Signatures of TQFTs and trace fields of two-bridge knots

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Abstract

Let $0 < s < r$ be coprime odd integers. We show that the Frobenius algebras governing the signatures of SO_3 TQFTs at the root $q = \exp(i\pi s/r)$ contain (and are often equal to) the trace field of the two-bridge knot of parameters (r, s). This gives an intriguing relationship between these two a priori unrelated objects of low-dimensional topology.

1 Introduction

Given $q \in \mathbb{C}$, a root of unity of order $2r$ with r odd, we consider the family of $\mathrm{SU}_2/\mathrm{SO}_3$ quantum representations of mapping class groups of surfaces indexed by $\pm q$, as constructed for instance in [BHMV]. Take $\Lambda = \{0, 1, \dots, r-2\}$ and $\Lambda_+ = \Lambda \cap 2\mathbb{Z}$. These representations take the following form for any closed compact oriented surface $S_{g,n}$ of genus g with n marked points:

$$\rho_q^\lambda : \mathrm{Mod}(S_{g,n}) \to \mathrm{PU}(W_q(S_{g,n}, \lambda)).$$

Here $\mathrm{Mod}(S_{g,n})$ denote the mapping class group of $S_{g,n}$ fixing the marked points and $\lambda \in \Lambda^n$ (Λ_+^n in the SO_3-case) is a coloring of the marked points. The Hermitian vector space $W_q(S, \lambda)$ is a modular functor: this means that it enjoys many compatibility properties with respect to usual operations on surfaces, as cutting along simple curves, see for instance [DM, Section 4].

As a function of q, the dimension of $W_q(S_{g,n}, \lambda)$ only depends on r: it is given by the celebrated Verlinde formula. On the contrary, its signature is a subtle function of q not much studied until the recent work of B. Deroin and the author, see [DM]. One of its main results is that cohomological invariants of the above representations, amongst which the signature appears at 0-th order, form a semi-simple cohomological field theory.

In this article, we consider only signatures: in particular, the present article is completely elementary and does not require any expertise in TQFT. The result of [DM] states that signatures are part of a 1+1 semi-simple TQFT, or equivalently are governed by a Frobenius algebra, denoted by V_q (V_q^+ for the SO_3 case).

Let us give the precise statement: take V_q (resp. V_q^+) to be the formal $\mathbb{Q}$-vector space with basis Λ (resp. Λ_+). We denote by e_λ the corresponding basis and define $\varepsilon(e_\lambda)=1$ if $\lambda=0$ and $\varepsilon(e_\lambda)=0$ otherwise. We will put in Section 3 a structure of semi-simple commutative algebra on V_q such that the bilinear form $\eta(x,y)=\varepsilon(xy)$ is non-degenerate on V_q and the following holds:

$$\mathrm{Sign}(W_q(S_{g,n},\lambda))=\varepsilon(\Omega^g e_{\lambda_1}\cdots e_{\lambda_n}).$$

Here, we have written $\Omega=\sum_i x_i y_i\in V_q$ where $\eta^{-1}=\sum x_i\otimes y_i\in V_q\otimes V_q$. The same statement holds for SO_3: just replace V_q with the subalgebra V_q^+ generated by e_λ for $\lambda\in\Lambda_+$ and Ω with $\Omega^+=\frac{1}{2}\Omega$.

Although explicit formulas were provided in [DM], the structure of the Frobenius algebras V_q and V_q^+ remained mysterious. Indeed, the main applications of our work concentrated on the case $r=5$ where we had $V_q^+=\mathbb{Q}[x]/(x^2-x-1)$ for $q=\exp(i\pi/5)$ and $V_q^+=\mathbb{Q}[x]/(x^2+x+1)$ for $q=\exp(3i\pi/5)$ (here we have put $e_0=1$ and $e_2=x$).

The main theorem of this article identifies V_q^+ with a completely different algebra, related to the geometry of two-bridge knots. I confess that I don't have a conceptual explanation of this result which came as a surprise from numerical experiments.

Let $0<s<r$ be coprime odd integers and consider the two-bridge knot $K(r,s)$ with parameter (r,s): it is characterized by the fact that its ramified double cover is the lens space $L(r,s)$, see [M, Chapter 9]. We set

$$P(r,s)=\{\rho:\pi_1(S^3\setminus K(r,s))\to\mathrm{SL}_2(\mathbb{C}),\rho(m)\text{ is parabolic}\}/\mathrm{SL}_2(\mathbb{C})$$

Here m stands for any meridian of $K(r,s)$ and the quotient is by conjugation. This set is a 0-dimensional algebraic variety defined over $\mathbb{Q}$. Hence $\mathbb{Q}[P(r,s)]$, the algebra of functions on $P(r,s)$ is a commutative $\mathbb{Q}$-algebra. We call it the Riley algebra as it is isomorphic to $\mathbb{Q}[x]/R(x)$ where R is the Riley polynomial of $K(r,s)$, see Section 3.3.

