SINGULAR INTERSECTIONS OF SUBGROUPS AND CHARACTER VARIETIES

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Abstract. We prove a global local rigidity result for character varieties of 3-manifolds into SL_2. Given a 3-manifold with toric boundary M satisfying some technical hypotheses, we prove that all but a finite number of its Dehn fillings $M_{p/q}$ are globally locally rigid in the following sense: every irreducible representation $\rho : \pi_1(M_{p/q}) \to \mathrm{SL}_2(\mathbb{C})$ is infinitesimally rigid, meaning that $H^1(M_{p/q}, \mathrm{Ad}_\rho) = 0$.

This question arose from the study of asymptotics problems in topological quantum field theory developed in [2]. The proof relies heavily on recent progress in diophantine geometry and raises new questions of Zilber-Pink type. The main step is to show that a generic curve lying in a plane multiplicative torus intersects transversally almost all subtori of codimension 1. We prove an effective result of this form, based mainly on a height upper bound of Habegger.

1. Introduction

Let M be a compact connected oriented 3-manifold without boundary. For any integer k called level, the quantum Chern-Simons theory associated to the group SU_2 and the level k gives an invariant $Z_k(M) \in \mathbb{C}$ called Witten-Reshetikhin-Turaev invariant. This invariant was introduced in [14] as a path integral, and constructed rigorously by Reshetikhin and Turaev using the representation theory of the quantum group $U_q sl_2$, see [12]. Formally, one can write

$$Z_k(M) = \int e^{ik\mathrm{CS}(A)} \mathcal{D}A.$$

In this expression, A is a 1-form on M with values in the Lie algebra su_2 and

$$\mathrm{CS}(A) = -\frac{1}{4\pi} \int_M \mathrm{Tr}(\mathrm{A} \wedge \mathrm{dA} + \frac{2}{3} \mathrm{A} \wedge [\mathrm{A} \wedge \mathrm{A}]).$$

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The measure $\mathcal{D}A$ is of course ill-defined but Witten understood its cut-and-paste properties from which Reshitikhin and Turaev constructed the invariant rigorously. Applying formally the stationary phase expansion to this path integral, it localizes around the critical points of the Chern-Simons functional which correspond to the flat connections, that is 1-forms A satisfying $dA+\frac{1}{2}[A\wedge A]=0$. The gauge equivalence classes of such connections correspond to conjugacy classes of representations $\rho:\pi_1(M)\to \mathrm{SU}_2$. Witten obtained formally the following asymptotic expansion:

$$Z_k(M)=\sum_\rho e^{\frac{i\pi}{4}m(\rho)+ik\mathrm{CS}(\rho)}\sqrt{T(M,\rho)}+O(k^{-1/2}). \tag{1.0.1}$$

In this formula, ρ runs over the conjugacy classes of irreducible representations from $\pi_1(M)$ to SU_2, $m(\rho)$ is an element of $\mathbb{Z}/8\mathbb{Z}$ called spectral flow and $T(M,\rho)$ is the Reidemeister torsion of M twisted by the representation Ad_ρ of $\pi_1(M)$ on su_2.

This formula is proved in very few cases, one of the difficulties being that the Reidemeister torsion is defined only for those irreducible representations ρ for which the space $H^1(M,\mathrm{Ad}_\rho)$ vanishes. The space $H^1(M,\mathrm{Ad}_\rho)$ can be identified to the Zariski tangent space of the character variety $X(M)$ at χ_ρ (see Section 2). Hence, a necessary condition for the Witten asymptotic formula to make sense is that the character variety is reduced of dimension 0.

If M is a compact, connected and oriented 3-manifold with toric boundary, we call Dehn surgery the result of gluing back to M a solid torus. Let ϕ be a homeomorphism from ∂M to $\partial D^2\times S^1$ reversing the orientation. It is well known that the homeomorphism type of the manifold $M\cup_\phi D^2\times S^1$ only depends on the homotopy class of the simple curve $\gamma=\phi^{-1}(S^1\times\{1\})\subset\partial M$. Hence we will denote by M_γ this 3-manifold without boundary and call it the Dehn filling of M with slope γ.

In [2], Charles and the first author proved that the Witten conjecture holds for M_γ if $M=S^3\setminus V(K)$ is the complement of a tubular neighborhood of the figure eight knot K, the linking number $\mathrm{lk}(\gamma,K)$ is not divisible by 4 and the character variety $X(M_\gamma)$ is reduced of dimension 0. The strategy adopted in that article should generalize to any knot provided one has a strong version of the AJ-conjecture and some information on the Reidemeister torsion.

The condition on the character variety to be reduced appeared as a technical point which was hard to check even in the case of the figure eight knot. However, we prove in this article that for a broad class of varieties M, this condition is satisfied for all but a finite number of slopes γ. More precisely, we show:

Theorem 1.1. *Suppose that M is a compact connected oriented irreducible 3-manifold with toric boundary such that*

(i) *The map $r : X(M) \to X(\partial M)$ induced by the inclusion $\partial M \subset M$ is proper.*
(ii) *The character variety $X(M)$ is reduced.*
(iii) *The image by r of the singular points of $X(M)$ are not torsion points of $X(\partial M)$ (see Section 2).*

Then for all but a finite number of slopes γ, the variety $X(M_\gamma)$ is reduced of dimension 0. Moreover, the number of exceptions can be effectively bounded.

It is well-known that $X(\partial M)$ is the quotient of a 2-dimensional torus $\mathbb{G}_\mathrm{m}^2$ by the involution $\sigma(x, y) = (x^{-1}, y^{-1})$. Denote by $\pi : \mathbb{G}_\mathrm{m}^2 \to X(\partial M)$ the quotient. In the setting of the previous theorem, the variety $C = \pi^{-1}r(X(M))$ is a plane curve defined by the so-called A-polynomial, see [4]. The following notions will be central to the proof of Theorem 1.1.

