THE FIRST JOHNSON SUBGROUPS ACT ERGODICALLY ON SU_2-CHARACTER VARIETIES

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Abstract. We show that the first Johnson subgroup of the mapping class group of a surface Σ of genus greater than one acts ergodically on the moduli space of representations of $\pi_1(\Sigma)$ in SU_2. Our proof relies on a local description of the latter space around the trivial representation and on the Taylor expansion of trace functions.

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1. Introduction and statements

Let Σ be a compact orientable surface and G a compact semi-simple Lie group. The space of all homomorphisms $\mathrm{Hom}(\pi_1(\Sigma), G)$ will be denoted $\mathcal{R}(\Sigma, G)$. We will consider then the quotient $\mathcal{M}(\Sigma, G) = \mathcal{R}(\Sigma, G)/G$ by the conjugacy G-action. Let $\mathrm{Mod}(\Sigma)$ denote the mapping class group of the surface Σ, namely the group of isotopy classes of homeomorphisms preserving the orientation of Σ fixing the boundary. The group $\mathrm{Aut}^+(\pi_1(\Sigma))$ of automorphisms of $\pi_1(\Sigma)$ preserving the orientation acts on $\mathcal{R}(\Sigma, G)$ by left composition and induces an action of the mapping class group $\mathrm{Mod}(\Sigma)$ on $\mathcal{M}(\Sigma, G)$.

Recall that, if Σ is closed and orientable then the dense and open subset of non-singular points of $\mathcal{M}(\Sigma, \mathrm{SU}_2)$ has a $\mathrm{Mod}(\Sigma)$-invariant symplectic structure, which was defined by Goldman in [2]. This induces a volume form on the non-singular part of $\mathcal{M}(\Sigma, \mathrm{SU}_2)$ and thus a $\mathrm{Mod}(\Sigma)$-invariant measure on $\mathcal{M}(\Sigma, \mathrm{SU}_2)$.

The main purpose of this paper is to understand the dynamical properties of the $\mathrm{Mod}(\Sigma)$ action, with respect to this invariant measure. The first result in this direction is due to Goldman (see [3]) who proved that $\mathrm{Mod}(\Sigma)$ acts ergodically on $\mathcal{M}(\Sigma, \mathrm{SU}_2)$. This ergodicity statement was further extended to all compact connected Lie groups by Pickrell and Xia ([11]).

Definition 1.1. The first Johnson subgroup $\mathcal{K}(\Sigma)$ is the subgroup of $\mathrm{Mod}(\Sigma)$ generated by the Dehn twists along separating simple curves on Σ.

Johnson proved (see [9]) that $\mathcal{K}(\Sigma)$ is a normal subgroup of infinite index of the Torelli subgroup of $\mathrm{Mod}(\Sigma)$, which is the subgroup of mapping classes of homeomorphisms acting trivially on the surface homology.

Theorem 1.1. *Let Σ be a closed orientable surface of genus $g \geq 2$. Then the Johnson subgroup $\mathcal{K}(\Sigma)$ acts ergodically on $\mathcal{M}(\Sigma, \mathrm{SU}_2)$.*

This solves affirmatively Conjecture 1.8 of Goldman from [3]. Previously the conjecture has been proved only for surfaces of genus $g = 1$ with two boundary components (for almost all boundary monodromies) following a different approach, by Goldman and Xia (see [8]).

Remark 1.1. It is sufficient to show that the lift of $\mathcal{K}(\Sigma)$ in $\mathrm{Aut}^+(\pi_1(\Sigma))$ acts ergodically on $\mathcal{R}(\Sigma, \mathrm{SU}_2)$, for a closed orientable surface Σ of genus $g \geq 2$.

The proof can be extended with only minor modifications to the case where $G = \mathrm{SU}_2 \times \mathrm{SU}_2 \times \cdots \times \mathrm{SU}_2$ is the direct product of k factors. Therefore we obtain:

Theorem 1.2. *Let Σ be a closed orientable surface of genus $g \geq 2$. Then the Johnson subgroup $\mathcal{K}(\Sigma)$ acts ergodically on $\mathcal{M}(\Sigma, \mathrm{SU}_2) \times \mathcal{M}(\Sigma, \mathrm{SU}_2) \times \cdots \times \mathcal{M}(\Sigma, \mathrm{SU}_2)$, by means of the diagonal action. In particular, the action of $\mathcal{K}(\Sigma)$ on $\mathcal{M}(\Sigma, \mathrm{SU}_2)$ is weakly mixing.*

Remark 1.2. Although the result is stated here for $G = \mathrm{SU}_2$, the proof could be adapted to simply connected compact groups. Observe also that the action of $\mathcal{K}(\Sigma)$ on $\mathcal{M}(\Sigma, \mathrm{U}_1)$ is trivial, contrary to the action of $\mathrm{Mod}(\Sigma)$, which is known to be ergodic.