Theorem 1. *Let r,s be coprime odd integers satisfying $0<s<r$ and set $q=\exp(\frac{i\pi s}{r})$. There is an isomorphism of algebras*

$$\mathbb{Q}[P(r,s)]\simeq V_q^+.$$

When $s>1$, the knot $K(r,s)$ is hyperbolic and the monodromy of the hyperbolic structure on $S^3\setminus K(r,s)$ appears as a complex point of $P(r,s)$. This means that the trace field of $K(r,s)$ is a factor of $\mathbb{Q}[P(r,1)]$. On the contrary, the knot $K(r,1)$ is a torus knot, hence is not hyperbolic. However it has a parabolic representation with traces in $\mathbb{Q}(\cos(\frac{2\pi}{r}))$ which is then a factor of $\mathbb{Q}[P(r,1)]$. The Hermitian TQFT corresponding to $q=\exp(i\pi/r)$ is unitary and its signature is equal to its dimension. The Frobenius algebra in this case is the usual Verlinde algebra.

In [DM], we observed without proof that amongst all roots of unity of order r, there is an involution corresponding to pairs of Frobenius algebras which are isomorphic as algebras. This phenomenon has now a nice explanation: it corresponds to the fact that $K(r,s)$ and $K(r,s^*)$ are isotopic as knots where s^* is the inverse of s modulo r.

The reader may notice that the previous theorem only states an isomorphism of algebras. Indeed, there is no natural structure of Frobenius algebra on $\mathbb{Q}[P(r,s)]$. In the next theorem, we identify this structure in terms of Riley polynomials: for simplicity, we only state the case of V_q in the introduction.

Theorem 2. *In the settings of Theorem 1, the Frobenius algebra on V_q has the following explicit form. Let $\varepsilon_n = (-1)^{\lfloor ns/r \rfloor}$ for $n = 1, \ldots, r-1$ and define the coprime and monic polynomials P_{r-2}, P_{r-1} of respective degrees $r-2, r-1$ such that the following holds:*

$$\frac{P_{r-1}}{P_{r-2}} = X + \cfrac{\varepsilon_1 \varepsilon_2}{X + \cfrac{\varepsilon_2 \varepsilon_3}{X + \cfrac{\cdots}{\cdots + \cfrac{\varepsilon_{r-2}\varepsilon_{r-1}}{X}}}}$$

Then $V_q = \mathbb{Q}[X]/(P_{r-1})$ and for any $f \in V_q$ one has:

$$\varepsilon(f) = \sum_x \mathrm{Res}_x \left(\frac{f P_{r-2}}{P_{r-1}} \right).$$

Finally, P_{r-1} is related to the Riley polynomial by the formula $P_{r-1}(X) = R(X^2)$.

The last statement has already been obtained in [JK]. This theorem shows that the Frobenius algebra structure involves the "continued fractions structure" of Riley polynomials. Its proof makes use of classical techniques of orthogonal polynomials. Moreover, the natural appearance of residues is a good sign for finding a spectral curve governing the cohomological field theories described above, which was the initial motivation of this work.

In the SO_3 case, one can write $V_q^+ = \mathbb{Q}[X]/(\chi)$ using the factorization $R(-X^2) = \chi(X)\chi(-X)$ which has also been obtained in [JK], see Section 4. In the present article, this factorization follows from the properties of the element $e_{r-3} \in V_q^+$. This element has been used in [BHMV] in order decompose the TQFT vector spaces associated to SU_2. It was also very useful in [DM] in order to compute the first term in the R-matrix associated to the CohFT. A lot of questions remain open, for which we indicate some sparse results.

Simplicity

For many values of (r,s), the algebra $\mathbb{Q}[P(r,s)] \simeq V_q^+$ is simple, i.e. is a number field, shown in yellow in Figure 1 (non-simple cases are shown in orange). The most appealing question is whether this algebra is always simple when r is prime, for any value of s. This is well-known if the cyclotomic polynomial of order r is irreducible modulo 2 (because the algebra modulo 2 is independent of s), the remaining cases are still open. It also seems that for a fixed even integer k, the algebra $\mathbb{Q}[P(r,r-k)]$ is simple provided that r is big enough. We prove it in the case $k=2$ by analysing the roots of the polynomial χ defined above.

Proposition 1. *For any odd r, the algebras $\mathbb{Q}[P(r, r-2)] \simeq V^+_{-\exp(2i\pi/r)}$ are number fields. They have no real embeddings if $r \equiv 1[4]$ and one real embedding if $r \equiv -1[4]$.*

Signatures

The algebra V_q^+, beyond governing signatures of TQFTs, has its own signature denoted by $r_1(V_q^+)$. It is the signature of the bilinear form $(x, y) \mapsto \mathrm{Tr}_{V_q^+}(xy)$ and is equal to the number of real embeddings of V_q^+ if it is a number field. It would be interesting to relate this signature to the signature of the form η on V_q: this one is given by the formula

$$\mathrm{Sign}(\eta) = \sum_{n=1}^{r-1} \varepsilon_n = 2\,\mathrm{Sign}(\eta^+)$$

which is also equal to the standard signature of the knot $K(r, s)$, see Section 4.3. We will prove in Section 4.3 the inequality

$$|\,\mathrm{Sign}(\eta^+)| \leq r_1(V_q^+). \tag{1}$$

I thought for a while that this inequality was an equality until counterexamples were found by P.V. Koseleff for $r \geq 39$. It would be nice to find an exact formula, maybe using Dedekind sums. Observe also that when $s > 1$, the trace field of $K(r, s)$ has at least one complex embedding due to the hyperbolic structure. Hence $r_1(V_q^+) < \dim V_q^+ = \frac{r-1}{2}$ in all those cases.

