}{a}\right) = \frac{ac}{b}\Delta\left(\frac{c^2}{1-a^2}, \frac{a^2}{1-c^2}\right)$$

Lemma 1.12 (Brick Identity)**.** *For any positive a, b, c with $a^2 + b^2 + c^2 = 1$ we have*

$$\frac{ac}{b}\Delta\left(\frac{c^2}{1-a^2}, \frac{a^2}{1-c^2}\right) + \frac{ba}{c}\Delta\left(\frac{a^2}{1-b^2}, \frac{b^2}{1-a^2}\right) + \frac{cb}{a}\Delta\left(\frac{b^2}{1-c^2}, \frac{c^2}{1-b^2}\right) \sim 0 \text{ mod prisms}$$

1.7. **Homological algebra.** We have proved that the (3 term) DSJ complex really is a chain complex; Theorem 1.7 will be proved if we can show that the DSJ complex is exact. This will be accomplished by exhibiting a quasi-isomorphism to another chain complex — the *Cathelineau complex* — whose homology is more easily calculated. This quasimorphism goes via an intermediate complex (with maps both to the DSJ complex and the Cathelineau complex) that we introduce now.

The relevant fragment of the intermediate complex for now is the part in degrees ≤ 3 and is defined as

$$\mathbb{R}[(\mathbb{R}^+)^3]_3 \oplus \mathbb{R}[(\mathbb{R}^+)^2]_3 \to \mathbb{R}[(\mathbb{R}^+)^2]_2 \oplus \mathbb{R}[\mathbb{R}^+]_2 \to \mathbb{R}[\mathbb{R}^+]_1 \to \Omega^1_{\mathbb{R}/\mathbb{Z}} \to 0$$

where for a set X and an index i (the degree), the expression $\mathbb{R}[X]_i$ means the $\mathbb{R}$-vector space generated by symbols $[x]_i$. Let's denote this intermediate complex $\mathbf{C}$.

The differential δ in degree 0 is the zero map, in degree 1 is $\delta_1[x]_1 := dx/x$, and in degree 2 is

$$\delta_2[x,y]_2 := [x]_1 + [y]_1 - [xy]_1, \quad \delta_2[x]_2 := x[x]_1 + (1-x)[1-x]_1$$

The definition of δ_3 is rather elaborate; first of all

$$\delta_3[x,y,z]_3 = [y,z]_2 - [xy,z]_2 + [x,yz]_2 - [x,y]_2$$

whereas the component of $\delta_3[x,y]_3$ in $\mathbb{R}[(\mathbb{R}^+)]_2$ is

$$[x]_2 - [y]_2 + (1-x)\left[\frac{(1-y)}{(1-x)}\right]_2 - y\left[\frac{x}{y}\right]_2$$

and the component of $\delta_3[x,y]_3$ in $\mathbb{R}[(\mathbb{R}^+)^2]_2$ is

$$(y-x)\left[y, \frac{(y-x)}{y}\right]_2 + x\left[y, \frac{x}{y}\right]_2 - (y-x)\left[(1-x), \frac{(y-x)}{(1-x)}\right]_2 - (1-y)\left[(1-x), \frac{(1-y)}{(1-x)}\right]_2$$

Lemma 1.13. *With these definitions,* $\mathbf{C}$ *is a chain complex.*

Proof. This is a computation. First of all,

$$\delta^2[x,y]_2 = \frac{dx}{x} + \frac{dy}{y} - \frac{d(xy)}{xy} = 0 \text{ and } \delta^2[x]_2 = dx + d(1-x) = d1 = 0$$

Secondly, $\delta^2[x,y,z]_3 = 0$ because δ on $[x,y,z]_3$ and its image are two successive terms of the bar complex of the group $\mathbb{R}^+$. Finally,

$$\delta_2\left[[x]_2 - [y]_2 + (1-x)\left[\frac{(1-y)}{(1-x)}\right]_2 - y\left[\frac{x}{y}\right]_2\right] = x[x]_1 + (1-x)[1-x]_1 - y[y]_1 - (1-y)[1-y]_1$$
$$+ (1-y)\left[\frac{(1-y)}{(1-x)}\right]_1 + (y-x)\left[\frac{(y-x)}{(1-x)}\right]_1 - x\left[\frac{x}{y}\right]_1 - (y-x)\left[\frac{(y-x)}{y}\right]_1$$

whereas

$$\delta_2(1-y)\left[(1-x), \frac{(1-y)}{(1-x)}\right]_2 = (1-y)[1-x]_1 + (1-y)\left[\frac{(1-y)}{(1-x)}\right]_1 - (1-y)[1-y]_1$$
$$\delta_2(y-x)\left[(1-x), \frac{(y-x)}{(1-x)}\right]_2 = (y-x)[1-x]_1 + (y-x)\left[\frac{(y-x)}{(1-x)}\right]_1 - (y-x)[y-x]_1$$
$$\delta_2(y-x)\left[y, \frac{(y-x)}{y}\right]_2 = (y-x)[y]_1 + (y-x)\left[\frac{(y-x)}{y}\right]_1 - (y-x)[y-x]_1$$
$$\delta_2 x\left[y, \frac{x}{y}\right]_2 = x[y]_1 + x\left[\frac{x}{y}\right]_1 - x[x]_1$$

and therefore $\delta^2[x,y]_3 = 0$. $\square$

The most important reason to introduce this complex is the following:

Theorem 1.14 (Exactness)**.** *The complex* $\mathbf{C}$ *is exact in degrees* ≤ 2.

This will follow from the comparison of $\mathbf{C}$ with the Cathelineau complex, but for now we defer the proof.

The complex $\mathbf{C}$ maps to the DSJ complex by a chain map t. Define $t_0 = \mathrm{id}/2$ on $\Omega^1_{\mathbb{R}/\mathbb{Z}}$, define $t_3 = 0$, and define[1]

$$t_1[x]_1 = \cot(\alpha(x)) \otimes \alpha(x), \quad t_2[x,y]_2 = \Delta(x,y), \quad t_2[x]_2 = 0$$

where $\sin^2\alpha(x) := x$ and $\Delta(x,y)$ is the normalized orthoscheme with $\alpha = \alpha(x)$ and $\gamma = \alpha(y)$ as in § 1.6. Note that $t_1[x]_1 = t_1(x)$ with the notation from § 1.6.

Lemma 1.15. *The map* t *is a chain map.*

Proof. We must show $Jt_1 = t_0\delta_1$, $Dt_2 = t_1\delta_2$ and $0 = t_2\delta_3$. We check this on generators.

$$Jt_1[x]_1 = J\cot(\alpha(x)) \otimes \alpha(x) = \frac{\cot(\alpha(x))d\sin(\alpha(x))}{\cos(\alpha(x))} = \frac{d\sin^2(\alpha(x))}{2\sin^2(\alpha(x))} = \frac{dx}{2x} = t_0\delta_1[x]_1$$

Likewise,

$$Dt_2[x,y]_2 = D\Delta(x,y) = t_1[x]_1 - t_1[xy]_1 + t_1[y] = t_1\delta_2[x,y]_2$$

as we calculated in § 1.6. Furthermore

$$\sin^2(\alpha(1-x)) = 1 - x = \cos^2(\alpha(x)) = \sin^2(\alpha(x)^\vee)$$

[1]for $0 \leq x, y \leq 1$

where the notation $\alpha^\vee := \pi/2 - \alpha$. Thus $\alpha(1-x) = \alpha(x)^\vee$ and therefore

$$\begin{aligned} t_1\delta_2[x]_2 &= \sin^2(\alpha(x))\cot(\alpha(x)) \otimes \alpha(x) + \sin^2(\alpha(x)^\vee)\cot(\alpha(x)^\vee) \otimes \alpha(x)^\vee \\ &= \sin(\alpha(x))\cos(\alpha(x)) \otimes (\alpha(x) + \alpha(x)^\vee) = 0 = Dt_2[x,y]_2 \end{aligned}$$

The equation $t_2\delta_3[x,y,z]_3 = 0$ is equivalent to Sydler's Lemma 1.10, and $t_2\delta_3[x,y]_3 = 0$ is equivalent to the Schläffli identity Lemma 1.11 (after substituting $(y-x)$, $(1-y)$ and x for x, y and z). □

We may now prove Theorem 1.7 assuming Theorem 1.14.

Proof. The first step is to show that there exists a function $h : (0,1) \to \mathcal{P}^1(\mathbb{R}^3)$ satisfying the following properties:

(1) $h(a) + h(b) - h(ab) = \Delta(a,b)$;
(2) if $a + b = 1$ then $ah(a) + bh(b) = 0$.

We may extend h to $\mathbb{R}[\mathbb{R}^+]_1 = \mathbf{C}_1$ by $h(1/a) = -h(a)$ and $\mathbb{R}$-linearity, and then the defining properties of h say exactly that for this extension, $h\delta = t_2$ on $\mathbf{C}_2$. By Theorem 1.14 the map δ is injective on $\mathbf{C}_2/\delta\mathbf{C}_3$ so if we write (arbitrarily) $\mathbf{C}_1 = A \oplus B$ where $A = \delta\mathbf{C}_2$ and $\delta : B \to \Omega^1_{\mathbb{R}/\mathbb{Z}}$ is an isomorphism we may define h to be 0 on B, and use the isomorphism $\delta^{-1} : A \to \mathbf{C}_2/\delta\mathbf{C}_3$ to define $h = t_2\delta^{-1}$ on A. The fact that t is a chain map (which implicitly uses Sydler's Lemma and the Schläffli identity) implies that t_2 is well-defined on $\mathbf{C}_2/\delta\mathbf{C}_3$. So this defines h with the desired properties (note the definition depends on a splitting of $\mathbf{C}_1$ as $A \oplus B$ which is not canonical).

The next step is to use h to define a function $\phi : \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z} \to \mathcal{P}^1(\mathbb{R}^3)$ which is a left-inverse to D. Obviously the existence of ϕ will prove that D is injective. Let's suppose we can find ϕ satisfying $h = \phi t_1$ on $\mathbf{C}_1$. Then $t_2 = h\delta = \phi t_1 \delta = \phi D t_2$ so ϕD is a left-identity for t_2. This means that D is injective on the image of t_2 which is all of $\mathcal{P}^1(\mathbb{R}^3)$ because $\mathcal{P}^1(\mathbb{R}^3)$ is generated by normalized orthoschemes, which are in the image of $\mathbb{R}[(\mathbb{R}^+)^2)]_2$.

So we are reduced to finding ϕ satisfying $h = \phi t_1$. We can try to define ϕ by the formula $\phi(1 \otimes \alpha) = \tan(\alpha)h(\sin^2(\alpha))$. To see that this is well-defined we first check that $\phi(1 \otimes \alpha^\vee) = -\phi(1 \otimes \alpha)$. This is true because

$$\tan(\alpha^\vee)h(\sin^2(\alpha^\vee)) + \tan(\alpha)h(\sin^2(\alpha)) = \frac{\cos^2(\alpha)h(\cos^2(\alpha)) + \sin^2(\alpha)h(\sin^2(\alpha))}{\sin(\alpha)\cos(\alpha)} = 0$$

by the second property of h.

Next we must check that $\phi(1 \otimes \theta) + \phi(1 \otimes \theta') = \phi(1 \otimes (\theta + \theta'))$. First, note that for any a, b, c with $a^2 + b^2 + c^2 = 1$ we have by the first property of h and the Brick identity

$$(*) \quad \frac{ac}{b}\left[h\left(\frac{c^2}{1-a^2}\right) + h\left(\frac{a^2}{1-c^2}\right) - h\left(\frac{a^2c^2}{(1-a^2)(1-c^2)}\right)\right] + \text{ cyclic permutations} = 0$$

By the second property of h we have identities of the form

$$\frac{c^2}{1-a^2}h\left(\frac{c^2}{1-a^2}\right) + \frac{b^2}{1-a^2}h\left(\frac{b^2}{1-a^2}\right) = 0 \text{ and cyclic permutations}$$

which after multiplying by $(1-a^2)a/bc$ is of the form

$$\frac{ac}{b}h\left(\frac{c^2}{1-a^2}\right)+\frac{ba}{c}h\left(\frac{b^2}{1-a^2}\right)=0 \text{ and cyclic permutations}$$

Using this, we may cancel six terms in $(*)$ in pairs and obtain the identity

$$(**)\quad \frac{ac}{b}h\left(\frac{a^2c^2}{(1-a^2)(1-c^2)}\right)+\text{ cyclic permutations}=0$$

Let's define θ and θ' by

$$\sin^2(\theta):=\frac{a^2c^2}{(1-a^2)(1-c^2)} \text{ and } \sin^2(\theta'):=\frac{b^2a^2}{(1-b^2)(1-a^2)}$$

Then $\tan(\theta)=ac/b$ and $\tan(\theta')=ba/c$, and furthermore

$$\begin{aligned}\frac{b^2c^2}{(1-b^2)(1-c^2)}&=\sin^2(\theta)\sin^2(\theta')\frac{(1-a^2)^2}{a^4}\\&=\sin^2(\theta)\sin^2(\theta')\frac{(1-\tan(\theta)\tan(\theta'))^2}{(\tan(\theta)\tan(\theta'))^2}=\sin^2(\pi-\theta-\theta')\end{aligned}$$

and therefore equation $(**)$ is precisely equal to the desired identity $\phi(1\otimes\theta)+\phi(1\otimes\theta')=\phi(1\otimes(\theta+\theta'))$ and ϕ is well-defined.

(Still need to show kernel of J is image of D). □

1.8. **Sydler's Lemma.** We now present the proof of Sydler's Lemma (i.e. Theorem 1.10). This is a rather terrifying exercise in solid Euclidean geometry; we follow Jessen [4] and Schwartz [6] very closely, though we slightly modify their notation to emphasize a (combinatorial) symmetry.

