STRUCTURE THEOREMS FOR ACTIONS OF HOMEOMORPHISM GROUPS

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Abstract. We give an orbit classification theorem and a general structure theorem for actions of groups of homeomorphisms and diffeomorphisms on manifolds, reminiscent of classical results for actions of (locally) compact groups. As applications, we give a negative answer to Ghys' "extension problem" for diffeomorphisms of manifolds with boundary, a classification of all homomorphisms $\mathrm{Homeo}_0(M) \to \mathrm{Homeo}_0(N)$ when $\dim(M) = \dim(N)$ (and related results for diffeomorphisms), and a complete classification of actions of $\mathrm{Homeo}_0(S^1)$ on surfaces. This resolves several problems in a program initiated by Ghys, and gives definitive answers to questions of Militon, Hurtado and others.

1. Introduction

Let M be a topological or smooth manifold. In this article we study actions of $\mathrm{Homeo}_0(M)$ and $\mathrm{Diff}^r_0(M)$ (the identity components of the group of homeomorphisms or C^r diffeomorphisms of M, respectively) on other manifolds and finite dimensional spaces. This can be considered the analog of *representation theory* for these large transformation groups; and our work gives structure theorems and rigidity results towards a classification of all such actions.

There are many natural examples of continuous actions of $\mathrm{Homeo}_0(M)$ and $\mathrm{Diff}^r_0(M)$ on other manifolds. For instance, each group obviously acts on the product manifold $M \times N$ for any manifold N, as well as the configuration space $\mathrm{Conf}_n(M)$ of n points in M for any n. In some cases, depending on the topology of M, the tautological action of $\mathrm{Homeo}_0(M)$ or $\mathrm{Diff}^r_0(M)$ on M also lifts to an action on covers of M. Bundles over M give further examples; in particular, $\mathrm{Diff}_0(M)$ acts naturally on the unit tangent bundle of M, the projectivized unit tangent bundle of M, and various jet bundles over M.

It is a longstanding question to what extent this is an exhaustive list. Rubin, in a 1989 paper [23], asked generally if there were "any reasonable assumptions" under which "the embeddability of $\mathrm{Homeo}(X)$ in $\mathrm{Homeo}(Y)$ will imply that X is some kind of continuous image of Y." Implicit in work of Whittaker [29] and Filipkiewicz [7] in the 60s and 80s on automorphisms of $\mathrm{Homeo}_0(M)$ and $\mathrm{Diff}^r_0(M)$, and isomorphisms among such groups, is the problem to classify the *endomorphisms* of these groups. More precise questions (which we will recall – and answer – later) were posed by Ghys in [8], and more recently by Militon [18] and Hurtado [11].

Recent work of Hurtado and the second author [11, 14] shows that any abstract homomorphism between groups of C^∞ diffeomorphisms, or from $\mathrm{Homeo}_0(M)$ to any separable topological group (of which all homeomorphism and diffeomorphism groups of manifolds are examples), are necessarily continuous. Thus, in this work, we study only continuous actions, but in many cases can draw conclusions for abstract homomorphisms as well.

1.1. **Results.** Our results establish precise settings in which the constructions given above (configuration spaces, covers, bundles, and their quotients) do form an exhaustive list of all possible actions; and we apply these structure theorems to various existence and classification problems for actions. The first major tool is a description of the possible structures of orbits.

Theorem 1.1 (Orbit Classification Theorem)**.** *For any action of* $\mathrm{Homeo}_0(M)$ *on a finite-dimensional CW complex, every orbit is either a point or the continuous injective image of a cover of a configuration space* $\mathrm{Conf}_n(M)$ *for some* n*. For any continuous action of* $\mathrm{Diff}^r_0(M)$ *on a finite-dimensional CW complex by homeomorphisms, every orbit is either a point or a continuous injective image of a cover of the* r*-jet bundle over* $\mathrm{Conf}_n(M)$ *quotient by a linear group.*

As above, $\mathrm{Conf}_n(M)$ denotes the configuration space of n distinct, unlabeled points in M. If $M = S^1$ or $M = \mathbb{R}$, we mean the configuration space of unlabeled points together with a cyclic or linear order. PConf_n denotes the (pure) configuration space of labeled points, so $\mathrm{PConf}_n(\mathbb{R}) = \mathrm{Conf}_n(\mathbb{R})$. Since $\mathrm{Conf}_n(M)$, and the k-jet bundles (for $k \leq r$) are manifolds on which $\mathrm{Homeo}_0(M)$ and $\mathrm{Diff}^r_0(M)$ act transitively, these examples of orbits do indeed occur. The challenge is to *a)* determine which covers can appear (see Section 2 for discussion and relationship with "point pushing" problems) and *b)* determine how orbits of these types can be glued together, i.e. how they partition a fixed manifold or CW complex N on which $\mathrm{Diff}^r_0(M)$ acts. The solution to these problems, in various specific contexts, is the focus of the three applications below.

Application I: Structure theorem in small codimension cases. We prove the following results for homeomorphism groups, which can be seen as analogs to the classical theory for actions of compact groups on manifolds as in [2].

Our first result answers a question of Ghys in the C^0 case and gives a counterexample to (but almost a proof of) Conjecture 1.1 of [18].

Theorem 1.2. *Let* M *be a connected, closed manifold and* N *a manifold with* $\dim(N) \leq \dim(M)$*. If there is a nontrivial homomorphism* $\rho : \mathrm{Homeo}_0(M) \to \mathrm{Homeo}(N)$*, then* $\dim(N) = \dim(M)$ *and* ρ *is induced by lifting homeomorphisms to a collection of admissible covers of* M*, and mapping these covers into* N*.*

Examples of such lifts to covers abound – for instance, if $M = S_g$ is a surface of genus $g \geq 2$, then $\mathrm{Homeo}_0(S_g)$ lifts to act on every cover of M. For a concrete example, by fixing a hyperbolic metric on S_g, the universal cover can be identified with the Poincaré disc, and the lifted action of $\mathrm{Homeo}_0(S_g)$ extends to a continuous action on the closed disc, pointwise fixing the boundary. Embedding this disc in another 2-manifold S (or in S_g itself) produces a nontrivial homomorphism $\mathrm{Homeo}_0(S_g) \to \mathrm{Homeo}_0(S)$. This is a counterexample to Conjecture 1.1 of [18]. See Section 3 for further discussion.

We also prove a similar result for continuous actions of diffeomorphism groups. Here we show that one may have only finitely many embedded finite covers of M in N defining the nontrivial orbits of the action. This gives a new proof of a theorem of Hurtado.

Theorem 1.2 has the following generalization in higher codimension.

Theorem 1.3. *Let M and N be connected, closed manifolds such that $\dim(N) < 2\dim(M)$. If there is an action of $\mathrm{Homeo}_0(M)$ on N without global fixed points, then N has the structure of a generalized flat bundle over M. When $\dim(N) - \dim(M) < 3$, the fiber F is a manifold as well.*

See Section 4 for the definition of generalized flat bundles. With more regularity, we obtain a stronger result.

Theorem 1.4 (Structure theorem for group actions by diffeomorphisms)**.** *Suppose M is a connected, closed manifold and N is a manifold with $\dim(N) < 2\dim(M)$. If there exists a continuous, nontrivial action $\mathrm{Diff}_0^s(M) \to \mathrm{Diff}_0^r(N)$, then $r \leq s$, the action is fixed point free, and N is a C^r-manifold bundle over M.*

A basic, nontrivial example of such is the action of $\mathrm{Diff}^s(M)$ by C^{s-1} diffeomorphisms on the projectivized tangent bundle of M.

Application II: Extension problems. Let W be a manifold with boundary M. If W has a C^r structure, then there is a natural "restrict to the boundary" map

$$\mathrm{Res}^r(W, M) : \mathrm{Diff}_0^r(W) \to \mathrm{Diff}_0^r(M)$$

which is surjective. We will omit r from the notation when $r = 0$. The *extension problem*, introduced by Ghys in [8], asks whether $\mathrm{Res}^r(W, M)$ has a *group theoretic section* ρ such that $\mathrm{Res}^r(W, M) \circ \rho$ is the identity map. In general, one expects that the answer may depend both on r and on the topology of W. One example where this question has a positive answer is for homeomorphisms of balls and spheres. Let $\mathbb{D}^{n+1}$ be the $n+1$ dimensional ball and S^n the n-sphere. Then

$$\mathbb{D}^{n+1} = \{(x, r) | x \in S^n, r \in [0, 1]\}/(x, 0) \sim (y, 0)$$

and there is a standard "coning off" action $C : \mathrm{Homeo}_0(S^n) \to \mathrm{Homeo}(\mathbb{D}^{n+1})$ defined by $C(f)(x, r) = (f(x), r)$. (Going forward, we refer to this as *coning.*) We answer [15, Question 3.18] in the following theorem.

Theorem 1.5. *Suppose $\dim(M) > 1$, and $\pi_1(M)$ has no nontrivial action on the interval (e.g. a group generated by torsion). Then $\mathrm{Res}^0(W, M)$ has a section if and only if $M = S^n$ and $W = \mathbb{D}^{n+1}$.*

Theorem 1.6. *For $n > 1$, any section of* $\mathrm{Res}(\mathbb{D}^{n+1}, S^n)$ *is conjugate to the standard coning. In fact, any nontrivial action of* $\mathrm{Homeo}_0(S^n)$ *on* $\mathbb{D}^{n+1}$ *is conjugate to the standard coning.*

We actually prove a more general result than Theorem 1.5, see Section 5.

Ghys [8] posed the extension problem for the genus g handlebody H_g in the smooth category (see also [15, Question 3.15, Question 3.19]). As a consequence of Theorem 1.4, we give a complete answer for continuous section of $\mathrm{Res}^r(W, M)$ that applies to any manifold with boundary pair M, W, and any regularity of at least C^1.

Corollary 1.7 (No differentiable extensions)**.** *Let W be a manifold with boundary M, and $r \geq 1$. Then* $\mathrm{Res}^r(W, M) : \mathrm{Diff}^r_0(W) \to \mathrm{Diff}^r_0(M)$ *does not have a continuous section. If $r = \infty$, any section is continuous by [11], and this hypothesis may be removed.*

Application III: $\mathrm{Homeo}_0(S^1)$ **actions on surfaces.** In contrast with Theorem 1.5, there are infinitely many non-conjugate extension actions of $\mathrm{Homeo}_0(S^1)$ on $\mathbb{D}^2$. For example, in addition to the standard coning, one may take the action on the open annulus $\mathrm{PConf}_2(S^1)$, with one end naturally compactified to a circle and the other to a point. Section 7 is devoted to the general classification problem posed by Militon in [18]. We prove the following.

Theorem 1.8. *There is an explicit family of actions* $\rho_{K,\lambda} : \mathrm{Homeo}_0(S^1) \to \mathrm{Homeo}_0(\mathbb{D}^2)$ *such that any nontrivial homomorphism* $\rho : \mathrm{Homeo}_0(S^1) \to \mathrm{Homeo}_0(\mathbb{D}^2)$ *is conjugate to* $\rho_{K,\lambda}$ *for some* K, λ.