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2 Generalities on Frobenius algebras

In this article, a Frobenius algebra is a finite dimensional commutative $\mathbb{Q}$-algebra V endowed with a linear form $\varepsilon : V \to \mathbb{Q}$ such that the bilinear form $\eta(x, y) = \varepsilon(xy)$ is non-degenerate.

There is a natural element $\Omega \in V$ constructed as follows: write $\eta^{-1} = \sum_i x_i \otimes y_i \in V \otimes V$. Then $\Omega = \sum_i x_i y_i$. In the standard equivalence between Frobenius algebras and $1+1$ TQFTs, V is the image of the circle, ε corresponds to capping off with a disc and Ω corresponds to the punctured torus.

The Frobenius algebra is said to be semi-simple if V is semi-simple as an algebra, that is, isomorphic to a product of number fields. Here is a useful equivalent formulation: let $M_x \in \mathrm{End}(V)$ be the operation of multiplication by $x \in V$ and set $\mathrm{Tr}_V(x) = \mathrm{Tr}\, M_x$. The algebra V is semi-simple if and only if the bilinear form $(x, y) \mapsto \mathrm{Tr}_V(xy)$ is non-degenerate. In this case there is a unique

$r\backslash s$	1	3	5	7	9	11	13	15	17	19	21
3	1										
5	2	0									
7	3	1	1								
9	4		0	0							
11	5	1	1	-1	1						
13	6	2	0	0	2	0					
15	7			1		1	1				
17	8	2	2	2	0	-2	0	0			
19	9	3	1	1	1	1	3	1	1		
21	10		2			0	0		2	0	
23	11	3	1	1	-1	1	-1	-3	1	1	1

Figure 1: Table of $\mathrm{Sign}(\eta^+)$: yellow for simple cases, orange for non-simple ones. In all shown cases, the inequality (1) is an equality.

invertible element $\alpha \in V$ such that $\varepsilon(x) = \mathrm{Tr}_V(\alpha x)$ for all $x \in V$ and one has moreover $\alpha^{-1} = \Omega$, see [DM, Section 5.1.1].

The Frobenius algebras of this article arise in the following form. Let $P \in \mathbb{Q}[X]$ be a polynomial with simple roots and $h \in \mathbb{Q}(X)$ be a rational function without poles at roots of P. Then we set $V = \mathbb{Q}[X]/(P(X))$ and define for $f \in \mathbb{Q}[X]$:

$$\varepsilon(f) = \sum_{z, P(z)=0} \mathrm{Res}_z \frac{fh}{P}.$$

This can be written equivalently

$$\varepsilon(f) = \sum_{z, P(z)=0} \frac{f(z)h(z)}{P'(z)} = \mathrm{Tr}_{V/\mathbb{Q}}\left(\frac{fh}{P'}\right)$$

so that one has $\alpha = \frac{h}{P'}$ and $\Omega = \frac{P'}{h}$.

In this case the invariant $\langle S_g \rangle$ associated to a genus g surface is given by

$$\langle S_g \rangle = \varepsilon(\Omega^g) = \mathrm{Tr}_V\, \Omega^{g-1} = \frac{1}{2i\pi} \int_C \frac{P'(z)^g dz}{P(z)h(z)^{g-1}}$$

where C is a contour with index 1 around the zeroes of P and 0 around the poles of h.

Remark 1. Notice that h is part of the necessary data to define a Frobenius algebra. For instance, if one sets $h = 1$, we get $\Omega = P'$ and $N(\Omega) = \mathrm{Disc}(P)$.

3 Signed Verlinde algebras for SU_2

Let q be a root of unity of order $2r$ with r odd: in the sequel we will choose $q = \exp(i\pi s/r)$ with $0 < s < r$ where s is odd and prime to r. We use below the notation of [BHMV].

We set $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ and $[n]! = [n][n-1]\cdots[1]$. We will denote by $\varepsilon_n \in \{-1, 0, 1\}$ the sign of $[n]$. We check that $\varepsilon_0 = \varepsilon_r = 0$, $\varepsilon_1 = 1$ and

$$\varepsilon_n = \mathrm{Sign}\left(\frac{\sin(\pi n s/r)}{\sin(\pi s/r)}\right) = (-1)^{\lfloor ns/r \rfloor}.$$

Moreover, as $q^r = -1$, we get $[r-n] = [n]$ and $\varepsilon_n = \varepsilon_{r-n}$.