The decomposition involves six auxiliary pyramids that Schwartz calls *pseudoprisms*. A pseudoprism has the following form: let XYZ be an isosceles triangle in a horizontal plane π, with the apex at Z. Let V and W lie vertically above Y and Z respectively, where $|VY|=2|WZ|$. If U is the midpoint of XV and U' is the midpoint of VY then the tetrahedron $WUU'V$ may be rotated through angle π about the edge WU so that vertex V is rotated to X, and the pseudoprism may thereby be decomposed into a prism. See Figure 1.

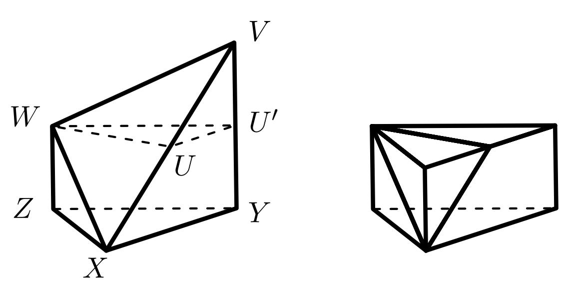

FIGURE 1. A pseudoprism is equidecomposable with a prism.

We aim to show the identity $T(a,b)+T(ab,c) \sim T(a,c)+T(ac,b)$ modulo prisms. This is accomplished by exhibiting two decompositions of a common polyhedron P as follows:

$$P = T(a,b) + T(ab,c) + ACDIH + CDIHD' + IHD'LK$$

and

$$P = T(a,c) + T(ac,b) + AC'D'I'H + C'D'I'HD + I'HDL'K$$

The six pyramids $ACDIH, \cdots, I'HDL'K$ are all pseudoprisms, and therefore equivalent to prisms. These decompositions are 'algebraic' in the sense that (depending on a, b, c) the polyhedra might overlap each other and some of them should be thought of as having negative orientation.

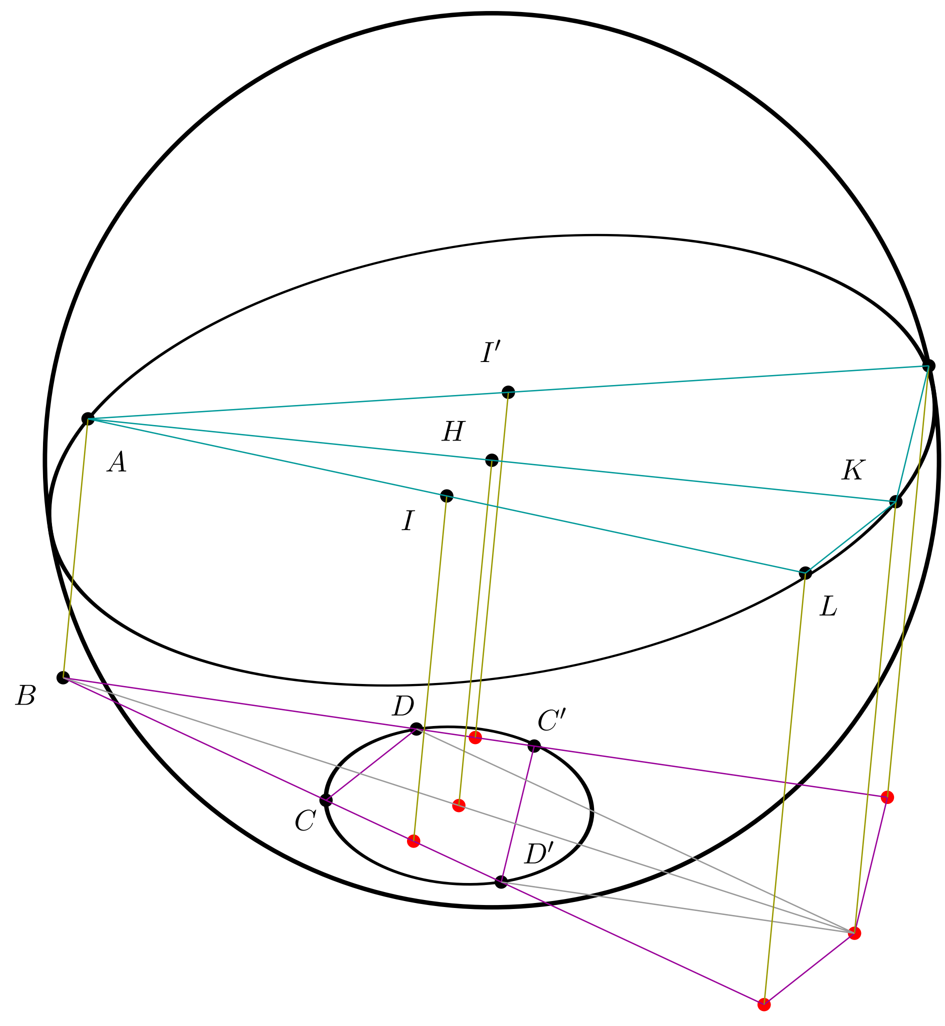

FIGURE 2. The location of vertices. H is the center of a sphere S containing a great circle through $ALKL'$. Red dots are the orthogonal projections of $IHI'LKL'$ to the plane π containing $BCDC'D'$. The points $CDC'D'$ lie on a circle which is the intersection of π with the sphere S centered at H.

There are 12 important vertices in the construction; see Figure 2. Four of them ($ABHK$) play the same role in both decompositions. The other eight ($CDIL$, $C'D'I'L'$) are interchanged in pairs by a combinatorial involution represented by the 'dash' superscript, corresponding to the algebraic involution interchanging the roles of b and c.

The four orthoschemes are $T(a,b) := ABCD$, $T(ab,c) := ADKL'$, $T(a,c) := ABC'D'$ and $T(ab,c) := AD'KL$. The polyhedron P has nine faces $ABDL'$, $ABD'L$, $ALKL'$, BDD', $L'DK$, LKD', $DD'H$, DKH, $D'KH$ and is invariant under the combinatorial involution. Thus one needs only to verify the decomposition in one case or the other.

The vertices $BCDC'D'$ are all in a plane π, which we may take to be the x–y plane. BCD' and BDC' are collinear, DC is perpendicular to BD' and $D'C'$ is perpendicular to BC'.

The vertices $AIHI'LKL'$ are in another plane. H is the midpoint of AK, I is the midpoint of AL, and I' is the midpoint of AM. The vertex H is the center of a sphere S which contains $ACDC'D'LKL'$.

For each vertex not in the x–y plane, let $p(\cdot)$ denote its vertical projection. Then $p(A) = B$, $p(I')$ is the midpoint of both $C'D$ and $Bp(L')$, and $p(I)$ is the midpoint of both CD' and $Bp(L)$. Finally, $p(H)$ is the center of a circle containing $CDC'D'$, and is the midpoint of $p(A)p(K) = Bp(K)$.

It is straightforward to verify that the pieces in the decomposition are orthoschemes and pseudoprisms with the desired geometry. One may verify the decomposition formally by comparing boundaries of the terms. To actually 'see' the decomposition is more challenging. Figure 3 shows the first decomposition, for a particular choice of a, b, c for which this decomposition may be realized by embedded disjoint polyhedra, and is possible to verify visually from the Figure.

The second decomposition (for the same choice of a, b, c) has some polyhedra negatively oriented and is harder to grasp visually; compare Figure 4.

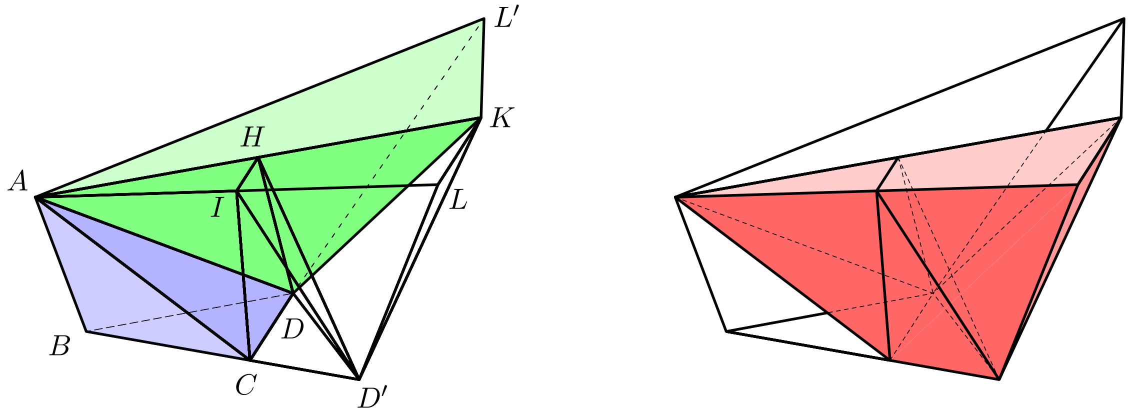

FIGURE 3. The orthoschemes $T(a,b) := ABCD$ (in blue) and $T(ab,c) := ADKL'$ (in green) and three pseudoprisms $ACDIH$, $CDIHD'$ and $IHD'LK$ (in red).

1.9. The Cathelineau complex.

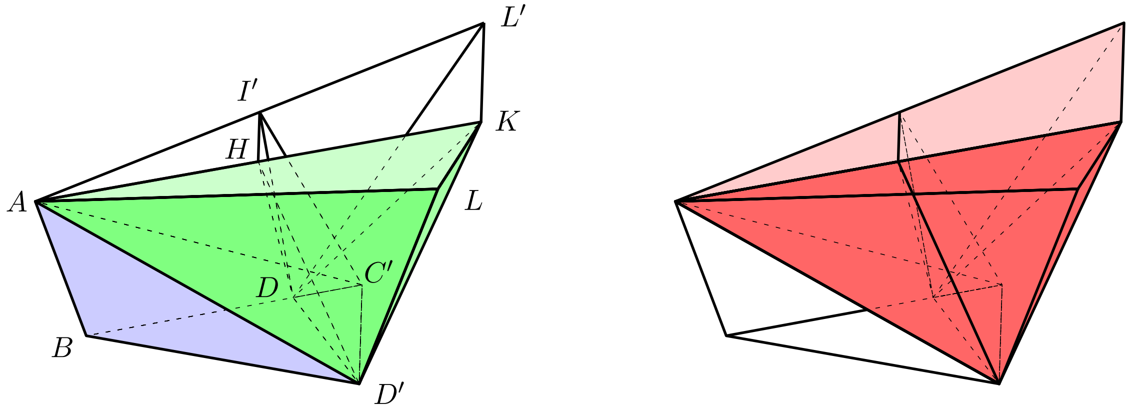

FIGURE 4. The orthoschemes $T(a,c) := ABC'D'$ (in blue) and $T(ab,c) := AD'KL$ (in green) and three pseudoprisms $AC'D'I'H$, $C'D'I'HD$ and $I'HDL'K$ (in red).

Definition 1.16 (The vector space β)**.** Let β denote the real vector space generated by symbols $[a]$ for $a \in \mathbb{R} - \{0,1\}$ subject to the following relations:

(1) $[a] = [1-a]$;
(2) $[1/a] = -[a]/a$;
(3) the *entropy equation*

$$[a] - [b] + (1-a)\left[\frac{(1-b)}{(1-a)}\right] - b\left[\frac{a}{b}\right] = 0$$

With this definition, we may now define the Cathelineau complex:

Definition 1.17 (Cathelineau complex)**.** The Cathelineau complex is the complex

$$0 \to \beta \xrightarrow{D} \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}^* \xrightarrow{L} \Omega^1_{\mathbb{R}/\mathbb{Q}} \to 0$$

where $D([a]) = a \otimes a + (1-a) \otimes (1-a)$ and L is the 'logarithmic derivative' $L(a \otimes b) = a(db/b)$.

Lemma 1.18. *The map D is well-defined.*

Proof. We must show that it vanishes on the defining relations of β. Evidently

$$D([a]) = a \otimes a + (1-a) \otimes (1-a) = D([1-a])$$

and

$$\begin{aligned} D([1/a]) &= 1/a \otimes 1/a + (a-1)/a \otimes (a-1)/a \\ &= -1/a \otimes a + (1-a)/a \otimes a - (1-a)/a \otimes (1-a) = D(-[a]/a) \end{aligned}$$

where we used the relations $x \otimes y = x \otimes (-y)$ and $x \otimes y/z = x \otimes y - x \otimes z$ in $\mathbb{R} \otimes \mathbb{R}^*$. Finally, D vanishes on the left-hand side of the entropy equation by a brute force calculation[2]. This calculation is essentially the same as the proof the $\delta^2 = 0$ on $\mathbb{R}[(\mathbb{R}^+)^2]_3$ that we gave in Lemma 1.13, except that because the target group is $\mathbb{R} \otimes \mathbb{R}^*$ rather than $\mathbb{R}[\mathbb{R}^+]$ we may assume multiplicativity in the $\mathbb{R}^*$ factor, and there is no need for a term analogous to the component of $\delta_3[x,y]_3$ in $\mathbb{R}[(\mathbb{R}^+)^2]_2$. □

[2] there is a scissors congruence proof of this fact

Analogous to Lemma 1.9 we have

Lemma 1.19. *L is surjective and $LD = 0$.*

Proof. For any $t \otimes t$ we have $L(t \otimes t) = dt$ so L is surjective. Furthermore, for any $[a]$ we have

$$LD([a]) = L(a \otimes a + (1-a) \otimes (1-a)) = da + d(1-a) = d1 = 0$$

∎

2. Hyperbolic Scissors Congruence

2.1. **Group homology.** Let G be a group and M a (left) $\mathbb{Z}[G]$-module (hereafter we just say 'a G-module'). Tensoring a G-module with $\mathbb{Z}$ (thought of as a trivial G module) defines a functor from G-modules to abelian groups which is right-exact. The left derived functors are the *homology of G*.