Remark 1.3. One can ask if a similar result holds true when Σ is a compact surface with boundary $\partial\Sigma$. In this case we consider the action of $\mathcal{K}(\Sigma)$ on the symplectic leaves $\mathcal{M}(\Sigma, \mathrm{SU}_2; (\alpha_i)_{i \in H_0(\partial\Sigma)})$ of $\mathcal{M}(\Sigma, \mathrm{SU}_2)$, consisting of those classes of representations whose conjugacy classes on each circle of $\partial\Sigma$ are the fixed α_i. We have to require that the genus $g \geq 2$ or $g = 1$ and the number of boundary components is at least 3. Our proof does not seem to extend to this case as we need that the trivial representation belongs to the space of representations considered.

Remark 1.4. The present proof heavily uses the symplectic structure on $\mathcal{M}(\Sigma, \mathrm{SU}_2)$. Notice that when Σ is closed but non-orientable the space $\mathcal{M}(\Sigma, \mathrm{SU}_2)$ still admits an invariant volume form (actually it is enough to have an invariant measure class) and the action of $\mathrm{Mod}(\Sigma)$ was proved to be ergodic in [10] for all but a few explicit small surfaces. However the present proof does not seem to extend to the non-orientable case.

The proof of Theorem 1.1 and hence the paper is organized in the following way. In the first section, we describe the local structure of the representation space around the trivial representation. Then we compute the Taylor expansion of trace functions, in particular we show that the first non trivial term in the expansion of the trace function of a separating curve has order 4. In Section 4, we show that these trace functions are generating around the trivial representation, in the sense that their differentials generate the cotangent space. Then we conclude our proof in Section 5 by an argument similar to the one in [7].

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2. Local structure of representation spaces around the trivial representation

We recall the following result due to Goldman and Millson (see [6]) for general Kähler manifolds, which for the case of surfaces was already obtained by Arms, Marsden and Moncrief [1]. This case is detailed in the appendix of [5].

Proposition 2.1. *Let Σ be a compact Riemann surface and G a compact Lie group.*

Let P be flat principal G-bundle over Σ and denote by $F(P)$ the space of flat connections on P. Given a flat connection A on P, let $Z^1(\Sigma, \mathrm{Ad}\, P)$ denote the space of infinitesimal deformations of A inside $F(P)$.

Then, there exists an analytical diffeomorphism between a neighborhood of A in $F(P)$ and a neighborhood of 0 in the subset of $Z^1(\Sigma, \mathrm{Ad}\, P)$ consisting of $\mathrm{Ad}\, P$-valued 1-forms η such that $[\eta, \eta]$ is exact.

We reformulate this result as it was stated in the corollary from ([5] p.143), by specializing to the case where P is the trivial bundle and A the trivial connection.

Given a base point x in Σ, we denote by $\mathcal{R}(\Sigma, G)$ the variety $\mathrm{Hom}(\pi_1(\Sigma, x), G)$ and by $\mathrm{Hol} : F(P) \to \mathcal{R}(\Sigma, G)$ the holonomy representation.

Definition 2.1. We denote by C the tangent cone at the identity representation, namely the set of elements u in $H^1(\Sigma,\mathfrak{g})$ satisfying $[u,u]=0$. Identifying $H^1(\Sigma,\mathfrak{g})$ with $\mathrm{Hom}(H_1(\Sigma,\mathbb{R}),\mathfrak{g})$ we define $C_{\mathrm{irr}}\subset C$ as the set of surjective maps.

We have then:

Proposition 2.2. *Let Σ be a compact Riemann surface and G a compact Lie group. There is an analytic function $F: H^1(\Sigma,\mathfrak{g})\to\Omega^1(\Sigma,\mathfrak{g})$ which satisfies*

1. *$F(0)=0$ and $D_0F(u)$ is closed and cohomologous to u.*
2. *If $[u,u]=0$ then $dF(u)+\frac{1}{2}[F(u),F(u)]=0$.*
3. *F maps C_{irr} to flat connections with irreducible monodromy.*
4. *The map $\mathrm{Hol}\circ F: C\to\mathcal{R}(\Sigma,G)$ is a real analytic diffeomorphism in the neighborhood of 0.*
5. *The map F is equivariant with respect to the adjoint action of G.*

Remark 2.1. The proof of this theorem uses harmonic theory and is an application of the implicit function theorem to the Kuranishi map in $\Omega^1(\Sigma,\mathfrak{g})$, see [5]. The space of 1-forms are topologized using Sobolev s-norms where s is sufficiently large so that the connections are at least C^2.