Let $i, j, k \in \{0, \ldots, r-2\}$. The triple (i, j, k) is said r-admissible if it satisfies

$$i \leq j + k, j \leq i + k, k \leq i + j, i + j + k \text{ is even and } i + j + k \leq 2r - 4. \quad (2)$$

For such a triple, one writes $i = b + c, j = a + c, k = a + b$ and sets

$$\langle i, j, k \rangle = (-1)^{a+b+c} \frac{[a+b+c+1]![a]![b]![c]!}{[a+b]![a+c]![b+c]!}.$$

We then define the Frobenius algebra V_q as a $\mathbb{Q}$-vector space with basis $e_0, \ldots, e_{r-2}$ endowed with a symmetric bilinear form $\eta : V_q^2 \to \mathbb{Q}$ for which e_i is orthogonal and verifies $\eta(e_i, e_i) = \mathrm{Sign}((-1)^i[i+1]) = (-1)^i \varepsilon_{i+1}$. We also endow it with a trilinear symmetric form $\omega : V_q^3 \to \mathbb{Q}$ defined by

$$\omega(e_i, e_j, e_k) = \mathrm{Sign}\langle i, j, k \rangle \text{ if } (i, j, k) \text{ is } r\text{-admissible, } 0 \text{ otherwise.}$$

The product $\cdot : V_q \times V_q \to V_q$ defined by $\omega(x, y, z) = \eta(x \cdot y, z)$ endows V_q with a structure of Frobenius algebra, as shown in [DM].

One checks by a direct computation the following formula where we put also $e_{-1} = e_{r-1} = 0$:

$$e_1 e_n = -\varepsilon_n \varepsilon_{n+1} e_{n-1} + e_{n+1}.$$

It is natural to introduce polynomials which satisfy the same recursion formula, that is we set $P_0 = 1, P_1 = X$ and

$$X P_n = -\varepsilon_n \varepsilon_{n+1} P_{n-1} + P_{n+1} \quad (3)$$

A direct induction shows that P_n is even (resp.) odd if n is even (resp. odd). These polynomials are obtained as the determinant of the upper-left minor of size n of the following matrix of size $r-1$:

$$M = \begin{pmatrix} X & -\varepsilon_1\varepsilon_2 & & & \\ 1 & X & -\varepsilon_2\varepsilon_3 & & \\ & & \ddots & & \\ & & 1 & X & -\varepsilon_{r-2}\varepsilon_{r-1} \\ & & & 1 & X \end{pmatrix}$$

As $r-1$ is even, one has $P_{r-1}(X) = P_{r-1}(-X)$: this shows that P_{r-1} is indeed the characteristic polynomial of the endomorphism of multiplication by

e_1. This construction shows that the map $\Phi : \mathbb{Q}[X] \to V_q$ defined by $\Phi(P) = P(e_1)$ induces an algebra isomorphism

$$\mathbb{Q}[X]/(P_{r-1}) \simeq V_q$$

which moreover satisfies $\Phi(P_n) = e_n$. For convenience, we will set $x = \Phi(X) = e_1$ so that we can write $e_n = P_n(x)$.

Lemma 1. *The polynomial P_{r-1} has simple roots, equivalently, the algebra V_q is semi-simple.*

Proof. To start, recall that one can write $P_{r-1}(X) = R(X^2)$. It is sufficient to show that R has simple roots: we will prove it by showing that its discriminant is odd. That is, we prove that the roots of R in $\overline{\mathbb{F}}_2$ are distinct. We observe that the recursion relation (3) defining P_n is the same modulo 2 as the one we obtain for $s = 1$ where $\varepsilon_1 = \cdots = \varepsilon_{r-1} = 1$. This gives the formula:

$$XU_n = U_{n+1} - U_{n-1}.$$

This polynomial satisfies the equation $U_n(i(t + t^{-1})) = i^n \frac{t^{n+1} - t^{-n-1}}{t - t^{-1}}$. Observe that one still has $U_{r-1}(x) = R(x^2)$ modulo 2. As R has degree $\frac{r-1}{2}$, we are reduced to finding $(r-1)/2$ distinct roots for U_{r-1}.

As r is odd, there are r distinct r-th roots of unity in $\overline{\mathbb{F}}_2$. As the Frobenius map $x \mapsto x^2$ is an isomorphism, there are as many roots of order $2r$, denoted by $1 = \zeta_1, \ldots, \zeta_r$. Roots of U_{r-1} have the form $i(\zeta_j + \zeta_j^{-1})$ for $j = 2, \ldots, r$. The involution $\zeta_j \mapsto \zeta_j^{-1}$ only fixes $\zeta_1 = 1$ and one has $\zeta_j + \zeta_j^{-1} = \zeta_k + \zeta_k^{-1}$ if and only if $j = k$. This finishes the proof of the lemma. □

Remark 2. This proof is a variant of the proof that the Riley polynomial has simple roots, see [R].

3.1 Decomposition into even and odd parts

Denote by V_q^+ the subspace generated by the e_n where n is even. The Frobenius algebra structure on V_q induces a Frobenius algebra structure on V_q^+ which we call the SO_3 signed Verlinde algebra of parameter q.

Remark 3. *In [DM], it corresponds to the Frobenius algebra associated with $SO(3)$ with the root $-q$, which has order r.*

Indeed, its basis is given by $e_0, e_2, \ldots, e_{r-3}$ and signs $\eta(e_{2i}, e_{2i}) = \varepsilon_{2i+1}$. The formula for ω is the same. As changing q to $-q$ does not change the quantum integers, the conclusion follows.