Definition 1.2. *Let C and C' be two curves in $\mathbb{G}_\mathrm{m}^2$. We say that C intersects C' transversally at $P \in C \cap C'$ if the two curves are smooth at P with distinct tangent lines. We define the singular intersection $C \cap_\mathrm{sing} C'$ of C and C' as the set of all points $P \in C \cap C'$ where the two curves are smooth with equal tangent lines.*

For any couple of relatively prime integers (p, q), let $H_{p,q}$ be the subtorus of $\mathbb{G}_\mathrm{m}^2$ defined by the equation $x^p y^q = 1$. Through some standard argument in character varieties, we reduce the proof of the previous theorem to show that C intersects transversally $H_{p,q}$ for almost all (p, q).

This fact is connected to recent questions in diophantine geometry surrounding the Zilber-Pink conjecture. Ineffectively, it follows from the 1999 bounded height property of Bombieri, Masser and Zannier (see Theorem 1 in [1]). Effective versions of the latter were worked out by Habegger over the years (see appendix B1 of [5], Theorem 7 in [6] and [7]), allowing us to give an explicit upper bound on the maximal size of a couple (p, q) such that C has non-empty singular intersection with a translate of $H_{p,q}$.

This upper bound might be of interest for the applications of Theorem 1.1 in topology. It only depends on quantities that can be computed from an equation $f(x, y) = 0$ defining C in $\mathbb{G}_\mathrm{m}^2$. The polynomial

f is involved through its total and partial degrees and its logarithmic Weil height $h(f)$ (see section 3 for the definition).

Theorem 1.3. *Let C be a curve in A with defining equation $f(x, y) = 0$ for an irreducible polynomial $f \in \bar{\mathbb{Q}}[X, Y]$. Let $\delta = \deg(f)$, $\delta_x = \deg_x(f)$ and $\delta_y = \deg_y(f)$. Assume C is not a translate of a subtorus. Then, for any translate $\gamma H_{p,q}$ with non-empty singular intersection with C, the quantity $\max(|p|, |q|)$ is at most*

$$\delta^3 \exp\left((6.10^5 + 1)\delta^4 \max\left(\delta_x \delta_y, h(f)\right)\right) .$$

In particular, the union $C^{\{1\}}$ of all singular intersections of the form $C \cap_{\mathrm{sing}} H_{p,q}$ is a finite set.

We also prove a mild strengthening of the last sentence that looks like a perfect analogue of the Zilber-Pink conjecture in the context of plane singular intersections.

In its multiplicative form –that is, when the ambient space is a multiplicative torus $T = \mathbb{G}_{\mathrm{m}}^n$– the Zilber-Pink conjecture predicts what happens to a subvaritety X when intersected to the union of all algebraic subgroups of fixed codimension m (see [16] and [11] for the original conjectures and [15] for a recent panorama of the subject).

Under the assumption that X is not contained in a proper algebraic subgroup of T, it is the statement that the subsets $X^{[m]}$ of X defined by

$$X^{[m]} = \bigcup_{\substack{\operatorname{codim} H = m \\ \zeta \text{ torsion}}} X(\bar{\mathbb{Q}}) \cap \zeta H(\bar{\mathbb{Q}}) \tag{1.3.1}$$

are not Zariski-dense in X for $m \geq \dim X + 1$, where the union runs over all subtori H of codimension m and all torsion points ζ of T.

Note that, in the particular case of a curve C lying in $\mathbb{G}_{\mathrm{m}}^2$, the assumption on C means precisely that C is not a translate of a subtorus by a torsion point. In comparison with the hypotheses of Theorem 1.3, this is weaker, yet it turns out to be sufficient for the finiteness of $C^{\{1\}}$. Under this assumption, we can even prove the finiteness of a slightly larger subset of C. Its definition is derived from formula (1.3.1) for $C^{[1]}$ by changing all intersections for singular intersections.

Theorem 1.4. *If C is a curve in $\mathbb{G}_{\mathrm{m}}^2$ that is not a translate of a subtorus by a torsion point, then*

$$C^{\{1,\mathrm{tor}\}} = \bigcup_{\substack{p \wedge q = 1 \\ \zeta \text{ torsion}}} C(\bar{\mathbb{Q}}) \cap_{\mathrm{sing}} \zeta H_{p,q}(\bar{\mathbb{Q}})$$

is a finite subset of C*.*

It is well known that, in the Zilber-Pink conjecture, the codimension value $m = \dim X + 1$ is optimal for Zariski non-density. If m is decreased further, then $X^{[m]}$ contains $X^{[\dim X]}$ that is dense in X for all X. In this respect, the main feature of Theorem 1.4 is to show that positive multiplicity of intersection can make up for a codimension drop among the H's: going from $C^{[2]}$ to the larger subset $C^{[1]}$ generates infinitely many new points, but restricting to the case of positive multiplicity yields $C^{\{1,\mathrm{tor}\}}$ and finiteness is recovered.

This line of thought goes further than the case of plane curves and makes sense in a more general framework, leading to new conjectures of Zilber-Pink type. These generalizations will be addressed in a separate article.

Finally, the last topic we study here is the relation between subsets of the form $C^{\{1,\mathrm{tor}\}}$ and $C^{[2]}$, showing that the first can be seen as a subset of the second type for a Zilber-Pink-like problem that takes place in a slightly different ambient space (see Theorem 3.6 and Remark 3.7).