Following Militon, we also know exactly which of the actions $\rho_{K,\lambda}$ are conjugate (see Section 7), so this gives a complete classification of actions of $\mathrm{Homeo}_0(S^1)$ on the disc. A similar classification may be obtained by the same methods for actions of $\mathrm{Homeo}_0(S^1)$ on other orientable surfaces. This proves [18, Conjecture 2.2].

Further context and historical remarks. Applicaitons I and II are motivated by [8], where Ghys framed the extension problem and also asked whether the existence of a nontrivial homomorphism $\mathrm{Diff}_0(M) \to \mathrm{Diff}_0(N)$ (when M and N are closed manifolds) implied that $\dim(M) \leq \dim(N)$. This was partially answered in [13] and definitively answered by Hurdado [11], who also gave a complete description of possible homomorphisms $\mathrm{Diff}_0(M) \to \mathrm{Diff}_0(N)$ in the case where the dimensions are equal. Our work gives a different approach to this question. Analogous questions for groups of homeomorphisms appear in [18] and [14, Question 5.5.]; these are all answered by Theorems 1.2 and 1.3.

In [18], Militon classified actions of $\mathrm{Homeo}(S^1)$ on the closed annulus and the torus. We give a shorter proof of this result and answer the remaining cases of the disc, sphere, and (half)-open annulus. (There are no nontrivial actions of $\mathrm{Homeo}(S^1)$ on surfaces of higher genus.) While we do not pursue this here, our techniques apply directly to classify actions on nonorientable surfaces as well. In this paper also appears the above mentioned conjecture that for a closed manifold M, any nontrivial homomorphism $\mathrm{Homeo}_0(M) \to \mathrm{Homeo}_0(M)$ is an inner automorphism. Our Theorem 1.2 proves that this is true when M does not have infinite admissible covers, or true with the further assumption that the action of $\mathrm{Homeo}_0(M)$ on M is transitive. Counterexamples in the general case include all manifolds which admit metrics of negative curvature.

Structure of the paper.

- In Section 2, we establish a "small quotient subgroup theorem" and our Orbit Classification Theorem.
- In Section 3, we classify homomorphisms between homeomorphism or diffeomorphism groups when $\dim(M) = \dim(N)$.
- In Section 4, we discuss how orbits fit together and prove a structure theorem in the C^0 category.
- In Section 5, we study the extension problem in the homeomoprhism case.
- In Section 6, we prove a structure theorem for orbits in the C^r category, $r \geq 1$ and study the extension problem in the differentiable case.
- In Section 7, we classify actions of $\mathrm{Homeo}_0(S^1)$ on $\mathbb{D}^2$ and other surfaces.

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2. Small quotient subgroups and classification of orbits

Definition 2.1. Let G be a topological group, and $A \subset G$ a subgroup. We say that A has *small quotient* if there exists $n \in \mathbb{N}$ such that, for any continuous injective map of an n-disc $\mathbb{D}^n \to G$, the projection $\mathbb{D}^n \to G \to G/A$ is non-injective. If A has small quotient, the *codimension* of A is the minimum n such that there exists a continuous injective map $\mathbb{D}^n \to G$ that descends to an injective map to G/A.

Before stating our main theorem, we give two basic properties.

Observation 2.2. *(Properties of small quotient subgroups)*

1. *If $A \subset B \subset G$ are subgroups and A has small quotient, then B also has small quotient, with codimension bounded above by* $\mathrm{codim}(A)$*.*
2. *If $H \subset G$ is a subgroup and A has small quotient in G, then $A \cap H$ has small quotient in H, and the codimension of A in G is bounded below by the codimension of $A \cap H$ in H.*

Proof. The first item is just the observation that the projection map $G \to G/B$ factors through $G \to G/A \to G/B$. The second follows from the fact that $H/(A \cap H)$ embeds in G/A. □

In this section, M always denotes a topological manifold. For simplicity we will typically assume all manifolds are connected. Many of the results apply to more general classes of topological spaces, but we focus on manifolds since this is our intended application.

Definition 2.3. A subgroup $G \subset \mathrm{Homeo}(M)$ has the *fragmentation property* if, for any open cover of M, G can be generated by homeomorphisms supported on elements of the cover. We say G has *local simplicity* if, for any relatively compact ball $B \subset M$, the subgroup of G consisting of elements supported on B is path connected and (algebraically) simple.

It follows from deep work of Edwards–Kirby [5] that if M is a compact topological manifold, then $\mathrm{Homeo}_0(M)$ has the fragmentation property and is locally simple. For noncompact M, the group $\mathrm{Homeo}_c(M)$ of compactly supported homeomorphisms isotopic to the identity through compactly supported isotopies also has these properties. Work of Mather [16, 17] and Thurston [27] establishes the fragmentation property for the diffeomorphism groups $\mathrm{Diff}^r_c(M)$ of any smooth manifold and the local simplicity for $\mathrm{Diff}^r_c(M)$ when $1 \leq r \leq \infty$ and $r \neq \dim(M) + 1$. Whether $\mathrm{Diff}^{\dim(M)+1}_c(M)$ is simple is a famous open question in the field.

The first step towards our orbit classification theorem is the following result on closed, small quotient subgroups.

Lemma 2.4. *Let $G \subset \mathrm{Homeo}(M)$ be a locally simple group, and $A \subset G$ a closed subgroup with small quotient in G. Then there exists a ball $B \subset M$ such that A contains all homeomorphisms in G supported on B.*

Proof. Let $n = \mathrm{codim}(A) + 1$ and fix n disjoint closed balls in M. Let $G_i \subset G$ denote the subgroup of homeomorphisms with support in the ith ball. Since G_i and G_j commute whenever $i \neq j$, we may identify the product $G' := G_1 \times \ldots \times G_n$ with a subgroup of G. Let $A' = A \cap G'$, this is also a closed subgroup. By Observation 2.2, the subgroup A' has codimension less than n in G'. Let $p_i : A' \to G_i$ be the natural projection, and let $A_i = p_i(A')$. Our goal is to show that some G_i is contained in A.

Observation 2.2 (1) implies that $A_1 \times \ldots \times A_n \supset A'$ has codimension less than n in G'. Let $\overline{A_i}$ denote the closure of A_i in G_i. Then we have

$$G'/(\overline{A_1} \times \ldots \times \overline{A_n}) = G_1/\overline{A_1} \times \ldots \times G_n/\overline{A_n}.$$

If $\overline{A_i} \neq G_i$, then $G_i/\overline{A_i}$ contains an injective image of $[0, 1]$. This is because $G_i/\overline{A_i}$ is a path-connected Hausdorff space. Thus, if $G_i \neq \overline{A_i}$ for all i, we would have an embedded $\mathbb{D}^n$ in $G_1/A_1 \times \ldots \times G_n/A_n$, contradicting the bound on codimension. We conclude that $G_i = \overline{A_i}$ for some i. Reindexing if needed, we assume $i = 1$.

Let $p : A' \to G_2 \times \ldots \times G_n$ denote the product map $p_2 \times \ldots \times p_n$, and let $K = \mathrm{Ker}(p)$. Note that $K = A \cap G_1$, which is a closed subgroup of G_1. Since A has small quotient in G, this means that $K = A \cap G_1$ has small quotient in G_1, so is in particular nontrivial. We have that $p_1(K) = K$ and $p_1(A') = A_1$, so $p_1(K)$ is a nontrivial normal subgroup of A_1. In other words, A_1 is contained in the normalizer $N_{G_1}(K)$ of K in G_1. It is a basic fact that the normalizer of a closed subgroup of a topological group is closed (if $H \subset G$ is closed, then $N_G(H) = \bigcap_{h \in H} \{g \in G \mid ghg^{-1} \in H\}$ is an intersection of closed sets because $g \mapsto ghg^{-1}$ is continuous), so $N_{G_1}(K)$ is a closed subgroup containing A_1, i.e. equal to G_1. This means that K is normal inside G_1. By local simplicity, $G_1 = K = A \cap G_1$, which is what we needed to show. □

Adding the hypotheses that G has the fragmentation property and A acts transitively, we have the following immediate consequence of Lemma 2.4.

Corollary 2.5. *Let $G \subset \mathrm{Homeo}(M)$ be a locally simple group with the fragmentation property. If $A \subset G$ is a closed subgroup with small quotient that acts transitively on M, then $A = G$.*

Now we deal with the case where the action of A is nontransitive.

Lemma 2.6. *Suppose $G \subset \mathrm{Homeo}(M)$ is a locally simple group with the fragmentation property, $A \subset G$ is a closed subgroup with small quotient in G, and assume that G has the additional property that for all $x, y \in M$, and ball B containing x and y, there exists a 1-parameter subgroup f_t supported on B and such that $f_1(y) = x$. Then there exists a finite set X such that A setwise preserves X and acts transitively on $M - X$; or in the case $\dim(M) = 1$, acts transitively on connected components of $M - X$.*

Proof. By Corollary 2.5, we may assume A is nontransitive, so there are points $x, y \in M$ such that $y \notin A \cdot x$. Let $\{f_t\}_{t \in \mathbb{R}} \subset G$ be a 1-parameter subgroup G consisting of homeomorphisms supported on a small neighborhood of x such that $f_1(y) = x$. In particular, this flow f_t is not a subgroup of A. Since A is closed, $\mathbb{R} \cap A$ is a closed, proper subgroup of $\mathbb{R}$, so either trivial or isomorphic to $\mathbb{Z}$. In the first case $\{f_t\}/(\{f_t\} \cap A)$ is $\mathbb{R}$, and in the second case it is S^1; either contains an embedded 1-dimensional disc.

More generally, suppose we can find points x_i and y_i, $1 \leq i \leq m$, where $y_i \notin A \cdot x_i$. In the case where M is 1-dimensional, assume that $y_1 < x_1 < ... < y_m < x_m$ with respect to the ordering induced by $\mathbb{R}$ in some local coordinate chart. Then we can take a continuous homomorphism $\mathbb{R}^n \to G$, where the ith factor is a flow whose time 1 map takes y_i to x_i, with support disjoint from the other factors. Then $\mathbb{R}^n \cap A$ is a closed subgroup of $\mathbb{R}^n$, and each element of $\mathbb{R} \times \ldots \mathbb{R} \times \{1\} \times \mathbb{R} \ldots \times \mathbb{R}$ (where the 1 is in the ith coordinate position) takes y_i to x_i, so is not an element of A. We know that every linear subspace of $\mathbb{R}$ has to intersect one of $\mathbb{R} \times \ldots \mathbb{R} \times \{1\} \times \mathbb{R} \ldots \times \mathbb{R}$ because at least one coordinate should be nonzero. Therefore $\mathbb{R}^n \cap A \subset \mathbb{R}^n$ does not contain any linear subspaces which means that $\mathbb{R}^n \cap A \subset \mathbb{R}^n$ is a discrete subgroup. Thus, $\mathbb{R}^n/(\mathbb{R}^n \cap A)$ is n-dimensional; and if D is a small topologically embedded n-disc in $\mathbb{R}^n$, then the projection of D to the quotient G/A will be injective.