Writing the matrix M in the basis $(e_0, e_2, \ldots, e_{r-3}, e_1, e_3, \ldots, e_{r-2})$ we compute:

$$P_{r-1}(X) = \det(M) = \det \begin{pmatrix} XI & A \\ B & XI \end{pmatrix} = \det(X^2 - AB).$$

On the other hand, AB is precisely the matrix of e_1^2 acting on V_q^+. We get from this the equality $P_{r-1}(X) = R(X^2)$ where $R(t) = \det(t - e_1^2)|_{V_q^+}$ is the characteristic polynomial of e_1^2 acting on V_q^+.

In the sequel, we will need other simple computations involving the first two and last two elements of the basis.

Lemma 2. *Let us write*

$$e_0 = 1, \quad e_1 = x, \quad e_2 = y, \quad e_{r-3} = \varepsilon_2 w, \quad e_{r-2} = \iota.$$

One has the following formulas

$$x^2 = y - \varepsilon_2 = -w^2, \quad \iota e_n = (-1)^n \varepsilon_{n+1} e_{r-2-n}, \quad \iota^2 = -1.$$

Proof. This comes directly from the definition of the algebra structure so that we leave it to the reader. We simply observe that $(r-2, m, n)$ is r-admissible if and only if $m + n = r - 2$ hence the product $e_{r-2}e_n$ only involves e_{r-2-n} with a sign that can be computed explicitly. □

Remark 4. There is a nice analogy between the formula $V_q = V_q^+[\iota]$ and the more standard one: $\mathbb{C} = \mathbb{R}[i]$.

3.2 The element Ω

Recall that any Frobenius algebra has a special element Ω which is the image of a punctured torus. It can be computed from any orthogonal basis $e_1, \ldots, e_n$ by the formula

$$\Omega = \sum_{i=1}^{n} \frac{e_i^2}{\eta(e_i, e_i)}.$$

In the case of V_q with its standard basis, this gives $\Omega = \sum_{n=0}^{r-2} (-1)^n \varepsilon_{n+1} e_n^2$. The element corresponding to V_q^+ is $\Omega^+ = \sum_{n=0}^{(r-3)/2} \varepsilon_{2n+1} e_{2n}^2$. Separating even and odd terms in the sum and using Lemma 2 gives $\Omega = 2\Omega^+$. Our purpose here is to give an alternative description:

Proposition 2. *The element Ω satisfies the following equation:*

$$\Omega = -P'_{r-1}(x) P_{r-2}(x).$$

Proof. This is a consequence of the Christoffel-Darboux formula for orthogonal polynomials, see [C]. We reproduce the (easy) proof for the convenience of the reader. Multiplying Equation (3) by $P_n(Y)$ and exchanging the roles of X and Y we get:

$$\begin{cases} XP_n(X)P_n(Y) = -\varepsilon_n \varepsilon_{n+1} P_{n-1}(X) P_n(Y) + P_{n+1}(X) P_n(Y) \\ YP_n(X)P_n(Y) = -\varepsilon_n \varepsilon_{n+1} P_{n-1}(Y) P_n(X) + P_{n+1}(Y) P_n(X) \end{cases}$$

Multiplying by ε_{n+1} and taking the difference gives

$$\varepsilon_{n+1}(X-Y)P_n(X)P_n(Y)=Q_n(X,Y)+Q_{n-1}(X,Y)$$

where $Q_n(X,Y)=\varepsilon_{n+1}(P_{n+1}(X)P_n(Y)-P_{n+1}(Y)P_n(X))$. Summing the left hand side after multiplication by $(-1)^n$ hence gives a telescopic sum yielding the so-called Christoffel-Darboux formula:

$$\sum_{n=0}^{r-2}(-1)^n\varepsilon_{n+1}P_n(X)P_n(Y)=-\frac{P_{r-1}(X)P_{r-2}(Y)-P_{r-1}(Y)P_{r-2}(X)}{X-Y}$$

Letting X go to Y we get the so-called confluent form:

$$\sum_{n=0}^{r-2}(-1)^n\varepsilon_{n+1}P_n(X)^2=-(P'_{r-1}(X)P_{r-2}(X)-P_{r-1}(X)P'_{r-2}(X))$$

Mapping this equation to V_q yields the proposition as P_{r-1} goes to 0. □

As a consequence, we have $\Omega=-P'_{r-1}(x)P_{r-2}(x)=-\iota P'_{r-1}(x)$.
In particular one can compute $\varepsilon:\mathbb{Q}[X]/(P_{r-1})\to\mathbb{Q}$ with the formula

$$\varepsilon(f)=\mathrm{Tr}_{V_q}(\Omega^{-1}f)=\mathrm{Tr}_{V_q}(\frac{\iota f}{P'_{r-1}(x)})=\sum_{P_{r-1}(z)=0}\mathrm{Res}_z(\frac{P_{r-2}f}{P_{r-1}}).$$