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2. Character variety and a reduction

Let Γ be a finitely generated group. We denote by $R(\Gamma)$ the algebraic variety of all representations $\rho : \Gamma \to \mathrm{SL}_2(\bar{\mathbb{Q}})$. This variety is generally used over $\mathbb{C}$ by topologists whereas it is actually defined over $\mathbb{Z}$. We adopt here the field $\bar{\mathbb{Q}}$ which is more convenient for our purposes. The group $\mathrm{SL}_2(\bar{\mathbb{Q}})$ acts on $R(\Gamma)$ by $g.\rho = g\rho g^{-1}$: we denote the algebraic quotient by $X(\Gamma) = R(\Gamma)//\mathrm{SL}_2(\bar{\mathbb{Q}})$. We refer to [10, 3] for the general theory and collect here some facts.

(i) Given a representation $\rho \in R(\Gamma)$ we define its character $\chi_\rho : \Gamma \to \bar{\mathbb{Q}}$ by the formula $\chi_\rho(\gamma) = \mathrm{Tr}\,\rho(\gamma)$. As a set, $X(\Gamma)$ is the quotient of $R(\Gamma)$ by the relation $\rho \sim \rho'$ iff $\chi_\rho = \chi_{\rho'}$. This justifies the name *character variety*.

(ii) If ρ, ρ' are two elements of $R(\Gamma)$ with $\chi_\rho = \chi_{\rho'}$ and ρ irreducible, then ρ and ρ' are conjugated.

(iii) The algebra of regular functions on $X(\Gamma)$ is generated by the so-called *trace functions* defined for any $\gamma \in \Gamma$ by $f_\gamma(\rho) = \mathrm{Tr}\,\rho(\gamma)$.

(iv) A representation $\rho \in R(\Gamma)$ is reducible if and only if for all $\alpha, \beta \in \Gamma$ one has $f_{[\alpha,\beta]}(\rho) = 2$. In particular the set of reducible characters is Zariski-closed in $X(\Gamma)$ and is denoted by $X^{\mathrm{red}}(\Gamma)$ whereas its complement is denoted by $X^{\mathrm{irr}}(\Gamma)$.

(v) At an irreducible representation ρ, there is a natural isomorphism $T_{\chi_\rho} X(\Gamma) \simeq H^1(\Gamma, \mathrm{Ad}_\rho)$.

(vi) If $\Gamma = \mathbb{Z}^2$, we consider the morphism $\pi : \mathbb{G}_{\mathrm{m}}^2 \to X(\Gamma)$ mapping (x, y) to the character of the representation $\rho_{x,y}$ defined by

$$\rho_{x,y}(a, b) = \begin{pmatrix} x^a y^b & 0 \\ 0 & x^{-a} y^{-b} \end{pmatrix}.$$

It is well-known that π induces an isomorphism between the quotient of $\mathbb{G}_{\mathrm{m}}^2$ by the involution $\sigma(x, y) = (x^{-1}, y^{-1})$ and $X(\Gamma)$. In particular, we will denote by $X(\Gamma)_{\mathrm{tor}}$ the image by π of the torsion points of $\mathbb{G}_{\mathrm{m}}^2$.

(vii) If $\phi : \Gamma \to \Gamma'$ is a group homomorphism, it induces an algebraic morphism $\phi^* : X(\Gamma') \to X(\Gamma)$.

If M is a connected compact oriented manifold, we set $X(M) = X(\pi_1(M))$. If M is a surface or a 3-manifold as in Theorem 1.1, then it is an Eilenberg-Maclane space, which means that there is a natural isomorphism $H^1(\pi_1(M), \mathrm{Ad}_\rho) \simeq H^1(M, \mathrm{Ad}_\rho)$. Let $i : \partial M \to M$ be the inclusion morphism: it induces a map $i_* : \pi_1(\partial M) \to \pi_1(M)$. We denote by r the map $(i_*)^* : X(M) \to X(\partial M)$ induced by the inclusion and call it the restriction map.

If M is a connected compact oriented 3-manifold with toric boundary, one can understand representations of M_γ for a given slope $\gamma \subset \partial M$ in the following way: by Van-Kampen theorem, the fundamental group of M is the amalgamated product $\pi_1(M) \underset{\pi_1(\partial M)}{*} \pi_1(D^2 \times S^1)$. Moreover, the map $\pi_1(\partial M) \to \pi_1(D^2 \times S^1)$ is surjective with kernel generated by γ hence one has $\pi_1(M_\gamma) = \pi_1(M)/\langle\gamma\rangle$ where $\langle\gamma\rangle$ is the normal closure of γ.

In particular, a representation $\rho : \pi_1(M_\gamma) \to \mathrm{SL}_2(\bar{\mathbb{Q}})$ is the same as a representation of $\rho : \pi_1(M) \to \mathrm{SL}_2(\bar{\mathbb{Q}})$ such that $\rho(\gamma) = 1$. In terms of character varieties, $X(M_\gamma)$ fits in the following diagram (which may not be cartesian):

$$\begin{array}{ccccc} & & X(M_\gamma) & & \\ & \swarrow & & \searrow & \\ X(M) & & & & X(D^2\times S^1) \\ & \overset{r}{\searrow} & & \overset{r'}{\swarrow} & \\ & & X(\partial M) & & \end{array}$$

The image of r' is the projection of a subtorus of $\mathbb{G}_{\mathrm{m}}^2$ by the map π. We will reduce Theorem 1.1 to considerations on the intersection of $\pi^{-1}r(X(M))$ with subtori of $\mathbb{G}_{\mathrm{m}}^2$. We start with a technical lemma.