Thus, if A has codimension at most n, then the maximal value of m for which we may find such a set of points is $m = n$. Fix such a maximal collection of points $x_1, \dots, x_m, y_1, ..., y_m$. In the case where $\dim(M) \geq 2$, maximality implies that $M - \{x_1, \dots, y_m\}$ is contained in a single orbit $\mathcal{O}$. Set $X = M - \mathcal{O}$, which is a subset of $\{x_1, \dots, y_m\}$, hence finite. By definition of orbit, A preserves $\mathcal{O}$ and the complement X, which is what we needed to show.

In the case $\dim(M) = 1$, maximality implies that each connected component of $M - \{x_1, \dots, y_m\}$ is contained in a single orbit, and we may replace $\mathcal{O}$ with the union of these finitely many orbits and conclude as above. □

Returning to the situation of interest, suppose that $G = \mathrm{Homeo}_c(M)$ or $G = \mathrm{Diff}^r_c(M)$ (for $r \neq \dim(M) + 1$) and $A \subset G$ is a subgroup with small quotient. Then, combining Corollary and Lemma 2.6, we conclude that A is contained in the stabilizer of a finite set X, and A contains $\mathrm{Homeo}_c(M - X)$, or $\mathrm{Diff}^r_c(M - X)$ respectively. In particular, since the closure of $\mathrm{Homeo}_c(M - X)$ is the identity component $\mathrm{Stab}(X)_0$ of the (pointwise) stabilizer of X in $\mathrm{Homeo}_c(M)$, we have proved the following.

Theorem 2.7. *Let M be a manifold, and $A \subset \mathrm{Homeo}_c(M)$ a closed subgroup with small quotient. Then there is a finite set $X \subset M$ such that $\mathrm{Stab}(X)_0 \subset A \subset \mathrm{Stab}(X)$, and $\mathrm{Homeo}_c(M)/A$ is homeomorphic to an intermediate cover of $C_X = \mathrm{Homeo}_c(M)/\mathrm{Stab}(X)_0 \to \mathrm{Conf}_{|X|}(M) = \mathrm{Homeo}_c(M)/\mathrm{Stab}(X)$.*

In the diffeomorphism case, the closure of $\mathrm{Diff}^r_c(M - X)$ is the identity component of the group of diffeomorphisms with trivial r-jets $\mathrm{Stab}(X)^r_0$ at each point in X, i.e. the subgroup consisting of diffeomorphisms fixing X pointwise and with trivial derivatives up to order r at each point of X. (This can be shown by approximating a diffeomorphism with a trivial r-jet by a diffeomorphism that is the identity in a neighborhood of a point using a standard "blow-up" construction, see e.g. [25].)

Recall that an r-jet at $x \in M$ is an equivalence class of C^r embedding of $(\mathbb{R}^m, 0)$ into (M, x), where two such embeddings f and g are equivalent if the composition gf^{-1} has the same derivatives up to order r at 0 as the identity map. Similarly we define r-jet at $(x_1, ..., x_n) \in \mathrm{Conf}_n(M)$ to be an equivalence class of C^r embedding of a disjoint union of n copies of $(\mathbb{R}^m, 0)$ into $(M, x_1, ..., x_n)$ up to the same equivalence class with the natural topology. Let $\mathrm{Conf}^r_n(M)$ be the space of r-jets on $\mathrm{Conf}_n(M)$. There is a natural action of $\mathrm{Diff}^r_c(M)$ in $\mathrm{Conf}^r_n(M)$ and by observation, point stabilizer of this action is $\mathrm{Stab}(X)^r$. Let $J^r(m)$ be the jet group of $\mathbb{R}^m$. The bundle $\mathrm{Conf}^r_n(M) \to \mathrm{Conf}_n(M)$ is a principal bundle with structure group $J^r_n(m)$ a product of n copies of $J^r(m)$, and $\mathrm{Stab}(X) < \mathrm{Stab}(X)^r$ is a normal subgroup such that $\mathrm{Stab}(X)/\mathrm{Stab}(X)^r \cong J^r_n(m)$. Using this, we can show the following.

Theorem 2.8. *If $A \subset \mathrm{Diff}^r_c(M)$ is a closed subgroup with small quotient, then $\mathrm{Stab}(X)^r_0 \subset A \subset \mathrm{Stab}(X)$ for a finite set $X \subset M$, and $\mathrm{Diff}^r_c(M)/A$ has the structure of a cover of $\mathrm{Conf}^r_{|X|}(M)$ quotient by a linear group.*

Proof. When $r \neq \dim(M)+1$, the proof follows the same outline as in the proof of Theorem 2.7. The observation above implies that $\mathrm{Stab}(X)^r_0 \subset A \subset \mathrm{Stab}(X)$, so $A \cap \mathrm{Stab}(X)^r < \mathrm{Stab}(X)^r$ is a subgroup consisting of components of $\mathrm{Stab}(X)^r$. Therefore $\mathrm{Stab}(X)^r/A \cap \mathrm{Stab}(X)^r$ is discrete, which implies that $\mathrm{Diff}^r_c(M)/A \cap \mathrm{Stab}(X)^r$ is a cover of $\mathrm{Diff}^r_c(M)/\mathrm{Stab}(X)^r = \mathrm{Conf}^r_{|X|}(M)$.

Since $\mathrm{Stab}(X)^r < \mathrm{Stab}(X)$ is a normal subgroup, $(A \cap \mathrm{Stab}(X)^r) \subset A$ is also a normal subgroup. Therefore $A/(A \cap \mathrm{Stab}(X)^r) \subset \mathrm{Stab}(X)/\mathrm{Stab}(X)^r \cong J^r_n(m)$ is subgroup of the (linear) jet group. Thus, the quotient space $\mathrm{Diff}^r_c(M)/A$ is the quotient of $\mathrm{Diff}^r_c(M)/A \cap \mathrm{Stab}(X)^r$ by the linear group $A/(A \cap \mathrm{Stab}(X)^r)$, as we claimed.

In the exceptional case when $r = \dim(M)+1$, we consider instead the subgroup $\mathrm{Diff}^{r+1}_c(M) \subset \mathrm{Diff}^r_c(M)$. As before, by the small quotient property and local simplicity of $\mathrm{Diff}^{r+1}_c(M)$, we can conclude that

$$\mathrm{Stab}(X)^{r+1}_0 \subset A \cap \mathrm{Diff}^{r+1}_c(M) \subset \mathrm{Stab}(X)$$

Since the closure of $\mathrm{Stab}(X)^{r+1}_0$ in $\mathrm{Diff}^r_c(M)$ is $\mathrm{Stab}(X)^r_0$, we know that $\mathrm{Stab}(X)^r_0 \subset A \subset \mathrm{Stab}(X)$, and can run the same argument as above. □

Using this, we can quickly derive the orbit classification (Theorem 1.1) stated in the introduction. For this we need a version of the classical invariance of domain theorem, which for completeness we recall here.

Lemma 2.9 (Invariance of domain for finite-dimensional CW complex)**.** *An n-dimensional CW complex has no injective continuous image of $\mathbb{R}^{n+1}$.*

Proof. Assume that there is such an embedding $f : \mathbb{R}^{n+1} \to P$ for a CW complex P. Let D be a closed disk in $\mathbb{R}^{n+1}$ and B be the interior of D. The image $f(D) \subset P$ is compact so it only intersects finitely many cells. Find $x \in f(B)$ such that x lands on the maximal dimensional cell $C(x)$ within cells that intersect $f(D)$. By assumption on the maximal dimension, we know that $f^{-1}(C(x))$ is open in $\mathbb{R}^{n+1}$. This shows that there is an injective, continuous image of $\mathbb{R}^{n+1}$ inside $C(x)$, which contradicts the well known invariance of domain theorem for Euclidean spaces. □

Proof of Theorem 1.1. Suppose that $\mathrm{Homeo}_c(M)$ acts nontrivially on a finite-type CW complex N. By automatic continuity [14], this action is necessarily continuous. Then for any $x \in N$, the stabilizer G_x of x is a closed subgroup of $\mathrm{Homeo}_c(M)$, and $\mathrm{Homeo}_c(M)/G_x$ is continuously and injectively mapped to N with the orbit of x as the image. By Lemma 2.9, the stabilizer G_x is a small quotient subgroup (of codimension at most $\dim(N)$) so by Theorem 2.7, $\mathrm{Homeo}_c(M)/G_x$ is homeomorphic to a cover of $\mathrm{Conf}_n(M)$ for some n.

The same argument applies for a continuous action of $\mathrm{Diff}^r_c(M)$ on N, and we conclude that the orbit is the continuous, injective image of a cover of one of the spaces given in Theorem 2.8. □

Remark. For a fixed manifold M, and fixed $r > 0$, it would be interesting to give a complete classification of the spaces $\mathrm{Diff}_c^r(M)/A$ that occur in each dimension. Theorem 2.8 reduces this to a problem about understanding small quotient subgroups of the jet group.

2.1. **Admissible covers.** We now discuss which covers can occur as quotients of $\mathrm{Homeo}_c(M)$ by small quotient, closed subgroups. For a finite set $X \subset M$, let $\mathrm{Stab}(X)$ denote its (setwise) stabilizer, and $\mathrm{Stab}(X)_0$ the identity component of $\mathrm{Stab}(X)$. There is a fiber bundle $\mathrm{Stab}(X) \to \mathrm{Homeo}_c(M) \to \mathrm{Conf}_{|X|}(M)$. From the long exact sequence of homotopy groups, we have the following exact sequence

$$\pi_1(\mathrm{Homeo}_c(M)) \xrightarrow{ev} \pi_1(\mathrm{Conf}_{|X|}(M)) \xrightarrow{p} \pi_0(\mathrm{Stab}(X)) \to 1 \tag{1}$$

where the evaluation map ev takes a loop f_t in $\mathrm{Homeo}_0(M)$ to the path of configurations $f_t(X)$. Let E_X denote the image of ev. From the long exact sequence of homotopy groups, we know that

$$\pi_1(\mathrm{Conf}_{|X|}(M))/E_{|X|} \cong \mathrm{Stab}(X)/\,\mathrm{Stab}(X)_0$$

We also have the following bundle map.

$$\begin{array}{ccccc}
\mathrm{Stab}(X)/\,\mathrm{Stab}(X)_0 & \longrightarrow & \mathrm{Homeo}_c(M)/\,\mathrm{Stab}(X)_0 & \longrightarrow & \mathrm{Homeo}_c(M)/\,\mathrm{Stab}(X) \\
\| & & \| & & \| \\
\pi_1(\mathrm{Conf}_{|X|}(M))/E_{|X|} & \longrightarrow & C_X & \longrightarrow & \mathrm{Conf}_{|X|}(M)
\end{array}$$

where C_X is some cover of $\mathrm{Conf}_{|X|}(M)$ associated to the subgroup $E_{|X|}$ such that $\pi_1(C_X) = E_{|X|}$. By Theorem 2.7, if A is a closed, small quotient subgroup, then we have $\mathrm{Stab}(X)_0 \subset A \subset \mathrm{Stab}(X)$ for some finite set X. Therefore we know that $\mathrm{Homeo}_0(M)/A$ is some intermediate cover of $\mathrm{Conf}_{|X|}(M)$. Those intermediate covers that arise in this way will be called *admissible covers*, and C_X will be called the *maximal admissible cover*, with deck group is $\pi_1(\mathrm{Conf}_{|X|}(M))/E_{|X|}$.