Explicitly: let C_* be a free resolution of $\mathbb{Z}$ by G-modules

$$\cdots \to C_3 \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0$$

One standard choice for C_* is the *bar complex*, which is generated in degree n by expressions $[g_1|g_2|\cdots|g_n]$ with boundary operator

$$\partial[g_1|g_2|\cdots|g_n] = g_1[g_2|\cdots|g_n] + \sum_i (-1)^i [g_1|\cdots|g_i g_{i+1}|\cdots|g_n] \pm [g_1|g_2|\cdots|g_{n-1}]$$

Then $H_*(G, M)$ is the homology of the complex

$$\cdots \to C_3 \otimes M \to C_2 \otimes M \to C_1 \otimes M \to C_0 \otimes M \to 0$$

and $H^*(G, M)$ is the homology of the complex

$$0 \to \mathrm{Hom}_G(C_0, M) \to \mathrm{Hom}_G(C_1, M) \to \mathrm{Hom}_G(C_2, M) \to \cdots$$

2.2. $(\mathbb{X}, G)$ **Scissors Congruence.** Fix a space $\mathbb{X}$ with a G-action. Let $\chi : G \to \pm 1$ be a character. Define $\mathcal{P}^\chi(\mathbb{X}, G)$ (or just $\mathcal{P}$ if χ is understood) to be the free abelian group generated by ordered $(n+1)$-tuples $X := (x_0, x_1, \cdots, x_n)$ modulo relations

1. if X lie in a proper hyperplane, $X = 0$;
2. $X = \chi(g) g X$ for any $g \in G$;
3. for any $(n+2)$-tuple $(x_0, x_1, \cdots, x_{n+1})$ we have the identity

$$\sum_i (-1)^i (x_0, \cdots, \hat{x}_i, \cdots, x_{n+1}) = 0$$

If H is any subgroup of G then $\mathcal{P}(\mathbb{X}, H)$ is a G-module in the obvious way. Thus one has $H_0(G, \mathcal{P}(\mathbb{X}, 1)) = \mathcal{P}(\mathbb{X}, G)$.

2.3. **The Steinberg Complex.** Fix a field F and a dimension n. Let $\mathbb{P}^n$ be n-dimensional projective space over F. A *flag* is a strictly increasing sequence of linear subspaces $L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_k$ of $\mathbb{P}^n$, where L_k is strictly proper. Define $\mathbb{T}_n$ to be the simplicial complex with one k-simplex for each flag as above. Because L_k is strictly proper for each flag, the dimension of $\mathbb{T}_n$ is $(n-1)$.

Theorem 2.1 (Solomon–Tits)**.** *The space $\mathbb{T}_n$ is homotopic to a wedge of $(n-1)$-spheres*

Proof. In fact, one may give a more precise description of the abelian group $H_{n-1}(\mathbb{T}_n)$ as follows. An ordered $(n+1)$-tuple of points $X := (x_0, x_1, \cdots, x_n) \in \mathbb{P}^n$ and a permutation $\sigma \in S_{n+1}$ determines a maximal flag

$$F(X,\sigma) := \langle x_{\sigma(0)} \rangle \subset \langle x_{\sigma(0)}, x_{\sigma(1)} \rangle \subset \cdots \langle x_{\sigma(0)}, x_{\sigma(1)}, \cdots, x_{\sigma(n-1)} \rangle$$

and therefore an $(n-1)$-simplex in $\mathbb{T}_n$. The chain

$$A(X) := \sum_{\sigma} (-1)^{|\sigma|} F(X,\sigma)$$

(called the *apartment* associated to X) is evidently a simplicial sphere, combinatorially isomorphic to the boundary of the abstract 'simplex' with vertices $x_0, x_1, \cdots, x_n$.

Now, for any two maximal flags F_1, F_2 in general position, there is some (unique) X so that $F_1 = F(X, \mathrm{id})$ and $F_2 = F(X,\tau)$ where $\tau \in S_{n+1}$ is the 'opposite' element; i.e. the permutation taking i to $n-i$ for all i. The apartment $A(X)$ minus F_2 may be canonically contracted using the standard geodesic combing of $S_{n+1} - \tau$ associated to the standard generating set. If F_2' is another flag in general position with respect to F_1, and $F_1 = F(X', \mathrm{id})$, $F_2' = F(X', \tau)$ for some X', the contraction of $A(X') - F_2'$ agrees with the contraction of $A(X) - F_2$ on the overlap (if any). Thus if we fix a flag F, the complex $\mathbb{T}_n$ minus all simplices F' where F' is in general position relative to F, is contractible; and $\mathbb{T}_n$ is therefore homotopic to a wedge of spheres $A(X)$ for X ranging over all ordered $(n+1)$-tuples of points for which $F = F(X, \mathrm{id})$. □

The group $\mathrm{PGL}(n,F)$ acts on $\mathbb{T}_n$ and therefore also on $H_{n-1}(\mathbb{T}_n)$; this is called the *Steinberg Representation*, denoted hereafter by St. The character of this representation is (up to sign) the Lefschetz number associated to the action on (reduced) homology (i.e. the alternating sum of the traces on each individual homology group). This may be also be computed as the alternating sum of the traces on the chain groups (at least when these groups are finitely generated). This may be computed by enumerating the conjugacy classes of parabolic subgroups P (which are exactly those that stabilize some flag F), and inducing the character of the trivial representation of P up to $\mathrm{PGL}(n,F)$. In other words,

$$\chi_{\mathrm{St}} = \sum_P (-1)^{|P|} \mathrm{Ind}_P^{\mathrm{PGL}(n,F)} \mathbf{1}_P$$

2.4. **Steinberg module for $\mathbb{X}$.** Let's fix an n-dimensional space $\mathbb{X}$ with a G-action, and lets suppose we are given the notion of a *linear subspace* of X spanned by a finite set of elements. For example, if $\mathbb{X}$ is spherical, Euclidean or hyperbolic space of dimension n, such linear subspaces are simply the totally geodesic subspaces.

For such an $\mathbb{X}$ we may define a Steinberg module $\mathrm{St}_{\mathbb{X}}$ as the homology of the complex $\mathbb{T}_{\mathbb{X}}$ of proper flags in $\mathbb{X}$ where the flag associated to an ordered $(n+1)$-tuple of points

$X := (x_0, \cdots, x_n)$ is the flag of linear subspaces spanned by $x, \cdots, x_i$ for $i < n$. Exactly as in the proof of the Solomon–Tits theorem, the complex $\mathbb{T}_\mathbb{X}$ is homotopic to a wedge of spheres of dimension $(n-1)$ and its homology in dimension $(n-1)$ is $\mathrm{St}_\mathbb{X}$.

Lemma 2.2. *As a G module $\mathrm{St}_\mathbb{X}$ is isomorphic to $\mathcal{P}(\mathbb{X}^n, 1)$. Consequently $\mathrm{St}_\mathbb{X} = \mathcal{P}(\mathbb{X}^n, 1)$ and $\mathcal{P}(\mathbb{X}^n, G) = H^0(G, \mathrm{St}_\mathbb{X})$.*

Proof. Both $\mathrm{St}_\mathbb{X}$ and $\mathcal{P}(\mathbb{X}^n, 1)$ are generated by nondegenerate ordered $(n+1)$-tuples X: such symbols generate $\mathcal{P}(\mathbb{X}^n, 1)$ by definition, whereas to each X is associated an apartment $A(X)$ and its homology $[A(X)] \in H_{n-1}(\mathbb{T}_\mathbb{X}) = \mathrm{St}$.

$\mathcal{P}(\mathbb{X}^n, 1)$ is the quotient of the group of nondegenerate $(n+1)$-tuples by the $(n+2)$-*term relation* $\sum_i (-1)^i (x_0, \cdots, \hat{x}_i, \cdots, x_{n+1}) = 0$ for any $(n+2)$-tuple. We must show that the $(n+2)$-term relation holds in $\mathrm{St}_\mathbb{X}$, and that all relations in $\mathrm{St}_\mathbb{X}$ are consequences of this one.

Let's consider the chain complex C_* that computes $\mathrm{St}_\mathbb{X}$. Let's let $\bar{C}_*$ denote the complex whose m chains are arbitrary ordered $(m+1)$-tuples of elements of $\mathbb{X}$. This complex is evidently contractible, so its homology is trivial. The complex C_* is a quotient of the subcomplex of $\bar{C}_*$ in degree $< n$ by the degenerate simplices (those that generate degenerate flags). Thus $\mathrm{St}_\mathbb{X} = H_{n-1}(C_*)$ is the cokernel of $\bar{C}_{n+1} \to \bar{C}_n$ modulo degenerate flags. But this is precisely a presentation of $\mathcal{P}(\mathbb{X}^n, 1)$. ∎

2.5. **Hyperbolic Geometry.** Let $\mathbb{H}^n$ denote hyperbolic space of dimension n. There are three natural scissors congruence groups associated to $\mathbb{H}^n$:

1. $\mathcal{P}(\mathbb{H}^n)$, the group of (ordinary, finite) hyperbolic polyhedra mod isometry and subdivision;
2. $\mathcal{P}(\bar{\mathbb{H}}^n)$, the group of *semi-ideal* hyperbolic polyhedra (those in which some or all vertices are allowed to lie at infinity) mod isometry and subdivision; and
3. $\mathcal{P}(\partial\mathbb{H}^n)$, the group of *ideal* hyperbolic polyhedra (those in which all vertices are required to lie at infinity) mod isometry and subdivision.

A totally geodesic subspace of $\mathbb{H}^n$ extends in an obvious way to a 'subspace' of $\bar{\mathbb{H}}^n$. Conversely, any $(k+1)$-tuple of points in $\bar{\mathbb{H}}^n$ determines such a subspace, except when $k = 0$ and the point lies in $\partial\mathbb{H}^n$. We therefore have Tits complexes $\mathbb{T}_\mathbb{H}$, $\mathbb{T}_{\bar{\mathbb{H}}}$ and $\mathbb{T}_{\partial\mathbb{H}}$, and there are natural inclusions of the first and third complex into the second.

Let G denote the isometry group of $\mathbb{H}^n$. Each of the scissors congruence groups may be computed as H_0 of G with coefficients in the respective Steinberg modules. There are therefore natural maps $\mathcal{P}(\mathbb{H}^n) \to \mathcal{P}(\bar{\mathbb{H}}^n)$ and $\mathcal{P}(\partial\mathbb{H}^n) \to \mathcal{P}(\bar{\mathbb{H}}^n)$.

Theorem 2.3 (Dupont)**.** *For any $n \ge 2$ the map $\mathcal{P}(\mathbb{H}^n) \to \mathcal{P}(\bar{\mathbb{H}}^n)$ is an isomorphism. For any odd $n > 2$ the map $\mathcal{P}(\partial\mathbb{H}^n) \to \mathcal{P}(\bar{\mathbb{H}}^n)$ is an isomorphism mod 2-torsion, and (unconditionally) an isomorphism when $n = 3$.*

The proof depends on some standard facts in homological algebra we now state. First of all, if H is a subgroup of G and M is an H-module, there is the so-called *induced* G-module $\mathrm{Ind}_H^G M$ which is just $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M$. If g_j is a set of (left) coset representatives of H in G, we may think of $\mathrm{Ind}_H^G M$ as an abelian group as $\oplus_i M_i$ where each M_i is isomorphic to M, and $g \in G$ acts on $m \in M_i$ by $gm = hm \in M_j$ where $gg_i = g_j h$ for some unique coset representative g_j and $h \in H$. The first fact we need is known as *Shapiro's Lemma*:

Lemma 2.4 (Shapiro's Lemma)**.** *For any H subgroup of G and any H-module M,*

$$H_i(H,M) = H_i(G, \mathrm{Ind}_H^G M)$$

We only need this lemma in dimensions $i = 0,1$ where it follows immediately from a simple calculation.

The second fact we need is known as *Center Kills*:

Lemma 2.5 (Center Kills)**.** *If z is in the center of G, and for some ring R the element z acts on an $R[G]$-module M as multiplication by $\lambda \in R$ then $(\lambda - 1)$ annihilates $H_*(G,M)$.*

This lemma follows from the elementary fact that for any $x \in G$, the 'conjugation' action of x on chains

$$x \cdot [g_1|\cdots|g_n]m \to [g_1^x|g_2^x|\cdots|g_n^x]xm$$

is the identity on homology. One may see this by an explicit chain homotopy from this map to the identity map, given by

$$[g_1|\cdots|g_n]m \to \sum_i (-1)^i [g_1|\cdots|g_i|x^{-1}|g_{i+1}^x|\cdots|g_n^x]m$$

whence the conjugation action of z acts on $H_*(G,M)$ both as multiplication by λ and as the identity.

Let us now prove Theorem 2.3

Proof. First we show that $\mathcal{P}(\mathbb{H}^n) \to \mathcal{P}(\bar{\mathbb{H}}^n)$ is an isomorphism. The map $\mathbb{T}_{\mathbb{H}}^n \to \mathbb{T}_{\bar{\mathbb{H}}^n}$ is an inclusion of chain complexes, and the cokernel is the chain complex $\oplus_p \mathbb{T}_{\bar{\mathbb{H}}_p^n}$ where for $p \in \partial\mathbb{H}^n$, the complex $\mathbb{T}_{\bar{\mathbb{H}}_p^n}$ is spanned by flags F beginning with the singleton p.