Lemma 2.1. *Let M be a manifold satisfying the assumptions of Theorem 1.1, then every irreducible component of $X(M)$ has dimension 1.*

Proof. From now, we denote the local system Ad_ρ with a subscript ρ. Let Y be an irreducible (reduced) component of $X(M)$ and χ_ρ be a smooth point of it. A standard argument involving Poincaré duality (see [8] p. 42) shows that the rank of the map $i^* : H^1_\rho(M) \to H^1_\rho(\partial M)$ is half the dimension of $H^1_\rho(\partial M)$. By Poincaré duality again, rk $i^* = \dim H^0_\rho(\partial M) \in \{1,3\}$. As r is proper, $r(Y)$ is a subvariety of the 2-dimensional variety $X(\partial M)$. This shows that $r(Y)$ has dimension 1 and because r is proper, Y also has dimension 1. □

Proposition 2.2. *Let M be a manifold satisfying the assumptions of Theorem 1.1 and fix an homeomorphism between ∂M and $S^1 \times S^1$. A slope γ corresponds to a pair (p,q) of relatively prime integers.*

Given χ_ρ a character of $X(M_{p/q})$, we denote by the same letter its restriction to $X(M)$. By the above remarks, $r(\chi_\rho) = \pi(x,y)$ for some $(x,y) \in \bar{\mathbb{Q}}^2$ with $x^p y^q = 1$.

Case 1: *$x \neq \pm 1$ or $y \neq \pm 1$.*

In that case, one can suppose that up to conjugation $\rho \circ i^* = \rho_{x,y}$. One has $H^1(M_{p/q}, Ad_\rho) = (i^*)^{-1} T_{x,y} H_{p,q}$ where $i^* : H^1(M, Ad_\rho) \to H^1(\partial M, Ad_{\rho_{x,y}})$ is induced by the restriction map.

In particular, if χ_ρ is a smooth point of $X(M)$ and $\pi^{-1}r(X(M))$ intersects $H_{p,q}$ transversally, then $H^1(M_{p/q}, Ad_\rho) = 0$.

Case 2: *$x = \pm 1$ and $y = \pm 1$.*

If χ_ρ is a smooth point of $X^{\mathrm{irr}}(M)$, then ρ factors through at most 1 Dehn filling $M_{p/q}$.

Proof. The main point is a consequence of the Mayer-Vietoris sequence applied to the decomposition $M_{p/q} = M \cup D^2 \times S^1$ given below.

$$H^0_\rho(M) \oplus H^0_\rho(D^2 \times S^1) \to H^0_\rho(S^1 \times S^1) \to \\ H^1_\rho(M_{p/q}) \to H^1_\rho(M) \oplus H^1_\rho(D^2 \times S^1) \to H^1_\rho(\partial M)$$

By the assumption $(x, y) \neq (\pm 1, \pm 1)$, the first map is onto and the result follows from the fact that the map $H^1_\rho(D^2 \times S^1) \to H^1_\rho(S^1 \times S^1)$ is the differential of the inclusion $H_{p,q} \subset A$.

In the second case, if χ_ρ is a smooth point of $X^{\mathrm{irr}}(M)$, then Poincaré duality implies that $H^0_\rho(\partial M)$ has dimension 1 and hence $\rho \circ i^*$ is a parabolic non-central representation. More explicitly, one has $\rho(a, b) = \pm \begin{pmatrix} 1 & au + bv \\ 0 & 1 \end{pmatrix}$ for some $(u, v) \neq (0, 0)$. Hence we can have $\rho(p, q) = 1$ for at most one slope $[p : q] \in \mathbb{P}^1(\mathbb{Q})$ and the result follows. □

Set $C = \pi^{-1}(r(X(M)) \subset \mathbb{G}_{\mathrm{m}}^2$. In the setting of Theorem 1.1, this is a curve defined by the so-called A-polynomial introduced in [4]. Applying Theorem 3.2 to C, we obtain that C is transverse to $H_{p,q}$ at smooth points of C for all but a finite number of slopes (p, q). Moreover by assumption, singular points of $X(M)$ do not map to torsion points of $X(\partial M)$ and hence belong to at most one subtorus. The only remaining case is a singular point (x, y) of C which is not a singular value of r. In the neighborhood of (x, y), C is a union of branches with non-trivial tangents. Removing these tangents from the list of admissible (p, q), we finally proved Theorem 1.1.

We would like to end this section with some comments on the topological meaning of the assumptions of Theorem 1.1.

(i) By Culler-Shalen theory (see [13]), the properness assumption of $r : X(M) \to X(\partial M)$ is implied by the assumption that M is small, meaning that it does not contain any closed incompressible surfaces (not boundary parallel). This assumption holds for a large family of knots such as 2-bridge knots. Such a hypothesis is necessary as the global local rigidity does not hold for instance for Whitehead doubles of knots.

(ii) The reducibility of the character variety is a notoriously hard question. There is no reason to believe that the character variety of a knot complement in S^3 is reduced, however, we do not know any counter-example.

(iii) The last assumption on singular points seems hard to check without knowing explicitly the character variety of M. However it is well-known that the singular points of $X(M)$ belonging to $X^{\mathrm{red}}(M)$ are encoded in the roots of the Alexander polynomial of M. Hence, the assumption contains in particular the fact that the Alexander polynomial does not vanish at roots of unity.

3. Plane curves

This section is devoted to the proof of Theorem 1.3. It relies heavily on the notions of *degree* and *height*.