3. Taylor expansion of trace functions

Let $\gamma:\mathbb{R}/\mathbb{Z}\to\Sigma$ a parametrized simple curve and set $G=\mathrm{SU}_2\subset M_2(\mathbb{C})$, where $M_2(\mathbb{C})$ denotes the algebra of 2-by-2 matrices with complex entries.

From now on, we will identify $H^2(\Sigma,\mathfrak{g})$ with $\mathfrak{g}$ by the evaluation on the fundamental class. We will denote by $\langle\cdot,\cdot\rangle$ the pairing between homology and cohomology. Choose once for all a norm $|\cdot|$ on $H^1(\Sigma,\mathfrak{g})$.

Proposition 3.1. *If γ is non separating:*

$$\mathrm{Tr}\,\mathrm{Hol}_\gamma F(u)=2+\frac{1}{4}\mathrm{Tr}\langle\gamma,u\rangle^2+O(|u|^3)$$

If γ is separating it decomposes the surface in two parts $\Sigma'\cup_\gamma\Sigma''$. Denote by $u=u'+u''$ the decomposition of u in $H^1(\Sigma,\mathfrak{g})=H^1(\Sigma',\mathfrak{g})\oplus H^1(\Sigma'',\mathfrak{g})$. We have

$$\mathrm{Tr}\,\mathrm{Hol}_\gamma F(u)=2+\frac{1}{16}\mathrm{Tr}[u'\wedge u']^2+O(|u|^5)$$

Proof. Given a connexion form $\alpha\in\Omega^1(\Sigma,\mathfrak{g})$, we consider the well-known formula:

$$\mathrm{Hol}_\gamma\alpha=\sum_{n\geq 0}\frac{1}{n!}\int_{\Delta^n}(\gamma^*\alpha)^n,$$

where $\int_{\Delta^n}\alpha^n=\int_{0<t_1<\cdots t_n<1}\beta(t_1)\cdots\beta(t_n)\mathrm{d}t_1\cdots\mathrm{d}t_n$ and $\alpha=\beta(t)\mathrm{d}t$.

From the identity $\mathrm{Tr}(\mathrm{Hol}_\gamma\alpha)^{-1}=\mathrm{Tr}(\mathrm{Hol}_\gamma\alpha)$, we observe that we can invert the parametrization of γ without changing the result. Writing $\beta(t)\mathrm{d}t=\gamma^*\alpha$ we compute:

$$\begin{aligned}\mathrm{Tr}(\mathrm{Hol}_\gamma\alpha) &= \sum_{n\geq 0}\frac{1}{n!}\mathrm{Tr}\int_{t_1<\cdots<t_n}\beta(1-t_1)\cdots\beta(1-t_n)\mathrm{d}t_1\cdots\mathrm{d}t_n.\\ &= \sum_{n\geq 0}\frac{1}{n!}\mathrm{Tr}\int_{t_1<\cdots<t_n}\beta(t_n)\cdots\beta(t_1)\mathrm{d}t_1\cdots\mathrm{d}t_n.\\ &= \sum_{n\geq 0}\frac{(-1)^n}{n!}\mathrm{Tr}\int_{t_1<\cdots<t_n}\beta(t_1)\cdots\beta(t_n)\mathrm{d}t_1\cdots\mathrm{d}t_n.\end{aligned}$$

This implies that only the even values of n contribute to the sum. Observe that

$$\operatorname{Tr}\int_{\Delta^2}(\gamma^*\alpha)^2 = \frac{1}{2}\operatorname{Tr}\int \beta(t_1)\beta(t_2)\mathrm{d}t_1\mathrm{d}t_2 = \frac{1}{2}\operatorname{Tr}\left(\int_\gamma \alpha\right)^2.$$

We now apply these formulas to $\alpha = F(u)$. Developing $F(u)$ into Taylor series we can write $F(u) = F_1(u) + F_2(u) + O(|u|^3)$ where $F_1(u)$ and $F_2(u)$ are 1-forms satisfying

(1) $\mathrm{d}F_1(u) = 0$.
(2) $F_1(u)$ is cohomologous to u.
(3) $\mathrm{d}F_2(u) + \frac{1}{2}[F_1(u), F_1(u)] = 0$.