For each $p \in \partial\mathbb{H}^n$ let $\mathbb{R}_p^{n-1} := \partial\mathbb{H}^n - p$ (here we are thinking of $\mathbb{H}^n$ in the upper half-space model with p at vertical infinity). Let F be any flag in $\bar{\mathbb{H}}_p^n$. Then every subspace F_i of F except the first is spanned by points at infinity; thus there is an isomorphism of complexes (of degree -1)

$$\oplus_p \mathbb{T}_{\bar{\mathbb{H}}_p^n} = \oplus_p \mathbb{T}_{\mathbb{R}_p^{n-1}}$$

The stabilizer G_p of p acts on $\mathbb{R}_p^{n-1}$ as the group of Euclidean *similarities* (i.e. isometries together with dilations). Furthermore, as a G-module, the homology of $\oplus_p \mathbb{T}_{\mathbb{R}_p^{n-1}}$ is just $\mathrm{Ind}_{G_p}^G \mathrm{St}_{\mathbb{R}_p^{n-1}}$. It follows that we have the following fragment of a long exact sequence

$$\to H_1(G_p, \mathrm{St}_{\mathbb{R}_p^{n-1}}) \to \mathcal{P}(\mathbb{H}^n) \to \mathcal{P}(\bar{\mathbb{H}}^n) \to H_0(G_p, \mathrm{St}_{\mathbb{R}_p^{n-1}}) \to 0$$

The conclusion follows from the fact (due to Dupont) that $H_*(G_p, \mathrm{St}_{\mathbb{R}_p^{n-1}})$ vanishes in every dimension, for $n \geq 2$. This follows from the argument of Center Kills together with the fact that the Euclidean scissors congruence groups $\mathcal{P}(\mathbb{E}^{n-1})$ decompose into a direct sum of $\mathbb{R}$-vector spaces on which a dilation by a factor λ acts as multiplication by λ^k, for some integer k between 1 and $(n-1)$.

(In low dimensions this is clear: area gives an isomorphism $\mathcal{P}(\mathbb{E}^2) = \mathbb{R}$ by Bolyai–Gerwein, and λ multiplies areas by λ^2; likewise, λ acts on prisms in $\mathcal{P}(\mathbb{E}^3)$ as multiplication by λ^3 and on $\mathcal{P}^1(\mathbb{E}^3)$ as multiplication by λ by Lemma 1.6).

Now let's consider $\mathcal{P}(\partial\mathbb{H}^n) \to \mathcal{P}(\bar{\mathbb{H}}^n)$. As before we have an inclusion of Tits complexes, and the cokernel is $\oplus_p \mathbb{T}_{\bar{\mathbb{H}}_p^n}$ where now we take the sum over $p \in \mathbb{H}^n$. For each p the complex $\mathbb{T}_{\bar{\mathbb{H}}^n_p}$ is isomorphic to $\mathbb{T}_{\mathbb{RP}^{n-1}_p}$ where $\mathbb{RP}^{n-1}_p$ is the projectivization of the tangent space $T_p\mathbb{H}^n$. As before the homology of $\oplus_p \mathbb{T}_{\bar{\mathbb{H}}^n_p}$ as a G-module is just $\mathrm{Ind}_{G_p}^G \mathrm{St}_{\mathbb{RP}^{n-1}_p}$ and we get an analogous long exact sequence

$$\to H_1(G_p, \mathrm{St}_{\mathbb{RP}^{n-1}_p}) \to \mathcal{P}(\partial\mathbb{H}^n) \to \mathcal{P}(\bar{\mathbb{H}}^n) \to H_0(G_p, \mathrm{St}_{\mathbb{RP}^{n-1}_p}) \to 0$$

For p finite the group G_p is just $O(n)$, and its action on $\mathbb{RP}^{n-1}_p$ is the standard action. When n is odd the (central) antipodal map $-1 \in O(n)$ acts on an $(n-1)$-sphere in an orientation-reversing way; by Center Kills it follows that $H_*(G_p, \mathrm{St}_{\mathbb{RP}^{n-1}_p})$ is 2-torsion. In dimension 3 one knows that $\mathcal{P}(\partial\mathbb{H}^3)$ and $\mathcal{P}(\bar{\mathbb{H}}^3)$ are divisible and the theorem follows. □

Remark 2.6. The surjectivity of $\mathcal{P}(\mathbb{H}^3) \to \mathcal{P}(\bar{\mathbb{H}}^3)$ and $\mathcal{P}(\partial\mathbb{H}^3) \to \mathcal{P}(\bar{\mathbb{H}}^3)$ is equivalent to the vanishing of $H_0(G_p, \mathrm{St}_{\mathbb{R}^2_p})$ and $H_0(G_p, \mathrm{St}_{\mathbb{RP}^2_p})$ for p ideal and finite respectively. We may give an elementary proof of this fact (which is really just a reformulation of Center Kills in geometric terms) at least modulo torsion for the second map, as follows.

First we show surjectivity for the first map. This is equivalent to the fact that any 3d semi-ideal polyhedron P is (stably) scissors congruent to a finite polyhedron. Let's prove this when P is a simplex with a single ideal vertex p (that we may place at vertical infinity). The intersection of P with a fixed horosphere π centered at p is a 2d Euclidean triangle P_π. If we act on P by a dilation (in the upper half-space model) by a factor of 2 we get a congruent polyhedron P' whose intersection with the same horosphere π is a Euclidean triangle P'_π which is congruent to four isometric copies of P_π. Thus P' minus four translates of P is an algebraic sum of finite polyhedra, i.e. $3P$ is in the image of $\mathcal{P}(\mathbb{H}^3)$. Dilating by a factor of 3 shows that $8P$ is in the image of $\mathcal{P}(\mathbb{H}^3)$, and since 3 and 8 are coprime, it follows that P itself is in the image of $\mathcal{P}(\mathbb{H}^3)$. By subdivision and induction, surjectivity is proved.

Next we show surjectivity for the second map (mod torsion). This is equivalent to the fact that any 3d semi-ideal polyhedron P is (stably) scissors congruent to an ideal polyhedron. Let's prove this when P is a simplex with a single finite vertex p. Call the other (ideal) vertices x_0, x_1, x_2 and let $-x_0, -x_1, -x_2$ be the vertices on $\partial\mathbb{H}^3$ so that $x_i, p, -x_i$ are collinear for each i.

Consider the six ideal simplices $(x_i, x_{i+1}, x_{i+2}, -x_i)$, $(x_i, -x_{i+1}, -x_{i+2}, -x_i)$, indices taken mod 3. The union of these ideal simplices is equal to the ideal octagon spanned by all the $\pm x_i$, except that it overlaps with degree three the two semi-ideal simplices (x_0, x_1, x_2, p) and $(-x_0, -x_1, -x_2, p)$. These latter two simplices are both isometric to P (the first one *is* P). It follows that $4P$ is equal to an algebraic sum of ideal polyhedra, so that $4P$ is in the image of $\mathcal{P}(\partial\mathbb{H}^3)$. By subdivision and induction, surjectivity is proved, mod torsion.

2.6. **Ideal scissors congruence.** We now focus attention on $\mathcal{P}(\partial\mathbb{H}^3)$. We make use of the standard identification $\partial\mathbb{H}^3 = \mathbb{CP}^1$ in such a way that the group of (orientation-preserving) isometries is $\mathrm{PSL}(2,\mathbb{C})$. The generators of $\mathcal{P}(\partial\mathbb{H}^3)$ are ideal hyperbolic simplices; each such simplex may be moved by an isometry to have (ordered) vertices at $0, 1, \infty, z$ where $z \in \mathbb{C} - \{0,1\}$, so $\mathcal{P}(\partial\mathbb{H}^3)$ is generated by symbols $[z]$.

Permuting the vertices gives the relations $[1-z] = [1/z] = -[z]$. Five ideal vertices span five different ideal simplices; thus there is the following *5-term relation*:

$$[a] - [b] + \left[\frac{b}{a}\right] - \left[\frac{1-b}{1-a}\right] + \left[\frac{1-b^{-1}}{1-a^{-1}}\right] = 0$$

Finally, the condition that simplices should be nondegenerate imposes $[\bar{z}] = -[z]$. This gives a presentation of $\mathcal{P}(\partial\mathbb{H}^3)$.

2.6.1. *Cross-ratios.* There is a natural interpretation of the arguments of the 5-term relation in terms of cross-ratios. Recall that for a non-degenerate 4-tuple $a, b, c, d \in \mathbb{CP}^1$ the cross-ratio $(a, b; c, d)$ is the expression

$$(a, b; c, d) = \frac{(a-b)(c-d)}{(a-d)(c-b)} \in \mathbb{C} - \{0, 1\}$$

(Note that there are several different conventions for the definition of cross-ratio; we adhere to this convention in what follows). The cross-ratio is invariant under the diagonal action of $\mathrm{PSL}(2, \mathbb{C})$ on the arguments. Note that $(a, 0, 1, \infty) = a$, and that under the permutation action of S_4 on the arguments the value factors through $S_4 \to S_3$ acting by permutations on

$$\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda-1}{\lambda}, \frac{\lambda}{\lambda-1}$$

In other words, cross-ratio invariant under permutations of the arguments by elements of the Klein 4-group.

If we fix 5 points $z_0, z_1, \cdots, z_4$ in $\mathbb{CP}^1$, the 5 term relation has the form

$$\sum (-1)^i \left[(z_0 \cdots \hat{z}_i \cdots z_4)\right] = 0$$

2.6.2. *Volume and Dehn invariant.* One has two natural (additive) invariants on $\mathcal{P}(\partial\mathbb{H}^3)$, namely volume, and the Dehn invariant. The Dehn invariant is defined as usual on $\mathcal{P}(\mathbb{H}^3)$; to extend it to $\mathcal{P}(\bar{\mathbb{H}}^3)$ and then to restrict it to $\mathcal{P}(\partial\mathbb{H}^3)$ we must make use of an observation of Thurston.

If P is a semi-ideal hyperbolic polyhedron, and $p \in P$ is an ideal vertex, then a horosphere centered at p intersects P in a Euclidean polygon, whose angles therefore sum to an integer multiple of π. Thus one may *renormalize* the lengths of the edges e of P by (arbitrarily) cutting off a horoball neighborhood of each ideal vertex, and letting ℓ_e denote the length of the resulting (finite) segments. The expression $D(P) := \sum_e \ell_e \otimes \alpha_e$ is independent of the choice; this is the Dehn invariant, and it defines a homomorphism from $\mathcal{P}(\partial\mathbb{H}^3)$ to $\mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z}$ just as in Euclidean 3-space. In terms of generators $[z]$ one has the formula

$$D([z]) = -2\left(\log|z| \otimes \arg(1-z) - \log|1-z| \otimes \arg(z)\right)$$

Volume is the other natural (additive) invariant. Let's compute this invariant on $[z]$. Schläffli's formula says that for a family of (combinatorially equivalent) finite 3d hyperbolic polyhedra one has

$$d\mathrm{Vol} = \frac{-1}{2}\sum_e \ell_e d\alpha_e$$

For semi-ideal polyhedra one has a similar formula using renormalized lengths ℓ_e associated to any horoball truncation.

Thus

$$\begin{aligned} d\mathrm{Vol}([z]) &= \log|z| d\arg(1-z) - \log|1-z| d\arg z \\ &= d(\log|z|\arg(1-z)) - \arg(1-z) d\log|z| + \log|1-z| d\arg(z) \end{aligned}$$

Integrating this expression, and using the fact that Vol goes to 0 as $z \to 0$,

$$\mathrm{Vol}([z]) = \log|z|\arg(1-z) - \mathrm{Im}\left[\int_0^z \frac{\log(1-z)}{z} dz\right]$$

This is a real-analytic function of $z \in \mathbb{C} - \{0,1\}$ called the *Bloch–Wigner Dilogarithm*, and in the sequel we denote it $\mathcal{L}_2(z)$.

2.7. **Dilogarithms.** Apart from the factor of $\log|z|\arg(1-z)$, the Bloch–Wigner Dilogarithm $\mathcal{L}_2(z)$ is the imaginary part of the (ordinary) Dilogarithm

$$\mathrm{Li}_2(z) := -\int_0^z \frac{\log(1-z)}{z} dz$$

which is multivalued on $\mathbb{C} - \{0,1\}$.

More generally one has the k-polylogarithms for integers $k \geq 1$:

$$\mathrm{Li}_k(z) := \sum_{n \geq 1} \frac{z^n}{n^k}$$

Note $\mathrm{Li}_1(z) = -\log(1-z)$ and $\mathrm{Li}_k(1) = \zeta(k)$ where ζ is the Riemann zeta function. The series for $\mathrm{Li}_2(1)$ converges very slowly; however Euler discovered the functional equation

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \mathrm{Li}_2(1) - \log z \log(1-z)$$

from which one may compute $\mathrm{Li}_2(1)$ numerically from the (rapidly converging) series for $\mathrm{Li}_2(1/2)$.

Abel generalized Euler's formula to a 5-term relation for Li_2 whose imaginary part gives the 5-term relation for volumes of ideal hyperbolic simplices.

2.8. **The Bloch Group.** Omitting the last defining relation for $\mathcal{P}(\partial\mathbb{H}^3)$ gives the *Bloch Group.*

Definition 2.7 (Bloch Group)**.** Let F be any field. Define $B_2(F)$ to be the free abelian group generated by symbols $[a]$ for $a \in F$ subject to the 5-term relation

$$[a] - [b] + \left[\frac{b}{a}\right] + \left[\frac{1-a}{1-b}\right] - \left[\frac{1-a^{-1}}{1-b^{-1}}\right]$$

Lemma 2.8. *If we define $\langle a\rangle := [a] + [1/a]$ then one has*

$$\langle a\rangle - \langle b\rangle + \langle b/a\rangle = 0$$

and $\langle a\rangle$ is 2-torsion for any a.