Recall that the *degree* $\deg(P)$ of a point $P = (x_0 : \ldots : x_n)$ of $\mathbb{P}_n(\bar{\mathbb{Q}})$ is defined as the minimal degree of a number field containing a system of homogeneous coordinates of P. Equivalently, assuming for simplicity that $x_0 \neq 0$,

$$\deg(P) = \left[\mathbb{Q}\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) : \mathbb{Q}\right] .$$

The *logarithmic Weil height* $h(P)$ of P is defined as follows. Let K be a number field containing a system of homogeneous coordinates $(x_0, \ldots, x_n)$ of P. At any place v of K, there is a unique absolute value $|\cdot|_v$ associated to v such that $|p|_v \in \{1/p, 1, p\}$, for any prime number p. Let K_v (*respectively* $\mathbb{Q}_v$) be the completion of K (*resp.* $\mathbb{Q}$) with respect to this absolute value (*resp.* the absolute value induced by $|\cdot|_v$). Then, $h(P)$ is given by the formula:

$$h(P) = \sum_v \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log\left(\max_{0 \leq i \leq n} |x_i|_v\right) ,$$

where the sum runs over all places of K. Because of the normalization factors $[K_v : \mathbb{Q}_v]/[K : \mathbb{Q}]$, the right-hand side neither depend on K, nore on the system of homogenous coordinates $(x_0, \ldots, x_n)$ that is chosen for P. Therefore, $h(P)$ is well-defined.

Then, for any non-zero vector v of $\bar{\mathbb{Q}}^{n+1}$, we define the *projective height of* v as the quantity $h(P_v)$, where P_v is the point of $\mathbb{P}_n(\bar{\mathbb{Q}})$ with homogenous coordinates v. Finally, we define the height $h(f)$ of a polynomial f with coefficients in $\bar{\mathbb{Q}}$ as the projective height of its vector of coefficients.

Roughly speaking, the height of a point measures its arithmetic complexity. For example if $P \in \mathbb{P}_1(\mathbb{Q})$ is a rational point written under the form $P = (p : q)$, where p and q are coprime integers, the definition above yields

$$h(p : q) = \log(\max(|p|, |q|)) .$$

In particular, for any positive real number M, there are only finitely many rational points P of the projective line such that $h(P) \leq M$.

More generally, Northcott's theorem asserts that if the degree and height are bounded over a subset S of $\mathbb{P}_n(\bar{\mathbb{Q}})$, then S is a finite set. This crucial fact is one of the main feature of the height.

As we will see, it can be used to obtain a second proof of Theorem 1.3, except for the upper bound on $\max(|p|,|q|)$ which follows from a different route. This estimate will be derived from a height upper bound of Habegger through the following lemma.

Lemma 3.1. *Let $f_1, f_2 \in \bar{\mathbb{Q}}[X,Y]$ be polynomials of total degree N_1 and N_2 and let $N = \max(N_1, N_2)$. Let $\mathcal{F}$ be the family of coefficients appearing in either f_1 or f_2 and let $h(\mathcal{F})$ be the projective height of $\mathcal{F}$. Then, for any point $P = (x,y)$ of $\mathbb{A}^2(\bar{\mathbb{Q}})$,*

$$h(f_1(P) : f_2(P)) \leq N h(x : y : 1) + h(\mathcal{F}) + \log \binom{N+2}{2} .$$

Proof. Let $f_1(X,Y) = \sum_{k+\ell \leq N_1} a_{k,\ell} X^k Y^\ell$ and let v be a place of a number field K containing both coordinates of P and the family $\mathcal{F}$. Then

$$|f_1(P)|_v \leq \varepsilon(v) \max_{k,\ell}(|a_{k,\ell}|_v) \max(1, |x|_v, |y|_v)^{N_1} , \tag{3.1.1}$$

where

$$\varepsilon(v) = \begin{cases} \binom{N_1+2}{2} & \text{if } v \,|\, \infty , \\ \\ 1 & \text{otherwise} . \end{cases}$$

Assume for simplicity that $N_1 \geq N_2$, so $f_2(X,Y)$ is again of the form $\sum_{k+\ell \leq N_1} b_{k,\ell} X^k Y^\ell$, for some $b_{k,l} \in \bar{\mathbb{Q}}$. Then an upper bound similar to (3.1.1) holds for $|f_2(P)|_v$. The only difference is the $a_{k,\ell}$'s are replaced by the $b_{k,\ell}$'s. Therefore,

$$\max(|f_1(P)|_v, |f_2(P)|_v) \leq \varepsilon(v) \max\{|f|_v \mid f \in \mathcal{F}\} \max(1, |x|_v, |y|_v)^{N_1}$$

and it follows that

$$\begin{aligned} h(f_1(P) : f_2(P)) &\leq N_1 h(x : y : 1) + h(\mathcal{F}) + \sum_{v \,|\, \infty} \log(\varepsilon(v)) \\ &= N_1 h(x : y : 1) + h(\mathcal{F}) + \log \binom{N_1+2}{2} , \end{aligned}$$

which concludes the proof as $N_1 = \max(N_1, N_2)$. ∎

We are now ready to prove our upper bound. From now on, we let $A = \mathbb{G}_\mathrm{m}^2$ be the ambient multiplicative torus. Recall that, for any curve C lying in A, we denote by $C^{\{1\}}$ the union of all singular intersections of the form $C \cap_{\mathrm{sing}} H_{p,q}$, where (p, q) varies among all couples of relatively prime integers.

Theorem 3.2. *Let C be a curve in A with defining equation $f(x, y) = 0$ for an irreducible polynomial $f \in \bar{\mathbb{Q}}[X, Y]$. Let $\delta = \deg(f)$, $\delta_x = \deg_x(f)$ and $\delta_y = \deg_y(f)$. Assume C is not a translate of a subtorus. Then, for any translate $\gamma H_{p,q}$ with non-empty singular intersection with C, the quantity* $\max(|p|, |q|)$ *is at most*

$$\delta^3 \exp\left((6.10^5 + 1)\delta^4 \max\left(\delta_x\delta_y, h(f)\right)\right) .$$

Moreover, $C^{\{1\}}$ is a finite set of effectively bounded degree and height.

