REIDEMEISTER TORSION OF TWO-BRIDGE KNOTS AND SIGNATURES OF TQFT

JULIEN MARCHÉ AND SEOKBEOM YOON

ABSTRACT. We establish an explicit relation between the adjoint Reidemeister torsion of the two-bridge knot $K(p, q)$ at any parabolic representation and the Frobenius algebra governing the signatures of SU_2-TQFT vector spaces at the root $\zeta = \exp(i\pi q/p)$. As applications, (a) we prove that the inverse sum of torsions is constant (i.e., independent of p and q); and (b) we show that along sequences of roots of the form $\zeta_n = \exp\left(i\pi \frac{a+bn}{c+dn}\right)$, the signatures have the same asymptotic behavior as the Verlinde formula.

CONTENTS

1. INTRODUCTION

Let p and q be coprime odd integers satisfying $0 < q < p$ and denote by $K(p, q)$ the two-bridge knot with these parameters. Independently, setting $\zeta = \exp(i\pi q/p)$, one can consider the SU_2-TQFT with defining root ζ. For any genus g, it gives a Hermitian vector space $\mathcal{V}_\zeta(S_g)$ endowed with a projective unitary representation of

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the mapping class group $\mathrm{Mod}(S_g)$. This article establishes a tight relation between, on the one hand, the space of parabolic $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(S^3 \setminus K(p,q))$ and the adjoint Reidemeister torsion, and, on the other hand, a Frobenius algebra governing the signatures of the various vector spaces $\mathcal{V}_\zeta(S_g)$. It completes the work initiated in [Mar23] and provides applications both to the computation of the Reidemeister torsion itself and to the study of the asymptotic behavior of the signatures for a sequence ζ_n of roots converging to a given root ζ.

In both the introduction and the main part of the paper, we have tried to separate, as much as possible, the knot-theoretic aspects from the TQFT aspects for the reader's benefit. Before proceeding further, we caution that our proofs are computational in nature: we do not yet have a conceptual explanation for the relations we uncover. Finding such an explanation remains an interesting open problem.

1.1. **About two-bridge knots.** We denote by $\mathcal{X}$ the set of conjugacy classes of irreducible representations $\rho : \pi_1(S^3 \setminus K(p,q)) \to \mathrm{SL}_2(\mathbb{C})$ which send any (hence every) meridian to a non-trivial matrix with trace 2. This set is a finite affine algebraic set defined over $\mathbb{Z}$, hence its coordinate ring $\mathbb{Q}[\mathcal{X}]$ is a finite commutative $\mathbb{Q}$-algebra. Riley showed in [Ril72] that it is semi-simple.

Given $[\rho] \in \mathcal{X}$ and any $n \geq 1$, we define ρ_n to be the composition of ρ with the $(n+1)$-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. We refer to $\rho = \rho_1$ as the tautological representation and to ρ_2 as the adjoint representation. By choosing appropriate generators of the twisted cohomology groups $\mathrm{H}^*(S^3 \setminus K(p,q), \rho_n)$, one can define the Reidemeister torsion $\tau_n(\rho)$ as a complex number depending on $[\rho] \in \mathcal{X}$. By construction, the map $\rho \mapsto \tau_n(\rho)$ is conjugation-invariant and algebraic, yielding an element $\tau_n \in \mathbb{Q}[\mathcal{X}]$. Throughout the paper, we consider τ_n up to sign to avoid unnecessary complications.

It will be technically more convenient to replace $\mathcal{X}$ with a ramified 2-cover $\hat{\mathcal{X}}$, and hence $\mathbb{Q}[\mathcal{X}]$ with the corresponding quadratic extension $\mathbb{Q}[\hat{\mathcal{X}}]$. Topologically, this amounts to recalling that $K(p,q)$ is doubly amphichiral, meaning that there exist two commuting involutions of S^3 which preserve $K(p,q)$ while reversing its orientation. Using this fact, we construct an orbifold quotient of $S^3 \setminus K(p,q)$ whose orbifold fundamental group G fits in the (split) exact sequence

$$0 \to \pi_1(S^3 \setminus K(p,q)) \to G \to Q_8 \to 0$$

where Q_8 is the usual quaternion group of order 8. Note that similar groups have already been considered in [LS13]. Then $\hat{\mathcal{X}}$ is the set of conjugacy classes of representations $\rho : G \to \mathrm{SL}_2(\mathbb{C})$ whose restriction to $\pi_1(S^3 \setminus K(p,q))$ defines an element of $\mathcal{X}$.

Concretely, we will show that there is an isomorphism $\mathbb{Q}[\hat{\mathcal{X}}] \cong \mathbb{Q}[X]/(P_{p-1}(X))$ where $P_k \in \mathbb{Z}[X]$ is the sequence of polynomials given by

$$P_0 = 1, \ P_1 = X \text{ and } P_k = \epsilon_k X P_{k-1} + P_{k-2} \text{ where } \epsilon_k = (-1)^{\lfloor \frac{kq}{p} \rfloor}.$$

We set $V = \mathbb{Q}[\hat{\mathcal{X}}]$ and $V^+ = \mathbb{Q}[\mathcal{X}]$ to simplify notation, and denote by x and $\mathcal{P}_k$ the image of X and P_k, respectively, in V. It turns out that V^+ is spanned by $\mathcal{P}_k$ for even $0 \leq k \leq p-3$, and that V is the quadratic extension of V^+ obtained by adjoining $\mathcal{P}_{p-2}$, which is a square root of -1 in V.

The recurrence relation defining the sequence P_k is typical for orthogonal polynomials. Indeed, there exists a non-degenerate bilinear form $\eta : V \times V \to \mathbb{Q}$ for which the basis $\mathcal{P}_0, \ldots, \mathcal{P}_{p-2}$ is orthogonal. It is given by $\eta(x,y) = \epsilon(xy)$ where

$\epsilon : V \to \mathbb{Q}$ is the linear form satisfying $\epsilon(\mathcal{P}_0) = 1$ and $\epsilon(\mathcal{P}_k) = 0$ for $0 < k < p - 1$. The data of η, or equivalently of ϵ, endows V with the structure of a commutative semi-simple Frobenius algebra. At first glance, this appears unrelated to the topological definition of V, however, one of our main theorems is that this additional structure arises from the adjoint Reidemeister torsion. To explain this, recall that for any Frobenius algebra (V, η), one defines an element $\Omega = \sum x_i y_i$ where $\eta^{-1} = \sum x_i \otimes y_i \in V \otimes V$. When V is semi-simple, ϵ can be recovered from Ω by the formula $\epsilon(x) = \mathrm{Tr}_{V/\mathbb{Q}}(\Omega^{-1} x)$.

Theorem A (Theorems 2.8, 2.14 and Equation (31)). *Let $0 < q < p$ be coprime odd integers and V be the Frobenius algebra described above with associated element Ω. Denoting by τ_n the Reidemeister torsion of $S^3 \setminus K(p, q)$ with coefficients ρ_n, one has*

$$\tau_1 = \frac{2}{x^2 \mathcal{P}_{\ell-1}} \quad and \quad \tau_2 = \frac{\Omega}{4x^2 \mathcal{P}_{\ell-1}^2}$$

where $0 < \ell < p$ is the unique odd integer satisfying $q\ell \equiv \pm 1 \pmod{p}$.

The appearance of $\mathcal{P}_{\ell-1}$ deserves some explanation. By the classification of two-bridge knots, $K(p, q)$ and $K(p, \ell)$ are isotopic. Denoting by $\mathcal{X}^*, V^*, \ldots$ all the previous data associated to $K(p, \ell)$, it follows that $\hat{\mathcal{X}}$ and $\hat{\mathcal{X}}^*$ are isomorphic as algebraic varieties, hence V and V^* are isomorphic as algebras. More precisely, we will show that the isomorphism $V^* \leftrightarrow V$ sends x^* to $x\mathcal{P}_{\ell-1}$ and x to $x^* \mathcal{P}_{q-1}^*$. As τ_n is a topological invariant, the formulas provided in V and V^* should match. This is obvious for τ_1, if one writes it as $\tau_1 = \frac{2}{xx^*}$. In the case of τ_2, we have instead:

$$\tau_2 = \frac{\Omega}{4x(x^*)^2} = \frac{\Omega^*}{4x^* x^2}.$$

This yields the reciprocity formula $x\Omega = x^* \Omega^*$, showing that the Frobenius algebra structures on V and V^* are indeed distinct. In addition, this formula will play a crucial role in our study of the asymptotics of the signature of TQFT.

As a first application of our formulas, we will prove the following two equations, where the latter extends the result of [Yoo22] to the parabolic case and has a quantum-physical motivation; see [GKY21]. Note that the condition $q \neq 1$ in the following theorem implies that the two-bridge knot $K(p, q)$ is hyperbolic, and that the theorem fails when $q = 1$.

Theorem B (Theorems 2.16 and 2.17). *Let $0 < q < p$ be coprime odd integers with $q \neq 1$ and $\mathcal{X}$ be the parabolic character variety of $S^3 \setminus K(p, q)$. Then one has*

$$\sum_{\rho \in \mathcal{X}} \frac{1}{\tau_1(\rho)} = \pm 1 \quad and \quad \sum_{\rho \in \mathcal{X}} \frac{1}{\tau_2(\rho)} = 0.$$

1.2. **About signatures of TQFT.** Fix an odd integer p and let $\mathbb{Q}(\zeta)$ be the cyclotomic field of order $2p$, where ζ is a primitive $2p$-th root of unity. We set $\Lambda = \{0, 1, \ldots, p - 2\}$ and $\Lambda^+ = \{0, 2, \ldots, p - 3\}$ which we use as "colors" indexing marked points. In this article, we consider the $\mathrm{SU}_2/\mathrm{SO}_3$-TQFT as a collection of $\mathbb{Q}(\zeta)$-vector spaces $\mathcal{V}_p(S_{g,n}; \lambda)$ where $S_{g,n}$ is a surface of genus g with n marked points and $\lambda \in \Lambda^n$ is a color of the marked points. These are endowed with an Hermitian form defined over $\mathbb{Q}(\zeta)$ and an action of the pure mapping class group $\mathrm{PMod}(S_{g,n})$ by projective isometries. A key property is that these vector spaces satisfy compatibility conditions under gluing. If S is a surface and $\gamma \subset S$ is a simple

closed curve disjoint from the marked points, then a new (possibly disconnected) surface S' is obtained by cutting S along γ, capping off the resulting boundaries with discs, and adding a new marked point in each disc. When γ is non-separating, the compatibility means an isomorphism

$$\mathcal{V}_p(S; \lambda) = \bigoplus_{\mu \in \Lambda} \mathcal{V}_p(S'; \lambda, \mu, \mu)$$

which is compatible with both the Hermitian structure and the mapping class group action. There is a similar statement for the separating case.

In this article, we embed $\mathbb{Q}(\zeta)$ into $\mathbb{C}$ by setting $\zeta = \exp(i\pi\frac{q}{p})$ and define $\mathbb{C}$-vector spaces as $\mathcal{V}_\zeta(S_{g,n}; \lambda) = \mathcal{V}_p(S_{g,n}; \lambda) \otimes \mathbb{C}$ whose signature is denoted by

$$\sigma_g(\tfrac{q}{p}; \lambda) = \operatorname{Sign} \mathcal{V}_\zeta(S_{g,n}; \lambda).$$

Since any surface can be decomposed into pairs of pants, the aforementioned compatibility allows one to reduce the computation of the signature to the sphere with at most three marked points, by introducing the so-called signed Verlinde Frobenius algebra [DM22]. It is a $(p-1)$-dimensional vector space $V = \bigoplus_{k \in \Lambda} \mathbb{Q}e_k$ equipped with a symmetric bilinear form $\eta : V \times V \to \mathbb{Q}$ and a trilinear symmetric form $\omega : V \times V \times V \to \mathbb{Q}$, given by

$$\eta(e_j, e_k) = \sigma_0(\tfrac{q}{p}; j, k) = \delta_{jk}(-1)^j \epsilon_{j+1},$$
$$\omega(e_j, e_k, e_l) = \sigma_0(\tfrac{q}{p}; j, k, l) \in \{-1, 0, 1\}.$$

Here δ_{jk} denotes the Kronecker delta. It was shown in [DM22] that there is a unique structure of semi-simple algebra on V so that one has $\omega(x, y, z) = \eta(xy, z)$ for any $x, y, z \in V$. Moreover, it was shown in [Mar23] that sending e_k to $\mathcal{P}_k$ defines an algebra isomorphism from V to $\mathbb{Q}[\hat{\mathcal{X}}]$. This is why we have already denoted it by V in the first part of the introduction. The initial motivation of this article was to extend the isomorphism $V \simeq \mathbb{Q}[\hat{\mathcal{X}}]$ to an isomorphism of Frobenius algebras. To achieve this, one needs to give a topological meaning to the element Ω. It was a pleasant surprise that the adjoint Reidemeister torsion turns out to play exactly this role.

One can recovers the signatures from the Frobenius algebra V by means of the following formula, which holds for any surface $S_{g,n}$ colored by $\lambda = (\lambda_1, \ldots, \lambda_n)$:

$$\sigma_g(\tfrac{q}{p}; \lambda) = \epsilon(\Omega^g e_{\lambda_1} \cdots e_{\lambda_n}) = \operatorname{Tr}_{V/\mathbb{Q}}(\Omega^{g-1} e_{\lambda_1} \cdots e_{\lambda_n}).$$

There is a similar SO_3-TQFT for which the set of colors is restricted to Λ^+. The corresponding Frobenius algebra is the subalgebra $V^+ \subset V$ consisting of even colors. As $\Omega \in V^+$, this subalgebra naturally inherits a Frobenius algebra structure. One often prefers the SO_3-TQFT because it has smaller dimension and better properties, such as irreducibility and the existence of an integral basis. In this article, however, we restrict to the SU_2-case, as their signatures only differ by a power of 2.

1.3. Asymptotics of the signature along roots of unity.
Let us first recall that the dimension of $\mathcal{V}_p(S_g)$ is given by the famous Verlinde formula (we set $n = 0$, i.e., we consider surfaces with no marked points from now on):

$$\dim \mathcal{V}_p(S_g) = \left(\frac{p}{2}\right)^{g-1} \sum_{k=1}^{p-1} \sin\left(\frac{k\pi}{p}\right)^{2-2g}.$$

One can easily deduce from the formula the asymptotic behavior

$$\dim \mathcal{V}_p(S_g) \underset{p \to \infty}{\sim} 2 \left(\frac{p^3}{2\pi^2} \right)^{g-1} \zeta(2g - 2),$$

which admits a beautiful interpretation; see, e.g., [Wit91]. For motivational purposes, we briefly recall this. Regarding the vector space $\mathcal{V}_p(S_g)$ as the geometric quantization of the character variety

$$\mathcal{M}(S_g) = \mathrm{Hom}(\pi_1(S_g), \mathrm{SU}_2)/\mathrm{SU}_2,$$

one can endow $\mathcal{M}(S_g)$ with a complex structure and construct a holomorphic line bundle L over $\mathcal{M}(S_g)$ such that

$$\mathcal{V}_p(S_g) \simeq \mathrm{H}^0(\mathcal{M}(S_g), L^{p-2}).$$

Since $c_1(L)$ represents the Atiyah-Bott-Goldman symplectic (Kähler) structure on $\mathcal{M}(S_g)$, the higher cohomology groups vanish for sufficiently large p, and the Hirzebruch-Riemann-Roch theorem yields

$$\dim \mathcal{V}_p(S_g) = \int_{\mathcal{M}(S_g)} e^{p\, c_1(L)} \mathrm{Todd}(X) \underset{p \to \infty}{\sim} p^{3g-3} \mathrm{Vol}(\mathcal{M}(S_g)),$$

together with the non-trivial fact that the Verlinde formula is a polynomial in p.

It is then natural to ask whether a similar behavior occurs for the signature along sequences of roots of unity. Numerical experiments show that the answer depends strongly on the specific sequence of roots of unity. Our final main theorem provides an explicit asymptotic formula for the signatures.