We have the following lifting property.

Proposition 2.10. *For a point $b \in M$, let $C_b \to M$ be an admissible cover. Then the natural lifting is a well-defined homomorphism $L : \mathrm{Homeo}_c(M) \to \mathrm{Homeo}(C_b)$ such that the action L is transitive.*

The map $\pi_1(\mathrm{Conf}_{|X|}(M)) \to \mathrm{Stab}(X)/\,\mathrm{Stab}(X)_0$ from Equation (1) is a generalization of the "point pushing" map, familiar from the study of mapping class groups of surfaces with a marked point. A loop α_t, $t \in [0, 1]$, in $\mathrm{Conf}_{|X|}(M)$ based at X can be thought of an isotopy of the points X, which can be extended to an ambient isotopy α'_t in $\mathrm{Homeo}_0(M)$ or in $\mathrm{Diff}_0(M)$. Though this extension is not canonical, the component of α'_1 in $\mathrm{Stab}(X)$ is well defined and gives the map described above.

In general, the image E_X of ev is contained in $\pi_1(\mathrm{PConf}_{|X|}(M)) \subset \pi_1(\mathrm{Conf}_{|X|}(M))$. It is a central subgroup of $\pi_1(\mathrm{Conf}_{|X|}(M))$, since if γ is a loop in $\pi_1(\mathrm{Conf}_n(M))$ based at

$X = \{x_1, \dots x_n\}$, and f_t a loop in $\mathrm{Homeo}_0(M)$, then $f_t(\gamma)$ gives an isotopy between γ and $ev(f_t)\gamma ev(f_t)^{-1}$, so these two elements of $\pi_1(\mathrm{Conf}_n(M))$ agree.

Low dimensional examples. In low dimensions, it is possible to give a complete description of which covers occur. If $\dim(M) \leq 3$, then homeomorphisms are all isotopic to smooth diffeomorphisms, so the inclusion of $\mathrm{Homeo}_0(M)$ into $\mathrm{Diff}_0(M)$ is a homotopy equivalence. This was proved by Munkres [20, Theorem 6.3] and Smale and Whitehead [28, Cor. 1.18] for surfaces, and in dimension 3 follows from work of Cerf [3] and Hatcher's proof of the Smale conjecture [10]. Many of the results cited below were originally proved in the smooth category, but we give the equivalent statement for homeomorphism groups.

- **Dimension 1.** It is an easy exercise to see that $\pi_1(\mathrm{Homeo}_0(S^1)) = \mathbb{Z}$ and also $\pi_1(\mathrm{PConf}_n(S^1)) = \mathbb{Z}$. Thus, all admissible covers are intermediate covers of the covering $\mathrm{PConf}_n(S^1) \to \mathrm{Conf}_n(S^1)$.
- **Higher genus surfaces.** For a surface S_g of genus $g > 1$, Earle-Eells [4] showed that $\mathrm{Homeo}_0(S_g)$ is contractible. Therefore $E_X = 1$ for any X and all covers of $\mathrm{Conf}_n(S_g)$ are admissible.
- **Sphere.** By Smale [24] we have $\pi_1(\mathrm{Homeo}_0(S^2)) \cong \pi_1(SO(3)) = \mathbb{Z}/2$. For $n \geq 3$, the center of $\pi_1(\mathrm{Conf}_n(S^2))$ is $\mathbb{Z}/2$ (see [6, §9.1]), which is exactly the image of ev.
- **2-torus.** We have $\pi_1(\mathrm{Homeo}_0(T^2)) \cong \pi_1(T^2) = \mathbb{Z}^2$ (see [4]). As remarked above, the induced map $\pi_1(T^2) \to \pi_1(\mathrm{Homeo}_0(T^2)) \stackrel{ev}{\to} \pi_1(\mathrm{PConf}_n(T^2))$ has image in the center of $\pi_1(\mathrm{Conf}_n(T^2))$. It is injective, since composition with the forgetful map (forgetting all but one point of x) gives a map $\pi_1(\mathrm{PConf}_n(T^2)) \to \pi_1(\mathrm{PConf}_1(T^2)) = \pi_1(T^2)$ on which this is clearly an isomorphism. Moreover, since kernel of this map $\pi_1(\mathrm{PConf}_n(T^2)) \to \pi_1(T^2) = \mathbb{Z}^2$ is center-free, we can conclude that the image of ev is the center of $\pi_1(\mathrm{Conf}_n(T^2))$. With this observation, we have recovered the following theorem of Birman.

Corollary 2.11 (Birman, see [21] Proposition 4.2 and [1])**.** *The center of* $\pi_1(\mathrm{Conf}_n(T^2))$ *is* $\mathbb{Z}^2$

Higher dimensions. When $\dim(M) \geq 3$, we know that $\pi_1(\mathrm{PConf}_n(M)) = \pi_1(M) \times ... \times \pi_1(M)$. By connecting two points in X by a path, we can see that the image lands in the diagonal. This gives the following.

Observation 2.12. *When* $\dim(M) \geq 3$*, the image* E_n *is the diagonal of* E_1*.*

In special cases, the geometry of M can give some insight into the image of E_1. For instance, if M is a compact manifold admitting a metric of negative curvature, then $\pi_1(M)$ is center-free, so every cover is admissible. By contrast, if M is a compact Lie group, since $M < \mathrm{Homeo}_0(M)$ and the projection to M is a self-homeomorphism of M, we know that the image E_1 is everything. So if $\dim(M) \geq 3$ and M is a Lie group, then E_n is the whole diagonal.

3. Application: actions of $\mathrm{Homeo}(M)$ on N when $\dim(M) \leq \dim(N)$

In [11], Hurtado classifies the homomorphisms $\mathrm{Diff}_0(M) \to \mathrm{Diff}_0(N)$ when M is closed and $\dim(M) \leq \dim(N)$. He shows that they are all constructed by lifting diffeomorphisms to some cover of M and then injectively, continuously mapping these covers disjointly in N. In this section, we prove the analogous result for homeomorphism groups, and also give an alternative ending to Hurtado's proof.

Theorem 3.1. *Let M be a closed manifold and N a manifold such that $\dim(M) = \dim(N)$. If $\rho : \mathrm{Homeo}_0(M) \to \mathrm{Homeo}(N)$ is a homomorphism, then there exists a countable collection of admissible covers M_i of M, and injective and continuous maps $\phi : M_i \to N$ with disjoint images, such that ρ is supported on $\bigcup_i \phi(M_i)$ and conjugate to the lifted action on each copy of M_i.*

Proof. Let ρ be as in the statement of the Theorem. By simplicity of $\mathrm{Homeo}_0(M)$, either ρ is faithful, or it is trivial (in which case the theorem is vacuously true). By automatic continuity [14], ρ is continuous. Take any point $x \in N$ not globally fixed by the action, and let G_x denote its stabilizer. Then $\mathrm{Homeo}_c(M)/G_x$ is continuously and injectively mapped to N with the orbit of x as the image. By Theorem 2.7 and our assumption on dimension, this is a covering of $\mathrm{Conf}_1(M) \cong M$ mapping injectively continuously in N (and in particular we must have $\dim(M) = \dim(N)$). Since orbits are pairwise disjoint, this proves the theorem. □

More generally, we have the following.

Theorem 3.2. *Let M be a compact manifold, and N a manifold such that $\dim(M) = \dim(N)$. If $\rho : \mathrm{Diff}_0^r(M) \to \mathrm{Homeo}(N)$ is a nontrivial continuous homomorphism, then there exist a countable collection of covers M_i of M, and continuous injective maps $\phi : M_i \to N$ with disjoint images, such that ρ is supported on $\bigcup_i \phi(M_i)$ and conjugate to the lifted action on each copy of M_i.*

Furthermore, if N has the structure of a C^s manifold for some $s \geq 0$, and the image of ρ lies in $\mathrm{Diff}^s(M)$, then $s \leq r$ and ϕ is a C^s embedding; if $s \geq 1$ and N is connected and compact, then N is a single finite cover of M.

Proof. Let $n = \dim(M) = \dim(N)$. By Theorem 2.8, we know that the minimum dimension of a nontrivial orbit of $\mathrm{Diff}_0^r(M)$ acting on N is n, in which case the action comes from a lift of the natural action of $\mathrm{Diff}_0^r(M)$ on $\mathrm{Conf}_1(M) = M$ to a finite cover.

Now suppose the image of ρ lies in $\mathrm{Diff}^s(M)$, and let ρ_i be the restriction of ρ to $\phi_i(M_i) \subset M$. Let $p_i : M_i \to M$ be projection. Then $p_i\rho_i$ is a continuous injection $\mathrm{Diff}_0^r(M) \to \mathrm{Diff}_c^s(M)$ obtained by a conjugacy induced from $p_i\phi_i$. By a theorem of Takens [26], one can conclude that $r \geq s$ and the conjugacy is of class C^s, hence ϕ_i is also.

If N is compact, then only finite covers can occur as nontrivial orbits. This can be seen by considering a diffeomorphism f with support on a small ball in M and nontrivial derivative at some point z; if M_i were an infinite cover, then $\phi(M_i)$ having compact closure would mean that we could find points x_n and z_n both converging to the same point in N, but where the derivative of $\phi(f)$ at x_n was trivial, and at z_n pulls back to df_z. Since $\phi(M_i)$ is both closed and open in N, we conclude there is only one orbit. □

Combining Theorem 3.2 with Hurtado's continuity result for diffeomorphisms of [11, Theorem 1.2] reproves Theorem 1.3 in [11] when M is compact.

4. Generalized flat bundle structure for actions by homeomorphisms

In this section, we give the proof of Theorem 1.3.

Definition 4.1. Let B, F be topological spaces. A space E is a (generalized) flat bundle with base space B and fiber F if there exists a homomorphism $\phi : \pi_1(B) \to \mathrm{Homeo}(F)$ such that

$$E = (\tilde{B} \times F)/\pi_1(B)$$

where the quotient is by the diagonal action of $\pi_1(B)$ via deck transformations on the universal cover $\tilde{B}$ and by ϕ on F.

Each generalized flat bundle is, in particular, a topological F-bundle over B via the map $p : E \to B$ induced by projection to the first factor in $\tilde{B} \times F$. If B and F are topological manifolds, then this is just the usual definition of a topological *flat* or *foliated* bundle. If B and F are smooth manifolds and ϕ is an action by diffeomorphisms, this agrees with the differential geometric notion of E admitting a flat connection.

Proof of Theorem 1.3. Let $\rho : \mathrm{Homeo}_0(M) \to \mathrm{Homeo}(N)$ be an action without global fixed point. Since an orbit is is a subset of N which has dimension $< 2\dim(M)$, no orbit can be a cover of $\mathrm{Conf}_n(M)$ for $n > 1$. Therefore every orbit is the image of a cover of $\mathrm{Conf}_1(M) = M$.