This formula is particularly nice due to the following continued fraction expansion which follows directly from Equation (3) by dividing by P_n and applying induction:

$$\frac{P_{r-1}}{P_{r-2}}=X+\cfrac{\varepsilon_1\varepsilon_2}{X+\cfrac{\varepsilon_2\varepsilon_3}{\dots+\cfrac{\varepsilon_{r-1}\varepsilon_{r-2}}{X}}}.$$

3.3 Relation with 2-bridge knots

Recall that the fundamental group of the two-brige knot $K(r,s)$ where r,s are coprime odd integers satisfying $0<s<r$ has a Wirtinger presentation given by

$$G=\pi_1(S^3\setminus K(r,s))=\langle u,v|wu=vw\rangle,\quad w=u^{\varepsilon_1}v^{\varepsilon_2}\cdots u^{\varepsilon_{r-2}}v^{\varepsilon_{r-1}}.$$

Following Riley (see [R] or [KM]), any representation $\rho:G\to\mathrm{SL}_2(\mathbb{C})$ such that $\rho(u)$ is parabolic can be conjugated such that

$$\rho(u)=\begin{pmatrix}1&1\\0&1\end{pmatrix},\quad \rho(v)=\begin{pmatrix}1&0\\x&1\end{pmatrix}.$$

A direct computation shows that ρ defined as above satisfies the relation defining G if and only if the upper left entry of $\rho(w)$ vanishes. Taking x a formal variable, we denote by $R(x)$ this coefficient and call R the Riley polynomial of parameters (r,s). This will not create a conflict of notation thanks to the following proposition.

Proposition 3. *The polynomial R defined above satisfies*

$$P_{r-1}(X) = R(X^2).$$

Proof. Multiplying the lines of M by ε_i and conjugating by the diagonal matrix with entries $\varepsilon_1, \varepsilon_1\varepsilon_2, \cdots$, we find

$$P_{r-1}(X) = \varepsilon_1 \cdots \varepsilon_{r-1} K_{r-1}(\varepsilon_1 X, \dots, \varepsilon_{r-1} X)$$

where we denote by K_n the continuant given by

$$K_n(x_1, \dots, x_n) = \det \begin{pmatrix} x_1 & -1 & & & \\ 1 & x_2 & -1 & & \\ & & \ddots & & \\ & & 1 & x_{n-1} & -1 \\ & & & 1 & x_n \end{pmatrix}$$

The symmetry $\varepsilon_i = \varepsilon_{r-i}$ even implies that $\varepsilon_1\varepsilon_2 \cdots \varepsilon_{r-1} = 1$. The formula $K_n(x_1, \dots, x_n) = x_1 K_{n-1}(x_2, \dots, x_n) + K_{n-2}(x_3, \dots, x_n)$ and a direct recursion gives the identity

$$\begin{pmatrix} K_n(x_1, \dots, x_n) & K_{n-1}(x_1, \dots, x_{n-1}) \\ K_{n-1}(x_2, \dots, x_n) & K_{n-2}(x_2, \dots, x_{n-1}) \end{pmatrix} = \begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} x_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n & 1 \\ 1 & 0 \end{pmatrix}.$$

Replacing x_i with $\varepsilon_i x$, we pack the matrices two by two and compute

$$\begin{pmatrix} \varepsilon_1 x & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_2 x & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 + \varepsilon_1\varepsilon_2 x^2 & \varepsilon_1 x \\ \varepsilon_2 x & 1 \end{pmatrix}$$

Conjugating with $g = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$ we can write it as follows:

$$g \begin{pmatrix} \varepsilon_1 x & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_2 x & 1 \\ 1 & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} 1 + \varepsilon_1\varepsilon_2 x^2 & \varepsilon_1 \\ \varepsilon_2 x^2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varepsilon_2 x^2 & 1 \end{pmatrix}.$$

Putting all together gives

$$g \begin{pmatrix} K_{r-1}(\varepsilon_1 x, \dots, \varepsilon_{r-1} x) & K_{r-2}(\varepsilon_1 x, \dots, \varepsilon_{r-2} x) \\ K_{r-2}(\varepsilon_2 x, \dots, \varepsilon_{r-1} x) & K_{r-3}(\varepsilon_2 x, \dots, \varepsilon_{r-2} x) \end{pmatrix} g^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\varepsilon_1} \begin{pmatrix} 1 & 0 \\ x^2 & 1 \end{pmatrix}^{\varepsilon_2} \cdots$$

Taking the upper left entry on both sides finally proves the proposition:

$$K_{r-1}(\varepsilon_1 x, \dots, \varepsilon_{r-1} x) = R(x^2) = P_{r-1}(x).$$

∎

4 Signed Verlinde algebras for SO_3

Let us concentrate on the SO_3 case: one has $\Omega^+ = \frac{1}{2}\Omega = -\frac{1}{2}P'_{r-1}(x)P_{r-2}(x)$. As $P_{r-1}(x) = R(x^2)$, $P_{r-2}(x) = \iota$ and $\iota x = -w$, we get

$$\Omega^+ = -xR'(x^2)\iota = wR'(-w^2).$$

It follows that w plays a prominent role in V_q^+: let us define $\chi(t) = \det(t - w)|_{V_q^+}$, the characteristic polynomial of w. We recall that we have set $R(t) = \det(t - e_1^2)|_{V_q^+}$ and $x = e_1$. A simple computation shows that w has the following matrix in the standard basis of V_q^+:

$$W = \begin{pmatrix} & & & & \varepsilon_{r-1} \\ & & & \varepsilon_{r-3} & -\varepsilon_{r-2} \\ & & \iddots & \iddots & \\ & \varepsilon_4 & -\varepsilon_5 & & \\ \varepsilon_2 & -\varepsilon_3 & & & \end{pmatrix}$$