Theorem C (Theorem 3.1). *Let $q_n = a + bn$ and $p_n = c + dn$, where n is odd and $a, b, c, d \in \mathbb{Z}$ satisfy $ad - bc = 1$ with a, d odd, b, c even, $0 \le b < d$ and $c > 0$. Then for any $g \ge 2$,*

* (1) *the signature $\sigma_g(\frac{q_n}{p_n})$ is a polynomial in n, and*
* (2) *under a technical condition (H), there exists a semi-simple d-dimensional Frobenius algebra (W, Ω_W) depending on a, b, c, d such that*

$$\sigma_g \left(\frac{q_n}{p_n} \right) \underset{n \to \infty}{\sim} 2 \left(\frac{n^3}{2\pi^2} \right)^{g-1} \zeta(2g - 2) \mathrm{Tr}_{W/\mathbb{Q}}(\Omega_W^{g-1}).$$

In the introduction, we do not describe the Frobenius algebra (W, Ω_W) or the technical condition (H); details can be found in Section 3. We expect that (H) holds for all a, b, c, d as in the theorem, and we have verified it for all $0 \le b < d < 100$. For example, when $(a, b, c, d) = (3, 2, 4, 3)$, one finds

$$W = \mathbb{Q}[X]/(4X^3 + 16X^2 + 23X + 12) \quad \text{and} \quad \Omega_W = 28X^2 + 80X + 65$$

so that

Genus g	1	2	3	4	5	6
$\lim\limits_{n \to \infty} \frac{\sigma_g(q_n/p_n)}{\dim \mathcal{V}_{p_n}(S_g)}$	$\frac{1}{3^0}$	$\frac{1}{3^3}$	$\frac{1}{3^6}$	$\frac{1345}{3^9}$	$\frac{1762}{3^{12}}$	$\frac{2241}{3^{15}}$

The proof of the above theorem is rather involved, so we outline its main steps for the reader's convenience.

Step 1: One has $\sigma_g(q_n/p_n) = \mathrm{Tr}_{V_n^*/\mathbb{Q}}(\Omega_n^{g-1})$ where V_n^* is the Frobenius algebra associated with q_n/p_n. Recall that $V_n^* \simeq V_n$ as algebras, hence the trace can be computed in V_n instead. Then, by construction, the pair (p_n, q_n) is

reciprocal to the pair (p_n, d). It will be much more convenient to work in V_n, as q_n is replaced by the constant d.

Step 2: We will show that in V_n, the polynomial P_{p_n-1} (and certain other polynomials used in the computation of the signature) arises as a specializations of a 2-variable polynomial Q; explicitly, one has

$$P_{p_n-1}(it + it^{-1})(t - t^{-1})^d = i^{n+1}Q(t, t^n).$$

Step 3: We consider the plane algebraic curve $\mathcal{C} = \{(u, v) \in (\mathbb{C}^*)^2 \,|\, Q(u, v) = 0\}$. The signature can be expressed as the sum of a rational function to the power $g - 1$ over a finite set $\mathcal{C}[n] = \{(u, v) \in \mathcal{C} \,|\, v = u^n\}$. We then study the distribution of these points as $n \to \infty$ and show that the main contribution to the signature concentrates around the singular points $\{\pm 1\}^2 \subset \mathcal{C}$.

Step 4: We analyze the branches of $\mathcal{C}$ at the singular points $(\pm 1, \pm 1)$. Under the assumption (H), exactly d branches pass through each singular point, and the points of $\mathcal{C}[n]$ accumulate along each branch in a regular way. The sum of these contributions, raised to the power $g - 1$, is accounted for in the term $\mathrm{Tr}_{W/\mathbb{Q}} \Omega^{g-1}$.

To end this section, we advise the reader that there is a parallel article which also study the asymptotic behavior of the signature function $\frac{q}{p} \mapsto \sigma_g(\frac{q}{p})$, but in a different regime. Contrary to the case of this article, the authors consider sequences $\frac{q_n}{p_n}$ converging to a (very) irrational limit. The order of convergence is then p_n^{2g-2} instead of p_n^{3g-3} and the limit is expected to have modular properties, see [MM25].

1.4. **Back to two-bridge knots.** To end this introduction, let us explain how our last theorem arises naturally from the knot-theoretical side. We do not attempt to make the argument precise.

The sequence of two-bridge knots $K(p_n, q_n)$ can be drawn in its Conway normal form starting from the continued expansion of p_n/q_n. It is known (see e.g. [Khi64]) that there are integers $a_1, \ldots, a_k$ (independent of n) such that for any n

$$\frac{p_n}{q_n} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_k + n - 1}}}$$

We observe in Figure 1 that this sequence of knots can be obtained by performing $\frac{1}{m}$-surgery on the trivial component L_2 of the link L, where $m = \frac{n-1}{2}$ for odd n. In the figure, boxes indicate twist regions, and L_2 is the trivial knot encircling the last twist region. This gives the key intuition for the last theorem.

We consider the character variety $\mathcal{X}_L$ of representations $\rho : \pi_1(S^3 \setminus L) \to \mathrm{SL}_2(\mathbb{C})$ which send a meridian m_1 of the non-trivial component L_1 to a non-trivial parabolic matrix. Denoting by m_2 and l_2 the meridian and longitude of L_2, respectively, the knot group of the surgered knot is obtained by adding the relation $m_2 l_2^{-m} = 1$. The character variety $\mathcal{X}_L$ is expected to be 1-dimensional and to project birationally onto the character variety of the boundary T_2 of a tubular neighborhood of L_2. The latter variety is parametrized by a pair $(u, v) \in (\mathbb{C}^*)^2$ corresponding to a representation $\rho_{u,v} : \pi_1(T_2) \to \mathrm{SL}_2(\mathbb{C})$ given by

$$\rho_{u,v}(l_2) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad \rho_{u,v}(m_2) = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}.$$
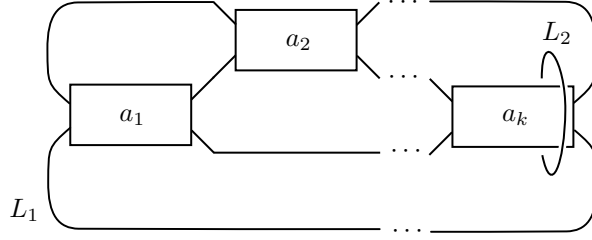
FIGURE 1. The Conway normal form of $[a_1, a_2, \ldots, a_k]$ with a trivial component.

The image of $\mathcal{X}_L$ in $(\mathbb{C}^*)^2$ is then a plane curve $\mathcal{C}$ given by the vanishing of a polynomial analogous to the A-polynomial of a knot. The parabolic character variety $\mathcal{X}_n$ of $K(p_n, q_n)$ maps to the set of points $\mathcal{C}[m] \subset \mathcal{C}$ satisfying $v = u^m$.

To end this rough picture, recall that one can define the Reidemeister torsion τ_L of $S^3 \setminus L$ with coefficients ρ_2 for $[\rho] \in \mathcal{X}_L$. This is a meromorphic function on $\mathcal{X}_L$ which behaves well under surgery, meaning that there is a formula expressing the collection of elements Ω_n in terms of the function τ_L evaluated at points in $\mathcal{C}[m]$. This expression happens to have poles for $u, v = \pm 1$, and the resulting expression for the signature will localize on those poles.

It is then natural to expect that the Frobenius algebra W, explicitly described in Section 3, can be described from the parabolic character variety of L, i.e., from representations which send both m_1 and m_2 to parabolic elements. We leave this for future investigation.

2. REIDEMEISTER TORSIONS OF TWO-BRIDGE KNOTS

2.1. Topological setting.
Let $0 < q < p$ be two coprime odd integers and $K(p, q)$ be the two-bridge knot of parameters (p, q). It is characterized by the property that its double branched cover is the lens space $L(p, q)$. Precisely, the knot $K(p, q)$ is defined as follows. Set

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \ : \ |z_1|^2 + |z_2|^2 = 2\}$$

and let $R : S^3 \to S^3$ be the homeomorphism defined by

$$R(z_1, z_2) = (\zeta z_1, \zeta^q z_2), \quad \zeta = \exp\left(\frac{2\pi i}{p}\right).$$

The lens space $L(p, q)$ is the quotient of S^3 by the free action of $\mathbb{Z}/p\mathbb{Z}$ generated R. The complex conjugation $C(z_1, z_2) = (\overline{z_1}, \overline{z_2})$ satisfies $CRC^{-1} = R^{-1}$ and hence extends the $\mathbb{Z}/p\mathbb{Z}$-action to an action of the dihedral group $D_p = \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. This D_p-action is no longer free, and the resulting quotient is homeomorphic to S^3, with ramification locus given by the knot $K(p, q)$.

A standard way to visualize the above construction is to decompose S^3 into two solid tori T_1 and T_2 defined by the inequalities $|z_1| \leq |z_2|$ and $|z_2| \leq |z_1|$,

respectively. Note that their common boundary is a torus given by $|z_1| = |z_2| = 1$. The quotient of T_1 by $\mathbb{Z}/p\mathbb{Z}$ is also a solid torus

$$T_1/(\mathbb{Z}/p\mathbb{Z}) = \left\{ (re^{i\theta}, z_2) \,\middle|\, 0 \le r \le 1, \ -\frac{\pi}{p} \le \theta \le \frac{\pi}{p}, \ r^2 + |z_2|^2 = 2 \right\}/_\sim$$

where $\sim$ means that the discs bounded by $z_1 = \zeta^{\frac{1}{2}}$ and $z_1 = \zeta^{-\frac{1}{2}}$ are identified by R; see Figure 2. If we further quotient $T_1/(\mathbb{Z}/p\mathbb{Z})$ by the conjugation C, we obtain a ball with some ramification. Precisely, the quotient of T_1 by D_p is a ball B_1 with ramification locus consisting of two arcs (blue arcs in Figure 2):

(1) $$\{(1, t) \,|\, -1 \le t \le 1\} \quad \text{and} \quad \{(\zeta^{\frac{1}{2}}, t\zeta^{\frac{q}{2}}) \,|\, -1 \le t \le 1\}.$$

Similarly, the quotient T_2/D_p is also a ball B_2 containing two ramified arcs. Gluing two balls B_1 and B_2 yields the 3-sphere, in which the ramified arcs combine to form the knot $K(p, q)$.
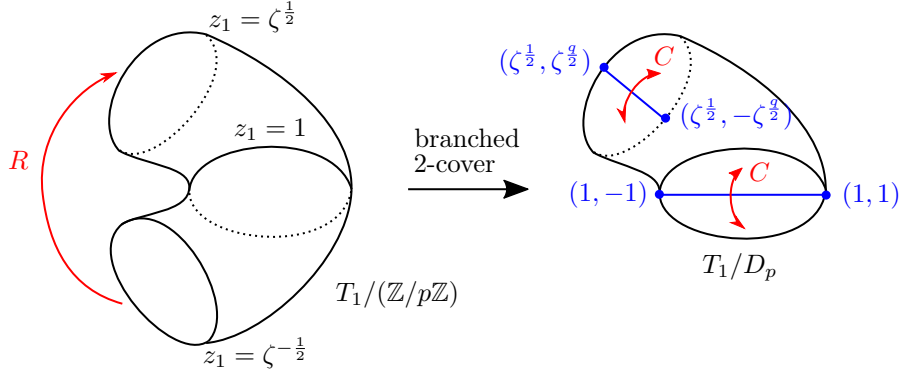

FIGURE 2. The quotients of T_1 by $\mathbb{Z}/p\mathbb{Z}$ and D_p.

If we project the ball B_1 onto a planar square so that the two ramified arcs in (1) are mapped to its vertical edges, the two ramified arcs in B_2 are attached to the vertical edges with slope q/p, as illustrated in Figure 3. The resulting diagram is called Schubert's normal form of $K(p, q)$. Letting u and v be the Wirtinger generators associated with the vertical edges, we obtain the following presentation of the knot group $G = \pi_1(S^3 \setminus K(p, q))$:

(2) $$G = \langle u, v \,|\, wuw^{-1}v^{-1} \rangle, \quad w = u^{\epsilon_1} v^{\epsilon_2} \cdots u^{\epsilon_{p-2}} v^{\epsilon_{p-1}}$$

where $\epsilon_k = (-1)^{\lfloor \frac{kq}{p} \rfloor}$ for any $k \in \mathbb{Z}$ and the relator in (2) is obtained by traversing the diagram once.

By interchanging the roles of B_1 and B_2, we obtain another presentation of G, which coincides with the presentation (2) of $\pi_1(S^3 \setminus K(p, \ell))$. Here $0 < \ell < p$ is the unique odd integer satisfying $q\ell \equiv \pm 1$ modulo p. This follows from the fact that the exchange map $E(z_1, z_2) = (z_2, z_1)$ induces a homeomorphism between $L(p, q)$ and $L(p, \ell)$ when $q\ell \equiv 1$, and the composition $E \circ F$, where $F(z_1, z_2) = (z_1, \overline{z_2})$, does the same when $q\ell \equiv -1$. In either case, $K(p, q)$ and $K(p, \ell)$ are isotopic, and thus their knot groups are isomorphic.

The knot $K(p, q)$ is doubly amphichiral, meaning that there are two commuting involutions h_1 and h_2 induced by the transformations $(z_1, z_2) \mapsto (z_1, -z_2)$ and $(z_1, z_2) \mapsto (-z_1, z_2)$, respectively. Each h_i fixes a circle that intersects the knot in two points, and the two circles meet at two points. Choosing a base point of
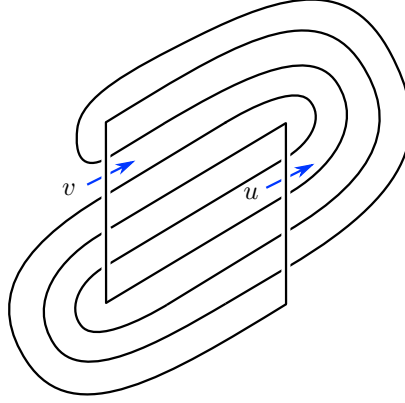
FIGURE 3. Schubert's normal form of $K(5, 3)$.

G at one of the intersection points of the two circles, the involutions h_1 and h_2 induce automorphisms h_1^* and h_2^* of G, respectively, which commute with each other. Explicitly, they are given by

$$(3) \qquad \begin{cases} h_1^*(u) = u^{-1} \\ h_1^*(v) = v^{-1} \end{cases}, \quad \begin{cases} h_2^*(u) = v^{-1} \\ h_2^*(v) = u^{-1} \end{cases}.$$

It is not trivial from the presentation (2) that h_1^* and h_2^* are well-defined, one of the key properties of the sequence ϵ_k is that it is palindromic in the sense that $\epsilon_k = \epsilon_{p-k}$ for all $0 < k < p$.

2.2. **Parabolic character varieties.** In this section, we consider the character variety of irreducible representations

$$\rho : G = \pi_1(S^3 \setminus K(p, q)) \to \mathrm{SL}_2(\mathbb{C})$$

which are *parabolic*, by which we mean that each meridian is mapped to a non-trivial matrix with trace 2. It is well-known that this is an affine algebraic set $\mathcal{X}$ defined over $\mathbb{Z}$ whose complex points can up to conjugation be put in the form

$$\rho(u) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(v) = \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix}.$$

Note that X is nonzero, as u and v are conjugate in G. The coordinate ring $\mathbb{Q}[\mathcal{X}]$ of $\mathcal{X}$ is generated by $\mathrm{Tr}\,\rho(u)$, $\mathrm{Tr}\,\rho(v)$, and $\mathrm{Tr}\,\rho(uv)$; hence, it has the single generator X. More precisely, it is shown in [Ril72] that $\mathbb{Q}[\mathcal{X}]$ is isomorphic to $\mathbb{Q}[X]/(R)$ where $R \in \mathbb{Z}[X]$ is called the Riley polynomial of the knot $K(p, q)$.

For the purpose of this article, it is more elegant to adopt a symmetric setting by instead taking

$$(4) \qquad \rho(u) = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}, \quad \rho(v) = \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix}.$$

Note that X in this setting is also non-zero, as meridians are sent to non-trivial matrices. This symmetric setting can be understood conceptually in the following way. Recall from Section 2.1 that $S^3 \setminus K(p, q)$ admits two involutions h_1 and h_2.

The quotient of $S^3 \setminus K(p, q)$ by these involutions is an orbifold $\mathcal{O}(p, q)$, where its (orbifold) fundamental group fits into the exact sequence

$$0 \to G \to \pi_1(\mathcal{O}(p, q)) \to (\mathbb{Z}/2\mathbb{Z})^2 \to 0 \,.$$

The above sequence splits. In addition, it follows from Equation (3) that

(5) $$\begin{cases} h_1 u h_1^{-1} = u^{-1} \\ h_1 v h_1^{-1} = v^{-1} \end{cases}, \quad \begin{cases} h_2 u h_2^{-1} = v^{-1} \\ h_2 v h_2^{-1} = u^{-1} \end{cases} .$$

We then add an extra global $\mathbb{Z}/2\mathbb{Z}$ isotropy group to $\pi_1(\mathcal{O}(p, q))$ by considering the fiber product

$$\hat{G} = \pi_1(\mathcal{O}(p, q)) \underset{(\mathbb{Z}/2\mathbb{Z})^2}{\times} Q_8 \,.$$

Here Q_8 is the standard quaternion group and $Q_8 \to (\mathbb{Z}/2\mathbb{Z})^2$ is the abelianization map. The group $\hat{G}$ is a central extension of $\pi_1(\mathcal{O}(p, q))$, and the involutions h_1 and h_2 lift to elements $\hat{h}_1$ and $\hat{h}_2$ in Q_8, respectively, whose squares are equal to the central element of order 2.