Proof. The proof follows from the 5-term relation. It follows that $\langle\cdot\rangle$ is multiplicative. Since also $\langle a\rangle = \langle a^{-1}\rangle$ by definition it follows that $\langle a\rangle$ is 2-torsion. □

Lemma 2.9. *The element $[a] + [1-a]$ is independent of a.*

Proof. Substitute $a, b \to 1-b, 1-a$ and collect terms from two copies of the 5-term relation. $\square$

The element $C_F := [a] + [1-a]$ satisfies

$$3C_F = [a] + [1-a] + \left[\frac{1}{a}\right] + \left[1 - \frac{1}{a}\right] + \left[\frac{a}{a-1}\right] + \left[\frac{1}{1-a}\right]$$
$$= \langle a \rangle + \langle 1-a \rangle + \langle 1-a^{-1} \rangle = \langle \frac{-a(1-a)^2}{a} \rangle = \langle -1 \rangle$$

In a field in which $x^2+1=0$ and $x^2-x+1=0$ have solutions $C_F=0$ but (for instance) in $B_2(\mathbb{R})$ the element C_F is 6-torsion.

2.9. **The Bloch Complex.** There is a *coproduct* $\Delta : B_2(F) \to \Lambda^2 F^*$ defined by $\Delta[z] = z \wedge (1-z)$. We shall see that this vanishes on the 5-term relation, and is therefore well-defined.

This fits into a (right-exact) complex

$$B_2(F) \xrightarrow{\Delta} \Lambda^2 F^* \to K_2^M(F) \to 0$$

where $K_2^M(F)$ is the Milnor K_2 of F, which for now we can take to be the cokernel of Δ by definition.

There is a commutative diagram

$$\begin{array}{ccc} B_2(\mathbb{C}) & \xrightarrow{\Delta} & \Lambda^2\mathbb{C}^* \\ \downarrow & & \downarrow \\ \mathcal{P}(\partial\mathbb{H}^3) & \xrightarrow{D} & \mathbb{R}\otimes\mathbb{R}/\pi\mathbb{Z} \end{array}$$

The first vertical map is the obvious one, where one thinks of $\mathcal{P}(\partial\mathbb{H}^3)$ as the quotient of the Bloch group by $[\bar{z}] = -[z]$. The second vertical map comes from the ($\mathbb{R}$-linear) isomorphism $\mathbb{C}^* = \mathbb{R}^+ \oplus \mathbb{R}/2\pi\mathbb{Z}$ which exhibits

$$\Lambda^2\mathbb{C}^* = \Lambda^2\mathbb{R}^+ \oplus \mathbb{R}^+ \otimes \mathbb{R}/\pi\mathbb{Z} \oplus \Lambda^2\mathbb{R}/\pi\mathbb{Z}$$

where $\mathbb{R}^+ \otimes \mathbb{R}/\pi\mathbb{Z}$ may be singled out as the -1 eigenspace for complex conjugation. Here the first factor $\mathbb{R}^+$ may be thought of multiplicatively; taking the logarithm of this factor gives the map $\mathbb{R}^+ \otimes \mathbb{R}/\pi\mathbb{Z} \to \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z}$. Another way to say all this is that the second horizontal row is the -1 eigenspace of complex conjugation on the first horizontal row.

2.9.1. Δ *is well-defined.* It remains to show that Δ is well-defined. I.e.:

Lemma 2.10. Δ *vanishes on the 5-term relation.*

We give two proofs of this fact. The first is purely computational.

Proof. In terms of the cross-ratio we want to show for any $z_0, \cdots, z_4$ that

$$\Delta\left(\sum (-1)^i \left[(z_0 \cdots \hat{z}_i \cdots z_4)\right]\right) = 0$$

Now, one may check that

$$\Delta[(a,b;c,d)] = \frac{(a-b)(c-d)}{(a-d)(c-b)} \wedge \frac{(a-c)(b-d)}{(a-d)(b-c)}$$

using the fact that $(a,c;b,d) = 1-(a,b;c,d)$. The wedge product here is with respect to F^* thought of as a multiplicative group; thus if we define

$$A(x_0,x_1,x_2) := \sum_i (x_i - x_{i+1}) \wedge (x_{i+1} - x_{i+2})$$

indices taken mod 3, then we have the identity

$$\Delta[(a,b;c,d)] = A(a,b,c) - A(a,b,d) + A(a,c,d) - A(b,c,d)$$

And the desired relation follows from this. □

Morally speaking, $\Delta[\cdot]$ is the 'coboundary' of A, and Δ of the 5-term relation is the coboundary of this coboundary, which is why it is 0.

The next proof hints at a deeper connection to Cluster Algebras, that we shall develop in the next subsection.

Proof. Consider a sequence f_n defined by $f_{n+2} := (1+f_{n+1})/f_n$. Then one may check that whatever f_1 and f_2 are, $f_5 = (1+f_1)/f_2$ and $f_6 = f_1$, so that the sequence is always periodic of period 5. Now,

$$\Delta(\sum[-f_i]) = \sum(-f_i) \wedge (1+f_i) = \sum f_i \wedge f_{i-1}f_{i+1} = (\prod f_i) \wedge (\prod f_i) = 0$$

By a calculation one may check that this identity is equivalent to the vanishing of Δ on the 5-term relation. □

2.10. **Cluster Algebras.** A *quiver* Q is a finite directed graph. We label the vertices by variables which for the moment we take to lie in F^* for some field F. We assume there are no loops, and at most one edge (of either orientation) between any two vertices.

Associated to Q is an element W_Q in $\Lambda^2 F^*$ which is the sum of terms $a \wedge b$ for all oriented edges in Q from a vertex labeled a to a vertex labeled b.

There is an operation on quivers called mutation, defined as follows.

(1) Choose a vertex v (by abuse of notation let's also refer to the label as v) and reverse the orientation of all arrows with an endpoint at v.
(2) Add all new composable edges (i.e. if there are oriented edges from x to y and from y to z add an oriented edge from x to z if none exists).
(3) Cancel pairs of edges with the same endpoints and opposite orientations.
(4) Change the label on vertex v to $(\prod_{x\to v} x + \prod_{v \to y} y)/v$ (here $\prod_{x\to v} x$ means the product, over all incoming edges in Q to v, of the label of the initial vertex).

Proposition 2.11. *(1) any sequence of mutations on Q will produce quivers in which vertices are labeled by rational functions of the original vertex variables in such a way that only monomials arise in the denominators; and*
(2) the set of possible (labeled) quivers obtained from Q by repeated mutation is finite if and only if one of them is a simply laced Dynkin diagram.

The collection of labeled quivers obtainable from some quiver by repeated mutation form the vertices of a directed graph in which directed edges correspond to mutation. Let Q and Q' be two elements obtained by mutation at a vertex v of Q. One may easily compute

$$W_{Q'} - W_Q = \Delta\left(\frac{\prod_{x\to v} x}{\prod_{v\to x} x}\right)$$

It follows that for Q_i the terms in a cycle of mutations one has $\sum W_{Q_i} = 0$.

The second proof of Lemma 2.10 encodes this identity for a 5-term cycle in the A_2 cluster algebra, starting at Q a single directed edge from f_1 to f_2.

2.11. **Rogers Identity.** Rogers Identity is concerned with the relationship between the Bloch group of $\mathbb{C}$ and the Bloch group of $\mathbb{C}(z)$. For each point $p \in \mathbb{CP}^1$ we let v_p be the discrete valuation on $\mathbb{C}(z)$ associated to p; i.e. $v_p(f)$ is the order of vanishing of f at p if $f(p)$ is finite, or the negative of the order of vanishing of $1/f$ if p is a pole of f.

Theorem 2.12 (Rogers Identity). *for $f(z) \in \mathbb{C}(z)$ with $f(\infty)$ finite there is the following identity in $B_2(\mathbb{C}(z))$:*

$$[f(z)] - [f(\infty)] = \sum_{a \neq b \in \mathbb{CP}^1} v_a(f) v_b(1-f) \left[\frac{z-a}{b-a} \right]$$

Example 2.13. Let $f(z)$ be the cross-ratio

$$f(z) := (z, x_1; x_2, x_3) = \frac{(z-x_1)(x_2-x_3)}{(z-x_3)(x_2-x_1)}$$

Remember that $1 - (z, x_1; x_2, x_3) = (z, x_2; x_1, x_3)$ and that the only zero of $(z, x_1; x_2, x_3)$ is x_1 and the only pole is x_3. Thus this identity reduces to

$$\begin{aligned} [(z, x_1; x_2, x_3)] - [(\infty, x_1; x_2, x_3)] &= \left[\frac{z-x_1}{x_2-x_1} \right] - \left[\frac{z-x_1}{x_3-x_1} \right] - \left[\frac{z-x_3}{x_2-x_3} \right] \\ &= [(z, x_1; x_2, \infty)] - [(z, x_1; x_3, \infty)] - [(z, x_3; x_2, \infty)] \end{aligned}$$

which specializes to the 5 term relation associated to a 5-tuple of points in the projective line over $\mathbb{C}(z)$.

The coproduct maps $B_2(\mathbb{C}(z))$ to $\Lambda^2 \mathbb{C}(z)^*$, which may be composed with the natural map to $\Lambda^2(\mathbb{C}(z)^*/\mathbb{C}^*) = \Lambda^2 \mathrm{Div}^0(\mathbb{CP}^1)$. We remark that Rogers Identity is essentially equivalent to the exactness of the sequence

$$0 \to B_2(\mathbb{C}) \to B_2(\mathbb{C}(z)) \to \Lambda^2 \mathrm{Div}^0(\mathbb{CP}^1) \to 0$$

For any constant $a \in \mathbb{C} - \{0, 1\}$, both a and $1-a$ represent 0 in $\mathrm{Div}^0(\mathbb{CP}^1)$, so $B_2(\mathbb{C})$ is in the kernel of the map to $\Lambda^2 \mathrm{Div}^0(\mathbb{CP}^1)$. This map is surjective, since

$$\left[\frac{z-a}{b-a} \right] \to \frac{z-a}{b-a} \wedge \frac{z-b}{b-a} = ([a] - [\infty]) \wedge ([b] - [\infty])$$

and $\mathrm{Div}(\mathbb{CP}^0)$ is generated by expressions of the form $[a] - [\infty]$. Conversely, any expression $\sum [f_i(z)]$ in $B_2(\mathbb{C}(z))$ may be expressed, via Rogers Identity, as a sum of terms of the form $[(z-a)/(b-a)]$ over the zeros and poles a, b of the f_i and $1 - f_i$ respectively. Any cancellation of the image of this sum in $\Lambda^2 \mathrm{Div}^0(\mathbb{CP}^1)$ arises from cancellation of suitable zeros and poles of the $f_i(z)$, which in turn may be realized by instances of the 5-term relation as in Example 2.13 modulo constant functions (i.e. elements of $B_2(\mathbb{C})$).

We now give the proof of Rogers Identity.

Proof. The proof is by induction on the degree of $f \in \mathbb{C}(z)$, by which we mean the sum of the degrees of the numerator and denominator when f is expressed as a ratio of coprime polynomials.

Suppose we write $f(z) = r_1(z)/r_2(z)$. Since the r_j are coprime, we can find distinct points $x_1, x_2, x_3 \in \mathbb{C}$ so that

$$r_1(x_1) = 0, \quad r_2(x_2) = 0, \quad (r_1 - r_2)(x_3) = 0$$

Let $\varphi(z) := (z, x_1; x_3, x_2)$ so that $\varphi(0) = x_1$, $\varphi(1) = x_3$ and $\varphi(\infty) = x_2$, and recall that we have already proved Rogers Identity for functions of the form φ (this is Example 2.13). By the 5-term relation

$$[f(z)] - [\varphi(z)] = -\left[\frac{\varphi(z)}{f(z)}\right] - \left[\frac{1 - f(z)}{1 - \varphi(z)}\right] + \left[\frac{1 - f(z)^{-1}}{1 - \varphi(z)^{-1}}\right]$$

and one may check that each term on the right has smaller degree than f. Collecting terms and some bookkeeping proves the theorem. $\square$

A Corollary of Rogers Identity is that the kernel of $\Delta : B_2(\mathbb{C}(z)) \to \Lambda^2\mathbb{C}(z)^*$ is contained in $B_2(\mathbb{C})$. The proof only used the fact that $\mathbb{C}$ is algebraically closed, to find suitable points x_1, x_2, x_3 for any $f(z) = r_1(z)/r_2(z)$. Generalizing this fact is:

Theorem 2.14 (Suslin)**.** *For any field F, if $x \in B_2(F(z))$ is in the kernel of Δ then $x \in B_2(F)$ modulo torsion.*

Another Corollary of Rogers Identity is the fact that $B_2(\mathbb{C})$ is divisible:

Corollary 2.15. *$B_2(\mathbb{C})$ is divisible. In fact for any z*

$$[z^n] = n\left(\sum_{j=1}^{n}[z\zeta^j]\right)$$

where $\zeta = e^{2\pi i/n}$.

Proof. By Rogers Identity we may write $[(1 - z^n)/(1 - z)]$ as a sum of linear terms. Thus the desired identity holds in $B_2(\mathbb{C}(z))$ and therefore also in $B_2(\mathbb{C})$ for any specific z not equal to 0 or an nth root of unity. $\square$

In fact, Suslin shows:

Theorem 2.16 (Suslin)**.** *$B_2(\mathbb{C})$ is uniquely divisible.*

Suslin proves this by defining a map 'division by n' which takes $[a] \in B_2(\mathbb{C})$ to $\sum_{\alpha^n = a}[\alpha]$. The difficulty is to show that this is well-defined (i.e. that the image of the 5-term relation is zero). This can be done by an explicit calculation when $n = 2$.