From the equality $w^2 = -x^2$, we get $\det(t^2 - w^2) = \det(t^2 + x^2)$ hence $\det(t - w)\det(-t - w) = \det(-t^2 - x^2)$. It follows that

$$\chi(t)\chi(-t) = R(-t^2) = P_{r-1}(it).$$

Hence, differentiating the above equation we get $\Omega^+ = -\frac{1}{2}\chi'(w)\chi(-w)$ in particular we get on V_q^+:

$$\varepsilon(f) = -2\sum_{\chi(t)=0} \mathrm{Res}_t \frac{f(t)}{\chi(t)\chi(-t)} = -2\sum_{\chi(t)=0} \mathrm{Res}_t \frac{f(t)}{R(-t^2)}.$$

As one has $V_q^+ = \mathbb{Q}[t]/(\chi(t))$, it is useful to study the polynomial χ. Its constant coefficient is ± 1, showing that w is a unit. Moreover, numerical experiments show that it has quite small coefficients that we compute in the next section.

4.1 Two examples

4.1.1 The case s=1

Set $q = \exp(\frac{i\pi}{r})$. In this case, as explained in Lemma 1, one has $P_{r-1} = U_{r-1}$ where $U_n(i(t + t^{-1})) = i^n \frac{t^{n+1} - t^{-n-1}}{t - t^{-1}}$. This proves that V_q is the sub-algebra of $\mathbb{Q}[t]/(t^{2r} - 1)$ generated by $i(t + t^{-1})$. In particular, the list of embeddings of V_q in $\mathbb{C}$ is given by $x \mapsto 2i\cos(\frac{k\pi}{r})$ for $k = 1, \ldots, r-1$. This also shows that V_q^+ is totally real.

A direct computation gives $P_{r-2}(x) = i^{r-2}$ and $P'_{r-1}(x) = \frac{2ri^r}{(t - t^{-1})^2}$, hence:

$$\Omega = \frac{-2r}{(t - t^{-1})^2}.$$

From the formulas of Section 2, we get

$$\langle S_g \rangle = \mathrm{Tr}_{V_q}(\Omega^{g-1}) = \big(\frac{r}{2}\big)^{g-1} \sum_{k=1}^{r-1} \sin(\frac{k\pi}{r})^{2-2g}.$$

We recover the standard Verlinde formula, as the TQFT for $q = \exp(i\pi/r)$ is unitary and its signature coincides with its dimension. We also observe that V_q is simple if and only if r is prime. This is due to the decomposition $t^{2r} - 1 = \prod_{d|2r} \Phi_d$ where Φ_d are the cyclotomic polynomials.

4.2 The case $s = r - 2$

This case is opposite to the previous one and behave somewhat more simply than the general case so that we include an analysis of it, yielding a proof of Proposition 1. Set $q = \exp(\frac{i\pi(r-2)}{r}) = -\exp(\frac{2i\pi}{r})$.

Proposition 4. *Setting $\chi(t) = \det(t - W)$, the characteristic polynomial of W one has:*

1. $\mathrm{Tr}(W) = 1$.
2. *The eigenvalues of W have positive real parts.*
3. *The polynomial χ is irreducible over $\mathbb{Q}$.*
4. *χ have no real roots if $r = 1[4]$ and one real root if $r = -1[4]$.*

Proof. For this specific root q, the signs $\varepsilon_1, \dots, \varepsilon_{r-1}$ are alternating except for the two middle ones: this gives a very simple matrix W for w acting on V_q^+: if $r = 4n + 1$, the first n colums of W have -1s, the last n columns have 1s. If $r = 4n + 3$, the first $n + 1$ rows have 1s, the last n rows have -1s.

It follows that $\mathrm{Tr}(W) = 1$. Moreover, W is almost antisymmetric: $M + M^T = 2E$ where E has only one non zero entry at $(n + 1, n + 1)$ where it is equal to 1.

Let X be a normalized eigenvector for W, i.e. $X^*X = 1$ where we write $X^* = \overline{X}^T$. One has $X^*WX = \lambda$ and $X^*W^*X = \overline{\lambda}$. Summing the two gives

$$\lambda + \overline{\lambda} = 2X^*EX = 2|X_{n+1}|^2.$$

This gives the positivity of the real part of λ. Moreover, if $X_{n+1} = 0$, one can solve the linear system $WX = \lambda X$ step by step and prove that $X = 0$. Hence $\mathrm{Re}(\lambda) > 0$ as claimed.

Suppose that there is a non trivial decomposition $\chi = PQ$. As χ is monic, P and Q are also monic with integral coefficients. One has $1 = \sum_{P(\lambda)=0} \mathrm{Re}(\lambda) + \sum_{Q(\lambda)=0} \mathrm{Re}(\lambda)$. The two terms are strictly positive integers summing to 1, a contradiction.