Definition 2.1. The *parabolic character variety* $\hat{\mathcal{X}}$ of $\hat{G}$ is the algebraic set of conjugacy classes of representations $\rho : \hat{G} \to \mathrm{SL}_2(\mathbb{C})$ such that the its restriction to G is irreducible and parabolic, and $\rho(\hat{h}_1^2) = \rho(\hat{h}_2^2) = -I$.

For any ρ in $\hat{\mathcal{X}}$, the last condition implies that its restriction to Q_8 is irreducible, because any reducible representation of Q_8 satisfies $\rho(\hat{h}_i^2) = I$. Hence, there is a unique conjugation so that Q_8 has its usual irreducible representation in $\mathrm{SL}_2(\mathbb{C})$:

(6) $$\rho(\hat{h}_1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho(\hat{h}_2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho(\hat{h}_1 \hat{h}_2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} .$$

Proposition 2.2. The conjugations described in Equation (5) force $\rho(u)$ and $\rho(v)$ to have the symmetric form given in (4) up to exchanging the role of u and v.

Proof. From the fact $\hat{h}_1 u \hat{h}_1^{-1} = u^{-1}$, we deduce that $\rho(u)_{11} = \rho(u)_{22}$. Here A_{ij} denotes the (i, j)-entry of A. Combined with the condition $\mathrm{Tr}\, \rho(u) = 2$, this implies $\rho(u)_{11} = \rho(u)_{22} = 1$. Since $\det \rho(u) = 1$, it follows that either $\rho(u)_{12}$ or $\rho(u)_{21}$ is zero. The same reasoning applies to v, showing that $\rho(v)$ is upper or lower triangular with ones on the diagonal. Since conjugating with h_2 changes u to v^{-1}, $\rho(u)$ and $\rho(v)$ can not be upper or lower triangular simultaneously, one must be upper and the other lower. Finally, the fact that the conjugation by $\hat{h}_1 \hat{h}_2$ exchanges u and v forces the non-zero off-diagonal entries of $\rho(u)$ and $\rho(v)$ to be equal. This results in Equation (4) up to exchanging the role of u and v. $\square$

As with any character variety, the coordinate ring $\mathbb{Q}[\hat{\mathcal{X}}]$ of $\hat{\mathcal{X}}$ is generated by trace functions $\mathrm{Tr}\, \rho(\gamma)$ for $\gamma \in \hat{G}$. As we have explicit generators of $\hat{G}$, it is easy to check that $\mathbb{Q}[\hat{\mathcal{X}}]$ is generated by x and i; for instance, we have $\mathrm{Tr}\, \rho(\hat{h}_2 u) = -x$ and $\mathrm{Tr}\, \rho(u \hat{h}_1 \hat{h}_2) = ix$. In Proposition 2.4 below, we will prove that i can be expressed as a polynomial in x, implying that $\mathbb{Q}[\hat{\mathcal{X}}]$ has the single generator x.

Given formal variables $X_1, X_2, \ldots, X_k$, we define the continuants

$$K(X_1, \ldots, X_k) = \det \begin{pmatrix} X_1 & -1 & & & \\ 1 & X_2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & X_{k-1} & -1 \\ & & & 1 & X_k \end{pmatrix} \in \mathbb{Z}[X_1, \ldots, X_k].$$

The following are well-known properties of continuants. We omit the proofs, as they follow readily by routine induction arguments.

Lemma 2.3. The continuants $K(X_1, \ldots, X_k)$ satisfy

$$(7) \qquad K(X_1, \ldots, X_k) = X_k K(X_1, \ldots, X_{k-1}) + K(X_1, \ldots, X_{k-2}),$$

$$K(X_1, \ldots, X_k) = K(X_k, \ldots, X_1),$$

$$(8) \qquad \begin{pmatrix} X_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} X_k & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} K(X_1, \ldots, X_k) & K(X_1, \ldots, X_{k-1}) \\ K(X_2, \ldots, X_k) & K(X_2, \ldots, X_{k-1}) \end{pmatrix}$$

for any $k \geq 1$. In addition,

$$(-1)^k K(X_1, \ldots, X_{i-1}) K(X_{i+k+2}, \ldots, X_{i+j})$$
$$= K(X_1, \ldots, X_{i+j}) K(X_{i+1}, \ldots, X_{i+k}) - K(X_1, \ldots, X_{i+k}) K(X_{i+1}, \ldots, X_{i+j})$$

holds for any $i \geq 1$, $k \geq 0$, and $j \geq k+1$.

We set

$$(9) \qquad P_k = K(\epsilon_1 X, \ldots, \epsilon_k X), \quad Q_k = K(\epsilon_2 X, \ldots, \epsilon_k X) \in \mathbb{Z}[X]$$

for $k \geq 1$ and extend both to all $k \in \mathbb{Z}$ by using the recursion (7); for instance,

$$P_0 = 1, \quad P_{-1} = 0, \quad Q_0 = 0, \quad Q_1 = 1.$$

Equation (8) says that for any $k \geq 1$

$$(10) \qquad \begin{pmatrix} P_k & P_{k-1} \\ Q_k & Q_{k-1} \end{pmatrix} = \begin{pmatrix} \epsilon_1 X & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \epsilon_k X & 1 \\ 1 & 0 \end{pmatrix}.$$

Grouping matrices in the right-hand side in pairs, we have

$$\begin{pmatrix} \epsilon_k X & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \epsilon_{k+1} X & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \epsilon_k X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \epsilon_{k+1} X & 1 \end{pmatrix} = \rho(u^{\epsilon_k} v^{\epsilon_{k+1}})$$

for any $k \geq 1$, implying that

$$(11) \qquad \begin{pmatrix} P_{2k} & P_{2k-1} \\ Q_{2k} & Q_{2k-1} \end{pmatrix} = \rho(u^{\epsilon_1} v^{\epsilon_2} \cdots u^{\epsilon_{2k-1}} v^{\epsilon_{2k}})$$

as well as

$$(12) \qquad \begin{pmatrix} P_{2k} & P_{2k+1} \\ Q_{2k} & Q_{2k+1} \end{pmatrix} = \rho(u^{\epsilon_1} v^{\epsilon_2} \cdots u^{\epsilon_{2k-1}} v^{\epsilon_{2k}} u^{\epsilon_{2k+1}}).$$

In particular, for $w = u^{\epsilon_1} v^{\epsilon_2} \cdots u^{\epsilon_{p-2}} v^{\epsilon_{p-1}}$, one has

$$(13) \qquad \rho(w) = \begin{pmatrix} P_{p-1} & P_{p-2} \\ Q_{p-1} & Q_{p-2} \end{pmatrix}.$$

It is shown in [Ril72] that the Riley polynomial is equal to $\rho(w)_{11} = P_{p-1}$. That is, the assignment (4) defines an $\mathrm{SL}_2(\mathbb{C})$-representation of G if and only if $P_{p-1}(X) = 0$.

Proposition 2.4. P_{p-2} satisfies $P_{p-2}^2 = -1$ in $\mathbb{Q}[X]/(P_{p-1})$.

Proof. Combining Equation (10) for $k = p - 1$ with the fact that $\epsilon_k = \epsilon_{p-k}$ for all $0 < k < p$, we deduce that $\rho(w)$ is symmetric, i.e., $P_{p-2} = Q_{p-1}$. Together with the fact that $\rho(w)$ has determinant 1, we have $P_{p-2}^2 = -1$ in $\mathbb{Q}[X]/(P_{p-1})$. □

The above proposition shows that the complex number i in (6) can be replaced by the polynomial $P_{p-2} \in \mathbb{Q}[X]/(P_{p-1})$. It follows that the coordinate ring $\mathbb{Q}[\hat{\mathcal{X}}]$ of $\hat{\mathcal{X}}$ has the single generator X.

Proposition 2.5. The coordinate ring $\mathbb{Q}[\hat{\mathcal{X}}]$ is isomorphic to $\mathbb{Q}[X]/(P_{p-1})$.

Proof. The coordinate ring $\mathbb{Q}[\mathcal{X}]$ of $\mathcal{X}$ is generated by $\operatorname{Tr} \rho(u)$, $\operatorname{Tr} \rho(v)$, and $\operatorname{Tr} \rho(uv)$ and is thus generated by X^2, subject to the relation given by the Riley polynomial. Namely, $\mathbb{Q}[\mathcal{X}] \simeq \mathbb{Q}[X^2]/(P_{p-1})$. On the other hand, $\mathbb{Q}[\hat{\mathcal{X}}]$ is obtained from $\mathbb{Q}[\mathcal{X}]$ by adding X and $i \in \mathbb{C}$, where the latter can be replaced by the polynomial P_{p-2} due to Proposition 2.4. Hence $\mathbb{Q}[\hat{\mathcal{X}}]$ is isomorphic to $\mathbb{Q}[X]/(P_{p-1})$. □

As Riley did in [Ril72] and the first author repeated in [Mar23], by reducing modulo 2, one can prove that the polynomial P_{p-1} has no multiple roots. In other words, $\mathbb{Q}[X]/(P_{p-1})$ is a semi-simple $\mathbb{Q}$-algebra.

2.3. **Computation of the torsion.** In this section, we compute the Reidemeister torsions of $K(p, q)$ associated with representations ρ_n defined as below. The computation proceeds in $V = \mathbb{Q}[\hat{\mathcal{X}}] = \mathbb{Q}[X]/(P_{p-1})$, and we denote by x, $\mathcal{P}_k$ and $\mathcal{Q}_k$ the image of X, P_k and Q_k in V, respectively.

Recall from Section 2.2 that the polynomial P_{p-1} is designed to give a parabolic representation $\rho : G \to \mathrm{SL}_2(V)$. The natural action of $\mathrm{SL}_2(V)$ on the n-th symmetric power $\mathrm{S}^n V^2$ of V^2 defines a representation $s_n : \mathrm{SL}_2(V) \to \mathrm{SL}(\mathrm{S}^n V^2)$. The composition $\rho_n = s_n \circ \rho$ is a representation of G into $\mathrm{SL}(\mathrm{S}^n V^2)$ and allows us to twist the usual cohomology of the knot complement by the local system $\mathrm{S}^n V^2$. Under some cohomological conditions specified below, the Reidemeister torsion with respect to ρ_n is well-defined (see e.g. [Por15]), and we simply denote it by

$$\tau_n \in V/\{\pm 1\}.$$

Note that the torsion is defined up to sign. This sign ambiguity can be resolved by fixing a homological order of the knot complement, but we will not address it in this paper.

The fact that the torsion is invariant under simple homotopy allows us to compute the torsion τ_n using the 2-dimensional complex associated with the presentation (2) of the knot group. Since this presentation has two generators u and v with a single relator $r = wuw^{-1}v^{-1}$, the corresponding chain complex is

$$(14) \qquad \mathrm{S}^n V^2 \xrightarrow{D^0} \mathrm{S}^n V^2 \oplus \mathrm{S}^n V^2 \xrightarrow{D^1} \mathrm{S}^n V^2.$$

The differentials D^i of the above complex are given as follows. Let F_2 be the free group with generators u and v, and let $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ be the usual Fox derivatives, i.e, the unique endomorphisms of $\mathbb{Z}[F_2]$ such that

$$\frac{\partial u}{\partial u} = 1, \quad \frac{\partial v}{\partial u} = 0, \quad \frac{\partial u}{\partial v} = 0, \quad \frac{\partial v}{\partial v} = 1$$

and

$$\frac{\partial ab}{\partial u} = \frac{\partial a}{\partial u} + a \frac{\partial b}{\partial u}, \quad \frac{\partial ab}{\partial v} = \frac{\partial a}{\partial v} + a \frac{\partial b}{\partial v} \quad \text{for any } a, b \in F_2.$$

Note that

$$(15) \qquad \frac{\partial a}{\partial u}(u-1) + \frac{\partial a}{\partial v}(v-1) = a - 1 \quad \text{for any } a \in F_2\,.$$

Abusing notation, we denote by $\rho_n : \mathbb{Z}[F_2] \to \mathbb{Z}[G] \to \mathrm{End}(\mathrm{S}^n V^2)$ the map sending a to $\rho_n(a)$. Then the differentials in the chain compelex (14) read

$$D^0(\xi) = \Big(\rho_n(u-1)\xi,\ \rho_n(v-1)\xi\Big), \quad D^1(\xi,\eta) = \rho_n\left(\frac{\partial r}{\partial u}\right)\xi + \rho_n\left(\frac{\partial r}{\partial v}\right)\eta.$$

Note that $D^1 \circ D^0 = 0$ follows from Equation (15) together with the fact that $\rho_n(r) = I$.

Let us further compute D^1. Recall that the word w is given by $w = u^{\epsilon_1} \cdots v^{\epsilon_{p-1}}$. A striaghtforward computation shows that

$$\frac{\partial w}{\partial u} = \sum_{k=1}^{\frac{p-1}{2}} \begin{cases} u^{\epsilon_1} v^{\epsilon_2} \cdots v^{\epsilon_{2k-2}} & \text{if } \epsilon_{2k-1} = 1 \\ -u^{\epsilon_1} v^{\epsilon_2} \cdots v^{\epsilon_{2k-2}} u^{\epsilon_{2k-1}} & \text{if } \epsilon_{2k-1} = -1 \end{cases},$$

$$\frac{\partial w}{\partial v} = \sum_{k=1}^{\frac{p-1}{2}} \begin{cases} u^{\epsilon_1} v^{\epsilon_2} \cdots u^{\epsilon_{2k-1}} & \text{if } \epsilon_{2k} = 1 \\ -u^{\epsilon_1} v^{\epsilon_2} \cdots u^{\epsilon_{2k-1}} v^{\epsilon_{2k}} & \text{if } \epsilon_{2k} = -1 \end{cases}.$$

Using Equations (11) and (12), one has

$$(16) \qquad \rho_n\left(\frac{\partial w}{\partial u}\right) = \sum_{k=1}^{\frac{p-1}{2}} \epsilon_{2k-1}\, s_n \begin{pmatrix} \mathcal{P}_{2k-2} & \mathcal{P}_{2k-2-\epsilon_{2k-1}} \\ \mathcal{Q}_{2k-2} & \mathcal{Q}_{2k-2-\epsilon_{2k-1}} \end{pmatrix},$$

$$(17) \qquad \rho_n\left(\frac{\partial w}{\partial v}\right) = \sum_{k=1}^{\frac{p-1}{2}} \epsilon_{2k}\, s_n \begin{pmatrix} \mathcal{P}_{2k-1-\epsilon_{2k}} & \mathcal{P}_{2k-1} \\ \mathcal{Q}_{2k-1-\epsilon_{2k}} & \mathcal{Q}_{2k-1} \end{pmatrix}.$$

Combining these equations with

$$(18) \qquad \frac{\partial r}{\partial u} = (1-v)\frac{\partial w}{\partial u} + w, \quad \frac{\partial r}{\partial v} = (1-v)\frac{\partial w}{\partial v} - 1\,,$$

which directly follows from $r = wuw^{-1}v^{-1}$, we obtain an explicit formula for D^1.

2.3.1. *Tautological representation.* In this section, we compute the torsion τ_1 associated with the tautological representation ρ_1. Note that s_1 is the identity map and thus $\rho_1 = \rho$.

Lemma 2.6. Writing the differentials D^i in matrix form, we have

$$D^0 = \begin{pmatrix} 0 & x \\ 0 & 0 \\ \hline 0 & 0 \\ x & 0 \end{pmatrix}, \quad D^1 = \left(\begin{array}{cc|cc} 0 & D_1^1 & D_2^1 & 0 \\ 0 & D_3^1 & D_4^1 & 0 \end{array} \right)$$

where $D_1^1 = \mathcal{P}_{p-2}$, $D_2^1 = -1$,

$$D_3^1 = \mathcal{Q}_{p-2} - x \sum_{k=1}^{\frac{p-1}{2}} \epsilon_{2k-1} \mathcal{P}_{2k-2-\epsilon_{2k-1}}\,, \quad D_4^1 = -x \sum_{k=1}^{\frac{p-1}{2}} \epsilon_{2k} \mathcal{P}_{2k-1-\epsilon_{2k}}\,.$$

Proof. It is clear from

$$\rho_1(u-1) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \quad \rho_1(v-1) = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$$

that the matrix form of D^0 is given as in the claim.