We first define a projection map p and fiber F. Let $y \in N$ be any point. Then $\mathrm{Stab}(y) \subset M$ is a closed subgroup with small quotient, so there exists a unique point $x \in M$ such that $\mathrm{Stab}(x)_0 \subset \mathrm{Stab}(y)$ by Theorem 2.7. This assignment defines a map $p : N \to M$, which is ρ-equivariant in the sense that $p\rho(f)(y) = f(p(y))$ for all $y \in N$ and $f \in \mathrm{Homeo}_0(M)$. We now show that p is continuous. To see this, let B be a small open ball in M, and let $H(B)$ denote the group of homeomorphisms supported on B. We claim that

$$p^{-1}(B) = N - \mathrm{Fix}(\rho(H(B)))$$

which is an open set, so proves continuity. To prove the claim, let $y \in p^{-1}(B)$, and take any $f \in \mathrm{Homeo}_0(M)$ supported on B such that $fp(y) \neq p(y)$. Then $y \notin \mathrm{Fix}(\rho(f))$. Conversely, if $y \notin p^{-1}(B)$ and f is supported on B, then $\rho(f)$ fixes y. This gives the desired equality.

Now fix a base point $b \in M$ and let $F := p^{-1}(b)$. Since $\rho(\mathrm{Stab}(b))$ preserves F, the action ρ defines a representation $\mathrm{Stab}(b) \to \mathrm{Homeo}(F)$. Also, for any $y \in F$, by definition of F, we have $\mathrm{Stab}(y) \subset \rho(\mathrm{Stab}(b)_0)$, so this representation factors through $\mathrm{Stab}(b)/\mathrm{Stab}(b)_0$, giving a homomorphism

$$\phi : \mathrm{Stab}(b)/\mathrm{Stab}(b)_0 \to \mathrm{Homeo}(F).$$

Let $E_b \subset \pi_1(M)$ be the image of the evaluation map for $X = \{b\}$ as defined in Section 2.1. Let $\Gamma := \operatorname{Stab}(b)/\operatorname{Stab}(b)_0 \cong \pi_1(M)/E_b$. Let C_b be the associated cover of M, and let $\pi : C_b \to M$ be the natural projection. Γ acts on C_b by deck transformations, and on F by ϕ. Note that the action of Γ on C_b is equivalent to the lifting action L of $\operatorname{Stab}(b) \subset \operatorname{Homeo}_c(M)$ defined in Proposition 2.10.

We claim that N is naturally homeomorphic to $(C_b \times F)/\Gamma$, with orbits of the action of $\operatorname{Homeo}_c(M)$ corresponding to the images of the "horizontal" sets $C_b \times \{y\}$ in the quotient. To see this, fix a basepoint $\tilde{b}$ in $\pi^{-1}(b)$. Let L denote the lifted action of $\operatorname{Homeo}_c(M)$ on C_b as in Proposition 2.10. For $y \in F$ and $m \in C_b$ let $f_m \in \operatorname{Homeo}_c(M)$ denote any homeomorphism satisfying $L(f_m)(m) = \tilde{b}$, and define a map $(C_b \times F) \to N$ by

$$(m, y) \mapsto \rho(f_m)^{-1}(y).$$

This is independent of the choice of homeomorphism f_m since if g_m is another such choice, then $g_m = hf_m$ for some $h \in \operatorname{Stab}(b)_0$. We claim that this descends to a well-defined map

$$I : (C_b \times F)/\Gamma \to N$$

which will be the desired homeomorphism.

To show I is well defined, we need to show that if $a \in \Gamma$, then $I(a(m), \phi(a)(y)) = I(m, y)$. For $\hat{a} \in \operatorname{Stab}(b)$ a representative of $a \in \Gamma$, we can take $f_{a(m)} = \hat{a} \circ f_m$ so we have

$$I(a(m), \phi(a)(y)) = \rho(f_m)^{-1}\rho(\hat{a})^{-1}(\phi(\hat{a})(y)) = I(m, y)$$

since ϕ agrees with ρ on F.

By construction, orbits of ρ are the images of level sets $C_b \times \{y\}$ in the quotient, and if $\rho(f_m)^{-1}(y) = \rho(f_{m'})^{-1}(y')$, for some $m, m' \in C_b$ and $y, y' \in F$, then $g := f_{m'}^{-1} f_m \in \operatorname{Stab}(b)$ and $\phi(g)(y) = y'$ and $L(g)(m) = m'$. This means that (m, y) is equivalent to (m', y'), so I is injective. Surjectivity follows from the fact that each orbit intersects F in at least one point, and the lifted action on C_b is transitive. Continuity of I and its inverse are also easily checked from the definition.

Now $(C_b \times F)/\Gamma$ is easily seen to be a flat bundle over M with fiber F, as we may write it as $(\tilde{M} \times F)/\pi_1(M)$ where the action of $\pi_1(M)$ is by deck transformations on the first factor, and by the action of $\pi_1(M) \to \pi_1(M)/E_b \to \operatorname{Homeo}(F)$ on F. This gives the desired flat bundle structure on N. □

Remark. In general, the fiber $F = p^{-1}(x)$ may not be a manifold. For a concrete example, Bing's "dog bone space" is a non-manifold space F such that $F \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4$. Then we may take $M = \mathbb{R}^n$, for $n \geq 3$ and a product action of $\operatorname{Homeo}_c(M)$ on $F \times M = \mathbb{R}^{n+3}$, for which the fiber will be F.

However, since the product of F with a ball is a manifold, we can conclude that F is a *generalized manifold* or *homology manifold* in the sense of [30, Chapter 8]. In the case where M has codimension 1 or 2, all homology manifolds are manifolds (see [2, Theorem 16.32]) so F is necessarily a manifold. See [30, Chapter 8] for further background.

5. Application: extension problems for homeomorphism groups

Now we will apply our results to the extension problem as introduced in [8] and further discussed in [15]. Recall from the introduction that, if W is a connected manifold with $\partial W = M$, there is a natural "restrict to the boundary" map

$$\mathrm{Res}(W, M) : \mathrm{Homeo}_0(W) \to \mathrm{Homeo}_0(M),$$

which is surjective, and the extension problem asks whether $\mathrm{Res}(W, M)$ has a group theoretic section. We will prove the following stronger version of Theorem 1.5.

Theorem 5.1. *Let M be a manifold of dimension at least 2 such that the deck group $\pi_1(M)/E_x$ of a maximal admissible cover of M for a singleton $\{x\}$ has no nontrivial action on $[0, 1)$. Then $\mathrm{Res}(W, M)$ has a section if and only if $M = S^n, W = \mathbb{D}^{n+1}$ and the extension is conjugate to the standard coning.*

Corollary 5.2. *The following manifolds have no section of $\mathrm{Res}(W, M)$, for any manifold W bounding M.*

1. *Any manifold $M \neq S^n$ such that the maximal admissible cover of M is a finite cover, such as when M is a compact Lie group.*
2. *Any manifold M such that $\pi_1(M)$ has no nontrivial action on $[0, 1)$, for example when $\pi_1(M)$ is*
   (a) *an arithmetic lattice of higher $\mathbb{Q}$-rank (Witte-Morris [31])*
   (b) *a group generated by torsion elements, such as the mapping class groups of a surface, a reflection group, etc.*

Remark. In contrast with Theorem 5.1, when $M = S^1$ and $W = \mathbb{D}^2$ there are infinitely many different, non-conjugate extensions. These are discussed and classified in Section 7 below.

Proof of Theorem 5.1. Assume that ρ is a section for $\mathrm{Res}(W, M)$. Define

$$W' = W - \mathrm{Fix}(\rho(\mathrm{Homeo}_0(M))).$$

By Theorem 1.3, there is a canonical flat bundle structure $F \to W' \to M$, where W' is foliated by orbits of the action ρ. Since $\dim(W) - \dim(M) = 1$, we know that the fiber F is a 1-dimensional manifold. Identifying $F = p^{-1}(b)$, where b is the basepoint for $\pi_1(M)$ in M, there is a unique connected component C of F that intersects $\partial W = M$, and it intersection with M is the singleton $\{b\}$. As in the proof of Theorem 1.3, the flat bundle structure is determined by the representation $\phi : \pi_1(M)/E_b \to \mathrm{Homeo}(F)$. Since the image of ϕ fixes $b \in \partial W$, it also preserves the component C. Since the action of ϕ on C is trivial, the connected component W'' of W' containing ∂W is homeomorphic to $(0, 1] \times M$, and in these coordinates ρ is the product action. Let $em : (0, 1] \times M \to W$ denote this embedding of W'' in W.

Fix $x \in M$, and let $r_n \to 0$ be a sequence in $(0, 1]$ such that $em(r_n, x)$ converges to some point $\alpha \in N$ as $n \to \infty$. Note that α is necessarily a fixed point of ρ.

Claim 5.3. *$em(r_n \times M) \to \alpha$ as $n \to \infty$; i.e., for any neighborhood U of α, there exists N such that $em(r_n \times M) \subset U$ for all $n > N$.*

Proof. Suppose for contradiction that U does not contain $em(r_n \times M)$ for all large n, so there exists a sequence $em(r_n, x_n) \in W'' - U$. Since M is compact, after passing to a subsequence we may assume that x_n converges to a point $y \in M$. Choose $f \in \mathrm{Homeo}_0(M)$ such that $f(x) = y$. Since $x_n \to y$, we may find a convergent sequence of homeomorphisms $f_n \to f$ such that $f_n(x_n) = y$. Then $\rho(f_n)em(r_n, x_n) = em(r_n, y) \notin U$. But $\rho(f^{-1})\rho(f_n)em(r_n, x_n) = em(r_n, x) \to \alpha$ and $\rho(f)(\alpha) = \alpha$. This contradicts continuity of ρ. □

Let B be any open ball around α and S be the boundary of B. We call a codimension 1 closed submanifold $X \subset W$ *separating* if $W - X$ has two components. If X is separating, the *exterior* component $\mathrm{Ext}(X)$ is the component of $W - X$ containing ∂W, and the *interior* $\mathrm{Int}(X)$ is the other. In particular, $B = \mathrm{Int}(S)$ and $W - \bar{B} = \mathrm{Ext}(B)$. Note that $em(r_n \times M)$ is separating and $\mathrm{Ext}(em(r_n \times M)) = em((r_n, 1) \times M)$ while $\mathrm{Int}(em(r_n \times M)) = W - em([r_n, 1) \times M)$. The following easy claim implies that whenever $em(r_n \times M) \subset B$, we have that $\mathrm{Int}(em(r_n \times M)) \subset B$.