2.12. **Symplectic Form on $\mathcal{M}_{0,n}$.** Let $\mathcal{M}_{0,n}$ denote the moduli space of n ordered distinct points in $\mathbb{CP}^1$ modulo the action of $\mathrm{PSL}(2, \mathbb{C})$. We define a natural symplectic form on this space. Let U denote the space of $2 \times n$ complex matrices

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

whose 2×2 minors $\Delta_{ij} := x_i y_j - x_j y_i$ are all nonzero (these Δ_{ij} are *Plücker coordinates* on U). The group $\mathrm{GL}(2, \mathbb{C})$ acts on U (by matrix multiplication on the left) and there is a commuting action of $(\mathbb{C}^*)^{n-1}$ which scales the columns. The quotient of this pair of actions is $\mathcal{M}_{0,n}$; the image of a matrix is the projective equivalence class of the sequence of points $(x_1/y_1, \cdots, x_n/y_n)$.

Now consider an n-gon whose vertices are labeled by vectors $v_i := \left(\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}\right)$. To each cyclically oriented triple of indices ijk we associate the expression

$$\omega_{ijk} := \frac{d\Delta_{ij}}{\Delta_{ij}} \wedge \frac{d\Delta_{jk}}{\Delta_{jk}} + \text{cyclic permutations}$$

Associated to a triangulation τ of the n-gon we define ω_τ to be the sum over all (oriented) triangles ijk in τ of ω_{ijk}.

Theorem 2.17. *The form ω_τ is independent of the triangulation.*

Remark 2.18. One should compare with Wolpert's Formula for the Weil–Petersson symplectic form on the moduli space of Riemann surfaces of fixed genus.

Proof. First of all, for any field F we defined $K_2^M(F)$ to be the cokernel of $\Delta : B_2(F) \to \Lambda^2 F^*$. There is a map $\Lambda^2 F^* \to \Omega^2_{F/\mathbb{Q}}$ defined by

$$x \wedge y \to \frac{dx}{x} \wedge \frac{dy}{y}$$

This vanishes on the image of Δ since

$$x \wedge (1-x) \to \frac{dx}{x} \wedge \frac{d(1-x)}{1-x} = 0$$

Now, define for each triple ijk the expression

$$W_{ijk} = \Delta_{ij} \wedge \Delta_{jk} + \Delta_{jk} \wedge \Delta_{ki} + \Delta_{ki} \wedge \Delta_{ij}$$

This is an element of $\Lambda^2 \mathbb{C}(U)^*$ where $\mathbb{C}(U)$ is the function field of U. Define likewise W_τ to be the sum of W_{ijk} over the triangles in some triangulation of the n-gon.

The map $\Lambda^2 \mathbb{C}(U)^* \to \Omega^2_{\mathbb{C}(U)/\mathbb{Q}}$ above takes W_τ to ω_τ so to show it is well-defined it suffices to show that any two triangulations give elements W_τ, $W_{\tau'}$ that differ by an element in the image of $B_2(\mathbb{C}(U))$.

As is well-known any two triangulations of a polygon differ by a sequence of 2–2 'flips', a move that switches the diagonal in some 4-gon $ijkl$ and changes W_τ by an expression of the form

$$W_{ijl} + W_{jkl} - W_{ijk} - W_{ikl}$$

On the other hand if v_i, v_j, v_k, v_l are the projective coordinates associated to the indices i, j, k, l then

$$\begin{aligned}\Delta[(v_i, v_j; v_k, v_l)] &= \frac{(v_i - v_j)(v_k - v_l)}{(v_i - v_l)(v_k - v_j)} \wedge \frac{(v_i - v_k)(v_j - v_l)}{(v_i - v_l)(v_j - v_k)} = \frac{\Delta_{ij}\Delta_{kl}}{\Delta_{il}\Delta_{kj}} \wedge \frac{\Delta_{ik}\Delta_{jl}}{\Delta_{il}\Delta_{jk}} \\ &= \Delta_{ij} \wedge \Delta_{ik} + \Delta_{ij} \wedge \Delta_{jl} + \Delta_{kl} \wedge \Delta_{ik} + \Delta_{kl} \wedge \Delta_{jl} \\ &\quad - \Delta_{il} \wedge \Delta_{ik} - \Delta_{il} \wedge \Delta_{jl} - \Delta_{kj} \wedge \Delta_{ik} - \Delta_{kj} \wedge \Delta_{jl} \\ &\qquad - \Delta_{ij} \wedge \Delta_{il} - \Delta_{ij} \wedge \Delta_{jk} - \Delta_{kl} \wedge \Delta_{il} - \Delta_{kl} \wedge \Delta_{jk} \\ &= W_{ijl} + W_{jkl} - W_{ijk} - W_{ikl}\end{aligned}$$

This proves the theorem. □

Remark 2.19. Associated to a triangulation τ of an n-gon we get an oriented graph Γ, by inscribing an oriented triangle σ' in each simplex σ of τ so that the vertices of σ' are the midpoints of σ. Each edge ij of the triangulation gets a Plücker coordinate Δ_{ij} and these label the vertices of Γ. Thinking of this as a quiver Q we get an element $W_Q \in \Lambda^2 F^*$ where F is the field containing the vertex labels, and $W_Q = W_\tau$. Internal mutations correspond in this language to 2–2 flips. We have already seen that the difference $W_{Q'} - W_Q$ of 2-forms associated to quivers differing by a mutation is the image under Δ of a suitable product of monomials in the vertex labels. This gives a calculation-free proof of Theorem 2.17.

2.13. **Milnor K-theory.** For F any field we may define *Milnor K-theory* to be the graded ring $K_*^M(F)$ which is the quotient of the tensor algebra $T(F^*)$ by the ideal generated by *Steinberg elements* $x \otimes (1-x)$. Thus $K_*^M(F) = \oplus_{i=0}^\infty K_i^M(F)$.

Example 2.20 (Low Degree). Milnor K-theory agrees with ordinary (algebraic) K theory in degrees 0 and 1. Thus

1. $K_0^M(F) = \mathbb{Z}$ because finite dimensional projective modules over a field are just vector spaces, which are classified by their dimension;
2. $K_1^M(F) = F^*$ because determinants of invertible matrices over F are just elements of F^*.

In degree 2 we have $K_2^M(F) = \otimes^2 F^*/\langle x \otimes (1-x)\rangle$. We claim this is isomorphic to $\Lambda^2 F^*/x \wedge (1-x)$, thus justifying our identifying $K_2^M(F)$ with the cokernel of $\Delta : B_2(F) \to \Lambda^2 F^*$ earlier. This isomorphism is a special case of Lemma 2.21 which we now prove.

Lemma 2.21 (Graded Commutative). *$K_*^M(F)$ is graded commutative. That is, for any A, B we have*

$$AB = (-1)^{deg(A)deg(B)} BA$$

Proof. By induction it suffices to show that $\{A, B\} = -\{B, A\}$. First we compute

$$\{A, -A\} = \{A, \frac{1-A}{1-A^{-1}}\} = \{A, 1-A\} - \{A, 1-A^{-1}\} = \{A^{-1}, 1-A^{-1}\} = 0$$

Next we compute

$$\{A, B\} + \{B, A\} + \{A, -A\} + \{B, -B\} = \{A, -AB\} + \{B, -AB\} = \{AB, -AB\} = 0$$

□

2.13.1. *The residue map.* For any field F with a discrete valuation V and residue field $\bar{F}$ there is a *residue* map $\mathrm{res}_V : K_n^M(F) \to K_{n-1}^M(\bar{F})$ defined as follows. Let π be a uniformizer in F^* (i.e. an element with $V(\pi) = 1$). Then every element of F^* can be written uniquely as $\pi^k \cdot u$ for $k \in \mathbb{Z}$ where u is a unit in the valuation ring (i.e. the subring where $V \geq 0$). Every $\{\pi^{k_1}u_1, \cdots, \pi^{k_n}u_n\} \in K_n^M(F)$ may be written (by expanding tensor products multiplicatively) as a sum of terms of two kinds:

(1) degree 0 in π: those of the form $\{u_1, \cdots, u_n\}$ all units; these map by res_V to 0; and
(2) degree 1 in π: those of the form $\{\pi, u_2, \cdots, u_n\}$; these map by res_V to $\{\bar{u}_2, \cdots, \bar{u}_n\}$ in $K_{n-1}^M(\bar{F})$.

Example 2.22. Let $F = \mathbb{C}(X)$, the field of rational functions on a curve X, and let $V = V_p = \mathrm{ord}_p$ for some $p \in X$. Then $\mathrm{res}_p : K_1^M(\mathbb{C}(X)) \to K_0^M(\mathbb{C})$ is just ord_p.

The residue maps V_p at different $p \in X$ are related by the following theorem. First note that for any element $\{f_1, f_2, \cdots, f_n\} \in K_n^M(\mathbb{C}(X))$ that $\mathrm{res}_p = 0$ for all but finitely many p. It follows that the infinite sum $\sum_p \mathrm{res}_p$ makes sense, and in fact:

Theorem 2.23 (Reciprocity)**.** *For any curve X, the map*

$$K_n^M(\mathbb{C}(X)) \xrightarrow{\sum_p \mathrm{res}_p} K_{n-1}^M(\mathbb{C})$$

is the zero map.

Example 2.24. $K_1^M(\mathbb{C}) = \mathbb{C}^*$. For any $\{f, g\} \in K_2^M(\mathbb{C}(X))$ and any $p \in X$ we can write $f = \pi^{k_1}u_1$, $g = \pi^{k_2}u_2$. Then

$$\{f, g\} = k_1\{\pi, u_2\} - k_2\{\pi, u_1\} + \{u_1, u_2\} + \{\pi^{k_1}, \pi^{k_2}\}$$

Thus $\mathrm{res}_p\{f, g\} = u_2(p)^{k_1}/u_1(p)^{k_2}$. If we define the *Weil symbol*

$$(f, g)_p := (-1)^{\mathrm{ord}_p(f)\mathrm{ord}_p(g)} \frac{g^{\mathrm{ord}_p(f)}}{f^{\mathrm{ord}_p(g)}}(p)$$

then $\prod_{p \in X}(f, g)_p = 1$. This is known as the *Weil reciprocity law.* It is easy to prove on $X = \mathbb{CP}^1$; in general, it may be proved by 'push down' (i.e. a suitable transfer map) to $\mathbb{CP}^1$ under some $X \to \mathbb{CP}^1$.

Theorem 2.23 for $F = \mathbb{C}(X)$ and $n = 3$ is known as the *Suslin Reciprocity Law.* We prove it in the special case $F = \mathbb{C}(t)$ (i.e. $X = \mathbb{CP}^1$). The group $K_3^M(\mathbb{C}(t))$ is generated by things of the form $\{f, g, h\}$ which may be expressed as a sum of expressions of the form $\{x - a, x - b, x - c\}$, $\{x - a, x - b, u\}$, $\{x_a, u_1, u_2\}$ and $\{u_1, u_2, u_3\}$. Proving the theorem for each of these expressions is easy; e.g.

$$\sum_p \mathrm{res}_p(\{x-a, x-b, x-c\}) = \{a-b, a-c\} + \{a-c, b-c\} + \{b-c, b-a\} = \{\frac{a-b}{a-c}, \frac{b-c}{a-c}\}$$

which is equal to 0 by the Steinberg relation.

For any curve X we may form the following commutative diagram

$$\begin{tikzcd} B_2(\mathbb{C}(X)) \otimes \mathbb{C}(X)^* \arrow[r] \arrow[d, "\sum_p \mathrm{res}_p"'] & \Lambda^3\mathbb{C}(X)^* \arrow[d, "\sum_p \mathrm{res}_p"'] \\ B_2(\mathbb{C}) \arrow[r] & \Lambda^2\mathbb{C}^* \end{tikzcd}$$

where the top horizontal map takes $[f]\otimes g \to f\wedge(1-f)\wedge g$. Suslin Reciprocity says that the right vertical map lies in the kernel of the map to $K_2^M(\mathbb{C})$ there is a lift $h:\Lambda^3\mathbb{C}(X)^*\to B_2(\mathbb{C})$. Goncharov conjectured that there is a lift which makes this diagram commute. This was proved by Rudenko. There is no known explicit formula for h in general, except when $X=\mathbb{CP}^1$, in which case one may take

$$h(f_1\wedge f_2\wedge f_3)=\sum_{a,b,c} V_a(f_1)V_b(f_2)V_c(f_3)[(a,b;c,\infty)]$$

2.14. **Cho–Kim formula.** Every finite 3-dimensional hyperbolic polyhedron is equal to an algebraic sum of ideal simplices; nevertheless it is challenging to give a direct formula for the volume of a finite hyperbolic 3-simplex T. One such formula is due to Cho–Kim.

Label the vertices of T from 1 to 4, let α_{ij} be the dihedral angles and ℓ_{ij} the edge lengths. Define an auxiliary function $CK(t)$ by

$$CK(t)=\frac{(t-e^{P_1})(t-e^{P_2})(t-e^{P_3})(t-e^{P_4})}{(t-1)(t-e^{H_1})(t-e^{H_2})(t-e^{H_3})}$$

where P_i is the perimeter of face i, and the H_i are the lengths of Hamiltonian cycles in the 1-skeleton.

The function $CK(t)$ takes the value 1 at 0 and ∞, and at two other complex numbers z_1, z_2. Then

$$\mathrm{Vol}(T)=\sum_\alpha \mathcal{L}_2\left(\frac{\alpha}{z_1}\right)-\mathcal{L}_2\left(\frac{\alpha}{z_2}\right)$$

where the sum is taken over all α equal to a zero or pole of $CK(t)$ (i.e. $\alpha=e^{P_i}$, $\alpha=1$ or $\alpha=e^{H_i}$).