Equation (15) for $a = r$ says that $\frac{\partial r}{\partial u}(u-1) + \frac{\partial r}{\partial v}(v-1) = 0$. Thus we can write a priori

$$\rho_1\left(\frac{\partial r}{\partial u}\right) = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}, \quad \rho_1\left(\frac{\partial r}{\partial v}\right) = \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix}.$$

It then follows from Equations (18) and (13) that $a = \mathcal{P}_{p-2} = D_1^1$, $c = -1 = D_2^1$, and from (16) and (17) that $b = D_3^1$ and $d = D_4^1$. $\qquad \square$

The above lemma implies that the chain complex (14) is acyclic if and only if $D_1^1 D_4^1 - D_2^1 D_3^1 \neq 0$. In the acyclic case, the torsion τ_1 can be computed directly by using the Cayley formula (see, e.g. [GKZ08, p.485]). Precisely, we have

$$(19) \qquad \tau_1 = \frac{\Delta_1}{\Delta_0} \quad \text{where} \quad \Delta_0 = x^2, \ \Delta_1 = D_1^1 D_4^1 - D_2^1 D_3^1.$$

Theorem 2.7. *The torsion of the two bridge knot $K(p,q)$ in its tautological representation ρ_1 is well-defined and given by*

$$\tau_1 = \frac{1}{x^2}\left(\mathcal{Q}_{p-2} - x \sum_{k=1}^{\frac{p-1}{2}} \epsilon_{2k-1}(\mathcal{P}_{2k-1} + \mathcal{P}_{2k-3}) \right).$$

Proof. The last equality of Lemma 2.3 implies that $\mathcal{P}_{p-2}\mathcal{P}_k = (-1)^k \mathcal{P}_{p-2-k}$ for any k. Hence we can rewrite

$$\sum_{k=1}^{\frac{p-1}{2}} \epsilon_{2k}\mathcal{P}_{p-2}\mathcal{P}_{2k-1-\epsilon_{2k}} = \sum_{k=1}^{\frac{p-1}{2}} \epsilon_{2k}\mathcal{P}_{p-1-2k+\epsilon_{2k}} = \sum_{k=0}^{\frac{p-3}{2}} \epsilon_{2k+1}\mathcal{P}_{2k+\epsilon_{2k+1}}.$$

In the last equality, we have substituted $2k$ with $p - 2k - 1$ and used the symmetry $\epsilon_{2k+1} = \epsilon_{p-2k-1}$. Then two sums that appear in

$$\Delta_1 = -x \sum_{k=1}^{\frac{p-1}{2}} \epsilon_{2k}\mathcal{P}_{p-2}\mathcal{P}_{2k-1-\epsilon_{2k}} + \mathcal{Q}_{p-2} - x \sum_{k=1}^{\frac{p-1}{2}} \epsilon_{2k-1}\mathcal{P}_{2k-2-\epsilon_{2k-1}}$$

add to $\sum_k \epsilon_{2k-1}(\mathcal{P}_{2k-1} + \mathcal{P}_{2k-3})$. Combining this with Equation (19), we obtain the desired formula of τ_1.

It remains to show that Δ_1 is invertible to prove that the chain complex (14) is acyclic and the torsion τ_1 is well-defined. This follows from Theorem 2.8 below, saying that $\Delta_1 = x^2\tau_1$ is invertible with the explicit inverse $\mathcal{P}_{\ell-1}/2$. $\qquad \square$

2.3.2. *Simplication of τ_1.* In this section, we show that the formula of τ_1 in Theorem 2.7 admits a remarkable simplification.

Theorem 2.8. *One has*

$$\tau_1 = \frac{2}{x^2 \mathcal{P}_{\ell-1}}$$

where $0 < \ell < p$ is the unique odd integer satisfying $q\ell \equiv \pm 1$ modulo p.

Prior to the proof, we record some lemmas.

Lemma 2.9. Let R_k be a sequence of polynomials satisfying the recursion

(20) $$R_k = \epsilon_k x R_{k-1} + R_{k-2}.$$

Then $R_k = R_0 P_k + R_{-1} Q_k$ for any k.

Proof. Recall that both P_k and Q_k satisfy the same recursion (20) with $P_{-1} = 0$, $P_0 = 1$, and $Q_{-1} = 1$, $Q_0 = 0$. It follows that R_k and $R_0 P_k + R_{-1} Q_k$ satisfy the same recursion with the same values for $k = 0$ and -1. Therefore, they coincide for any k. □

We henceforth set $\iota = \mathcal{P}_{p-2}$; by Proposition 2.4, we have $\iota^2 = -1$.

Lemma 2.10. One has $\mathcal{P}_{k+p} = (-1)^k \iota \mathcal{P}_k$ and $\mathcal{P}_{k+2p} = \mathcal{P}_k$ for any k.

Proof. Since $\epsilon_{k+p} = -\epsilon_k$, it follows that the sequence $P_{k+p}(-x) = (-1)^{k+p} P_{k+p}(x)$ satisfies the recursion (20). By Lemma 2.9, we have

$$(-1)^{k+p} P_{k+p} = -P_p P_k + P_{p-1} Q_k.$$

As $P_p = \epsilon_p x P_{p-1} + P_{p-2}$, we obtain the relation $\mathcal{P}_{k+p} = (-1)^k \iota \mathcal{P}_k$ in V. Applying it twice, one has $\mathcal{P}_{k+2p} = \mathcal{P}_k$. □

Lemma 2.11. One has $\mathcal{P}_{-2-k} = \mathcal{P}_k$ for any k.

Proof. Since $\epsilon_{-k} = -\epsilon_k$ for $p \nmid k$, the sequence P_{-2-k} satisfies the recursion (20) for $0 < k < p$. By Lemma 2.9, $P_{-2-k} = P_{-2} P_k + P_{-1} Q_k = P_k$ holds for $0 < k < p$, which also holds for $k = 0$. Using Lemma 2.10, the formula extends to all k in V. □

Lemma 2.12. One has $\mathcal{P}_{\ell'-1} \mathcal{Q}_{p-2} = -\iota(\mathcal{P}_{\ell'} + \mathcal{P}_{\ell'-2})$ where ℓ' is the unique odd integer satisfying $0 < \ell' < 2p$ and $\ell' q \equiv -1$ modulo $2p$.

Proof. One easily proves that $\epsilon_{\ell'+k} = \epsilon_k$ if $p \nmid k$. This implies that $P_{\ell'+k}$ satisfies the recursion (20) for $0 < k < p$ and thus, by Lemma 2.9,

$$P_{\ell'+k} = P_{\ell'} P_k + P_{\ell'-1} Q_k.$$

for $0 < k < p$. Taking $k = p - 2$, we have $P_{\ell'+p-2} = P_{\ell'} P_{p-2} + P_{\ell'-1} Q_{p-2}$ which, combined with Lemma 2.10, gives $-\iota \mathcal{P}_{\ell'-2} = \iota \mathcal{P}_{\ell'} + \mathcal{P}_{\ell'-1} \mathcal{Q}_{p-2}$. □

Lemma 2.13. For all $-1 \leq k < p$ one has

$$(P_{\ell'} - P_{\ell'-2}) P_k = P_{\ell'+k} - P_{\ell'-k-2}.$$

The same equation holds with a minus sign for $-p < k < 0$.

Proof. We prove it by induction on k: it holds for $k = -1$ as $P_{-1} = 0$ and obviously for $k = 0$. Take now $0 < k < p$ and suppose the formula holds for $k - 1$ and $k - 2$. Then one has

$$\begin{aligned}
(P_{\ell'} - P_{\ell'-2}) P_k &= \epsilon_k x (P_{\ell'} - P_{\ell'-2}) P_{k-1} + (P_{\ell'} - P_{\ell'-2}) P_{k-2} \\
&= \epsilon_k x (P_{\ell'+k-1} - P_{\ell'-k-1}) + P_{\ell'+k-2} - P_{\ell'-k} \\
&= \epsilon_k (\epsilon_{\ell'+k} (P_{\ell'+k} - P_{\ell'+k-2}) - \epsilon_{\ell'-k} (P_{\ell'-k} - P_{\ell'-k-2})) \\
&\quad + P_{\ell'+k-2} - P_{\ell'-k}.
\end{aligned}$$

As $\epsilon_{\ell'+k} = \epsilon_k$ and $\epsilon_{\ell'-k} = -\epsilon_k$ for $0 < k < p$, the last expression simplifies and gives the result. The same induction argument is applied for $-p < k < 0$. □

Proof of Theorem 2.8. First observe that if $0 < \ell' < p$, then $\ell = \ell'$; otherwise, $\ell = 2p - \ell'$. In either case, by Lemmas 2.10 and 2.11, we have $\mathcal{P}_{\ell-1} = \mathcal{P}_{\ell'-1}$.

Due to Theorem 2.7, it suffices to show that

$$\mathcal{P}_{\ell'-1}(\mathcal{Q}_{p-2} - xS) = 2 \quad \text{where } S = \sum_{k=1}^{\frac{p-1}{2}} \epsilon_{2k-1}(\mathcal{P}_{2k-1} + \mathcal{P}_{2k-3}).$$

Applying Lemma 2.12 with the formula $-x\mathcal{P}_{\ell'-1} = \mathcal{P}_{\ell'} - \mathcal{P}_{\ell'-2}$, the above equality is equivalent to

$$(21) \qquad (\mathcal{P}_{\ell'} - \mathcal{P}_{\ell'-2})S = 2 + \iota(\mathcal{P}_{\ell'} + \mathcal{P}_{\ell'-2}).$$

On the other hand, using the symmetry $\mathcal{P}_{2k-3} = \mathcal{P}_{-2k+1}$ and $\epsilon_{2k-1} = -\epsilon_{-2k+1}$, we can write

$$S = \sum_{k=1}^{\frac{p-1}{2}} (\epsilon_{2k-1}\mathcal{P}_{2k-1} - \epsilon_{-2k+1}\mathcal{P}_{-2k+1}) = \sum_{-p<k<p} \delta_k \epsilon_k \mathcal{P}_k$$

where $\delta_k = 1$ for $0 < k < p$ odd, $\delta_k = -1$ for $-p < k < 0$ odd, and $\delta_k = 0$ otherwise. As Lemma 2.13 states that $(\mathcal{P}_{\ell'} - \mathcal{P}_{\ell'-2})\mathcal{P}_k = \delta_k(\mathcal{P}_{\ell'+k} - \mathcal{P}_{\ell'-k-2})$ for odd $-p < k < p$, we have

$$(\mathcal{P}_{\ell'} - \mathcal{P}_{\ell'-2})S = \sum_{k:\text{odd}} (\epsilon'_k \mathcal{P}_{\ell'+k} - \epsilon'_k \mathcal{P}_{\ell'-k-2}) = \sum_{k:\text{even}} (\epsilon'_{k-\ell'} - \epsilon'_{\ell'-2-k})\mathcal{P}_k.$$

In this formula, $\epsilon'_k = \epsilon_k$ if $p \nmid k$, and $\epsilon'_k = 0$ otherwise; the sums are taken over $k \in \mathbb{Z}/2p\mathbb{Z}$, which are well-defined by Lemma 2.10. As $\mathcal{P}_{2k} = \mathcal{P}_{2p-2-2k}$ and $\mathcal{P}_{p-1} = 0$, we can pack these in the form

$$(22) \qquad (\mathcal{P}_{\ell'} - \mathcal{P}_{\ell'-2})S = \sum_{k=0}^{\frac{p-3}{2}} (u_k + v_k)\mathcal{P}_{2k}$$

where $u_k = \epsilon'_{2k-\ell'} - \epsilon'_{\ell'+2k}$ and $v_k = \epsilon'_{-2-2k-\ell'} - \epsilon'_{\ell'-2-2k}$.

- For $k = 0$, one has $\epsilon'_{-\ell'} + \epsilon'_{-2-\ell'} = 0$, $\epsilon'_{\ell'} = \epsilon'_{\ell'-2} = -1$, and $\mathcal{P}_0 = 1$. It follows that $(u_0 + v_0)\mathcal{P}_0 = 2$.
- For $k > 0$ one has $\epsilon_{\ell'+2k} = \epsilon_{2k} = \epsilon_{2k-\ell'}$ hence $u_k = 0$ unless $\ell' + 2k \equiv p$ or $2k - \ell' \equiv p \pmod{2p}$:

  If $\ell' < p$, then there is k_0 such that $\ell' + 2k_0 = p$, and one has $u_{k_0}\mathcal{P}_{2k_0} = \epsilon_{2k_0-\ell'}\mathcal{P}_{2k_0} = \epsilon_{p-2\ell'}\mathcal{P}_{p-\ell'} = -\mathcal{P}_{p-\ell'} = \iota\mathcal{P}_{\ell'-2}$;

  If $\ell' > p$, then there is k_0 such that $2k_0 - \ell' = -p$, and one has $u_{k_0}\mathcal{P}_{2k_0} = -\epsilon_{\ell'+2k_0}\mathcal{P}_{2k_0} = \epsilon_{\ell'-p}\mathcal{P}_{\ell'-p} = -\mathcal{P}_{\ell'+p} = \iota\mathcal{P}_{\ell'}$.
- Similarly, one has $v_k = 0$ for $k > 0$ unless $\ell' - 2 - 2k \equiv p$ or $-2 - 2k - \ell' \equiv p \pmod{2p}$:

  If $\ell' < p$, then there is k_0 such that $-2 - 2k_0 - \ell' = -p$, and one has $v_{k_0}\mathcal{P}_{2k_0} = -\epsilon_{\ell'-2-2k_0}\mathcal{P}_{2k_0} = \iota\mathcal{P}_{\ell'}$;

  If $\ell' > p$, then there is k_0 such that $\ell' - 2 - 2k_0 = p$, and one has $v_{k_0}\mathcal{P}_{2k_0} = \epsilon_{-2-2k_0-\ell'}\mathcal{P}_{2k_0} = \iota\mathcal{P}_{\ell'-2}$.

We conclude that in both cases $\ell' < p$ and $\ell' > p$, the sum in (22) results in $2 + \iota(\mathcal{P}_{\ell'} + \mathcal{P}_{\ell'-2})$. This proves Equation (21) and thus the theorem. □

2.3.3. *Adjoint representation.* In this section, we compute the torsion τ_2 associated with the adjoint representation $\rho_2 = s_2 \circ \rho$.

The symmetric power $S^2 V^2$ is 3-dimensional and the representation s_2 is known to be isomorphic to the adjoint representation $\mathrm{SL}_2(V) \to \mathrm{SL}(\mathrm{sl}_2(V))$, mapping g to the map $\xi \mapsto g\xi g^{-1}$. We denote by $\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3$ the standard basis of $S^2 V^2$ and by $\langle \cdot, \cdot \rangle : S^2 V^2 \otimes S^2 V^2 \to V$ the canonical invariant pairing on $S^2 V^2$.

The torsion τ_2 is our main case of interest. However, it provides extra difficulties due to the fact that the chain complex (14) is not acyclic. That is, $\mathrm{H}^*(M, S^2 V^2)$ does not vanish where M is the knot complement of $K(p, q)$ and the coefficient $S^2 V^2$ is twisted by $\rho_2 = s_2 \circ \rho$. More precisely, it is well-known that $\mathrm{H}^k(M, S^2 V^2)$ vanishes except for $k = 1, 2$. In this non-acyclic case, defining the torsion requires fixing a basis of the cohomology [Por97].

Let ∂M be the boundary torus of M and let $m \subset \partial M$ be a meridian of $K(p, q)$ homotopic to $u \in G$. The inclusion maps induce the following homomorphisms

$$(23) \qquad \mathrm{H}^1(M, S^2 V^2) \to \mathrm{H}^1(m, S^2 V^2), \quad \mathrm{H}^2(M, S^2 V^2) \to \mathrm{H}^2(\partial M, S^2 V^2).$$

It is well-known that when $K(p, q)$ is hyperbolic and ρ is a lift of the geometric representation, these two maps are isomorphism. In our case, this will follow from a computation.

Applying the Poincaré duality, combined with the invariant pairing of $S^2 V^2$, to the isomorphisms in (23), we obtain

$$(24) \qquad \mathrm{H}^1(M, S^2 V^2) \simeq \mathrm{H}^0(m, S^2 V^2)^*, \quad \mathrm{H}^2(M, S^2 V^2) \simeq \mathrm{H}^0(\partial M, S^2 V^2)^*.$$

By definition of the 0-th cohomology, $\mathrm{H}^0(m, S^2 V^2)$ and $\mathrm{H}^0(\partial M, S^2 V^2)$ are generated by elements of $S^2 V^2$ that are fixed by the restrictions $\rho|_m$ and $\rho|_{\partial M}$, respectively. It follows that both are 1-dimensional and may be assume to be generated by $\mathfrak{e}_1$. We specify $\mathfrak{e}_1$ as a basis for both $\mathrm{H}^0(m, S^2 V^2)$ and $\mathrm{H}^0(\partial M, S^2 V^2)$. This determines a basis of $\mathrm{H}^*(M; S^2 V^2)$ through the isomorphisms in (24). With this choice of basis, the torsion τ_2 is well-defined.