As $F(u) = O(|u|)$ we derive the expression:

$$\operatorname{Tr}(\operatorname{Hol}_\gamma F(u)) = 2 + \frac{1}{4}\operatorname{Tr}\left(\int_\gamma F(u)\right)^2 + \frac{1}{24}\operatorname{Tr}\int_{\Delta^4}\gamma^* F(u) + O(|u|^5).$$

As $F_1(u)$ is cohomologous to u we have $\int_\gamma F_1(u) = \langle\gamma, u\rangle$. This proves the first part of the proposition. Suppose now that γ separates and denote by Σ' the submanifold of Σ with $\partial\Sigma' = \gamma$.

We have $\int_\gamma F_1(u) = 0$ and $\int_\gamma F_2(u) = \int_{\Sigma'} dF_2(u) = -\frac{1}{2}\int_{\Sigma'}[F_1(u)\wedge F_1(u)]$. This shows the formula

$$\operatorname{Tr}(\operatorname{Hol}_\gamma F(u)) = 2 + \frac{1}{16}\operatorname{Tr}\left(\int_{\Sigma'}[F_1(u)\wedge F_1(u)]\right)^2 + \frac{1}{24}\operatorname{Tr}\int_{\Delta^4}\gamma^* F_1(u)^4 + O(|u|^5).$$

To conclude we observe that the left hand side is invariant by a gauge transformation, that is we can replace $F(u)$ by $gF(u)g^{-1} - (\mathrm{d}g)g^{-1}$ for some $g : \Sigma \to G$. Writing $g = \exp(\xi)$ for some $\xi : \Sigma \to \mathfrak{g}$, we compute that an infinitesimal gauge transformation maps $F_1(u)$ to $F_1(u) - \mathrm{d}\xi$. We conclude that the right hand side is invariant by such transformations and hence depends only on the cohomology class u of $F_1(u)$. Taking a 1-form α_1' cohomologous to u and such that $\gamma^*\alpha_1' = 0$, we get the result of the proposition. □

4. Non-separating trace functions around the trivial representation

For any oriented separating curve $\gamma \subset \Sigma$, denote by $g_\gamma : H^1(\Sigma, \mathfrak{g}) \to \mathbb{R}$ the map defined by

$$g_\gamma(u) = \operatorname{Tr}[u' \wedge u']^2$$

where Σ' is the subsurface of Σ so that $\partial\Sigma' = \gamma$ and $u = u' + u''$ is the decomposition of u in $H^1(\Sigma, \mathfrak{g}) = H^1(\Sigma', \mathfrak{g}) \oplus H^1(\Sigma'', \mathfrak{g})$.

Identifying $H^1(\Sigma, \mathfrak{g})$ to $\operatorname{Hom}(H_1(\Sigma, \mathbb{R}), \mathfrak{g})$ we define $C_{\mathrm{irr}} \subset C$ as the subset of surjective elements.

Proposition 4.1. *For any surface Σ of genus $g > 1$ there is a finite set of separating curves S of cardinality $g(2g+1)(2g^2+g+1)/2$ and a neighborhood U of 0 in $H^1(\Sigma, \mathfrak{g})$ such for all u in $C_{\mathrm{irr}} \cap U$ and $v \in H^1(\Sigma, \mathfrak{g})$ we have:*

$$[v, u] = 0 \text{ and } D_u g_\gamma(v) = 0, \forall \gamma \in S \quad \Rightarrow \quad v = [\xi, u] \text{ for some } \xi \in \mathfrak{g}$$

This amounts to say that there is a neighborhood V of 0 in C/G such that the derivatives of the functions g_γ for γ in S generate the cotangent space at every surjective $u \in V$. The reason is that for any $u \in C_{\mathrm{irr}}$ we have:

$$T_u(C_{\mathrm{irr}}/G) = \{v \in H^1(\Sigma, \mathfrak{g}) \text{ such that } [u \wedge v] = 0\}/\{[\xi \wedge u] \text{ for } \xi \in H^0(\Sigma, \mathfrak{g})\}.$$

Proof. Let ω denotes the intersection product on $H_1(\Sigma,\mathbb{R})$. Let γ be a separating curve, Σ',Σ'' the corresponding subsurfaces and write $u = P(u) + (u - P(u))$ the decomposition of any u in $H^1(\Sigma,\mathfrak{g}) = H^1(\Sigma',\mathfrak{g}) \oplus H^1(\Sigma'',\mathfrak{g})$.