Claim 5.4. *If X, Y are disjoint, separating manifolds in a manifold W with boundary, and $Y \subset \mathrm{Int}(X)$, then $\mathrm{Int}(Y) \subset \mathrm{Int}(X)$.*

Proof. First, $(W - Y) \cap \mathrm{Ext}(X) = \mathrm{Ext}(X)$ which is connected. Then $\mathrm{Int}(Y) \cap \mathrm{Ext}(X)$ and $\mathrm{Ext}(Y) \cap \mathrm{Ext}(X)$ partition $(W - Y) \cap \mathrm{Ext}(X) = \mathrm{Ext}(X)$ into two connected components, so one of these sets must be empty. But $\mathrm{Ext}(X) \cap \mathrm{Ext}(Y)$ contains ∂W. Thus, $\mathrm{Int}(Y) \cap \mathrm{Ext}(X) = \emptyset$, so $\mathrm{Int}(Y) \subset \mathrm{Int}(X)$. □

From Claim 5.3, we obtain that for any ball B around α, there exists N such that $em(r_n \times M) \subset B$ for $n > N$. By Claim 5.4, we know that $\mathrm{Int}(em(r_n \times M)) \subset B$ for $n > N$. This also shows that the topological boundary of the set $em((0, 1] \times M)$ contains a unique point in the interior of W, which is the Hausdorff limit of the sets $em(x \times M)$ as $x \to 0$. From this it follows that W is the one point compactification of $(0, 1] \times M$. Theorem 5.1 now follows from the following proposition.

Proposition 5.5. *The one point compactification of $M \times (0, 1]$ is a manifold if and only if M is the sphere.*

The proof of this uses Poincaré conjecture and is a standard exercise: If N is the one-point compactifiction $M \times (0, 1] \sqcup \{\infty\}$ and this is assumed to be a manifold, one shows that the local homology groups $H_k(N, N - \infty; \mathbb{Z})$ are $\mathbb{Z}$ for $k = 0, \dim(M) + 1$ and 0 otherwise. This implies that M is a homology sphere. Since N agrees with the cone on M, it is simply connected; since M is homotopic to $N - \infty$ it is simply connected provided $\dim(M) > 1$ (in the one dimensional case, the Proposition is trivial). The Poincaré conjecture then implies that M is a sphere. □

Since the 2-dimensional torus has no nontrivial admissible cover, we obtain the following corollary.

Corollary 5.6. $\mathrm{Res}(H_g, S_g)$ *does not have a section for* $g = 1$*.*

While we do not know whether this holds for $g > 1$, in the next section we will prove that no surface has a section in the category Diff.

Theorem 5.1 has a further generalization as follows.

Theorem 5.7. *Let* M, W *be closed connected manifolds with* $\dim(W) = \dim(M)+1 \geq 3$*, and suppose that the deck group* $\pi_1(M)/E_x$ *of a maximal admissible cover of* M *for a singleton* $\{x\}$ *has no nontrivial action on* S^1*. There exists a nontrivial action of* $\mathrm{Homeo}_0(M)$ *on* W *if and only if either*

(1) $W = M \times S^1$ *and the action is trivial on the* S^1 *factor, or*
(2) $M = S^n, W = S^{n+1}$*, and the action is by doubling the standard coning.*

Proof of Theorem 5.7. Assume there exists a nontrivial homomorphism $\rho : \mathrm{Homeo}_0(M) \to \mathrm{Homeo}(W)$. As in the proof of Theorem 5.1 the assumption on $\pi_1(M)/E_x$ means that every connected component of $W - \mathrm{Fix}(\rho(\mathrm{Homeo}_0(M)))$ is a trivial bundle over M. If ρ has no fixed point, then the fiber is a closed 1 manifold since W is closed. Thus, $W \cong M \times S^1$ with the product action that is trivial on the S^1 factor.

If ρ has at least one fixed point, consider a point α in the frontier of $\mathrm{Fix}(\rho(\mathrm{Homeo}_0(M)))$ and a component W' of $W - \mathrm{Fix}(\rho(\mathrm{Homeo}_0(M)))$ whose closure contains α. Using the same argument as in the proof of Theorem 5.1, we may find a sequence $x_n \in W$ with $x_n \to \alpha$ such that each x_n has nontrivial orbit under ρ, and use this to conclude that α is an isolated fixed point, compactifying one end of W'. Since W' is a manifold, it must be an interval bundle over S^n, hence $M = S^n$. We have just shown that each fixed point of ρ is isolated, so there are only finitely many fixed points. The complement of the fixed set in W is therefore a connected, non-closed 1 manifold bundle over S^n, i.e. $(0, 1) \times S^n$; it is compactified by a single point at each end, so $W = S^{n+1}$. □

6. Application: bundle structure and the extension problem in the differentiable case

Recall that in Section 2 we proved the following.

Theorem 2.8. If $A \subset \mathrm{Diff}^r_c(M)$ is a closed subgroup with small quotient, then $\mathrm{Stab}(X)^r_0 \subset A \subset \mathrm{Stab}(X)$ for a finite set $X \subset M$, and $\mathrm{Diff}^r_c(M)/A$ has the structure of a cover of $\mathrm{Conf}^r_{|X|}(M)$ by a linear group.

In this section we use this to give a classification theorem for smooth actions, and in particular solve the extension problem of [8].

Theorem 6.1. *Let M be a connected, closed manifold and N a manifold with* $\dim(N) < 2\dim(M)$*. Suppose there is a nontrivial, continuous homomorphism* $\mathrm{Diff}_0^s(M) \to \mathrm{Diff}_0^r(N)$*, for $r \geq 1$. Then $r \leq s$, the action is fixed point free, and N is a C^r-manifold bundle over M.*

Remark. As an example illustrating the bound on dimension in the theorem above, note that the natural action of $\mathrm{Diff}_0^s(M)$ on the projectivized unit tangent bundle of an n-manifold M gives a nontrivial, continuous action on a $2n-1$-manifold.

Before giving the proof of Theorem 6.1 we state and prove the (negative) solution to the extension problem as a corollary.

Corollary 6.2. *Let M be a closed, connected manifold and let W be a compact manifold with $\partial W = M$. Then there is no continuous extension* $\mathrm{Diff}_0^r(M) \to \mathrm{Diff}_0^r(W)$ *for any* $1 \leq r \leq \infty$*, and no abstract extension* $\mathrm{Diff}_0^\infty(M) \to \mathrm{Diff}_0^\infty(W)$*.*

Proof. The first statement follows from Theorem 6.1 and the fact that a compact manifold W with boundary M does not have the structure of a bundle over M. For the second statement, we only additionally require that every extension $\mathrm{Diff}_0^\infty(M) \to \mathrm{Diff}_0^\infty(W)$ is automatically continuous, which is the statement of [11, Theorem 1.2]. □

Proof of Theorem 6.1. First, we treat the case where the action has no global fixed points. We will later show that this is always the case. Let $\rho : \mathrm{Diff}_0^s(M) \to \mathrm{Diff}_0^r(N)$ be an action without global fixed point. As in the proof of Theorem 1.3, we first define a projection map $p : N \to M$ where $p(y) = x$ if $\mathrm{Stab}(y) \subset \mathrm{Stab}(x)$. For $x \in M$, let $J_0(x) \subset \mathrm{Diff}_0^s(M)$ denote the subgroup of $\mathrm{Stab}(x)_0$ consisting of diffeomorphisms whose s-jet at x is trivial. By Theorem 2.8, we also know that $J_0(x) \subset \mathrm{Stab}(y)$. The same argument as in the proof of Theorem 1.3 shows that for any open ball B,

$$p^{-1}(B) = N - \mathrm{Fix}(\rho(H(B)))$$

where $H(B)$ denotes the diffeomorphisms supported on B, so p is continuous. Now fix a basepoint $b \in M$. The next step is to argue that the fiber $F := p^{-1}(b)$ is a C^r submanifold of N.

Fix local coordinates around b, identifying b with the origin in $\mathbb{R}^n$. Let $H \subset \mathrm{Stab}_0(b) \subset \mathrm{Diff}^\infty(M) \subset \mathrm{Diff}^s(M)$ be a subgroup such that the germ of each element of H at b agrees with an element of $\mathrm{SL}(n, \mathbb{R})$ in our local coordinates. We may additionally choose H so that the map $H \to \mathrm{SL}(n, \mathbb{R})$, given by taking the germ at b, has a continuous section in a neighborhood of the identity.

Since H fixes b, $\rho(H)$ preserves the fiber F, so acts on F by homeomorphisms. We claim that this action factors through the map $H \to \mathrm{SL}(n, \mathbb{R})$ given by taking germs at B. Indeed, if h_1, h_2 have the same germ at b, then $h_1 h_2^{-1} \in J_0(x)$, so acts trivially on F. We denote the induced action by $\rho' : \mathrm{SL}(n, \mathbb{R}) \to \mathrm{Homeo}(F)$. Since ρ is continuous and H was chosen to have a local continuous section, this action is continuous. Although we do not know *a priori* that F has a smooth structure and the action is by diffeomorphisms of F, each element $\rho'(g)$ extends to a C^r diffeomorphism of N.

If $\mathcal{O}$ is an orbit of $\rho'(H)$ that is not a singleton, then $\mathcal{O}$ is the quotient of $\mathrm{SL}(n,\mathbb{R})$ by some closed subgroup. Since it has dimension at most $n-1$, it follows that the dimension is equal to $n-1$, and $\mathcal{O}$ is either S^{n-1} or $\mathbb{R}P^{n-1}$. (This is classical, and follows from e.g. [12, Theorem II.3.2] by considering the action of the compact subgroup $\mathrm{SO}(n)$.) Since $F = p^{-1}(b)$ is preserved by ρ', our assumption on dimension means that each connected component of F is either pointwise fixed, or C^r diffeomorphic to S^{n-1} or $\mathbb{R}P^{n-1}$ with the standard action. Note that F may well be disconnected, this happens for example with the action of $\mathrm{Diff}(\Sigma)$ on the projectivized tangent bundle of $\widetilde{\Sigma}$, where Σ is a surface of genus at least 2.

Let $S \subset H$ be the subgroup whose germs lie in $\mathrm{SO}(n)$. Then S permutes the fibers over points near b, so there exists a neighborhood U of F such that $\mathrm{Fix}(\rho(S)) \cap U \subset F$, and the action of S on U factors through an action of $\mathrm{SO}(n)$. In the case where components of F are pointwise fixed, we may apply the local linearization theorem (see Theorem 1, §5.2 in [19]) which says that $\mathrm{Fix}_\rho(S)$ is a C^r submanifold and each point $x \in F$ admits a neighborhood U with a local coordinate chart $\phi : U \to V \subset \mathbb{R}^{\dim(N)}$ with $\phi(x) = 0$ and such that $F \cap U$ is mapped to a linear subspace of $\mathbb{R}^{\dim(N)}$ with $\phi\rho(f)\phi^{-1} = D\rho(f)$ for all $f \in S$. It follows that p is a surjective, proper C^r submersion, so $p : N \to M$ defines a C^r-bundle structure. Otherwise, the action of $\mathrm{SO}(n)$ on F is transitive, and that p defines a C^r-bundle follows from the classical slice theorem for compact transformation groups.

Since $\rho(\mathrm{Diff}^s_0(M))$ preserves fibers, it induces a faithful action on the base space M by C^r diffeomorphisms, which shows that $s \geq r$. This concludes the proof in the case where the action is fixed-point free.