Let h be Rudenko's function proving Goncharov's conjecture for $\mathbb{CP}^1$. Then

$$h(CK(t)\wedge(t,z_1;1,z_2)\wedge t)$$

is an explicit sum of ideal simplices algebraically realizing T in $\mathcal{P}(\bar{\mathbb{H}}^3)$.

The Dehn invariant of T is $\sum e^{\ell_{ij}}\wedge e^{i\alpha_{ij}}$ by definition. On the other hand, it is also equal to $\sum_p \mathrm{res}_p(CK(t)\wedge(t,z_1;1,z_2)\wedge t)$. Equating these two quantities gives a rather unexpected linear algebraic relation between the edge lengths and dihedral angles of T.

3. Hopf Algebras and Motivic Cohomology

3.1. **Spherical Scissors Congruence.** There are actually two different natural definitions of spherical scissors congruence. One, that we may call 'geometric spherical scissors congruence' and denote $\mathcal{P}(S^n)_g$ is generated by isometry classes of spherical simplices contained in an open half space, with relations the usual cut and paste along hyperplanes.

The other, that we may call 'algebraic spherical scissors congruence' and denote $\mathcal{P}(S^n)_a$ is generated in dimension n by ordered $(n+1)$ tuples of points $(x_0,\cdots,x_n)$ with relations

(1) (nondegeneracy): if the x_i lie in a hyperplane, $(x_0,\cdots,x_n)=0$;
(2) (boundary): for any $(n+2)$-tuple of points $\sum(-1)^i(x_0,\cdots,\hat{x}_i,\cdots,x_{n+1})=0$; and
(3) (group): for any $g\in \mathrm{O}(n+1)$ we have $(gx_0,\cdots,gx_n)=(-1)^{\mathrm{sign}(\det(g))}(x_0,\cdots,x_n)$.

In $\mathcal{P}(S^n)_a$ the boundary relation and nondegeneracy imply

$$(x_0, x_1, \cdots, x_n) = (-x_0, x_1, \cdots, x_n)$$

ands by induction

$$(x_0, x_1, \cdots, x_n) = (-x_0, -x_1, \cdots, -x_n)$$

But if n is even, the antipodal map $x \to -x$ has determinant -1 so the group relation implies that $\mathcal{P}(S^{2n})_a$ is 2-torsion, and in fact one may show that $\mathcal{P}(S^{2n})_a = 0$.

Algebraic spherical scissors congruence is more natural from the point of view of homological algebra, since one evidently has

$$\mathcal{P}(S^n)_a = H^0(\mathrm{O}(n+1), \mathrm{St}_{\mathbb{RP}^n})$$

For this reason one might want to refer to $\mathcal{P}(S^{2n})_a$ as *elliptic* scissors congruence (where elliptic geometry refers specifically to the geometry of $\mathrm{O}(n+1)$ acting on $\mathbb{RP}^n$).

There is a natural map $\mathcal{P}(S^n)_g \to \mathcal{P}(S^n)_a$ which is evidently surjective. Dupont–Sah show

Theorem 3.1 (Dupont–Sah; AG sequence)**.** *For any n there is an exact sequence*

$$0 \to \mathcal{P}(S^{n-1})_g \to \mathcal{P}(S^n)_g \to \mathcal{P}(S^n)_a \to 0$$

where $\mathcal{P}(S^{n-1})_g \to \mathcal{P}(S^n)_g$ is obtained by coning *a spherical simplex contained in the equator of S^n to the north pole.*

In particular, one obtains a coning isomorphism $\mathcal{P}(S^{2n-1})_g \to \mathcal{P}(S^{2n})_g$ for each n. The inverse is more or less explicit, and related to Gauss–Bonnet.

Remark 3.2. Product with an orthogonal interval defines a map $\mathcal{P}(\mathbb{E}^{n-1}) \to \mathcal{P}(\mathbb{E}^n)$. Jessen showed this is an isomorphism $\mathcal{P}(\mathbb{E}^3) \to \mathcal{P}(\mathbb{E}^4)$, and one may ask whether $\mathcal{P}(\mathbb{E}^{2n-1}) \to \mathcal{P}(\mathbb{E}^{2n})$ is an isomorphism for all n (this is open even for $n = 3$).

Because of Theorem 3.1 all the interesting information in geometric spherical scissors congruence is already contained in algebraic spherical scissors congruence; so from now on we restrict attention exclusively to the algebraic kind, and drop the subscript. Furthermore, we ignore torsion by tensoring everything over $\mathbb{Q}$.

Spherical scissors congruence plays a distinguished role compared to the hyperbolic and Euclidean variants because of the following structure theorem, due to Sah:

Theorem 3.3 (Sah; Hopf algebra)**.** *There is a natural structure of a commutative graded Hopf algebra on $\oplus_{n\ge 0}\mathcal{P}(S^{2n-1})$ where $\mathcal{P}(S^{-1}) := \mathbb{Q}$.*

We denote this Hopf algebra by $\mathcal{S}_*$ with $\mathcal{S}_n := \mathcal{P}(S^{2n-1})$.

Proof. We must define three (graded) operations on $\mathcal{S}_*$ — a *product*, a *coproduct* and an *antipode* — satisfying suitable compatibility conditions.

The product $m : \mathcal{S}_* \times \mathcal{S}_* \to \mathcal{S}_*$ is orthogonal join: given spherical simplices $A \subset S^{2n-1}$ and $B \subset S^{2m-1}$ their join $A * B \subset S^{2n+2m-1}$ is obtained by realizing S^{2n-1} and S^{2m-1} as the intersection of $S^{2n+2m-1} \subset \mathbb{R}^{2n+2m}$ with orthogonal coordinate subspaces $\mathbb{R}^{2n}$ and $\mathbb{R}^{2m}$, and taking the union of all geodesic segments joining A to B. This is commutative and associative, and the unit is the element $1 \in \mathcal{S}_0$.

The coproduct $\Delta : \mathcal{S}_n \to \oplus_{i=0}^n \mathcal{S}_i \otimes \mathcal{S}_{n-i}$ is the Dehn invariant: for $P \in \mathcal{P}(S^{2n-1})$ the component $\Delta_i(P) \in \mathcal{S}_i \otimes \mathcal{S}_{n-i}$ is

$$\Delta_i(P) = \sum_F F \otimes \mathrm{link}_P(F)$$

where the sum is taken over the $(2i-1)$-dimensional faces F of P, and $\mathrm{link}_P(F)$ means the intersection of an orthogonal linking sphere to F with P. We sometimes write $\Delta(P) = 1 \otimes P + P \otimes 1 + \Delta'(P)$ and refer to Δ' as the 'restricted coproduct'. This coproduct is cocommutative and coassociative (i.e. composing $\Delta \otimes 1$ with Δ is the same as composing $1 \otimes \Delta$ with Δ). One must also check the product and coproduct are compatible; i.e.

$$\begin{tikzcd}
\mathcal{S}_* \otimes \mathcal{S}_* \arrow[r, "\Delta \circ m"] \arrow[d, "\Delta\otimes\Delta"'] & \mathcal{S}_* \otimes \mathcal{S}_* \\
\mathcal{S}_* \otimes \mathcal{S}_* \otimes \mathcal{S}_* \otimes \mathcal{S}_* \arrow[r, "1\otimes\sigma\otimes 1"] & \mathcal{S}_* \otimes \mathcal{S}_* \otimes \mathcal{S}_* \otimes \mathcal{S}_* \arrow[u, "m\otimes m"']
\end{tikzcd}$$

Finally, the antipode $A : \mathcal{S}_* \to \mathcal{S}_*$ takes a spherical simplex to its polar dual (each vertex determines a dual hypersphere, and the polar dual is cut out by these hyperspheres). This satisfies that $m \circ A \otimes 1 \circ \Delta$ is projection to the ground field $\mathcal{S}_0 = \mathbb{Q}$. □

Associated to $\mathcal{S}_*$ is the *cobar complex*:

$$\mathcal{S}_* \xrightarrow{\Delta'} \mathcal{S}_* \otimes \mathcal{S}_* \xrightarrow{\Delta\otimes 1 - 1\otimes\Delta} \mathcal{S}_* \otimes \mathcal{S}_* \otimes \mathcal{S}_* \to \cdots$$

where the nth differential is $\sum_i (-1)^i 1 \otimes \cdots \Delta \cdots \otimes 1$. The homology consists of a collection of graded $\mathbb{Q}$-vector spaces, and we denote the nth graded piece of the i-dimensional homology by $H^i(\mathcal{S}_*)_n$.

The Dehn invariant makes the direct sums $\oplus\mathcal{P}(\mathbb{E}^{2n-1})$ and $\oplus\mathcal{P}(\mathbb{H}^{2n-1})$ into Hopf comodules $\mathcal{M}_*^{\mathbb{E}}$ and $\mathcal{M}_*^{\mathbb{H}}$ over $\mathcal{S}_*$, and there are associated cobar complexes

$$\mathcal{M}_* \xrightarrow{\Delta'} \mathcal{M}_* \otimes \mathcal{S}_* \xrightarrow{\Delta\otimes 1 - 1\otimes\Delta} \mathcal{M}_* \otimes \mathcal{S}_* \otimes \mathcal{S}_* \to \cdots$$

with homology $H^i(\mathcal{M}_*^{\mathbb{E}})_n$ and $H^i(\mathcal{M}_*^{\mathbb{H}})_n$.

Conjecture 3.4 (Goncharov)**.** *Let $H^i_{\mathcal{M}}(\mathbb{C}, \mathbb{Q}(n))$ denote motivic cohomology. Then*

(1) $H^i(\mathcal{S}_*)_n = H^i_{\mathcal{M}}(\mathbb{C}, \mathbb{Q}(n))^+$ where $+$ denotes the $+1$ eigenspace of complex conjugation;

(2) $H^i(\mathcal{M}_*^{\mathbb{H}})_n = H^i_{\mathcal{M}}(\mathbb{C}, \mathbb{Q}(n))^-$ where $-$ denotes the -1 eigenspace of complex conjugation; and

(3) $H^i(\mathcal{M}_*^{\mathbb{E}})_n = \Omega^i_{\mathbb{R}/\mathbb{Q}}$, the Kähler differentials of $\mathbb{R}$ over $\mathbb{Q}$.

3.2. Complexified Scissors Congruence.

Let Q be a smooth quadric in $\mathbb{CP}^{2n-1}$. Any two such are projectively equivalent. A quadric in $\mathbb{CP}^{2n-1}$ contains two families of $\mathbb{CP}^{n-1}$s, parameterized by special Fano varieties. An *orientation* α is a choice of one of these two families.

Example 3.5. When $n = 1$ a smooth quadric in $\mathbb{CP}^1$ is a pair of points and an orientation is a choice of one of these points.

When $n = 2$ a smooth quadric in $\mathbb{CP}^3$ is isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ and an orientation is a choice of the family $\mathbb{CP}^1 \times 1$ or $1 \times \mathbb{CP}^1$.

In coordinates, if Q is given by the vanishing of a quadratic form $\sum a_{ij}x_ix_j$ an orientation is a choice of a square root of $\det(a_{ij})$.

Definition 3.6 (Complexified Scissors Congruence)**.** Complexified Scissors Congruence is a graded $\mathbb{Q}$-vector space $\mathcal{G}_*$ where $\mathcal{G}_0 = \mathbb{Q}$, and where $\mathcal{G}_n$ is generated by tuples $(Q, \alpha, H_1, \cdots, H_{2n})$ where

(1) Q is a smooth quadric in $\mathbb{CP}^{2n-1}$ and α is an orientation on Q;
(2) the H_i are hyperplanes in $\mathbb{CP}^{2n-1}$ in general position;

subject to relations

(1) (orientation:) $(Q, \bar{\alpha}, H_1, \cdots, H_{2n}) = -(Q, \alpha, H_1, \cdots, H_{2n})$;
(2) (group:) for $g \in \mathrm{PGL}$ we have $(gQ, g\alpha, gH_1, \cdots, gH_{2n}) = (Q, \alpha, H_1, \cdots, H_{2n})$; and
(3) (boundary:): $\sum(-1)^i(Q, \alpha, H_0, \cdots, \hat{H}_i, \cdots, H_{2n}) = 0$.

Example 3.7. $\mathcal{G}_1 = \mathbb{C}^*$ modulo (multiplicative) torsion (thought of as a vector space over $\mathbb{Q}$). To see this observe that any oriented quadric Q consists of an ordered pair of points $Q_1, Q_2 \in \mathbb{CP}^1$, the hyperplanes H_1, H_2 are two more distinct points, and the cross ratio $(Q_1, H_1; Q_2, H_2)$ is a complete invariant. This invariant is well-defined, since for any H_1, H_2, H_3 we have

$$(Q_1, H_1; Q_2, H_2) - (Q_1; H_1; Q_2, H_3) + (Q_1, H_2; Q_2, H_3) = 0$$

Theorem 3.8. *$\mathcal{G}_2 \cong B_2(\mathbb{C})$.*

We give only the barest outline of a proof.

Proof. The first step is to show that $\mathcal{G}_2$ is generated by ideal simplices; i.e. those with vertices on Q. There are at least two plausible ways to do this; the first is via a complexification of the Cho–Kim formula and Rudenko's h function; the second is to modify Dupont's homological argument in the proof of Theorem 2.3.