Theorem 2.14. *The torsion of the two-bridge knot $K(p, q)$ in the adjoint representation ρ_2 and with the homological marking referred to above is given by*

$$\tau_2 = \frac{1}{2x^2 \mathcal{P}_{\ell-1}^2} \sum_{k=1}^{\frac{p-1}{2}} \epsilon_{2k} \mathcal{P}_{2k-1}^2$$

where $0 < \ell < p$ is the unique odd integer satisfying $q\ell \equiv \pm 1$ modulo p.

Proof. Let C^* be the chain complex (14) and H^* be its homology. We first define a quasi-isomorphism $f_* : C^* \to \mathrm{H}^*$.

$$
\begin{array}{ccccccc}
C^* : & & S^2 V^2 & \xrightarrow{D^1} & S^2 V^2 \oplus S^2 V^2 & \xrightarrow{D^2} & S^2 V^2 \\
& \downarrow f_* & & \downarrow & & \downarrow f_1 & & \downarrow f_2 \\
\mathrm{H}^* : & & 0 & \xrightarrow{\ 0\ } & V & \xrightarrow{\ 0\ } & V
\end{array}
$$

Note that $S^2 V^2$ is isomorphic to V^3 with basis $\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3$.

Since m is a 1-subcomplex of the 2-complex used to define C^*, one can choose f_1 by mapping (ξ, η) to $\langle \xi, \mathfrak{e}_1 \rangle$. In the standard basis, we have $f_1 = (0, 0, 1, 0, 0, 0)$.

For the second one, we consider $\partial M \simeq S^1 \times S^1$ and a cellular map homotopic to the inclusion $\partial M \hookrightarrow M$. The boundary torus has a cell decomposition corresponding to the presentation

$$\pi_1(\partial M) \simeq \mathbb{Z}^2 = \langle a, b \,|\, R \rangle, \quad R = aba^{-1}b^{-1}.$$

The natural morphism $\pi_1(\partial M) \to \pi_1(M)$ sends a to the meridian u and b to the (blackboard-framed) longitude w^*w. Here w^* is the word w read from right to left; see [Ril72, Prop.1]. Note that this morphism is well-defined due to the following equality that holds in F_2:

$$(25) \qquad [u, w^*w] = [g, r][r, w^*], \quad g = u^{\epsilon_0}v^{\epsilon_1} \cdots u^{\epsilon_{\ell'-1}}v^{\epsilon_{\ell'}}$$

where $0 < \ell' < 2p$ is the inverse of $-q$ modulo $2p$; see Lemma 2 in [Mar24]. This allows us to write down explicitly the morphism of complex inducing the restriction map $\mathrm{H}^*(M, \mathrm{S}^2V^2) \to \mathrm{H}^*(\partial M, \mathrm{S}^2V^2)$ as follows:

$$
\begin{array}{ccccc}
\mathrm{S}^2V^2 & \xrightarrow{(\rho_2(u-1),\rho_2(v-1))} & \mathrm{S}^2V^2 \oplus \mathrm{S}^2V^2 & \xrightarrow{(\rho_2(\frac{\partial r}{\partial u}),\rho_2(\frac{\partial r}{\partial v}))} & \mathrm{S}^2V^2 \\
\Big\downarrow{\scriptstyle =} & & \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \Psi} \\
\mathrm{S}^2V^2 & \xrightarrow{(\rho_2(a-1),\rho_2(b-1))} & \mathrm{S}^2V^2 \oplus \mathrm{S}^2V^2 & \xrightarrow{(\rho_2(\frac{\partial R}{\partial a}),\rho_2(\frac{\partial R}{\partial b}))} & \mathrm{S}^2V^2
\end{array}
$$

In this diagram, Φ is the Jacobian matrix of the transformation $(a, b) \mapsto (u, w^*w)$ and, Ψ sends ξ to $\rho_2(g)\xi - \rho_2(w^*)\xi$; see Equation (25). Recall that we chose the basis of $\mathrm{H}^0(\partial M, \mathrm{S}^2V^2)$ as $\mathfrak{e}_1$, hence the map $f_2 : \mathrm{S}^2V^2 \to V$ is given by

$$f_2(\xi) = \langle \rho_2(g)\xi, \, \mathfrak{e}_1 \rangle - \langle \rho_2(w^*)\xi, \, \mathfrak{e}_1 \rangle = \langle \xi, \, \rho_2(g)^{-1}\mathfrak{e}_1 \rangle - \langle \xi, \, \rho_2(w^*)^{-1}\mathfrak{e}_1 \rangle.$$

Recall from Equation (13) that $\rho(w)$ has entries $\mathcal{P}_{p-1} = 0$, $\mathcal{P}_{p-2} = \mathcal{Q}_{p-1}$, and $\mathcal{Q}_{p-2}$ where $\mathcal{P}_{p-2}^2 = -1$. From the symmetry between w and w^*, we have

$$(26) \qquad \rho(w^*) = \begin{pmatrix} \mathcal{Q}_{p-2} & \mathcal{P}_{p-2} \\ \mathcal{P}_{p-2} & 0 \end{pmatrix}.$$

It follows that $\rho_2(w^*)^{-1}\mathfrak{e}_1 = -\mathfrak{e}_3$. On the other hand, we have

$$\rho(g) = \begin{pmatrix} K(\epsilon_0 x, \ldots, \epsilon_{\ell'} x) & K(\epsilon_0 x, \ldots, \epsilon_{\ell'-1} x) \\ K(\epsilon_1 x, \ldots, \epsilon_{\ell'} x) & K(\epsilon_1 x, \ldots, \epsilon_{\ell'-1} x) \end{pmatrix} = \begin{pmatrix} * & * \\ \mathcal{P}_{\ell'} & \mathcal{P}_{\ell'-1} \end{pmatrix}.$$

It follows that $\rho_2(g)^{-1}\mathfrak{e}_1 = \mathcal{P}_{\ell'-1}^2\mathfrak{e}_1 + 2\mathcal{P}_{\ell'-1}\mathcal{P}_{\ell'}\mathfrak{e}_2 - \mathcal{P}_{\ell'}^2\mathfrak{e}_3$. Therefore,

$$f_2(\xi) = \langle \xi, \, \mathcal{P}_{\ell'-1}^2\mathfrak{e}_1 + 2\mathcal{P}_{\ell'-1}\mathcal{P}_{\ell'}\mathfrak{e}_2 - (\mathcal{P}_{\ell'}^2 - 1)\mathfrak{e}_3 \rangle,$$

showing that $f_2 = (*, *, \mathcal{P}_{\ell'-1}^2)$ in the matrix form. On the other hand, if $\ell \equiv -q^{-1}$ (mod p), then $\ell' = \ell$; otherwise, $\ell' = 2p - \ell$. In the latter case, the identity $\mathcal{P}_{2p-k} = \mathcal{P}_{k-2}$ implies that $\mathcal{P}_{\ell'-1} = \mathcal{P}_{2p-\ell-1} = \mathcal{P}_{\ell-1}$. Thus, in either cases, we have $\mathcal{P}_{\ell'-1} = \mathcal{P}_{\ell-1}$ and thus $f_2 = (*, *, \mathcal{P}_{\ell-1}^2)$

Finally, we compute the torsion τ_2 from the acyclic chain complex associated with the mapping cone of $f_* : C^* \to \mathrm{H}^*$:

$$\mathrm{S}^2V^2 \xrightarrow[D_0]{(u-1,v-1)} \mathrm{S}^2V^2 \oplus \mathrm{S}^2V^2 \xrightarrow[D_1]{(\frac{\partial r}{\partial u},\frac{\partial r}{\partial v},f_1)} \mathrm{S}^2V^2 \oplus V \xrightarrow[D_2]{(f_2,0)} V.$$

As computed, the differentials D^i have matrix form

$$D^0 = \left(\begin{array}{ccc} 0 & -x & -x^2 \\ 0 & 0 & 2x \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2x & 0 & 0 \\ -x^2 & x & 0 \end{array} \right), \quad D^1 = \left(\begin{array}{ccc|ccc} & \rho_2(\frac{\partial r}{\partial u}) & & & \rho_2(\frac{\partial r}{\partial v}) & \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right)$$

and $D^2 = \left(* \; * \; P^2_{\ell-1} \; 0 \right)$. To apply the Cayley formula, we extract the rows $1, 2, 5$ of D^0 to get a determinant Δ_0, we extract the (complementary) columns $3, 4, 6$ and the rows $1, 2, 6$ of D_1 to get a determinant Δ_1 and keep the third entry of D_2 to get Δ_2. Then the Cayley formula (see, e.g. [GKZ08, p.485]) says that

$$(27) \qquad\qquad \tau_2 = \frac{\Delta_1}{\Delta_0 \Delta_2}$$

where $\Delta_0 = 4x^3$, $\Delta_1 = -2x(\sum_{k=1}^{\frac{p-1}{2}} \epsilon_{2k} \mathcal{P}^2_{2k-1})$, and $\Delta_2 = \mathcal{P}^2_{\ell-1}$. Here Δ_1 is computed from (see Equation (18))

$$\rho_2 \left(\frac{\partial r}{\partial v} \right) = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 2x & 0 & 0 \\ x^2 & -x & 0 \end{array} \right) \left(\begin{array}{ccc} * & * & a \\ * & * & * \\ * & * & * \end{array} \right) - I = \left(\begin{array}{ccc} -1 & 0 & 0 \\ * & * & -2xa \\ * & * & * \end{array} \right)$$

where $a = -\sum_{k=1}^{\frac{p-1}{2}} \epsilon_{2k} \mathcal{P}^2_{2k-1}$, which follows from Equation (17). Substituting Δ_i into the Cayley formula (27), we obtain the desired formula.

To show acyclicity of the cone complex, it is sufficicient to show that no Δ_i vanish. The non trivial one is Δ_1, and its relation with the invertible element Ω in Section 2.5 will show that it is invertible; see Equation (31). $\qquad\square$

2.4. **Comparing reciprocal pairs.** Let $0 < \ell < p$ be the unique odd integer such that $\ell q \equiv \pm 1 \pmod{p}$. Recall from Section 2.1 that the knots $K(p, q)$ and $K(p, \ell)$ are isotopic. It follows that the orbifolds $\mathcal{O}(p, q)$ and $\mathcal{O}(p, \ell)$ are homeomorphic, and the algebras V and V^* associated to (p, q) and (p, ℓ), respectively, are isomorphic. It is useful to make this algebra isomorphism explicit for two reasons. First, it clarifies the denominator of the formula in Theorems 2.8 and 2.14, providing a topological explanation for the discrepancy between the torsions and the element Ω in Section 2.5 below that will be needed to compute signatures. Second, it provides one of the few ways to verify the formulas in these sections: since torsions are topological invariants, they should correspond under the isomorphism.

2.4.1. *Explicit isomorphism.*

Proposition 2.15. Let ℓ be the integer as above and let $V = \mathbb{Q}[X]/(P_{p-1})$ and $V^* = \mathbb{Q}[X^*]/(P^*_{p-1})$ the algebras associated to $K(p, q)$ and $K(p, \ell)$, respectively. Then the maps

$$V^* \to V, \; x^* \mapsto x\mathcal{P}_{\ell-1}(x) \quad \text{and} \quad V \to V^*, \; x \mapsto x^*\mathcal{P}^*_{q-1}(x^*)$$

are algebra isomorphisms, inverse to each other.

Proof. Recall from Section 2.1 that we can decompose the 3-sphere into two ball B_1 and B_2 where the knot $K(p, q)$ becomes two ramified arcs in each ball. Let us draw Schubert's normal form of $K(p, q)$ in the $\mathbb{R}^2$-plane so that its vertical edges run from $(0, 0)$ to $(0, 1)$ and from $(1, 0)$ to $(1, 1)$, respectively, as in Figure 4. From

Equation (4), we have $\operatorname{Tr} \rho(uv) = 2 + x^2$ where uv is a loop in the ball B_1 that encloses the two ramified arcs (the vertical edges of the normal form).
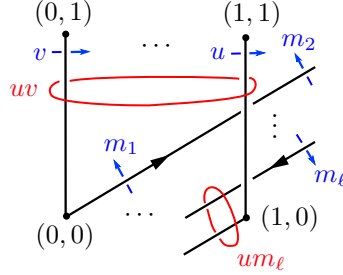

FIGURE 4. Wirtinger generators in the normal form.

In the other ball B_2, the two ramified arcs appear as arcs of slope q/p. If we follow the diagram from $(0,0)$ to $(1, 1/p)$, i.e., to the crossing just above $(1,0)$, the associated Wirtinger generator begins as v and is conjugated whenever it crosses a vertical edge. Let us denote this sequence of Wirtinger generators by $m_1, m_2, \ldots$ as in Figure 4. Then we have

$$m_k = (u^{\epsilon_1} v^{\epsilon_2} \cdots v^{\epsilon_{k-1}})^{-1} \, v \, (u^{\epsilon_1} v^{\epsilon_2} \cdots v^{\epsilon_{k-1}}).$$

When we arrive to the crossing at $(1, 1/p)$, the corresponding Wirtinger generator is m_ℓ and Equation (11) yields

$$\rho(m_\ell) = \begin{pmatrix} \mathcal{Q}_{\ell-2} & -\mathcal{P}_{\ell-2} \\ -\mathcal{Q}_{\ell-1} & \mathcal{P}_{\ell-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} \mathcal{P}_{\ell-1} & \mathcal{P}_{\ell-2} \\ \mathcal{Q}_{\ell-1} & \mathcal{Q}_{\ell-2} \end{pmatrix}$$
$$= \begin{pmatrix} 1 - x\mathcal{P}_{\ell-2}\mathcal{P}_{\ell-1} & -x\mathcal{P}_{\ell-2}^2 \\ x\mathcal{P}_{\ell-1}^2 & 1 + x\mathcal{P}_{\ell-2}\mathcal{P}_{\ell-1} \end{pmatrix}.$$

Since the vertical edge from $(1,0)$ to $(1,1)$ is attached with the arc of slope q/p at $(1,0)$, the loop in B_2 enclosing the two ramified arcs is given by um_ℓ as depicted in Figure 4, with

$$\rho(um_\ell) = \begin{pmatrix} 1 - x\mathcal{P}_{\ell-2}\mathcal{P}_{\ell-1} + x^2\mathcal{P}_{\ell-1}^2 & * \\ * & 1 + x\mathcal{P}_{\ell-2}\mathcal{P}_{\ell-1} \end{pmatrix}$$

and $\operatorname{Tr} \rho(um_\ell) = 2 + x^2\mathcal{P}_{\ell-1}^2$. As the homeomorphism between the orbifolds $\mathcal{O}(p,q)$ and $\mathcal{O}(p,\ell)$ is given by interchanging the role of B_1 and B_2, it swaps the loops uv and um_ℓ, and thus exchanges their traces $2+x^2$ and $2+x^2\mathcal{P}_{\ell-1}^2$. It follows that the map sending x^* to $x\mathcal{P}_{\ell-1}$ gives an isomorphism from V^* to V. Interchanging the role of B_1 and B_2, we obtain its inverse, sending $x \in V$ to $x^*\mathcal{P}_{q-1}^*(x^*) \in V^*$. □

2.4.2. *Cusp shape.* Let m and l be a meridian and a canonical longitude of $K(p,q)$, viewed as commuting elements in G and let $\rho_{\text{geom}} : G \to \mathrm{SL}_2(\mathbb{C})$ be a lift of the geometric representation, provided that $K(p,q)$ is hyperbolic. Up to conjugation, one can assume that

$$\rho_{\text{geom}}(m) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho_{\text{geom}}(l) = \begin{pmatrix} -1 & c \\ 0 & -1 \end{pmatrix}.$$

The element c is called the cusp shape of $K(p,q)$: it is a geometric invariant of the knot, controlling the Euclidean structure at infinity. By analogy, one can define the cusp shape associated to the tautological representation $\rho : G \to \mathrm{SL}_2(V)$. As the

blackboard-framed longitude is given by w^*w (recall the proof of Theroem 2.14), it follows from Equations (13) and (26) that

$$\rho(m) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \rho(l) = \begin{pmatrix} -1 & 2\iota\mathcal{Q}_{p-2} - 2x\sum_{k=1}^{p-1}\epsilon_k \\ 0 & -1 \end{pmatrix}.$$

Here the sum of the epsilons in $\rho(l)$ arises from the writhe correction. We derive

$$(28) \qquad c = \frac{2\iota\mathcal{Q}_{p-2}}{x} - 2\sum_{k=1}^{p-1}\epsilon_k \in V$$

which show a tight relation between $\mathcal{Q}_{p-2}$ and the cusp shape of $K(p,q)$. Computing the same element for $K(p,\ell)$ yields

$$\frac{\mathcal{Q}_{p-2}(x)}{x} = \frac{\mathcal{Q}_{p-2}^*(x^*)}{x^*}$$

where the isomorphism between V and V^* is used implicitly. Note that the sum of the epsilons in (28) does not change when we replace q by ℓ, since they are inverses of each other. Substituting $x^* = x\mathcal{P}_{\ell-1}(x)$ yields the following identity:

$$\mathcal{Q}_{p-2}^*(x^*) = \mathcal{Q}_{p-2}(x)\mathcal{P}_{\ell-1}(x) \in V.$$

We verified this identity by computer for many values of p and q.