Proof. Let $P = (x, y)$ be a point belonging to the singular intersection of C and a translate γH, with $H = H_{p,q}$; then, $T_P C = T_P(\gamma H)$. The tangent space $T_P(\gamma H)$ is the subspace of $T_P A$ defined by the following equation

$$(T_P(\gamma H)) \ : \ \frac{p}{x}dx + \frac{q}{y}dy = 0\,.$$

Similarly,

$$(T_P C) \ : \ \frac{\partial f}{\partial x}(P)dx + \frac{\partial f}{\partial y}(P)dy = 0\,.$$

Therefore, the equality $T_P C = T_P(\gamma H)$ means that the two vectors of partial derivatives

$$\left(\frac{p}{x}, \frac{q}{y}\right) \quad \text{and} \quad \left(\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P)\right)$$

are colinear.

Using height theory, we now derive that $\max(|p|, |q|)$ is bounded from above, thereby showing that the possible H's are finitely many. Indeed, p and q are coprime, so $\log(\max(|p|, |q|)) = h(p : q)$ is the logarithmic Weil height of the point $(p : q) \in \mathbb{P}^1(\bar{\mathbb{Q}})$. Then, colinearity of the vectors of partial derivatives means that the corresponding points of $\mathbb{P}^1(\bar{\mathbb{Q}})$ are equal or, equivalently,

$$(p : q) = \left(x\frac{\partial f}{\partial x}(P) : y\frac{\partial f}{\partial y}(P)\right) .$$

Hence, using lemma 3.1 with $f_1 = x\,\partial f/\partial x$ and $f_2 = y\,\partial f/\partial y$, we obtain the following upper bound:

$$h(p : q) \leq \delta h(x : y : 1) + h(\mathcal{F}) + \log\left(\frac{(\delta + 1)(\delta + 2)}{2}\right) ,$$

where $\mathcal{F}$ is the family of coefficients of the partial derivatives polynomials $\partial f/\partial x$ and $\partial f/\partial y$.

Here, we can obtain a slightly better log-term, with numerator $\delta(\delta+1)$ instead of $(\delta+1)(\delta+2)$, by using the relation $h(p:q) = h_{\mathrm{m}}(p/q)$ where the right-hand side is the height of the point p/q of $\mathbb{G}_{\mathrm{m}}(\bar{\mathbb{Q}})$. This height satisfies a triangle inequality relative to multiplication, so

$$\begin{aligned} h_{\mathrm{m}}\left(\frac{p}{q}\right) &\leq h_{\mathrm{m}}\left(\frac{p}{q}\frac{y}{x}\right) + h_{\mathrm{m}}\left(\frac{x}{y}\right) \\ &= h\left(\frac{p}{x} : \frac{q}{y}\right) + h(x:y) \\ &\leq h\left(\frac{\partial f}{\partial x}(P) : \frac{\partial f}{\partial y}(P)\right) + h(P) \end{aligned}$$

and then we apply our lemma to the partial derivatives of f instead of $x\,\partial f/\partial x$ and $y\,\partial f/\partial y$. We will use this sharper bound in the sequel.

Finally, any element of $\mathcal{F}$ is a product ka_α of a coefficient of $f(X) = \sum_\alpha a_\alpha x^{\alpha_1} y^{\alpha_2}$ and a positive integer k equal to α_1 or α_2 pending on which partial derivative of f is considered. In any case, we have $k \leq \max(\delta_x, \delta_y) \leq \delta$, so

$$h(\mathcal{F}) \leq h(f) + \log\delta$$

and therefore

$$h(p:q) \leq \delta h(x:y:1) + h(f) + \log\left(\frac{\delta^2(\delta+1)}{2}\right). \tag{3.2.1}$$

To conclude the proof, we use an explicit height upper bound for the points of $C^{[1]}$ due to Habegger (see Theorem B.1 in [5]). It reads as follows: for any point $P = (x, y)$ in $C^{[1]}$,

$$\max(h_{\mathrm{m}}(x), h_{\mathrm{m}}(y)) \leq 3.10^5\, \delta^3 \max\left(\delta_x\delta_y, h(f)\right). \tag{3.2.2}$$

Combining this result with inequality (3.2.1) and using the elementary fact that $h(x:y:1) \leq h_{\mathrm{m}}(x) + h_{\mathrm{m}}(y)$, we obtain

$$\begin{aligned} h(p:q) &\leq 6.10^5\delta^4 \max\left(\delta_x\delta_y, h(f)\right) + h(f) + \log\left(\frac{\delta^2(\delta+1)}{2}\right) \\ &\leq (6.10^5+1)\delta^4 \max\left(\delta_x\delta_y, h(f)\right) + \log(\delta^3) \end{aligned}$$

and taking exponentials of both sides gives the upper bound of the theorem.

Finally, boundedness of $h(P)$ implies boundedness of $\max(|p|, |q|)$, which proves finiteness of the H's. Restricting to the case $\gamma = 1$, *i.e.* $P \in C^{\{1\}}$, it implies finiteness of the P's, because the assumption on C guarantees that its intersection with any proper subtorus is finite.

Of course, the effective upper bound on $h(P)$ mentioned in our theorem is Habegger's bound (3.2.2). Whereas the effective upper bound on $\deg(P)$ follows from our bound on $\max(|p|,|q|)$ by applying Bézout's theorem to the intersection of C and $H_{p,q}$, which gives

$$\deg(P) \ll \max(|p|,|q|)\deg(C)[K_0:\mathbb{Q}]\,,$$

where K_0 is any number field such that C is defined over K_0 (see the proof of lemma 3.3 below for a detailed exposition of a similar argument).