A direct computation shows:

$$D_u g_\gamma(v) = 4\,\mathrm{Tr}([P(u)\wedge P(u)][P(u)\wedge P(v)])$$

Suppose that this quantity vanishes for all curves γ bounding a 1-holed torus Σ'. It implies that for all rank 2 lattices $\Lambda \subset H_1(\Sigma,\mathbb{Z})$ such that $\Lambda \oplus \Lambda^{\perp_\omega} = H_1(\Sigma,\mathbb{Z})$ (where $\Lambda^{\perp_\omega}$ is the symplectic orthogonal of Λ) we have

$$\mathrm{Tr}([P_\Lambda(u)\wedge P_\Lambda(u)][P_\Lambda(u)\wedge P_\Lambda(v)]) = 0$$

where P_Λ is the projection on $\mathrm{Hom}(\Lambda,\mathfrak{g})$ parallel to $\mathrm{Hom}(\Lambda^{\perp_\omega},\mathfrak{g})$.

Take (x,y) a symplectic base of Λ and denote by p_Λ the symplectic projection of $H_1(\Sigma,\mathbb{R})$ onto $\Lambda\otimes\mathbb{R}$. Identifying u and v to elements in $\mathrm{Hom}(H_1(\Sigma,\mathbb{R}),\mathfrak{g})$, $P_\Lambda(u)$ and $P_\Lambda(v)$ are identified with $u\circ p_\Lambda$ and $v \circ p_\Lambda$ respectively.

Moreover, $[u\wedge v] = c\circ(u\otimes v)\circ\omega^{-1}$ where $\omega^{-1}\in H_1(\Sigma,\mathbb{R})^{\otimes 2}$ represents the inverse symplectic product and $c:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ stands for the Lie bracket. We deduce from this formula the identity:

$$[P_\Lambda(u)\wedge P_\Lambda(v)] = [u(x),v(y)] - [u(y),v(x)]$$

Let $Q_{u,v} : H^1(\Sigma,\mathbb{R})\times H^1(\Sigma,\mathbb{R})\to\mathbb{R}$ be the map defined by

$$Q_{u,v}(x,y) = \mathrm{Tr}([u(x),u(y)]\,([u(x),v(y)] - [u(y),v(x)]))$$

We use the following lemma in order to reduce the statement to prove the vanishing of $Q_{u,v}$. We postpone its proof to the end of the section as we could not find a proof of it avoiding computations.

Lemma 4.1. *Let $u,v\in H^1(\Sigma,\mathfrak{g})$ such that u is surjective, $[u\wedge u] = [u\wedge v] = 0$ and $Q_{u,v} = 0$. Then there exists $\xi\in\mathfrak{g}$ so that $v = [\xi\wedge u]$.*

It remains to show that the vanishing of $Q_{u,v}$ can be detected by a finite number of rank 2 symplectic lattices in $H_1(\Sigma,\mathbb{Z})$. This is the content of the following lemma.

Let $S = \{x_i\otimes y_i, i\in I\}\subset S^2H_1(\Sigma,\mathbb{Z})$ be a set of vectors satisfying $\omega(x_i,y_i) = 1$. We say that S is quadratically generating if any quadratic form on $S^2H_1(\Sigma,\mathbb{R})$ vanishing on S vanishes everywhere.

Lemma 4.2. *There exists quadratically generating sets S. Moreover one can find such sets with cardinality $g(2g+1)(2g^2+g+1)/2$.*

Again we postpone the proof this lemma to the end of the section. Observe that the proposition follows by considering any quadratically generating set S. By the assumption of the proposition we have for any u,v and any $x_i\otimes y_i\in S$ the equality $Q_{u,v}(x_i,y_i) = 0$. The generating property implies that $Q_{u,v} = 0$ and Lemma 4.1 implies the result.

□

Proof of Lemma 4.1: Let $(\xi_i)_{i\in\mathbb{Z}_3}$ be a basis of $\mathfrak{g}$ so that $[\xi_i,\xi_{i+1}] = \xi_{i+2}$ and normalize the trace so that $\mathrm{Tr}(\xi_i\xi_j) = \delta_{ij}$ for all $i,j\in\mathbb{Z}_3$.

Write $u = \sum_i u_i\xi_i$ for $u_i\in H^1(\Sigma,\mathbb{R})$. The hypothesis on u imply that the u_i are non zero and mutually orthogonal. Fix a basis $(e_i)_{0\le i<n}$ of $H_1(\Sigma,\mathbb{R})$ so that $u_i = e_i^*$ for $i = 0,1,2$. Then $v = \sum_{i\in\mathbb{Z}_3}\sum_{j<n} v_i^l e_l^*\otimes\xi_i$, $x = \sum_l x_l e_l$ and $y = \sum_l y_l e_l$.