2.4.3. *Invariance of the adjoint torsion.* Let us comment the corresponding equation for the adjoint torsion τ_2. Anticipating sligthly the next section, define Ω so that we can write

$$\tau_2 = \frac{\Omega}{4x^2\mathcal{P}_{\ell-1}^2} = \frac{\Omega^*}{4(x^*)^2(\mathcal{P}_{q-1}^*)^2}$$

Here the second equality uses the isomorphism $V \simeq V^*$ implicitly, with the fact that $\ell^* = q$. Substituting $x = x^*\mathcal{P}_{q-1}^*$, the reciprocal of the relation $x^* = x\mathcal{P}_{\ell-1}$, we deduce that

$$(29) \qquad \Omega^*(x^*) = \Omega(x)\mathcal{P}_{\ell-1}(x)^{-2}$$

This shows that the element Ω transforms non trivially under the isomorphism between V and V^*, but in a similar way to $\mathcal{Q}_{p-2}$.

2.5. **Reidemeister torsions in Frobenius algebra.** It is shown in [Mar23] that $V = \mathbb{Q}[x]/(P_{p-1})$ has a natural semi-simple Frobenius algebra structure, meaning that it is endowed with a linear form $\epsilon : V \to \mathbb{Q}$, a bilinear symmetric non-degenerate form $\eta : V \times V \to \mathbb{Q}$, and an invertible element $\Omega \in V$ related by the following equations:

$$(30) \qquad \eta(x,y) = \epsilon(xy), \quad \epsilon(x) = \mathrm{Tr}_{V/\mathbb{Q}}(\Omega^{-1}x), \quad \Omega = \sum_k \frac{e_k^2}{\eta(e_k,e_k)}$$

where x and y are any elements of V, and e_k is an orthogonal basis for η. In our case, the natural basis $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{p-2}$ of V is orthogonal and satisfies $\epsilon(\mathcal{P}_k) = 0$ for $k > 0$ and $\epsilon(\mathcal{P}_0) = 1$. Moreover, $\eta(\mathcal{P}_k, \mathcal{P}_k) = (-1)^k\epsilon_{k+1}$ so that

$$\Omega = \sum_{k=0}^{p-2}(-1)^k\epsilon_{k+1}\mathcal{P}_k^2.$$

Let us write $\Omega = \Omega^+ - \Omega^-$ by decomposing the sum over even and odd indexes of the epsilons. Using Lemmas 2.10 and 2.11, we have

$$\Omega^+ = \sum_{k=1}^{(p-1)/2} -\epsilon_{2k}\mathcal{P}_{2k-1}^2 = \sum_{k=1}^{(p-1)/2} -\epsilon_{2k}\mathcal{P}_{p-2k-1}^2 = -\sum_{k=0}^{(p-3)/2} \epsilon_{p-2k-1}\mathcal{P}_{2k}^2.$$

As $\epsilon_{p-k} = \epsilon_k$, it follows that $\Omega^+ = -\Omega^-$, and hence $\Omega^+ = \frac{1}{2}\Omega$. By Theorem 2.14, we obtain

$$(31) \qquad \tau_2 = \frac{\Omega^+}{2x^2\mathcal{P}_{\ell-1}^2} = \frac{\Omega}{4x^2\mathcal{P}_{\ell-1}^2}.$$

This relation between τ_2 and Ω implies that the inverse sum of τ_2 vanishes; see Theorem 2.16 below. Note that a similar equation for non-parabolic representations was studied in [Yoo22]. See [GKY21] for the quantum-physical motivation behind such inverse sums of adjoint torsions.

Theorem 2.16. *Fix coprime odd integers p and q satisfying $1 < q < p$, that is, fix a hyperbolic knot-bridge knot $K(p,q)$ and let $\mathcal{X}$ be its parabolic character variety. Then*

$$\sum_{\rho \in \mathcal{X}} \frac{1}{\tau_2(\rho)} = 0.$$

Proof. For any $f \in \mathbb{Q}[\mathcal{X}]$, we have

$$\sum_{\rho \in \mathcal{X}} f(\rho) = \text{Tr}_{\mathbb{Q}[\mathcal{X}]/\mathbb{Q}}(f) = \frac{1}{2}\text{Tr}_{\mathbb{Q}[\hat{\mathcal{X}}]/\mathbb{Q}}(f).$$

Therefore, it suffices to compute the trace of $1/\tau_2$ in $V = \mathbb{Q}[\hat{\mathcal{X}}]$ over $\mathbb{Q}$. From Equations (30) and (31), one has

$$\begin{aligned}
\text{Tr}_{V/\mathbb{Q}}(1/\tau_2) &= \text{Tr}_{V/\mathbb{Q}}(4x^2\mathcal{P}_{\ell-1}^2/\Omega) \\
&= \epsilon(4x^2\mathcal{P}_{\ell-1}^2) \\
&= 4\eta(\epsilon_\ell x\mathcal{P}_{\ell-1}, \epsilon_\ell x\mathcal{P}_{\ell-1}) \\
&= 4\eta(\mathcal{P}_\ell - \mathcal{P}_{\ell-2}, \mathcal{P}_\ell - \mathcal{P}_{\ell-2})
\end{aligned}$$

As $\mathcal{P}_k$ is orthogonal with $\eta(\mathcal{P}_k, \mathcal{P}_k) = (-1)^k\epsilon_{k+1}$, one has

$$\eta(\mathcal{P}_\ell - \mathcal{P}_{\ell-2}, \mathcal{P}_\ell - \mathcal{P}_{\ell-2}) = (-1)^\ell(\epsilon_{\ell+1} + \epsilon_{\ell-1}) = 0.$$

The last equality follows from the fact that $q\ell \equiv \pm 1 \pmod{p}$, implying that $\epsilon_{\ell+1}$ and $\epsilon_{\ell-1}$ have different signs. This completes the proof. $\square$

The above theorem motivates us to consider the inverse sum of τ_1. Although we have no quantum-physical explanation for its simplicity—specifically, for its independence of p and q—Theorem 2.17 below shows that the inverse sum of τ_1 is ± 1.

Theorem 2.17. *Fix coprime odd integers p and q satisfying $1 < q < p$, that is, fix a hyperbolic knot-bridge knot $K(p,q)$ and let $\mathcal{X}$ be its parabolic character variety. Then*

$$\sum_{\rho \in \mathcal{X}} \frac{1}{\tau_1(\rho)} = \epsilon_{(\ell'-1)/2}$$

where $0 < \ell' < 2p$ is the unique odd integer satisfying $\ell'q \equiv -1$ modulo $2p$.

Proof. It follows from Theorem 2.8 that

$$\mathrm{Tr}_{V/\mathbb{Q}}(1/\tau_1) = \frac{1}{2}\,\mathrm{Tr}_{V/\mathbb{Q}}(x^2\mathcal{P}_{\ell'-1}) = \frac{1}{2}\,\mathrm{Tr}_{V/\mathbb{Q}}(x(\mathcal{P}_{\ell'-2} - \mathcal{P}_{\ell'})).$$

Recall that $\mathcal{P}_0, \ldots, \mathcal{P}_{p-2}$ form an orthogonal basis of V, and for i and $j \in \mathbb{Z}/2p\mathbb{Z}$, $\eta(\mathcal{P}_i, \mathcal{P}_j) = 0$ unless $i \equiv j$ or $i + j \equiv 2p - 2$, in which case this scalar product gives $(-1)^i \epsilon_{i+1}$. The trace of an endomorphism $M \in \mathrm{End}(V)$ is given by

$$\mathrm{Tr}_{V/\mathbb{Q}}(M) = \sum_{k=0}^{p-2} \eta(\mathcal{P}_k, M\mathcal{P}_k)\eta(\mathcal{P}_k, \mathcal{P}_k).$$

If we let M be the multiplication by $x(\mathcal{P}_{\ell'-2} - \mathcal{P}_{\ell'})$, then for $0 \le k \le p - 2$,

$$\begin{aligned}
M\mathcal{P}_k &= (\mathcal{P}_{\ell'-2} - \mathcal{P}_{\ell'})x\mathcal{P}_k \\
&= (\mathcal{P}_{\ell'-2} - \mathcal{P}_{\ell'})\epsilon_{k+1}(\mathcal{P}_{k+1} - \mathcal{P}_{k-1}) \\
&= \epsilon_{k+1}(-\mathcal{P}_{\ell'+k+1} + \mathcal{P}_{\ell'-k-3} + \mathcal{P}_{\ell'+k-1} - \mathcal{P}_{\ell'-k-1}).
\end{aligned}$$

Here the last equality follows from Lemma 2.13. Therefore, one has

$$\begin{aligned}
\mathrm{Tr}_{V/\mathbb{Q}}(M) &= \sum_{k=0}^{p-2} \epsilon_{k+1}\,\eta(\mathcal{P}_k, -\mathcal{P}_{\ell'+k+1} + \mathcal{P}_{\ell'-k-3} + \mathcal{P}_{\ell'+k-1} - \mathcal{P}_{\ell'-k-1})\,\eta(\mathcal{P}_k, \mathcal{P}_k) \\
&= \sum_{k=0}^{p-2} (-1)^k\,\eta(\mathcal{P}_k, -\mathcal{P}_{\ell'+k+1} + \mathcal{P}_{\ell'-k-3} + \mathcal{P}_{\ell'+k-1} - \mathcal{P}_{\ell'-k-1}).
\end{aligned}$$

Expanding the bilinear form η in the sum, we obtain four terms: $\eta(\mathcal{P}_k, -\mathcal{P}_{\ell'+k+1})$, $\eta(\mathcal{P}_k, \mathcal{P}_{\ell'-k-3})$, $\eta(\mathcal{P}_k, \mathcal{P}_{\ell'+k-1})$, and $\eta(\mathcal{P}_k, \mathcal{P}_{\ell'-k-1})$. We analyze the non-trivial contributions from each term individually. Recall that $\eta(\mathcal{P}_k, \mathcal{P}_i) \ne 0$ if and only if $k \equiv i$ or $k + i \equiv 2p - 2 \pmod{2p}$.

- First term: if $k \equiv \ell' + k + 2 \pmod{2p}$, then $\ell' = -1$ which we excluded. The only possibility is to have $k + \ell' + k + 1 \equiv 2p - 2 \pmod{2p}$ which appears only for $k = (2p - \ell' - 3)/2$. This contributes to the trace $-\epsilon_{k+1} = -\epsilon_{p-(\ell'+1)/2} = -\epsilon_{(\ell'+1)/2}$.
- Second term: if $k + \ell' - k - 3 \equiv 2p - 2 \pmod{2p}$, then ℓ' is congruent to 1 modulo p, which is impossible. If $k \equiv \ell' - k - 3 \pmod{2p}$, then $k = (\ell' - 3)/2$. This contributes $\epsilon_{(\ell'-1)/2}$.
- Third term: the case $k \equiv \ell' + k - 1 \pmod{2p}$ is impossible. The other case $k + \ell' + k - 1 \equiv 2p - 2 \pmod{2p}$ happens when $k = p - (\ell'+1)/2$ and contributes $\epsilon_{p-(\ell'-1)/2} = \epsilon_{(\ell'-1)/2}$.
- Fourth term: the case $k + \ell' - k - 1 \equiv -2 \pmod{2p}$ is excluded. It remains the case $k \equiv \ell' - k - 1 \pmod{2p}$ which happens when $k = (\ell' - 1)/2$ and contributes $-\epsilon_{(\ell'+1)/2}$.

Collecting the non-trivial terms, we conclude that the trace of M is equal to $\epsilon_{(\ell'-1)/2} - 2\epsilon_{(\ell'+1)/2}$. If we write $q\ell' = 2ps - 1$ for some integer s, it follows that $\epsilon_{(\ell'+1)/2} = (-1)^s$ and $\epsilon_{(\ell'-1)/2} = (-1)^{s+1}$, giving $\mathrm{Tr}_{V/\mathbb{Q}}(M) = 4(-1)^{s+1}$. Combining all the above, one has

$$\sum_{\rho \in \mathcal{X}} \frac{1}{\tau_1(\rho)} = \frac{1}{2}\,\mathrm{Tr}_{V/\mathbb{Q}}(1/\tau_1) = \frac{1}{4}\,\mathrm{Tr}_{V/\mathbb{Q}}(x(\mathcal{P}_{\ell'-2} - \mathcal{P}_{\ell'})) = (-1)^{s+1}.$$

This completes the proof. $\qquad\square$

Remark 2.18. For $q = 1$, we have $\ell' = 2p - 1$ and $\mathcal{P}_{\ell'-1} = \mathcal{P}_{-2} = 1$. In this case, a direct computation shows that $\mathrm{Tr}_{V/\mathbb{Q}}(1/\tau_1) = \mathrm{Tr}_{V/\mathbb{Q}}(x^2)/2 = 2 - p$.

3. Asymptotics of the signature function

3.1. **Statement of the theorem.** Let $\Gamma(2)$ be the principal congruence subgroup of level 2 in $\mathrm{SL}_2(\mathbb{Z})$. Fix $M \in \Gamma(2)$ and define the sequence (p_n, q_n) for odd n by

$$(32) \qquad \begin{pmatrix} q_n \\ p_n \end{pmatrix} = M \begin{pmatrix} 1 \\ n \end{pmatrix}.$$

Writing the matrix M as

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2),$$

we have odd integers a, d and even integers b, c with $q_n = a + bn$ and $p_n = c + dn$. As we want to study the asymptotics of signatures along this sequence (p_n, q_n), we further impose that $0 \leq b < d$. Then for n big enough, p_n and q_n are coprime odd integers satisfying $0 < q_n < p_n$. We observe that $\lim_n \frac{q_n}{p_n} = \frac{b}{d}$ is always the quotient of an even number by an odd one. Moreover, up to a shift of n, the sequence (p_n, q_n) only depends on b/d. With such a shift, we can moreover assume $c > 0$. We denote by $\Gamma_S(2)$ be the subset of $\Gamma(2)$ consisting of matrices with $0 \leq b < d$ and $c > 0$.