□

Theorem 3.2 is non trivial in the sense that both $C^{[1]}$ and

$$C^{\{1,A(\bar{\mathbb{Q}})\}} = \bigcup_{p\wedge q=1} \{P \in C_{\mathrm{reg}}(\bar{\mathbb{Q}}) \mid T_P C = T_P(PH_{p,q})\}$$

are infinite for all C. The first is obtained from $C^{\{1\}}$ by removing the tangency condition, while the second is the set of smooth points $P \in C(\bar{\mathbb{Q}})$ at which $T_P C$ coincides with the tangent line of a translated $H_{p,q}$ – without any multiplicative dependance assumption on P.

The case of $C^{[1]}$ is well known. One of the two projections π induces a dominant morphism over C, so $\pi(C)$ contains a dense open subset of $\mathbb{G}_{\mathrm{m}}$. In particular, it contains infinitely many torsion points and their inverse images form an infinite subset of $C^{[1]}$.

For $C^{\{1,A(\bar{\mathbb{Q}})\}}$, the argument is completely similar, up to a different choice of the dominant morphism π. Assuming $f(x,y)=0$ is a defining equation for C, with f irreducible, let $\pi=\sigma_C$ be the rational map from C to the dual projective line which sends any smooth point $P\in C(\bar{\mathbb{Q}})$ to the point

$$\sigma_C(P) = \left(x\frac{\partial f}{\partial x}(P) : y\frac{\partial f}{\partial y}(P)\right) \tag{3.2.3}$$

of $\mathbb{P}_1^*$. Note that it doesn't depend on the irreducible polynomial f chosen initially. Moreover, σ_C has the following property: for any smooth point $P=(x,y)$ of $C(\bar{\mathbb{Q}})$,

$$\sigma_C(P) = (\alpha:\beta) \iff (T_P C)\ :\ \alpha\frac{dx}{x}+\beta\frac{dy}{y}=0\,.$$

It follows that σ_C is constant if and only if C is a translated subtorus, in which case $\sigma_C(C)$ is a rational point and $C^{\{1,A(\bar{\mathbb{Q}})\}}=C$. Otherwise, σ_C is dominant, so its image contains a dense open subset of the line, hence infinitely many rational points. Finally, the inverse images of these points in C form an infinite subset of $C^{\{1,A(\bar{\mathbb{Q}})\}}$.

Hence, $C^{\{1,A(\bar{\mathbb{Q}})\}}$ is always Zariski-dense in C. However, under the assumption of Theorem 3.2, it is a sparse subset in the following sense.

Lemma 3.3. *Let C be a curve in A defined over a number field K_0 and assume C is not a translate of a subtorus. Then, for all $P \in C^{\{1,A(\bar{\mathbb{Q}})\}}$,*

$$\deg(P) \leq \deg(C)^2[K_0 : \mathbb{Q}]\,.$$

Proof. Assume first that C is defined over $\mathbb{Q}$, *i.e.* C can be defined by an equation $f(x, y) = 0$ with coefficients in $\mathbb{Q}$. Moreover, assume f is irreducible over $\bar{\mathbb{Q}}$, so $\deg(f) = \deg(C)$.

From formula (3.2.3), it follows that the rational map σ_C is also defined over $\mathbb{Q}$. Hence, so is its fiber over any rational point $(p : q)$ of $\mathbb{P}_1^*$. Therefore, if $P = (x, y)$ belongs to $\sigma_C^{-1}(p : q)$, then so does any Galois conjugate of P over $\mathbb{Q}$. Moreover, the number of distinct conjugates of P is precisely the number of embeddings of $\mathbb{Q}(x, y)$ in $\bar{\mathbb{Q}}$ over $\mathbb{Q}$, which equals the degree of P. Hence,

$$\deg(P) \leq |\sigma_C^{-1}(p : q)|$$

and the lemma thus follows from the estimate

$$|\sigma_C^{-1}(p : q)| \leq \deg(C)^2\,. \tag{3.3.1}$$

To prove (3.3.1), consider the irreducible components $C_1, \ldots, C_r$ of the algebraic subset of A defined by the vanishing of

$$g_{p,q}(x, y) = qx\frac{\partial f}{\partial x}(x, y) - py\frac{\partial f}{\partial y}(x, y)\,.$$

Then, $\sigma_C^{-1}(p : q)$ is the union of the intersections $C_{\mathrm{reg}} \cap C_i$ and

$$\sum_{i=1}^{r} \deg(C_i) \leq \deg(g) \leq \deg(f) = \deg(C)\,.$$

Using Bézout's theorem, it follows that

$$|\sigma_C^{-1}(a : b)| \leq \sum_{i=1}^{r} \deg(C)\deg(C_i) \leq \deg(C)^2\,,$$

which proves the claim if C is defined over $\mathbb{Q}$.

In general, C is defined over a number field K_0. Considering conjugates over K_0 instead of conjugates over $\mathbb{Q}$, the argument above yields

$$\deg(P) \leq [K_0(x, y) : K_0][K_0 : \mathbb{Q}] \leq |\sigma_C^{-1}(p : q)|[K_0 : \mathbb{Q}]$$

and the lemma follows from (3.3.1). □

Remark 3.4. In comparison with Theorem 3.2, this lemma provides a much sharper bound for the degrees of the points of $C^{\{1\}}$. Recall that the one from the theorem was derived from the fact that

$$\deg(P) \ll \max(|p|,|q|)\deg(C)[K_0:\mathbb{Q}]$$

combined with our estimate on $\max(|p|,|q|)$, thereby leading to a large upper bound depending on the height of C.