If this is accomplished, we can use the product structure $Q = \mathbb{CP}^1 \times \mathbb{CP}^1$ to write the four vertices of an ideal simplex as $(x_1, y_1), \cdots, (x_4, y_4)$, and then we define the image in the Bloch group to be $[(x_1, x_2; x_3, x_4)] - [(y_1, y_2; y_3, y_4)]$. It is then a long but routine calculation to verify that the resulting map is an isomorphism. □

Generalizing Sah's theorem one has:

Theorem 3.9 (Hopf algebra)**.** *There is a natural structure of a commutative graded Hopf algebra on $\mathcal{G}_*$.*

Proof. We simply define the product, coproduct and antipode, leaving the verification of its properties to the reader.

The product is defined as follows. Given $(Q, H_1, \cdots, H_{2n})$ and $(Q', H'_1, \cdots, H'_{2m})$ we let q be a quadratic form on $\mathbb{C}^{2n}$ for which the projectivization of its zero locus is Q, and likewise let h_i be the subspace whose projectivization is H_i; and define analogously q' on $\mathbb{C}^{2m}$ and subspaces h'_j. Then on $\mathbb{C}^{2n+2m}$ we can take the quadratic form $q \oplus q'$ and the subspaces $h_i \oplus \mathbb{C}^{2m}$ and $\mathbb{C}^{2n} \oplus h'_j$ and then projectivize the result.

The coproduct is defined by analogy with the Dehn invariant as follows. Given $S := (Q, H_1, \cdots, H_{2n})$ we enumerate the subsets $I \subset \{1, \cdots, 2n\}$ and define $H_I := \cap_{i \in I} H_i$. The 'faces' are terms

$$S_I := (H_I \cap Q, H_I \cap H_j \text{ for } j \notin I)$$

thought of as a quadric and a collection of hyperplanes in the projective space H_I. The 'angles' are terms

$$S^I := (H_I^\perp \cap Q, H_I^\perp \cap H_i \text{ for } j \in I)$$

all in the projective space $H_I^\perp$. Then

$$\Delta(S) = \sum_I S_I \otimes S^I$$

Finally the antipode takes $(Q, H_1, \cdots, H_n)$ to $(Q, (\cap_{k\neq 1} H_k)^\perp, \cdots, (\cap_{k\neq n} H_k)^\perp)$. □

3.3. **Orthoschemes.** We may usefully define orthoschemes in complexified scissors congruence as follows. An *orthoscheme* is an expression $(Q, H_1, \cdots, H_n)$ for which $H_i \perp H_j$ for $|i-j| > 1$. Let O_{n-1} denote the space of (generic) orthoschemes in $\mathbb{CP}^{n-1}$. Following Coxeter we may then prove:

Theorem 3.10. *There is a natural bijection* ort *between* $\mathcal{M}_{0,n+2}$ *and (generic) projective orthoschemes* O_{n-1} *in* $\mathbb{CP}^{n-1}$ *such that*

$$\begin{array}{ccc} \mathcal{M}_{0,n+2} & \xrightarrow{\text{ort}} & O_{n-1} \\ {\scriptstyle \textit{forget } i+2}\big\downarrow & & \big\downarrow{\scriptstyle \textit{ith face}} \\ \mathcal{M}_{0,n+1} & \xrightarrow{\text{ort}} & O_{n-2} \end{array}$$

Proof. Associated to $(x_0, \cdots, x_{n+1})$ in $\mathcal{M}_{0,n+2}$ we take lines $\ell_0, \cdots, \ell_{n+1}$ in $\mathbb{C}^2$. Let $\sigma : \oplus_i \ell_i \to \mathbb{C}^2$ be sum of vectors, and let E be the kernel; i.e. E is the space of tuples $(v_0, \cdots, v_{n+1})$ such that $v_i \in \ell_i$ and $\sum v_i = 0$. Thus each $e \in E$ defines an oriented polygon, and we may define a quadratic form q on E to be the (algebraic complex) 'area' enclosed by e.

Associated to $0 < i < j < n+1$ we get two subsets of indices

$$I := \{0, \cdots, i, j, \cdots, n+1\}, \quad J := \{i, \cdots, j\}$$

Define $E_I \subset E$ to be the subspace for which $v_k \neq 0$ only when $k \in I$ and E_J similarly. Then E_I and E_J are orthogonal with respect to q.

Now, define ort to be the map

$$\text{ort} : (x_0, \cdots, x_{n+1}) \to (Q, E_{012}^\perp, E_{123}^\perp, \cdots, E_{n-1,n,n+1}^\perp)$$

where Q is the projectivization of $q = 0$ in $\mathbb{P}(E)$ and each $E_{i,i+1,i+2}^\perp$ is the projectivization of the corresponding hyperplane in E. These hyperplanes are orthogonal when their indices are not adjacent, and the map ort has the desired functorial properties. □

3.4. **Volumes and periods.** There are several natural Hopf algebras that are conjectured to be isomorphic to $\mathcal{G}_*$, including:

(1) the Hopf algebra of framed mixed Tate motives;
(2) the Hopf algebra of framed mixed Tate structures of 'geometric origin';
(3) the Hopf algebra of multiple polylogarithms; and
(4) the Hopf algebra of pairs of simplices.

Before discussing this, let's consider what can be said about the image of $\mathcal{G}_*$ under certain natural maps.

Complexified scissors congruence makes sense over any field; we denote the resulting Hopf algebra by $\mathcal{G}_*(F)$. For simplicity let's discuss complexified scissors congruence over $\bar{\mathbb{Q}}$. Volume is well-defined as a real-valued function on hyperbolic scissors congruence classes. We would like to complexify this function, and somehow obtain

$$\mathcal{G}_n(\bar{\mathbb{Q}}) \xrightarrow{vol} \mathbb{C}$$

This is problematic even for $n = 2$, where we are asking for a complexification of the Bloch–Wigner Dilogarithm $\mathcal{L}_2$. Any natural complexification will have nontrivial monodromy around 0 and 1, and will therefore be multi-valued. Neumann [5] showed how to define an 'extended' Bloch group $\tilde{B}_2$, generated by symbols $[\tilde{z}]$ for $\tilde{z}$ a point in the universal abelian cover of $\mathbb{C} - \{0, 1\}$, and one may define in a natural way a *Chern–Simons* invariant on $\tilde{B}_2$ which may be thought of as 'imaginary volume' and takes values in $\mathbb{R}/4\pi^2\mathbb{Z}$.

One natural ring where one might try to define the image of a suitable complexified volume function is the ring of motivic periods:

Definition 3.11 (Motivic Period Ring)**.** The ring $\mathbf{P}_{\mathcal{M}}$ of motivic periods is generated by symbols $[X, D_1, D_2, \gamma, \omega]$ where

(1) X is a projective algebraic variety defined over $\bar{\mathbb{Q}}$;
(2) D_1 and D_2 are divisors;
(3) γ is a cycle in $H_n(X, D_2; \mathbb{Q})$; and
(4) ω is a class in algebraic de Rham cohomology $H^n(X, D_1)$;

which determine the formal 'period' which is the expression $\int_\gamma \omega$, quotiented by natural relations corresponding to

(1) change of variables;
(2) linearity; and
(3) Stokes theorem.

Addition and multiplication of periods are associated in an obvious way to disjoint union and the Künneth formula, so $\mathbf{P}_{\mathcal{M}}$ is a ring which comes with a natural evaluation map to $\mathbb{C}$ (the numerical value of the integral corresponding to the expression $\int_\gamma \omega$).

One almost has a map $\Psi : \mathcal{G}_*(\bar{\mathbb{Q}}) \to \mathbf{P}_{\mathcal{M}}$ which associates to $(Q, H_1, \cdots, H_{2n})$ in $\mathbb{CP}^{2n-1}$ a cycle $\gamma \in H_{2n-1}(\mathbb{CP}^{2n-1}, \cup H_i)$ and a volume form $\omega \in H^{2n-1}(\mathbb{CP}^{2n-1} - Q)$ invariant under projective transformations preserving Q. To write down a formula for ω we choose a quadratic form $q := \sum q_{ij}x_ix_j$ whose projectivized zero set is Q, and define

$$\omega := \pm i^n \sqrt{\det q} \cdot \frac{\sum (-1)^i x_i dx_0 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_{2n-1}}{q^n}$$

Note that the orientation on Q corresponds to the choice of a square root of $\det q$, which is necessary to define ω. Since there is no natural choice for the cycle γ the image is only well-defined in the image $\mathbf{P}_{\mathcal{M}}/(2\pi i)\mathbf{P}_{\mathcal{M}}$.

Conjecture 3.12. *The map $\Psi : \mathcal{G}_*(\bar{\mathbb{Q}}) \to \mathbf{P}_{\mathcal{M}}/(2\pi i)\mathbf{P}_{\mathcal{M}}$ is injective, and the image is equal to the periods of mixed Tate motives of geometric origin.*

3.5. **Mixed Hodge Structures.** Cohomology groups of algebraic varieties over $\mathbb{C}$ carry a *Mixed Hodge Structure*. This is a subtle package of linear algebra that generalizes the usual package known as a Hodge structure, which arises for smooth projective varieties.

Definition 3.13 (Pure Hodge Structure). A *pure Hodge structure of weight* n consists of an abelian group $H_\mathbb{Z}$ and a decomposition $H := H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C} = \oplus_{p+q=n} H^{p,q}$ for which complex conjugation takes $H^{p,q}$ to $H^{q,p}$.

One may equivalently recover the Hodge structure from a finite decreasing filtration of H by complex subspaces F^pH satisfying

$$F^pH \cap \overline{F^qH} = 0 \text{ and } F^pH \oplus \overline{F^qH} = H \text{ whenever } p+q = n+1$$

Then we may obtain a pure Hodge structure from such a filtration by $H^{p,q} = F^pH \cap \overline{F^qH}$. Conversely, for any pure Hodge structure we may obtain such a filtration by $F^pH = \oplus_{i \ge p} H^{i,n-i}$.

Example 3.14. If X is a compact Kähler manifold (for example if X is a smooth projective complex variety) then $H_\mathbb{Z} := H^n(X;\mathbb{Z})$ admits a canonical pure Hodge structure, given by identifying $H := H^n(X;\mathbb{C})$ with de Rham cohomology, and decomposing forms into their (p,q)-parts.

Definition 3.15 (Mixed Hodge Structure). A *mixed Hodge structure* is a triple $(H_\mathbb{Z}, W_*, F^*)$ consisting of the following:

1. a (finitely generated) abelian group $H_\mathbb{Z}$, and associated vector spaces $H_\mathbb{Q} := H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$ and $H_\mathbb{C} := H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}$;
2. a finite increasing (weight) filtration $0 = W_{-1} \subset W_0 \subset W_1 \subset \cdots \subset W_{2n} = H_\mathbb{Q}$;
3. a finite decreasing (Hodge) filtration $H_\mathbb{C} = F^0 \supset F^1 \supset F^2 \supset \cdots$

so that the induced filtration F^* on the graded pieces $\mathrm{gr}_k^W H := W_k \otimes \mathbb{C} / W_{k-1} \otimes \mathbb{C}$ are pure Hodge structures of weight k.

The induced filtration is defined in the obvious way, i.e.

$$F^p \mathrm{gr}_k^W H := (F^p \cap W_k \otimes \mathbb{C} + W_{k-1} \otimes \mathbb{C}) / W_{k-1} \otimes \mathbb{C}$$

Theorem 3.16 (Deligne). *The singular cohomology (in any dimension) of a complex algebraic variety has a canonical mixed Hodge structure.*

Mixed Hodge structures are compatible with the Künneth isomorphism and product in cohomology. They form an abelian category. A certain subclass of Mixed Hodge structures are of particular importance.

Definition 3.17 (Mixed Tate structure). A Mixed Hodge Structure is *Mixed Tate* (or Hodge–Tate) if

1. $\mathrm{gr}_i^W H = 0$ if i is odd; and
2. the pure Hodge structure on $\mathrm{gr}_{2k}^W H$ is concentrated in $H^{k,k}$.

A mixed Tate stucture is *n-framed* by a choice of $v_n \in \mathrm{gr}_{2n}^W H$ and $g_0 \in (\mathrm{gr}_0^W H)^*$.

Two n-framed mixed Tate structure are *equivalent* if .... Denote the set of equivalence classes of n-framed mixed Tate structures by $\mathcal{H}_n$, and let $\mathcal{H}_* := \oplus_{n=0}^\infty \mathcal{H}_n$.

Theorem 3.18 (Hopf algebra)**.** *$\mathcal{H}_*$ has the natural structure of a graded commutative Hopf algebra.*

Proof. See [1] for details. $\square$

References

[1] A. Beilinson, A. Goncharov, V. Schechtman and V. Varchenko, *Aomoto dilogarithms, mixed Hodge structures and motivic cohomology of a pair of triangles in the plane*, The Grothendieck Festschrift, Birkhauser, 1990, 135–172

[2] H. Coxeter, *Trisecting an Orthoscheme*, Computers Math. Applic. **17** No. 1-3 (1989), 59–71

[3] A. Goncharov, *Volumes of hyperbolic manifolds and mixed Tate motives*, JAMS **12** (1999), no. 2, 569–618

[4] B. Jessen, *The algebra of polytopes and the Dehn–Sydler theorem*, Math. Scand. **22** (1968), 241–256

[5] W. Neumann, *Extended Bloch group and the Cheeger–Chern–Simons class*, Geom. Topol. **8** (2004), 413–474

[6] R. Schwartz, *The Dehn–Sydler Theorem Explained* unpublished note, https://www.math.brown.edu/reschwar/Papers/dehn_sydler.pdf

[7] J.-P. Sydler, *Conditions nécessaires et suffisantes pour l'équivalence des polyèdres de l'espace euclidien à trois dimensions*, Comment. Math. Helv. **40** (1965), 43–80

University of Chicago, Chicago, Ill 60637 USA
E-mail address: dannyc@math.uchicago.edu