For the general case, suppose now that $\rho : \mathrm{Diff}^s_0(M) \to \mathrm{Diff}^r_0(N)$ is an action with $\mathrm{Fix}(\rho) \neq \emptyset$. Let N' be a connected component of $N - \mathrm{Fix}(\rho)$ such that the frontier of N' contains points in $\mathrm{Fix}(\rho)$. (Such a component exists because $\mathrm{Fix}(\rho)$ is closed, and N is connected.) Since z is a global fixed point, there is a homomorphism $\mathrm{Diff}^s(M) \to \mathrm{GL}(T_z)$ given by $f \mapsto D\rho(f)_z$. Since $\mathrm{Diff}^\infty(M)$ is simple, the restriction of this homomorphism to $\mathrm{Diff}^\infty(M)$ is trivial.

Using the fact that M is compact, we may find a point $b \in M$ such that there exists a sequence $\{x_n\} \subset p^{-1}(b) \cap N'$ converging to a point $z \in \mathrm{Fix}(\rho)$. As before, let $H \subset \mathrm{Stab}_0(b) \subset \mathrm{Diff}^\infty(M)$ be a group that is locally linear at b, and S a locally orthogonal subgroup. If infinitely many points in the sequence x_n are fixed by $\rho(H)$, then by local linearization there exists $h \in S \subset H$ such that $\rho(h)$ has derivative bounded away from the identity at x_n, for each such n. This contradicts our observation that $\rho(h)$ has trivial derivative at z. Otherwise, we have a sequence of orbits $\rho(S)(x_n) = \mathcal{O}_n$ of the action of H, each homeomorphic to S^{n-1} or $\mathbb{R}P^{n-1}$.

Claim 6.3. *$\mathcal{O}_n$ converges to z.*

Proof. If not, there exists $z_n \in \mathcal{O}_n$ such that $z_n = g_n(x_n)$ converges to another point $z' \neq z$. Since $\mathrm{SO}(n)$ is a compact group, we can take a subsequence such that $g_n \to g \in \mathrm{SO}(n)$. Since the action is continuous, we know that $z_n = g_n(x_n) \to g(x_n) \to z$ which contradicts that $z' \neq z$. □

Now to conclude the proof of the theorem, let $h \in H$ be a locally hyperbolic element. Since the action of $\rho(h)$ on $\mathcal{O}_n$ is conjugate to the standard action, the claim above implies there is a sequence of points converging to z at which $\rho(h)$ fixes and has derivative bounded away from the identity, again giving a contradiction. We conclude that there is no global fixed point for ρ. □

7. Application: actions of $\mathrm{Homeo}_0(S^1)$ on compact surfaces

In this section, we will classify actions of $\mathrm{Homeo}_0(S^1)$ on compact surfaces. We will give the proof for the disc $\mathbb{D}^2$ which proves [18, Conjecture 2.2]. A complete classification of actions on other surfaces can be obtained by essentially the same argument, giving in particular a new proof of the main theorem of [18].

7.1. **The statement.** We first give a general procedure to construct actions of $\mathrm{Homeo}_0(S^1)$ on $\mathbb{D}^2$. (Compare [18, §2].) Let $L = [0,1]$ be the orbit space of the standard $SO(2)$ action on $\mathbb{D}^2$, where $r \in [0,1]$ represents the circle of radius r. Let $K \subset L$ be a closed subset including 0. We use the convention $S^1 = \mathbb{R}/\mathbb{Z}$ in the following. Let $a_0 : \mathrm{Homeo}_0(S^1) \to \mathrm{Homeo}(\mathbb{R}/\mathbb{Z} \times [0,1])$ be defined by

$$a_0(f)(\theta, r) = (f(\theta), \tilde{f}(r + \tilde{\theta}) - \tilde{f}(\tilde{\theta}))$$

where $\tilde{f} \in \mathrm{Homeo}_0(\mathbb{R})$ is any lift of $f \in \mathrm{Homeo}_0(S^1)$ to $\mathrm{Homeo}_0(\mathbb{R})$ and $\tilde{\theta} \in \mathbb{R}$ represents a lift of $\theta \in \mathbb{R}/\mathbb{Z}$. Note that this is well defined and independent of the choice of lifts.

Let T^k be the kth power of a standard Dehn twist in the closed annulus $\mathbb{R}/\mathbb{Z} \times [0,1]$ given by $T^k(\theta, r) = (\theta + kr, r)$, and let $a_k(f) = T^k a_0(f)(T^k)^{-1}$. For example,

$$a_1(f)(\theta, r) = T^1 a_0(f)(\theta - r, r) = T^1(f(\theta - r), \tilde{f}(\tilde{\theta}) - \tilde{f}(\tilde{\theta} - r)) = (f(\theta), \tilde{f}(\tilde{\theta}) - \tilde{f}(\tilde{\theta} - r)).$$

The fact that both $a_0(f)$ and $a_1(f)$ have the same first coordinate is a pure coincidence. This is not true for a_k when $k \neq 0, 1$.

Let $\lambda : L - K \to \{0, 1\}$ be a function which is constant on each component. For $a < b \in [0,1]$, let $n_{a,b}$ be the affine normalization $n_{a,b} : S^1 \times [0,1] \to S^1 \times [a,b]$ given by $n_{a,b}(\theta, r) = (\theta, a + r(b-a))$. We denote by $\rho_{K,\lambda}$ the action such that, for each component (a,b) of $L - K$, the restriction of $\rho_{K,\lambda}$ to its closure is given by

$$\rho_{K,\lambda}|_{S^1 \times [a,b]}(f) = n_{a,b} \circ a_k(f) \circ n_{a,b}^{-1}.$$

It is easy to check that $\rho_{K,\lambda}$ is indeed a continuous group action, since the first coordinate of a_0 and a_1 is just the standard action on S^1, and the second coordinate is also continuous.

We prove the following.

Theorem 7.1 (Classification of $\mathrm{Homeo}_0(S^1)$ actions on the disc)**.** *Any nontrivial homomorphism $\rho : \mathrm{Homeo}_0(S^1) \to \mathrm{Homeo}_0(\mathbb{D}^2)$ is conjugate to $\rho_{K,\lambda}$ for some K, λ. Two homomorphisms $\rho_{K,\lambda}$ and $\rho_{K',\lambda'}$ are conjugate to each other if and only if there is a homeomorphism $h : [0,1] \to [0,1]$ such that $h(K) = K'$ and λ and $h \circ \lambda'$ agree on all but finitely many components of $[0,1] - K$ on each closed interval in the interior of $[0,1]$.*

The proof will also show that actions on the half-open annulus (up to conjugacy) are the same as those on the disc under the identification of $[0,1) \times S^1$ with $\mathbb{D}^2 - 0$. Actions on the open and closed annulus, the sphere, and the torus have an analogous classification which can be obtained by the same proof.

We will use the following classical result on $SO(2)$ actions.

Lemma 7.2 ([19] Ch 6.5)**.** *Any faithful, continuous action of $SO(2)$ on $\mathbb{D}^2$ is conjugate to the standard action by rotations.*

This is also true for the sphere and the (open, closed, or half-open) annulus, while all actions of $SO(2)$ on the torus $S^1 \times S^1$ are conjugate to rotation of one S^1 factor.

Now suppose that $\rho : \mathrm{Homeo}_0(S^1) \to \mathrm{Homeo}_0(\mathbb{D}^2)$ is a representation. Using the automatic continuity result of Rosendal and Solecki [22], we know that ρ is continuous, and by simplicity of $\mathrm{Homeo}_0(S^1)$ we may assume ρ is faithful. By Lemma 7.2, we can also assume that the restriction of ρ to $SO(2)$ agrees with the standard action by rotations. We will apply several successive conjugations to put ρ in the form given by Theorem 7.1.

7.2. **First conjugation: coning on a closed, invariant set.** Let G_0 denote the stabilizer of $0 \in S^1$. The first conjugacy will put $\mathrm{Fix}(\rho(G_0))$ on the x-axis, so that the restriction of ρ to the set $SO(2) \cdot \mathrm{Fix}(\rho(G_0))$ agrees with coning.

Lemma 7.3. *The fixed point O of $\rho(SO(2))$ is a global fixed point for $\rho(\mathrm{Homeo}_0(S^1))$.*

Proof. This lemma has a direct proof which is given in [18, Proposition 6.1]. An alternative quick argument can be obtained by quoting the general classification [9, Theorem 1.1], since the closed subgroup $\mathrm{Stab}(O)$ contains $SO(2)$ (in particular, it contains a nonconstant path), and also contains any element that commutes with a nontrivial element of $SO(2)$. □

Let $L = [0,1]$ be the orbit space of the standard $SO(2)$ action on $\mathbb{D}^2$, where $r \in [0,1]$ represents the circle of radius r, and let p denote the projection $p : \mathbb{D}^2 \to L = [0,1]$. With the exception of O, every other $\rho(\mathrm{Homeo}_0(S^1))$-orbit is 1 or 2-dimensional. Let $U \subset \mathbb{D}^2$ be the union of all the 1-dimensional orbits and O. Then $K := p(U) \subset [0,1]$ is a closed subset. (Each 2-dimensional orbit is open by invariance of domain.) Define $h_1' : K - \{0\} \to \mathbb{D}^2$ by

$$h_1'(x) = \mathrm{Fix}(\rho(G_0)) \cap p^{-1}(x).$$

Away from O, this function is uniquely determined by its projection to the S^1 coordinate, which we denote by $g_1' : K - \{0\} \to S^1$.

Lemma 7.4. *The functions h_1' and g_1' are continuous on $K - \{0\}$.*

Proof. Continuity of h_1' implies continuity of g_1'. For h_1', we need to show that for a closed set $C \subset \mathbb{D}^2$, the set $h_1'^{-1}(C)$ is closed. We have the following computation:

$$h_1'^{-1}(C) = p(\mathrm{Fix}(\rho(G_0)) \cap C).$$

Since $\mathrm{Fix}(\rho(G_0))$ is a closed set and p is a proper map, we know that $h_1'^{-1}(C)$ is a closed set as well. □

By the Tietze extension theorem, we can extend the function g_1' to a continuous function $g_1 : (0, 1] \to S^1$. Let G_0 denote the stablizer of $0 \in S^1 = \mathbb{R}/\mathbb{Z}$ in $\mathrm{Homeo}_0(S^1)$. Since $\rho(G_0)$ fixes $g_1(r)$ for $r \in K$, the action of $\rho(f)$ on $p^{-1}(r)$ agrees with $\rho(f)(\theta, r) = (f(\theta - g_1(r)) + g_1(r), r)$. Define a homeomorphism h_1 of $\mathbb{D}^2$ by

$$h_1(\theta, r) = (\theta + g_1(r), r)$$

and $h_1(O) = O$. Then $h_1^{-1} \circ \rho \circ h_1|_{p^{-1}(K)}$ is "coning", i.e. $h_1^{-1} \circ \rho(f) \circ h_1(\theta, r) = (f(\theta), r)$ whenever $r \in K$. From now on, we replace ρ with its conjugate $h_1^{-1} \circ \rho \circ h_1$.