To state the asymptotic of the signature $\sigma_g(\frac{q_n}{p_n})$, we construct an explicit Frobenius algebra W_M that depends on $M \in \Gamma_S(2)$ as follows. Set

$$(33) \qquad \alpha_r = \begin{cases} \lfloor \frac{rc}{d} \rfloor - \lfloor \frac{(r-1)c}{d} \rfloor & \text{for } 0 < r < d \\ \lfloor \frac{rc}{d} \rfloor - \lfloor \frac{(r-1)c}{d} \rfloor - 1 & \text{for } r = d \end{cases}.$$

Note that $\alpha_r \geq 0$ and $\alpha_1 + \cdots + \alpha_d = c - 1$. We define

$$N(\alpha_r) = i^{\alpha_r} \begin{pmatrix} \alpha_r + 1 + x & i^{-1}(\alpha_r + x) \\ i^{-1}(\alpha_r + x) & -\alpha_r + 1 - x \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}[i][x])$$

and set

$$\begin{pmatrix} -iH_1 \\ H_2 \end{pmatrix} = N(\alpha_1)\overline{N(\alpha_2)}N(\alpha_3)\cdots\overline{N(\alpha_{d-1})}N(\alpha_d) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

From the fact that $\alpha_1 + \cdots + \alpha_d = c - 1$ is odd, one checks that both H_1 and H_2 are integer polynomials of degree d. Similarly, we define an integer polynomial H_3 of degree b as

$$\begin{pmatrix} H_3 \\ * \end{pmatrix} = N(\alpha_1)\overline{N(\alpha_2)}N(\alpha_3)\cdots\overline{N(\alpha_b)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Provided that

(H) H_1 has no multiple roots and no common root with H_3,

we define the semi-simple Frobenius algebra W_M of degree d by

$$W_M = \mathbb{Q}[x]/(H_1) \quad \text{with} \quad \Omega_{W_M} = -H_1'H_2H_3^{-2}.$$

Theorem 3.1. *Fix $M \in \Gamma_S(2)$ and let (p_n, q_n) be the sequence defined in (32). Then for any $g \geq 2$,*

(1) the signature $\sigma_g(\frac{q_n}{p_n})$ is a polynomial in n, and

(2) under the assumption (H), its asymptotic satisfies

$$\sigma_g\left(\frac{q_n}{p_n}\right) \underset{n\to\infty}{\sim} 2\left(\frac{n^3}{2\pi^2}\right)^{g-1} \zeta(2g-2)\, \mathrm{Tr}_{W_M/\mathbb{Q}}\left(\Omega_{W_M}^{g-1}\right).$$

3.2. **Reciprocal Frobenius algebra.** Fix $M \in \Gamma_S(2)$ and (p_n, q_n) as in Theorem 3.1. As $dq_n - bp_n = 1$, d is the inverse of q_n modulo p_n. Let us denote by V_n^* the Frobenius algebra associated to (p_n, q_n) and V_n the one associated to (p_n, d). Here we adopt the notations from Section 2.1, except that the subscript n is added, and we attach $*$ to the algebra associated with (p_n, q_n). Recall from Section 2.4 that they are isomorphic as algebras and denoting by x^*, x their generators, one has the relations:
$$x^* = x\mathcal{P}_{q_n-1}, \quad x = x^*\mathcal{P}_{d-1}^*$$
together with (see Equation (29))
$$\Omega_n^* = \Omega_n \mathcal{P}_{q_n-1}^{-2}.$$
It follows that
$$\sigma_g\left(\frac{q_n}{p_n}\right) = \mathrm{Tr}_{V_n^*/\mathbb{Q}}\left((\Omega_n^*)^{g-1}\right) = \mathrm{Tr}_{V_n/\mathbb{Q}}\left((\Omega_n\mathcal{P}_{q_n-1}^{-2})^{g-1}\right).$$

Proposition 3.2. One has
$$\begin{pmatrix} P_{p_n-1} \\ P_{p_n-2} \end{pmatrix} = \begin{pmatrix} (-1)^{d+1}X & 1 \\ 1 & 0 \end{pmatrix}^{n+\alpha_d} \cdots \begin{pmatrix} -X & 1 \\ 1 & 0 \end{pmatrix}^{n+\alpha_2} \begin{pmatrix} X & 1 \\ 1 & 0 \end{pmatrix}^{n+\alpha_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
and
$$\begin{pmatrix} P_{q_n-1} \\ P_{q_n-2} \end{pmatrix} = \begin{pmatrix} (-1)^{b+1}X & 1 \\ 1 & 0 \end{pmatrix}^{n+\alpha_b} \cdots \begin{pmatrix} -X & 1 \\ 1 & 0 \end{pmatrix}^{n+\alpha_2} \begin{pmatrix} X & 1 \\ 1 & 0 \end{pmatrix}^{n+\alpha_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$
Here the integers α_r, defined in (33), are independent of n.

Proof. We first consider the ϵ-sequence $\epsilon_k = (-1)^{\lfloor \frac{dk}{dn+c} \rfloor}$ for $1 \leq k \leq p_n - 1$ associated to the Frobenius algebra V_n. It is a sequence of d alternating blocks of 1s and -1s. Precisely, to find the last term of the r-th block for $1 \leq r \leq d$, we compare dk and $r(dn + c)$: writing $k = rn + s$, this amounts to compare ds and rc. The last term of the r-th block is then $k_r = rn + \lfloor \frac{rc}{d} \rfloor$. In particular, the last term is $k_d = nd + c = p_n$. As we want to stop the sequence at $p_n - 1$, we set instead $k_d = p_n - 1$. We also set $k_0 = 0$ for notational convenience. Note that $k_r - k_{r-1} = n + \alpha_r$.

We now recall that $P_k = \epsilon_k X P_{k-1} + P_{k-2}$. Equivalently,
$$\begin{pmatrix} P_k \\ P_{k-1} \end{pmatrix} = \begin{pmatrix} \epsilon_k X & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_{k-1} \\ P_{k-2} \end{pmatrix}.$$
By the definition of k_r, one has $\epsilon_k = (-1)^r$ for all $k_{r-1} < k \leq k_r$. Since $k_r - k_{r-1} = n + \alpha_r$, we obtain the first formula.

On the other hand, as $ad - bc = 1$, we have $\frac{bc}{d} = a - \frac{1}{d}$ and thus $k_b = bn + \lfloor \frac{bc}{d} \rfloor = bn + a - 1 = q_n - 1$. It follows that P_{q_n-1} is obtained from the same formula as P_{p_n-1} by taking only the first b blocks. This proves the proposition. $\square$

Set $X = i(t + t^{-1})$ for a formal variable t. Then a direct computation shows that

(34) $$\begin{pmatrix} X & 1 \\ 1 & 0 \end{pmatrix}^k = \frac{i^k}{t - t^{-1}} \begin{pmatrix} t^{k+1} - t^{-k-1} & -i(t^k - t^{-k}) \\ -i(t^k - t^{-k}) & -(t^{k-1} - t^{-k+1}) \end{pmatrix}.$$

Changing the sign of X amounts to changing i to $-i$, hence

(35) $$\begin{pmatrix} -X & 1 \\ 1 & 0 \end{pmatrix}^k = \frac{i^{-k}}{t - t^{-1}} \begin{pmatrix} t^{k+1} - t^{-k-1} & i(t^k - t^{-k}) \\ i(t^k - t^{-k}) & -(t^{k-1} - t^{-k+1}) \end{pmatrix}.$$

Combining the above formulas with Proposition 3.2, we obtain the following proposition, which produces three two-variable polynomials. We will need more precise information about these polynomials, which we defer to the technical Lemma 3.7.

Proposition 3.3. There exist polynomials $Q_M, R_M, S_M \in \mathbb{Z}[U^{\pm 1}, V^{\pm 1}]$, depending only on $M \in \Gamma_S(2)$, such that

$$\begin{cases} P_{p_n-1}(it + it^{-1})(t - t^{-1})^d = i^{n+1}Q_M(t, t^n) \\ P_{p_n-2}(it + it^{-1})(t - t^{-1})^d = i^n R_M(t, t^n) \\ P_{q_n-1}(it + it^{-1})(t - t^{-1})^b = S_M(t, t^n) \end{cases}$$

for any odd n.

Proof. It follows from Proposition 3.2 and Equations (34), (35) that $P_{p_n-1}(it+it^{-1})$ can be expressed as a polynomial in t and t^n with coefficients in $\mathbb{Z}[i]$, multiplied by $i^{\sum_r(-1)^{r+1}(n+\alpha_r)}(t - t^{-1})^d = i^{n+\kappa}(t - t^{-1})^d$ where $\kappa = \sum_r(-1)^{r+1}\alpha_r$. As $\kappa \equiv \sum_r \alpha_r = c - 1 \equiv 1 \pmod 2$, one gets $\pm i^{n+1}(t - t^{-1})^d$. Remaining powers of i appear exactly in non-diagonal coefficients, and this property is stable by product. This proves the desired identity for P_{p_n-1} and P_{p_n-2}. For P_{q_n-1}, there is no term of the form i^n and the remaining power of i is $\alpha_1 - \alpha_2 + \cdots + \alpha_b \equiv a - 1 \equiv 0$ modulo 2. $\square$

3.3. The signature is a polynomial in n.
Let us prove in this section the first item of Theorem 3.1.

From [Mar23], one has $\Omega_n = -P_{p_n-2}P'_{p_n-1}$, hence

$$(36) \qquad \sigma_g\left(\frac{q_n}{p_n}\right) = \mathrm{Tr}_{V_n/\mathbb{Q}}\left(\left(-P_{q_n-1}^{-2}P_{p_n-2}P'_{p_n-1}\right)^{g-1}\right).$$

Recall from the Residue Formula (see e.g. [GH78]) that given any polynomial P with simple roots and any $Q \in \mathbb{Q}[x]$ one has

$$\mathrm{Tr}_{\mathbb{Q}[x]/P}(Q) = \sum_{P(x)=0} Q(x) = \sum_{x\in\mathbb{C}}\mathrm{Res}_x(\frac{QP'}{P}dx) = -\mathrm{Res}_{x=\infty}(\frac{QP'}{P}dx).$$

We cannot apply this directly to the computation of σ_g because we need to invert P_{q_n-1}, which creates new poles and invalidates the residue formula. To circumvent this problem, we use

$$\frac{x^*}{x} = P_{q_n-1}(x), \quad \frac{x}{x^*} = P^*_{d-1}(x^*)$$

from which we deduce

$$P_{q_n-1}(x)^{-1} = P^*_{d-1}(x^*) = P^*_{d-1}(xP_{q_n-1}(x)).$$

To simply the notation, let us set $Q(x) = P^*_{d-1}(xP_{q_n-1}(x))$. In order to apply Proposition 3.3 , we perform the change of variables $x = it + it^{-1}$. Since this maps $t = 0$ to $x = \infty$ and is non-ramified there, we can replace x by $it + it^{-1}$, $x = \infty$ by $t = 0$, and dx by $\frac{i(t-t^{-1})dt}{t}$. It follows that $\sigma_g(\frac{q_n}{p_n})$ is equal to

$$\mathrm{Res}_{t=0}\left((-1)^g\frac{Q(it + it^{-1})^{2g-2}P_{p_n-2}(it + it^{-1})^{g-1}P'_{p_n-1}(it + it^{-1})^g}{P_{p_n-1}(it + it^{-1})}\frac{i(t - t^{-1})dt}{t}\right).$$

On the other hand, from Proposition 3.3, one computes

$$(37) \qquad P'_{p_n-1}(it + it^{-1}) = i^n \left(\frac{t \frac{d}{dt} Q_M(t, t^n)}{(t - t^{-1})^{d+1}} - \frac{d(t + t^{-1}) Q_M(t, t^n)}{(t - t^{-1})^{d+2}} \right).$$

Substituting this into the above residue expression of $\sigma_g(\frac{q_n}{p_n})$, we obtain

$$(38) \qquad \sigma_g \left(\frac{q_n}{p_n} \right) = \mathrm{Res}_{t=0} \left(\frac{R(t, t^n)}{Q_M(t, t^n)(t - t^{-1})^m} \frac{dt}{t} \right)$$

for some integer m independent of n and some $R \in \mathbb{Z}[U^{\pm 1}, V^{\pm 1}]$.

Lemma 3.4. Let R and $S \in \mathbb{Z}[U^{\pm 1}, V^{\pm 1}]$ be two polynomials and suppose that the lowest V-coefficient of S is $(U - U^{-1})^\ell$ modulo $\mathbb{Z}[U^{\pm 1}]^\times$. Then

$$\mathrm{Res}_{t=0} \left(\frac{R(t, t^n)}{S(t, t^n)(t - t^{-1})^m} \frac{dt}{t} \right)$$

is a polynomial in n of order less than $m + \ell$.

Proof. It is sufficient to consider $R(U, V) = U^a V^b$ by the linearity of the residue. Let d be the lowest V-degree of S. Then, by the assumption, we can write

$$S(U, V) = \epsilon U^c (U - U^{-1})^\ell V^d (1 - V \tilde{S}(U, V))$$

where $\epsilon = \pm 1$ and $\tilde{S} \in \mathbb{Z}[U^{\pm 1}, V][[U]]$, hence

$$\frac{R(t, t^n)}{S(t, t^n)(t - t^{-1})^m} = \frac{t^{a+nb-c-nd+m+\ell}}{\epsilon(1 - t^n \tilde{S}(t, t^n))(t^2 - 1)^{m+\ell}}$$

$$= \epsilon^{-1} t^{a-c+m+\ell+n(b-d)} \sum_{i,j \geq 0} t^{ni} \tilde{S}(t, t^n)^i t^{2j} (-1)^{m+\ell} \binom{j + m + \ell - 1}{m + \ell - 1}.$$

We need to pick the constant coefficient in t: the key point is that only a finite number of $i's$ can contribute. Fix a value of i and expand $\tilde{S}(t, t^n)^i$. All terms have the form

$$C \sum_{j \geq 0} t^{A+Bn} t^{2j} \binom{j + m + \ell - 1}{m + \ell - 1}.$$

If $B < 0$ and $A + Bn$ is even, the constant term of this series is $\binom{\frac{A+Bn}{2} + m + \ell - 1}{m + \ell - 1}$. This is a polynomial in n (which is assumed to be odd) of degree $m + \ell - 1$. This completes the proof. $\qquad \square$

Lemma 3.7 below shows that Lemma 3.4 can be applied to Equation (38). It follows that the signature $\sigma_g(\frac{q_n}{p_n})$ is a polynomial in n, which proves the first part of Theorem 3.1.

3.4. **The leading term of the signature.** We now prove the second part of Theorem 3.1, starting again from Equation (36) that we rewrite in the following form:

$$\sigma_g \left(\frac{q_n}{p_n} \right) = \sum_{x \in Z_n} \left(\frac{-P_{p_n-2}(x) P'_{p_n-1}(x)}{P_{q_n-1}(x)^2} \right)^{g-1}$$

where $Z_n = \{x \in \mathbb{C} \mid P_{p_n-1}(x) = 0\}$. From the relation between P_{p_n-1} and Q_M in Proposition 3.3, one sees a direct relationship between the zeros of P_{p_n-1} and the zeros of Q_M. To be more precise, we define

$$\mathcal{C} = \{(u, v) \in (\mathbb{C}^*)^2 \mid Q_M(u, v) = 0\}, \quad \mathcal{C}[n] = \{(u, v) \in \mathcal{C} \setminus \{\pm 1\}^2 \mid v = u^n\}.$$

The map $\mathcal{C}[n] \to Z_n$ given by $(u, v) \mapsto i(u + u^{-1})$ is surjective and 2-to-1: two preimages are related by the involution $(u, v) \mapsto (u^{-1}, v^{-1})$.

Lemma 3.5. The set $\bigcup_{n \geq 1} \mathcal{C}[n]$ is relatively compact in $\mathcal{C}$.

Proof. This amounts to showing that points in $\mathcal{C}[n]$ cannot accumulate to ideal points of $\mathcal{C}$. Recall the well-known fact that such ideal points correspond to zeroes of the side polynomials of Q_M, that is, the one-variable polynomials associated with the sides of the Newton polygon of Q_M. To see this, fix an ideal point of $\mathcal{C}$ and take Puiseux coordinates of the form $u = s^p$ and $v = s^q F(s)$, where p, q are coprime and F is a convergent series with $F(0) \neq 0$. As $Q_M(s^p, s^q F(s)) = 0$ identically, degree considerations imply that the linear form $(i, j) \mapsto pi + qj$ should support a face of the Newton polygon of Q_M.

By Lemma 3.7 below, this polygon is an explicit parallelogram, so the only possibilities are $(p, q) \in \{(0, 1), (0, -1), (d, -c), (-d, c)\}$. In the first two cases, the Puiseux coordinates have the form $u = s^{\pm 1}$ and $v = F(s)$. As $F(0) \neq 0$, it is impossible to have $F(s) = s^{\pm n}$ in a neighborhood of 0. In the latter two cases, we have $u = s^{\pm d}$ and $v = s^{\mp c} F(s)$. Again, the equation $v = u^n$ forces $F(s) = s^{\pm nd \pm c}$, which is impossible for s sufficiently close to 0. This completes the proof. $\square$

Recall the formula $P_{p_n - 1}(it + it^{-1})(t - t^{-1})^d = i^{n+1} Q_M(t, t^n)$ of Proposition 3.3. Differentiating it at $(t, t^n) \in \mathcal{C}[n]$, one has

$$(39) \quad P'_{p_n - 1}(it + it^{-1}) i(t - t^{-1})^{d+1} = i^{n+1} t \partial_U Q_M(t, t^n) + i^{n+1} n t^n \partial_V Q_M(t, t^n).$$

Hence, setting

$$F_n(u, v) = (u - u^{-1})^{2b - 2d - 1} \frac{R_M(u, v)}{S_M(u, v)^2} \left(u \partial_U Q_M(u, v) + n v \partial_V Q_M(u, v) \right)$$

we can write

$$\sigma_g \left(\frac{q_n}{p_n} \right) = \frac{1}{2} \sum_{(u, v) \in \mathcal{C}[n]} F_n(u, v)^{g-1}.$$

Lemma 3.6. For any neighborhood W of $\{\pm 1\}^2 \subset \mathcal{C}$, there is a constant $C > 0$ such that $|F_n(u, v)| \leq Cn$ for any $(u, v) \in \mathcal{C}[n] \setminus W$.