From now on, $i\in\mathbb{Z}_3$ and $0\le l<n$. We have $u(x) = \sum_i x_i\xi_i$ and $v(x) = \sum_{i,l} v_i^l x_l\xi_i$ so that:

$$[u(x), u(y)] = \sum_i (x_i y_{i+1} - x_{i+1} y_i)\xi_{i+2}$$

$$[u(x), v(y)] = \sum_i y_l (x_i v^l_{i+1} - x_{i+1} v^l_i)\xi_{i+2}$$

This gives the identity

$$\sum_{i,l} \Big((x_i y_{i+1} - x_{i+1} y_i)(x_i y_l v^l_{i+1} - x_{i+1} y_l v^l_i - x_l y_i v^l_{i+1} + x_l y_{i+1} v^l_i) \Big) = 0$$

This biquadratic polynomial in x and y vanishes identically if and only if all its coefficient vanish.

- The coefficient of $x_i^2 y_{i+1} y_l$ for $l > 3$ is v^l_{i+1} showing that $v^l_i = 0$ for all i and $l > 3$.
- The coefficient of $x_i^2 y_{i+1}^2$ is $v^{i+1}_{i+1} + v^i_i$ showing that $v^i_i = 0$ for all i.
- The coefficient of $x_i^2 y_{i+1} y_{i-1}$ is $v^{i-1}_{i+1} + v^{i+1}_{i-1}$.

This show that the matrix v^j_i is antisymmetric so that the set of solutions have dimension 3 as the set of infinitesimal symmetries. □

Proof of Lemma 4.2: The vector space $\mathcal{Q}$ of quadratic forms on $S^2 H^1(\Sigma, \mathbb{R})$ has dimension $N = g(2g+1)(2g^2+g+1)/2$. Each symplectic lattice $\Lambda_i = \mathrm{Span}(x_i, y_i)$ acts linearly on $\mathcal{Q}$ by evaluation at $x_i \otimes y_i$. If these linear forms generate the dual of $\mathcal{Q}$, then a subset of size N of them will do.

It remains to prove that these linear forms indeed generate the dual of $\mathcal{Q}$. Let then $Q \in \mathcal{Q}$ such that, for any $x, y \in H_1(\Sigma, \mathbb{Z})$ satisfying $\omega(x, y) = \pm 1$ we have $Q(x, y) = 0$.

Let (x_0, y_0) be such an integral symplectic basis. Then the map $\mathrm{Sp}(2g, \mathbb{R}) \to \mathbb{R}$ sending A to $Q(Ax_0, Ay_0)$ is algebraic. This map vanishes on $\mathrm{Sp}(2g, \mathbb{Z})$, which is Zariski dense in $\mathrm{Sp}(2g, \mathbb{R})$ and hence, it vanishes identically. As the group $\mathrm{Sp}(2g, \mathbb{R})$ acts transitively on the set of pairs $(x, y) \in H_1(\Sigma, \mathbb{R})$ satisfying $\omega(x, y) = 1$, we get $Q(x, y) = 0$ for such pairs. As Q is homogeneous, we have the same result for all (x, y) such that $\omega(x, y) \neq 0$. By density, we get $Q = 0$. □

Let S be a set of simple curves γ_i bounding 1-holed tori in Σ. We will say that this set is quadratically generating if the corresponding 2-dimensional symplectic lattices are quadratically generating.

Proposition 4.2. *Let S be a quadratically generating set of curves, and $f_\gamma : \mathcal{M}(\Sigma, G) \to \mathbb{R}$ be the associated trace functions.*

For any conical neighborhood V of the subset of reducible representations in $\mathcal{M}(\Sigma, G)$ there is a neighborhood U of the trivial representation in $\mathcal{M}(\Sigma, G)$ so that for any $x \in U \setminus V$ the derivatives of f_γ at x, with $\gamma \in S$, generate the cotangent space $T^*_x \mathcal{M}(\Sigma, G)$.

Remark 4.1. Observe that $\mathcal{M}(\Sigma, G)$ is locally modeled on the cone C/G. By conical neighborhood V we mean that V contains some neighborhood invariant by scaling in the model. This is indeed independent on the real analytic diffeomorphism we choose.