Applying Northcott's theorem, it follows from boundedness of the degree that any subset of $C^{\{1,A(\bar{\mathbb{Q}})\}}$ of bounded height is finite. We will refer to this as a *Northcott property*. In particular, it implies that

$$C^{\{1\}} \subset C^{[1]} \cap C^{\{1,A(\bar{\mathbb{Q}})\}}$$

is a finite set, which gives a second proof of Theorem 3.2, except for the upper bound on $\max(|p|,|q|)$.

This second approach leads to the following result. Recall that a torsion variety is a translate of a subtorus by a torsion point.

Theorem 3.5. *Let C be a curve in A that is not a torsion variety. Then, the union $C^{\{1,\mathrm{tor}\}}$ of all singular intersections of C with 1-dimensional torsion subvarieties of A is a finite set of effectively bounded height.*

Proof. Assume first that C is not a translate of a subtorus. Then, the height is bounded over $C^{[1]}$ (see Theorem 1 in [1]), hence also on its subset $C^{\{1,\mathrm{tor}\}}$. As the latter is also a subset of $C^{\{1,A(\bar{\mathbb{Q}})\}}$, the Northcott property derived from lemma 3.3 proves the claim.

Now, assume C is a translate γH of a subtorus H of A. Then, because of the assumption on C, γ is non-torsion. Hence $C \neq \zeta H$ for all $\zeta \in A_{\mathrm{tor}}$, so $C \cap \zeta H = \emptyset$ because the two are distinct translates of the same subtorus. Finally, if $H' \neq H$ is a second subtorus, then $T_P(PH') \cap T_P(PH) = 0$ at any point $P \in A(\bar{\mathbb{Q}})$. Therefore, $C \cap_{\mathrm{sing}} \zeta H'$ is again empty, which gives $C^{\{1,\mathrm{tor}\}} = \emptyset$ and completes the proof of the theorem. $\square$

We conclude this section with an alternative formulation of Theorem 3.5 showing the equivalence of this statement to a tangential Zilber-Pink problem. It also explains the analogy between subsets of the form $C^{\{1,\mathrm{tor}\}}$ and $C^{[2]}$: the first correspond to a subset of the second form by considering a section over C of the dual projectivized tangent bundle $\mathbb{P}^*(TA)$. Recall that the fiber of $\mathbb{P}^*(TA)$ over a point $P \in A(\bar{\mathbb{Q}})$ parametrizes lines in T_PA.

For C smooth, the section of interest $\mathcal{C}$ is simply

$$\mathbb{P}^*(TC) = \{(P, [T_PC]) \mid P \in C(\bar{\mathbb{Q}})\}.$$

When C is singular, $\tilde{\sigma}_C(P) = (P, [T_P C])$ only defines a rational section of $\mathbb{P}^*(TA)$ over C. We thus define $\mathcal{C}$ as the Zariski closure of the image of $\tilde{\sigma}_C$ in $\mathbb{P}^*(TA)$. We call $\mathcal{C}$ the *tangent section* of $\mathbb{P}^*(TA)$ over C.

To make the connection between $\tilde{\sigma}_C$ and the σ_C defined previously, consider first the trivialization of the cotangent bundle T^*A (*resp.* the tangent bundle TA) associated to the global 1-forms dx/x and dy/y (*resp.* the global vector fields $x\partial/\partial x$ and $y\partial/\partial y$). Through this trivialization, the map $\pi : \mathbb{P}^*(TA) \to A$ can be identified to the first projection $A \times \mathbb{P}^*_1 \to A$ and our rational section $\tilde{\sigma}_C$ is then given by $\tilde{\sigma}_C(P) = (P, \sigma_C(P))$.

Finally, for any couple of relatively prime integers (p, q) and any torsion point ζ of A, let $\mathcal{H}^\zeta_{p,q}$ be the tangent section of $\mathbb{P}^*(TA)$ over the torsion variety $\zeta H_{p,q}$.

Theorem 3.6. *Let C be a curve in A and let $\mathcal{C}$ be the tangent section of $\mathcal{A} = \mathbb{P}^*(TA)$ over C. If $\mathcal{C}$ is not of the form $\mathcal{H}^\zeta_{p,q}$, then*

$$\mathcal{C}^{[2]} = \bigcup_{\substack{p \wedge q = 1 \\ \zeta \text{ torsion}}} \mathcal{C}(\bar{\mathbb{Q}}) \cap \mathcal{H}^\zeta_{p,q}$$

is a finite set.

Remark 3.7. In the formula above, intersections are usual intersections, not singular ones. Moreover $\mathrm{codim}(\mathcal{H}^\zeta_{p,q}, \mathcal{A}) = 2$, so the union of the theorem is indeed a subset of the type $\mathcal{C}^{[2]}$ for a Zilber-Pink-like problem in $\mathcal{A}$.

Proof. It suffices to show that there are finitely many smooth points P of C such that $\tilde{\sigma}_C(P) = (P, [T_P C])$ lands in the union of the $\mathcal{H}^\zeta_{p,q}$. But assuming $\tilde{\sigma}_C(P) \in \mathcal{H}^\zeta_{p,q}$ means precisely that there is a point Q in $C' = \zeta H_{p,q}$ such that

$$\tilde{\sigma}_C(P) = \tilde{\sigma}_{C'}(Q)\,.$$

In other words, $P = Q$ and C and C' have equal tangent lines at P, so $P \in C \cap_{\mathrm{sing}} C'$; finiteness thus follows from Theorem 3.5. □

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