Proof. Let us write $x = it + it^{-1}$. As $P_{p_n - 2}(x)^2 = -1$ for any $x \in Z_n$ (see Proposition 2.4), this term can readily be neglected. As (t, t^n) belongs to a compact subset of $\mathcal{C}$, the polynomials $U \partial_U Q_M(U, V)$ and $V \partial_V Q_M(U, V)$ are uniformly bounded on $\mathcal{C}[n]$. Moreover, by the third point of Lemma 3.7, the set of $(u, v) \in \mathcal{C}$ with $u = \pm 1$ is reduced to $\{\pm 1\}^2$. This means that there exists $\epsilon > 0$ such that for all $(u, v) \in \mathcal{C} \setminus W$ one has $|u^2 - 1| > \epsilon$. Combining these with Equation (39), we deduce that there is a constant $C > 0$ such that $|P'(iu + iu^{-1})| \leq Cn$ for all $(u, v) \in \mathcal{C}[n] \setminus W$.

It remains to deal with $P_{q_n - 1}(x)^{-2}$. If we first replace it with $P^*_{d-1}(x P_{q_n - 1}(x))^2$, we are reduced to showing that $P_{q_n - 1}(x)$ is uniformly bounded. To this aim, recall from Proposition 3.3 that $P_{q_n - 1}(x)(t - t^{-1})^b = S_M(t, t^n)$. The same argument as for $P'_{q_n - 1}$ shows that $P_{q_n - 1}(it + it^{-1})$ is uniformly bounded for $(t, t^n) \in \mathcal{C}[n] \setminus W$, proving the lemma. $\square$

Since the cardinality of $\mathcal{C}[n]$ is linear in n, from the above lemma, one has

$$\sum_{(u, v) \in \mathcal{C}[n] \setminus W} F_n(u, v)^{g-1} = O(n^g)$$

which is negligible compared to the expected n^{3g-3} for $g \geq 2$. Therefore, henceforth we concentrate on $\mathcal{C}[n] \cap W$.

Let $W^{\pm}$ be a neighborhood $(1, \pm 1) \in \mathcal{C}$. For parity reasons, the neighborhood of $(-1, \pm 1)$ will contribute in the same way to the sum. The previous lemma implies that

$$\sigma_g\left(\frac{q_n}{p_n}\right) = \sum_{(u,v) \in \mathcal{C}[n] \cap (W^+ \cup W^-)} F_n(u,v)^{g-1} + O(n^g)$$

Let $\overline{\mathcal{C}}$ be the normalization of $\mathcal{C}$ and suppose that the first projection $\overline{\mathcal{C}} \to \mathbb{C}$ is not ramified over 1. This implies that there exist $d - 1$ branches of $\mathcal{C}$ with Puiseux coordinates of the form $u = e^s, v = e^{F_j(s)}$ with $F_j(0) = 0$ and $F_j'(0) \neq 0$. By the symmetries of Q_M given in Lemma 3.7, the branches over (ϵ_u, ϵ_v) are parametrized by $u = \epsilon_u e^s, v = \epsilon_v e^{F_j(s)}$ for $\epsilon_u, \epsilon_v \in \{\pm 1\}$.

Recall that an element of $\mathcal{C}[n]$ in the j-th branch satisfies $\pm e^{F_j(s)} = \pm e^{ns}$. Equivalently, $F_j(s) = ns - i\pi k$ for some $k \in \mathbb{Z}$. As $F_j(s)$ is close to 0, there is a unique value of $k \neq 0$ which can solve this equation, reciprocally given $k \neq 0$, there is a unique solution $s_j^k(n)$ of this equation that we can expand into powers of $\frac{1}{n}$ in the form

$$s_j^k(n) = \frac{i\pi k}{n}\left(1 + F_j'(0)\frac{1}{n} + O(\frac{1}{n^2})\right).$$

Along such a sequence, we have $u = e^{\frac{i\pi k}{n} + O(\frac{1}{n^2})}$ and $v = (-1)^k e^{\frac{i\pi k}{n}F_j'(0) + O(\frac{1}{n^2})}$. Depending on the parity of k, this will converge to $(1,1)$ or $(1,-1)$. By the fourth point of Lemma 3.7, one sees that $F_j'(0)$ is a root of H_1. As H_1 has simple roots by assumption, it justifies our assumption on the ramification of $\mathcal{C}$ over 1.

One can reduce to the case $k > 0$: the case $k < 0$ corresponds to the involution $(u,v) \mapsto (u^{-1}, v^{-1})$ and gives the same contribution.

Let us compute the behavior of $F_n(u,v) \sim n(u - u^{-1})^{2b-2d-1}\frac{R_M(u,v)v\partial_V Q_M(u,v)}{S_M(u,v)^2}$ on this sequence.

- Directly, $(u - u^{-1})^{2b-2d-1} \sim \left(\frac{2i\pi k}{n}\right)^{2b-2d-1}$.
- By Lemma 3.7 point (6): $S_M(u,v)^2 \sim 2^{2b}\left(\frac{i\pi k}{n}\right)^{2b}H_3(F_j'(0))^2$.
- By Lemma 3.7 point (5): $R_M(u,v) \sim 2^d\left(\frac{i\pi k}{n}\right)^d H_2(F_j'(0))$.
- By Lemma 3.7 point (4): $v\partial_V Q_M(u,v) \sim 2^d\left(\frac{i\pi k}{n}\right)^{d-1}H_1'(F_j'(0))$.

Putting everything together gives

$$F_n(e^{s_j^k(n)}, e^{ns_j^k(n)}) \sim -\frac{n^3}{2(\pi k)^2}\frac{H_1'(F_j'(0))H_2(F_j'(0))}{H_3(F_j'(0))^2}.$$

Summing over $k > 0$ and over j gives the final formula

$$\sigma_g\left(\frac{q_n}{p_n}\right) \sim 2\left(\frac{n^3}{2\pi^2}\right)^{g-1}\zeta(2g-2)\sum_j\left(-\frac{H_1'(F_j'(0))H_2(F_j'(0))}{H_3(F_j'(0))^2}\right)^{g-1}.$$

3.5. **The main lemma.** Here we state and prove the main lemma used in the last two subsections.

Lemma 3.7. The polynomial Q_M satisfies the symmetries

$$Q_M(-U,V) = Q_M(U,V) = -Q_M(U,-V) = -Q_M(U^{-1}, V^{-1})$$

together with the following properties:

(1) The terms with highest (resp. lowest) power of V are $\pm V^d (U + U^{-1})^{d-1} U^c$ and $\pm V^{-d} (U + U^{-1})^{d-1} U^{-c}$ respectively.

(2) Its Newton polygon is the parallelogram of vertices $\pm (d-1, 0) \pm (c, d)$.

(3) $Q_M(\pm 1, V) = \pm 2^{d-1} (V - V^{-1})^d$.

(4) $Q_M(e^u, e^v) = 2^d u^d H_1(v/u)$ modulo terms in (u, v) of total order $< d$.

Similarly, the polynomials R_M and S_M satisfy

(5) $R_M(e^u, e^v) = 2^d u^d H_2(v/u)$ modulo terms in (u, v) of total order $< d$.

(6) $S_M(e^u, e^v) = 2^b u^b H_3(v/u)$ modulo terms in (u, v) of total order $< b$.

Proof. Before all, we provide an explicit formula for Q_M. Set $X = it + it^{-1}, U = t, V = t^n$, and $N_k = \begin{pmatrix} i(t + t^{-1}) & 1 \\ 1 & 0 \end{pmatrix}^k (t - t^{-1})$. We compute

$$N_{n+\alpha} = \begin{pmatrix} i^{n+\alpha} t^{n+\alpha+1} - i^{n+\alpha} t^{-n-\alpha-1} & i^{n+\alpha-1} t^{n+\alpha} - i^{n+\alpha-1} t^{-n-\alpha} \\ i^{n+\alpha-1} t^{n+\alpha} - i^{n+\alpha-1} t^{-n-\alpha} & i^{n+\alpha-2} t^{n+\alpha-1} - i^{n+\alpha-2} t^{-n-\alpha+1} \end{pmatrix}$$

$$= i^{n+\alpha} \begin{pmatrix} U^{\alpha+1} V - U^{-\alpha-1} V^{-1} & i^{-1} U^\alpha V - i^{-1} U^{-\alpha} V^{-1} \\ i^{-1} U^\alpha V - i^{-1} U^{-\alpha} V^{-1} & -U^{\alpha-1} V + U^{-\alpha+1} V^{-1} \end{pmatrix}$$

If we index the elements of these matrices by signs, say $\eta, \xi \in \{\pm 1\}$, the entry η, ξ of the matrices above are given by

$$(N_{n+\alpha})_{\eta, \xi} = i^{n+\alpha-1+\frac{\eta+\xi}{2}} \left(U^{\alpha + \frac{\eta+\xi}{2}} V - U^{-\alpha - \frac{\eta+\xi}{2}} V^{-1} \right).$$

It follows that setting $\mathcal{N} = N_{n+\alpha_1} \overline{N_{n+\alpha_2}} \cdots \overline{N_{n+\alpha_{d-1}}} N_{n+\alpha_d}$, its entry $\mathcal{N}_{\xi_0, \xi_d}$ equals:

$$\sum_{\xi_1, \ldots, \xi_{d-1}} i^{\sum_{r=1}^d (-1)^{r+1}(n + \alpha_r - 1 + \frac{\xi_{r-1} + \xi_r}{2})} \prod_{r=1}^d \left(U^{\alpha_r + \frac{\xi_{r-1} + \xi_r}{2}} V - U^{-(\alpha_r + \frac{\xi_{r-1} + \xi_r}{2})} V^{-1} \right)$$

We compute that the power of i reduces to $n + \kappa - 1 + \frac{\xi_d + \xi_0}{2}$ where $\kappa = \sum (-1)^{r+1} \alpha_r$ and the product can further be expanded as follows:

$$\sum_{\xi_1, \ldots, \xi_{d-1} \in \{\pm 1\}} \sum_{\eta_1, \ldots, \eta_d \in \{\pm 1\}} \left(\prod_{r=1}^d \eta_r \right) U^{\sum_{r=1}^d \eta_r (\alpha_r + \frac{\xi_{r-1} + \xi_r}{2})} V^{\sum_{r=1}^d \eta_r}.$$

Finally, we sum over $\xi_1, \ldots, \xi_{d-1}$ to obtain

(40)
$$\sum_{\eta_1, \ldots, \eta_d \in \{\pm 1\}} \left(\prod_{r=1}^d \eta_r \right) U^{\sum_{r=1}^d \eta_r \alpha_r + \frac{\eta_1 \xi_0 + \eta_d \xi_d}{2}} \prod_{r=1}^{d-1} \left(U^{\frac{\eta_r + \eta_{r+1}}{2}} + U^{-\frac{\eta_r + \eta_{r+1}}{2}} \right) V^{\sum_{r=1}^d \eta_r}.$$

To recover $Q_M(U, V)$ from this formula, we simply take $\xi_0 = \xi_d = +1$ and multiply the result by the sign $i^{\kappa-1}$. It follows readily from this expression that Q_M contains only even powers of U and odd powers of V. The symmetry $Q_M(U^{-1}, V^{-1}) = -Q_M(U, V)$ is clear from the change $\eta_r \mapsto -\eta_r$.

From this formula, one sees that the maximal power of V in Q_M is d and corresponds to putting $\eta_r = 1$ for all r: its coefficient is

$$i^{\kappa-1} U^{\sum \alpha_r + 1} (U + U^{-1})^{d-1} = i^{\kappa-1} U^c (U + U^{-1})^{d-1}.$$

The case of the minimal coefficient is similar and gives $i^{\kappa-1} U^{-c} (U + U^{-1})^{d-1}$.

Let us prove the second statement, we consider the linear form $L : \mathbb{Z}^2 \to \mathbb{Q}$ given by $L(i, j) = i - \frac{c}{d} j$. It takes the values $\pm (d-1)$ on the sides of the parallelogram expected as the Newton polygon of Q_M. Hence one needs to prove that any non

trivial monomial $U^i V^j$ appearing in Q_M satisfies $|L(i,j)| \leq d-1$. It reduces to taking a sequence $\eta_1, \ldots, \eta_d \in \{\pm 1\}$ and compute

$$L\Big(\sum_{r=1}^{d} \eta_r \alpha_r + \frac{\eta_1 + \eta_d}{2} + \sum_{r=1}^{d-1} \pm \frac{\eta_r + \eta_{r-1}}{2}, \ \sum_{r=1}^{d} \eta_r \Big)$$
$$= \sum_{r=1}^{d} \eta_r \alpha_r + \frac{\eta_1 + \eta_d}{2} + \sum_{r=1}^{d-1} \pm \frac{\eta_r + \eta_{r+1}}{2} - \Big(\sum_{r=1}^{d} \eta_r \Big) \frac{c}{d}.$$

Let S be the set of indices $0 < r < d$ for which $\eta_r \neq \eta_{r+1}$: one has

$$\Big| \sum_{r=1}^{d-1} \pm \frac{\eta_r + \eta_{r+1}}{2} \Big| \leq d - 1 - \#S$$

so that we are reduced to showing $|\sum \eta_r \alpha_r + \frac{\eta_1 + \eta_d}{2} - (\sum \eta_r) \frac{c}{d}| \leq \#S$. We do a discrete integration by parts in each sum.

On one hand: $\sum_{r=1}^{d} \eta_r \alpha_r = \sum_{r=1}^{d} \eta_r (\beta_r - \beta_{r-1}) = \sum_{r=0}^{d-1} (\eta_r - \eta_{r+1}) \beta_r + \eta_d (c-1)$ where $\beta_r = \lfloor \frac{rc}{d} \rfloor$ for $0 \leq r < d$ and $\beta_d = c - 1$. The sequence $\eta_r - \eta_{r+1}$ vanishes if $r \notin S$ so that this sum reads $2 \sum_{r \in S} \eta_r \beta_r + \eta_d (c-1)$.

On the other hand: $\sum_{r=1}^{d} \eta_r = \sum_{r=1}^{d-1} r(\eta_r - \eta_{r+1}) + d\eta_d = 2 \sum_{r \in S} r \eta_r + d\eta_d$. We are then reduced to

$$\Big| 2 \sum_{r \in S} \eta_r \Big(\beta_r - r\frac{c}{d} \Big) + \frac{\eta_1 - \eta_d}{2} \Big| \leq \#S.$$

As $-1 \leq \beta_r - \frac{rc}{d} \leq 0$ and the sequence η_r is alternating, the sum of two consecutive terms does not exceed 1 in absolute value. If $\eta_1 = \eta_d$, $\#S$ is even and the result follows. When $\eta_1 \neq \eta_d$ then $\#S$ is odd: either the first or the last term of the sum has a sign opposite to $\frac{\eta_1 - \eta_d}{2}$: again their sum does not exceed 1 in absolute value and the argument repeats and the second point is proved. We also observe that in case of equality, the sequence of ξs and the sequence of ηs should be constant: this shows that the non trivial coefficient on the slanted sides occur only at vertices.

Let us prove the third statement, we simply put $U = 1$ in Equation (40). We also observe from the expression just above that powers of U in Q_M have the form $\sum_{r=1}^{d} \eta_r (\alpha_r + \frac{\xi_{r-1} + \xi_r}{2})$ which is congruent modulo 2 to $\sum_{r=1}^{d} \alpha_r + \frac{\xi_0 + \xi_d}{2} = c$. Hence Q has only even powers of U and the statement follows.

Finally the last three statements are clear once we have observed that writing $U = e^u, V = e^v$ one has

$$i^{n+\alpha} \begin{pmatrix} U^{\alpha+1}V - U^{-\alpha-1}V^{-1} & i^{-1}U^\alpha V - i^{-1}U^{-\alpha}V^{-1} \\ i^{-1}U^\alpha V - i^{-1}U^{-\alpha}V^{-1} & -U^{\alpha-1}V + U^{-\alpha+1}V^{-1} \end{pmatrix}$$
$$= i^{n+\alpha} \begin{pmatrix} 2\sinh((\alpha+1)u + v) & \frac{2}{i}\sinh(\alpha u + v) \\ \frac{2}{i}\sinh(\alpha u + v) & -2\sinh((\alpha-1)u + v) \end{pmatrix}$$
$$= 2i^{n+\alpha} \begin{pmatrix} (\alpha+1)u + v & i^{-1}(\alpha u + v) \\ i^{-1}(\alpha u + v) & -(\alpha-1)u - v \end{pmatrix} + \text{higher order terms.}$$

Taking a product of d such matrices, the first order will be given by the product of the first orders. The formulas $(4, 5, 6)$ are direct consequences of the definition of H_1, H_2, H_3. $\qquad\square$

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Département de mathématiques et applications, Ecole Normale Supérieure, Université PSL, CNRS, Sorbonne Université, 75005 Paris, France

https://webusers.imj-prg.fr/~julien.marche/

Email address: julien.marche@ens.psl.eu

Department of Mathematics, Chonnam National University, Gwangju, South Korea

http://sites.google.com/view/seokbeom

Email address: sbyoon15@gmail.com