Proof. The proposition is equivalent to saying that for $\rho \in \mathcal{R}(\Sigma, G)$ close enough to the trivial representation but far enough from reducible representations, the intersection of the kernels of the derivatives of the functions f_γ, with $\gamma \in S$, is precisely the tangent space of the G action. Recall that the map $F : C \to \mathcal{R}(\Sigma, G)$ is a G-equivariant real analytic diffeomorphism, hence it is sufficient to prove the analogous claim for $f_\gamma \circ F$. From Proposition 3.1, we have at each point u of C:

$$f_\gamma(F(u)) = 2 + \frac{1}{16} g_\gamma(u) + O(|u|^5).$$

As F is a smooth function of u, we have

$$16 D_u(f_\gamma \circ F) = D_u g_\gamma + O(|u|^4). \tag{1}$$

By Proposition 4.1, there is a neighborhood U of 0 in C such that for all u in $C_{\mathrm{irr}} \cap U$, the derivatives of g_γ, $\gamma \in S$, generate the cotangent space of C at u, modulo the 3-dimensional space of elements of the form $[\xi, u]$, with $\xi \in H^0(\Sigma, \mathfrak{g})$. This means that we can find a set $S' \subset S$ of cardinal $6g-6$, such that the derivatives of g_γ, $\gamma \in S'$, are linearly independent at u. This is an open condition and thus it holds true for some open neighborhood $U_{S'}$ containing u. In particular $\{U_{S'}\}$, for $S' \subset S$ (of cardinal $6g-6$) forms an open covering of $C_{\mathrm{irr}} \cap U$. Taking an auxiliary euclidean structure in $H^1(\Sigma, \mathfrak{g})$, this last condition can be expressed by saying that the Gram determinant $\mathrm{Gram}(D_u g_\gamma)_{\gamma \in S'}$ of $(D_u g_\gamma)_{\gamma\in S'}$ relative to the euclidean structure is positive on $U_{S'}$. Thus $\Phi(u) = \sum_{S' \subset S} \mathrm{Gram}(D_u g_\gamma)_{\gamma\in S'}$ is positive on $C_{\mathrm{irr}} \cap U$.

Define $\Psi(u) = \sum_{S'\subset S} \mathrm{Gram}(16 D_u f_\gamma \circ F)_{\gamma \in S'}$. By reversing the argument above, we need to prove that there is a neighborhood U' of 0 in C so that for any $u \in U' \setminus V$ one has $\Psi(u) > 0$.

We observe that $D_u g_\gamma$ is cubic in u and so $\Phi(u)$ is homogeneous of degree $N = 36(g-1)$. By using Equation (1) and expanding the Gram determinant, we find that

$$\Psi(u) = \Phi(u) + O(|u|^{N+1}). \tag{2}$$

Let V be the conic neighborhood of $C \setminus C_{\mathrm{irr}}$ given in the statement and S be the compact set defined by $S = \{u \in C, |u| = 1\}$. As $S \setminus V$ is a compact subset of C_{irr}, there exists $\epsilon > 0$ so that $\Phi > \epsilon$ on $S \setminus V$. By homogeneity we get $|\Phi(u)| > \epsilon |u|^N$ on $C_{\mathrm{irr}} \setminus V$ which together with Equation (2) proves the result.

□

Remark 4.2. We will call an open set of the form $U \setminus \overline{V}$ given by the above proposition a "neighborhood of the trivial representation". We can use these neighborhoods to define a topology on $\mathcal{M}_{\mathrm{irr}}(\Sigma, G) \cup \{1\}$. A crucial observation is that these open sets are connected, as C_{irr} is locally connected around the trivial representation. In fact, according to Lemma 4.1 a point of C_{irr} is given by a triple of pairwise orthogonal elements of $H^1(\Sigma; \mathbb{R})$ and the symplectic group $\mathrm{Sp}(2g, \mathbb{R})$ acts transitively on the space of such triples.

5. Ergodicity of the Johnson subgroup action on $\mathcal{M}(\Sigma, \mathrm{SU}_2)$

Fix a set of separating curves S of genus one which are quadratically generating. Let $\mathcal{K}_1(\Sigma; S) \subset \mathcal{K}(\Sigma)$ denote the normal subgroup of $\mathrm{Mod}(\Sigma)$ generated by the Dehn twists along the curves $\gamma \in S$. Further define, for all $n \geq 1$, $\mathcal{K}_{n+1}(\Sigma; S)$ as being the normal subgroup in $\mathcal{K}_n(\Sigma; S)$ generated by the Dehn twists along curves in S. The result of Theorem 1.1 follows from the more general:

Proposition 5.1. *For each $n \geq 1$ the action of $\mathcal{K}_n(\Sigma, S)$ on $\mathcal{M}(\Sigma, \mathrm{SU}_2)$ is ergodic.*