7.3. **Building block: Irreducible actions.** Call an action of $\mathrm{Homeo}_0(S^1)$ on a surface *irreducible* if there are no zero or 1 dimensional orbits. The following is an easy consequence of Theorem 1.1.

Corollary 7.5. *Up to conjugacy, there are only two irreducible actions of* $\mathrm{Homeo}_0(S^1)$ *on connected surfaces: the standard action on* $\mathrm{Conf}_2(S^1)$ *and the standard action on* $\mathrm{PConf}_2(S^1)$.

Proof. Let S be a connected surface with an irreducible $\mathrm{Homeo}_0(S^1)$ action. Since S is 2-dimensional, by invariance of domain every orbit is an open subsurface of S, and by Theorem 1.1 each is homeomorphic to either $\mathrm{PConf}_2(S^1)$ or $\mathrm{Conf}_2(S^1)$. Since S is connected, it cannot be covered by disjoint open subsurfaces, so is either $\mathrm{PConf}_2(S^1)$ or $\mathrm{Conf}_2(S^1)$. □

Since $\mathrm{Conf}_2(S^1)$ is homeomorphic to the Möbius band, this orbit type does not occur when S is orientable. So we may focus on $\mathrm{PConf}_2(S^1)$, which is homeomorphic to an open annulus. These are precisely the 2-dimensional orbits that occur in Theorem 7.1.

7.4. **Second conjugation: 2-dimensional orbits.** Let $S^1 \times (a, b)$ be a ρ-invariant open annulus on which ρ is irreducible. By Corollary 7.5 there exists $h \in \mathrm{Homeo}(S^1 \times (a, b))$ such that

$$h \circ \rho(f) \circ h^{-1} = n_{a,b} \circ a_0(f) \circ n_{a,b}^{-1}$$

We also have $\rho(r_\theta) = r_\theta = a_0(r_\theta)$, and therefore

$$h \circ r_\theta = r_\theta \circ h$$

Writing h in coordinates as $h(\alpha, r) := (g(\alpha, r), k(\alpha, r))$, the coordinate functions satisfy

$$g(\alpha + \theta, r) = g(\alpha, r) + \theta$$

$$k(\alpha + \theta, r) = k(\alpha, r)$$

for any θ. This shows that $k(\alpha, r) = k(r)$ and $g(\alpha, r) = \alpha + g(r)$ where $g(r)$ and $k(r)$ are continuous functions of $r \in [a, b]$ such that $g(a) = g(b) = 0$ and $k(a) = a$ and $k(b) = b$. Since for each 2-dimensional orbit of the form $S^1 \times (a, b)$, we have that $k(a) = a, k(b) = b$ and $k : [a, b] \to [a, b]$ is increasing, these k glue together to give a continuous function, which extends to a continuous function that is the identity outside of the union of the 2-dimensional orbits of ρ. Abusing notation, denote this function also by k, and let $h_2(\theta, r) = (\theta, k(r))$. This defines a homeomorphism of $\mathbb{D}^2$. Going forward, we replace ρ with its conjugate $h_2 \circ \rho \circ h_2^{-1}$. This simplifies the form of the associated function h conjugating ρ to the a_0 action given by Corollary 7.5, and we now have

$$h(\alpha, r) = (g(r) + \alpha, r). \tag{2}$$

Say that a 2-dimensional orbit has *degree* d if this map $g : [a, b] \to S^1$ is isotopic relative to the boundary to a degree d map (Recall that $g(a) = g(b) = 0$, so the notion of degree makes sense here). Our next goal is to prove the following.

Lemma 7.6. *Let $J \subset (0, 1]$ be a closed interval bounded away from 0. Among the 2-dimensional orbits that intersect $J \times S^1$, only finitely many have degree not in $\{0, 1\}$.*

Recall that G_0 denotes the stabilizer of 0 in $\mathrm{Homeo}_0(S^1)$. In general, we will use the notation G_θ for the stabilizer of θ. There are two 1-dimensional orbits of G_0 in $\mathrm{PConf}_2(S^1)$. For the a_0 action, these orbits are $\{(-r, r) \mid r \in (0, 1)\}$ and $\{(0, r) \mid r \in (0, 1)\}$. Thus, the 1-dimensional orbits of $\rho(\mathrm{Homeo}_0(S^1))$ on an irreducible annulus $S^1 \times (a, b)$ are the sets $\{h(0, r) \mid r \in (0, 1)\}$ and $\{h(n_{a,b}(-r, r)) \mid r \in (0, 1)\}$.

Proof of Lemma 7.6. Suppose for contradiction that there are infinitely many 2-dimensional orbits $U_k = S^1 \times (a_k, b_k)$, indexed by $k \in \mathbb{N}$, of degree different from 0 or 1 inside a compact sub-annulus of $\mathbb{D}^2 - O$. Without loss of generality, we assume they all have degree > 1 (the case where the degree is negative is similar) and assume that a_k converges monotonically to some $r \in (0, 1)$.

For a fixed orbit $U = S^1 \times (a, b)$, the 1-dimensional orbits of $\rho(G_0)$ in U are the sets

$$\{h \circ n_{a,b}(-r, r) | r \in (0, 1)\} = \{(g(t), t) | t \in (a, b)\}$$

and

$$\{h \circ n_{a,b}(-r, r) | r \in (0, 1)\} = \{(g(t) - \tfrac{t-a}{b-a}, t) | t \in (a, b).\}$$

where g is a degree d map, and hence the map $t \mapsto g(t) - \frac{t-a}{b-a}$ has degree $d - 1 \geq 1$.

For convenience, we now switch to working on the universal cover. Since $\rho(G_0)$ acts on $\mathbb{D}^2 - O$ with fixed points, we may lift it to an action $\tilde{\rho}$ on the universal cover of $\mathbb{D}^2 - O$; then the 1-dimensional orbits in U_k are continuous curves from (m, a_k) to $(m + d, b_k)$ and from (m, a_k) to $(m + d - 1, b_k)$, for $m \in \mathbb{Z}$. See Figure 1.

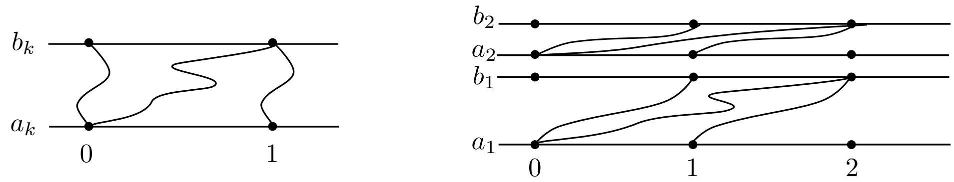

FIGURE 1. Orbits of G_0 in a degree 1 orbit for ρ (left) and in degree 2 orbits (right). Dots represent fixed points

Each 1-dimensional orbit of G_0 is canonically homeomorphic with $(0,1)$ via a map $\phi : \theta \mapsto \text{Fix}(G_0 \cap G_\theta)$. (The map ϕ depends on the orbit, but we suppress that notation for the time being.) Choose $\theta \in (0,1)$ and, let $x_k \in \{\theta\} \times (a_k, b_k)$ be a point contained in a 1-dimensional orbit of G_0. Let $\theta_k = \phi^{-1}(x_k)$. Pass to a subsequence so that θ_k converges to some $\theta_\infty \in S^1$. If $\theta_\infty \neq \theta$, then we may take $h \in G_0$ such that $h(\theta) \neq \theta$, but $h(\theta_k) = \theta_k$ for all k large. But since $\text{Fix}(\rho(h))$ is a closed set, it contains the point (θ, r), contradicting the fact that $h \notin G_\theta$. Thus, we conclude that $\theta_\infty = \theta$, i.e. every convergent "vertical" sequence of points $x_k = (\theta, r_k)$ contained in 1-dimensional orbits of G_0 corresponds, via the identifications defined by ϕ, to a sequence converging to the first coordinate θ. But this contradicts the existence of a sequence of orbits of degree > 1. □

7.5. **Final conjugation: normal form for 2-dimensional orbits.** We now conjugate the map on each 2-dimensional orbit so that it agrees on each orbit with a_0 or a_1. First, by Lemma 7.6, there are only finitely may orbits of degree neither 0 nor 1. Let h_3 be a homeomorphism that is supported on the union of these orbits and agrees with a power of a Dehn twist on each, conjugating the action on this orbit to the (normalized) action of a_0.

Let h_4 be a function supported on the union of the degree 0 orbits, conjugating each to a_0 (this is the function h from equation (2) in Section 7.4), and let h_5 be a function supported on the (closure of the) union of the degree 1 orbits, agreeing with $h(\theta, r) = (g(r) - \frac{r-a}{b-a} + \theta, r)$. We need to show that h_4 and h_5 are homeomorphisms; we give the details for h_4, the case of h_5 is completely analogous.

By construction, h_4 is continuous on each individual 2-dimensional orbit, and extends to a continuous function on the closure of each individual orbit. What we need to show is continuity at accumulation points of such orbits. Suppose that $S^1 \times (a_n, b_n)$ is a sequence of 2-dimensional orbits of degree 0 with $a_n \to r$ for some $r \neq 0$. Let g_n denote the function from equation (2) on $S^1 \times (a_n, b_n)$, and ϕ_n the identification with $(0,1)$ defined in the proof of Lemma 7.6 (where it was called ϕ). We need to show that g_n converges to the constant function 0. Recall that $\{(g_n(t) - \frac{t-a_n}{b_n-a_n}, t) | t \in (a_n, b_n)\}$ is a 1-dimensional orbit c_n. If g_n did not converge to 0, we could pass to a subsequence and find a sequence of points $t_n = \phi_n(\theta)$ such that $|g_n(t_n)| > \epsilon > 0$ where ϕ_n denotes the corresponding identification $\phi_n : \theta \to \text{Fix}(G_0 \cap G_\theta) \cap c_n$. But this contradicts the fact that $\phi_n(\theta)$ converges to (θ, r) as shown in Lemma 7.6. Thus, $h_3 \circ h_4 \circ h_5$ is a homeomorphism conjugating ρ into standard form.

7.6. **Characterization of conjugacy classes.** It remains only to show that two homomorphisms $\rho_{K,\lambda}$ and $\rho_{K',\lambda'}$ are conjugate to each other if and only if there is a homeomorphism $h : [0,1] \to [0,1]$ such that $h(K) = K'$ and λ and $h \circ \lambda'$ agree on all but finitely many components of $[0,1] - K$ on each closed interval in the interior of $[0,1]$. This is proved in [18, Proposition 2.3]. In brief, since K and K' are the union of 1-dimensional orbits, they are necessarily conjugate if the actions are. On any fixed two-dimensional orbit, there is a unique conjugacy between the two actions by Corollary 7.5, and if λ and $h \circ \lambda'$ differ on only finitely many components of $[0, c] - K$ for some $c < 1$, then these conjugacies can glue together to form a continuous homeomorphism. Conversely, if there is a conjugacy between the actions, one can identify the (necessarily finitely many) components on which they differ by looking at the image of a radial line under the conjugacy. Full details can be found in [18].

□

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