Let us prove the last point: using Descartes' rule of signs and the previous point, it is sufficient to show that the characteristic polynomial $\chi(t)$ has coefficients with alternating signs. The computation of those coefficient is deferred to Lemma 3 stated below. It remains to determine the sign of the contribution of the pair (A, B) to the characteristic polynomial. As $W_{ij} + W_{ji} = 0$ unless $i = j$, we find that if $|A|$ is even, its sign is $|A|/2$ whereas if $|A|$ is odd, its sign is $(|A| - 1)/2$, the same being true for B. Setting $a = |A|$ and $b = |B|$, one has $a + b = k$ and the sign of the contribution is

$$(-1)^{k(k+1)/2+a(a-1)/2+b(b-1)/2}.$$

This can be rewritten $(-1)^{a+b+ab}$. However, because of the conditions on A and B, a and b cannot be both odd integers so that $(-1)^{ab} = 1$ and we get the sign $(-1)^k$ as expected.

□

Lemma 3. *Let M be a matrix of size n with $M_{i,j} = 0$ if $i + j \notin \{n + 1, n + 2\}$. Its characteristic polynomial has the following expression:*

$$\det(t - M) = \sum_{k=0}^{n} t^{n-k} (-1)^{\frac{k(k+1)}{2}} \sum_{(A,B)} \prod_{(i,j)\in A\cup B} M_{ij}$$

In this formula, $A \subset \{(i, j), i + j = n + 1\}$ and $B \subset \{(i, j), i + j = n + 2\}$ satisfy the following properties.

1. *They are both invariant by the involution $(i, j) \mapsto (j, i)$.*
2. *Their images by the projections $(i, j) \mapsto i$ and $(i, j) \mapsto j$ are disjoint.*
3. *The sum of their cardinality is equal to k.*

Proof. The proof follows from a direct analysis of the expansion $\det(t - M) = \sum_{\sigma\in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} (t\delta_{i\sigma(i)} - M_{i\sigma(i)}) = \sum_{\sigma\in S_n} \varepsilon(\sigma) \sum_{I,J} t^{|I|} (-1)^{|J|} \prod_{j\in J} M_{j\sigma(j)}$. In the formula, I, J run over all partitions of $\{1, \ldots, n\}$ such that $\sigma(i) = i$ for all $i \in I$. By construction, σ permutes J. Writing $\sigma(j) + j = n + 1 + \delta_j$ for $\delta_j \in \{0, 1\}$, one finds that for any $j_1 < j_2 \in J$:

$$\sigma(j_2) - \sigma(j_1) = j_1 - j_2 + \delta_{j_2} - \delta_{j_1} < 0.$$

This proves that σ invert all pairs in J hence σ acts by puting all elements in J in reverse order. In particular σ is an involution and $\varepsilon(\sigma|_J) = (-1)^{k(k-1)/2}$ where $k = |J|$. The final sign of the contribution of the pair (I, J) is $(-1)^{k+k(k-1)/2}$ as expected. The last point to prove is the fact that A and B are stable by the involution $(i, j) \mapsto (j, i)$ but this comes from the fact that σ is an involution. □

4.3 On signatures

In this section we compare various notions of signatures that naturally appear in this context. Let V be a finite dimensional commutative $\mathbb{Q}$-algebra: for $t \in V$ we define $\mathrm{Sign}_V(t)$ to be the signature of the quadratic form

$$(x, y) \mapsto \mathrm{Tr}_V(xyt).$$

If V is a number field, let $\phi_1, \dots, \phi_{r_1} : V \to \mathbb{R}$ be the family of real embeddings of V. We have the formula

$$\mathrm{Sign}_V(t) = \sum_{i=1}^{r_1} \mathrm{Sign}\, \phi_i(t).$$

This gives in particular

$$\mathrm{Sign}_V(1) = r_1(V) \text{ and } |\mathrm{Sign}_V(t)| \leq r_1 \text{ for all } t \in V.$$

As one can write $\eta(x, y) = \mathrm{Tr}_V(\Omega^{-1}xy)$, the usual signature of the bilinear form η is given by

$$\mathrm{Sign}(\eta) = \mathrm{Sign}_V(\Omega^{-1}) = \mathrm{Sign}_V(\Omega).$$

Recall that the Frobenius algebra V_q and its even part V_q^+ are naturally endowed with a non-degenerate bilinear form η (resp. η^+) which have associated signatures. From the fact that the standard basis is orthogonal we immediately compute

$$\mathrm{Sign}(\eta) = \sum_{n=1}^{r-1} \varepsilon_n = 2\,\mathrm{Sign}(\eta^+)$$

On the other hand, the knot $K(r, s)$ also has a signature: it is by definition the signature of the matrix $M + M^T$ where M is a Seifert matrix of $K(s, r)$. One finds in [M, Theorem 9.3.6] the formula

$$\mathrm{Sign}(K(r, s)) = \sum_{n=1}^{r-1} \varepsilon_n$$

so that these signatures coincide.

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