Proof. For $n \geq 1$ we denote by $\mathcal{A}_n(S)$ the set of orbits of curves in S under the action of $\mathcal{K}_{n-1}(\Sigma; S)$, where $\mathcal{K}_0(\Sigma; S)$ states for $\mathrm{Mod}(\Sigma)$. Define the set:

$$U_n = \{\rho \in \mathcal{M}_{\mathrm{irr}}(\Sigma, G) \text{ such that } \mathrm{Span}\{D_\rho f_\gamma \text{ for } \gamma \in \mathcal{A}_n(S)\} = T^*_\rho \mathcal{M}(\Sigma, G)\}.$$

As this is an open condition U_n is open. Moreover U_1 is invariant by the mapping class group.

We will prove the claim by recurrence on n by means of a bootstrap argument. Consider first $n = 1$. Remark 4.2 directly implies that there is a connected neighborhood V of the trivial representation in $\mathcal{M}(\Sigma, G)$ so that $V \cap \mathcal{M}_{\rm irr}(\Sigma, G) \subset U_1$. Let U_1^0 be the unique connected component of U_1 which contains V. The open set U_1^0 is then non empty and invariant by the mapping class group. By the ergodicity of the mapping class group on $\mathcal{M}(\Sigma, {\rm SU}_2)$, proved by Goldman in [3], the complement of U_1^0 has measure 0. The end of the proof for $n = 1$ follows from the arguments of Goldman and Xia in [7]. Specifically, one key ingredient is the infinitesimal transitivity Lemma 3.2 from [7], which we state here for the sake of completeness:

Lemma 5.1. *Let X be a connected symplectic manifold and $\mathcal{F}$ be a set of functions such that their differential at all points of X span the cotangent space. Then the group generated by the Hamiltonian flows associated to the functions in $\mathcal{F}$ acts transitively on X.*

The Hamiltonian flow associated to the trace function f_γ is covered by the hamiltonian flow Φ_γ^t of $h_\gamma = \arccos(f_\gamma/2)$, the so-called Goldman twist flow (see e.g. [3, 7]) defined by γ. The flow Φ_γ gives a circle action of period π on an open dense subset of $\mathcal{M}(\Sigma, {\rm SU}_2)$. The action of the Dehn twist along the separating simple curve γ on $\mathcal{M}(\Sigma, {\rm SU}_2)$ is identified with the time h_γ of the Goldman twist flow.

This circle action is a rotation of angle h_γ and therefore the Dehn twist along γ acts ergodically on the orbit of $\rho \in \mathcal{M}(\Sigma, {\rm SU}_2)$ under the Goldman twist flow defined by γ, for all ρ with irrational $\frac{h_\gamma(\rho)}{\pi}$. Now, this implies (see [7, Proposition 5.4]) that any measurable function on $\mathcal{M}(\Sigma, {\rm SU}_2)$ which is invariant by the action of the Dehn twist along γ should be constant on the orbits of the Goldman twist flow defined by γ outside a nullset of $\mathcal{M}(\Sigma, {\rm SU}_2)$. Therefore, any measurable function on $\mathcal{M}(\Sigma, {\rm SU}_2)$ which is invariant by the group $\mathcal{K}_1(\Sigma; S)$ should be invariant by the group generated by the Hamiltonian flows associated to the functions f_γ, for $\gamma \in S$, almost everywhere. Then, by the transitivity Lemma 5.1 and Proposition 4.2 it must be constant on U_1^0 almost everywhere. This establishes the claim for $n = 1$.

Assume now that the claim holds for n. One can find a connected neighborhood V_n of the trivial representation in $\mathcal{M}(\Sigma, G)$ so that $V_n \subset U_n$ using again Remark 4.2. Let U_n^0 be the unique connected component of U_n which contains V_n. The open set U_n^0 is then non empty and invariant by the group $\mathcal{K}_n(\Sigma; S)$. By using the ergodicity of the $\mathcal{K}_n(\Sigma; S)$ action on $\mathcal{M}(\Sigma, {\rm SU}_2)$, which is the induction hypothesis, the complement of U_n^0 has measure 0. Then again the arguments from [7] imply that the $\mathcal{K}_{n+1}(\Sigma; S)$ action is ergodic. □

Remark 5.1. The subgroup $\Gamma(S)$ generated by the Dehn twists along curves in S is contained in the intersection $\cap_{n\geq 1} K_n(\Sigma; S)$, but we don't know whether the inclusion is strict. One also might speculate that $\Gamma(S)$ acts ergodically on $\mathcal{M}(\Sigma, {\rm SU}_2)$.

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