

# Automatic continuity for some homeomorphism groups and mapping class groups of noncompact manifolds

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## Abstract

We extend the proof of automatic continuity for homeomorphism groups of manifolds to non-compact manifolds and manifolds with marked points and their mapping class groups. Specifically, we show that, for any manifold  $M$  homeomorphic to the interior of a compact manifold, and a set  $X \subset M$  homeomorphic to the union of a Cantor set and finite set, the *relative homeomorphism group*  $\text{Homeo}(M, X)$  and the *mapping class group*  $\text{Homeo}(M, X)/\text{Homeo}_0(M, X)$  have the property that any homomorphism from such a group to any separable topological group is necessarily continuous.

## 1 Introduction

Finite type surfaces and their mapping class groups (groups of homeomorphisms up to isotopy) have been studied for decades; mapping class groups of infinite type surfaces are much less understood, but provide a promising approach study dynamics of group actions where existing tools fall short.

For instance, suppose that  $\Gamma$  is an infinite group acting on a surface  $S$ . If the action of  $\Gamma$  is not minimal, then there is an invariant open set  $O \subset S$ , which we may take to be the complement of the closure of an orbit. If the orbit closure (and hence the boundary of  $O$ ) is *not* totally disconnected, then the theory of prime ends can be used to extract information about the dynamics of the action of  $\Gamma$ . Classically, this theory requires  $O$  to be simply connected – and so its boundary not a single point – but several aspects generalize to the non-simply connected setting, as seen, for instance, in recent work of Koropecki, Le Calvez and Nassiri [3]. In the case when the orbit closure *is* totally disconnected, passing to a minimal set for the action of  $\Gamma$  on the orbit closure gives a  $\Gamma$ -invariant set  $X$  which is either finite or a Cantor set, and a nontrivial homomorphism from  $\Gamma$  to the *mapping class group*  $\text{Homeo}(S - X)/\text{Homeo}_0(S - X)$ , giving invariants for the action.

This applies just as well to higher dimensional manifolds as soon as one has an orbit of a group action with totally disconnected closure. A similar strategy was also used by Calegari to study infinite groups acting on  $\mathbb{R}^2$  with bounded orbits in [?]. Collapsing each bounded components of the complement of an orbit closure to a point produces an action of the group on a plane with an invariant finite or Cantor set, hence again a homomorphism to the mapping class group of  $\mathbb{R}^2 - X$ .

In addition to being a possible source of invariants for group actions, mapping class groups of infinite type surfaces or manifolds (such as the complement of a Cantor set example above) provide interesting examples of non-locally compact Polish topological groups. N. Vlamis asked us whether certain examples of these groups have the *automatic continuity* property: every homomorphism from such a group to any separable topological group is necessarily continuous. The purpose of this note is to demonstrate automatic continuity for a wide class of relative homeomorphism and mapping class groups of manifolds, including the relative Cantor set examples. To do this, we use a condition formulated by Rosendal and Solecki in [5].

**Definition 1.** A topological group  $G$  is *Steinhaus* if there is some  $n \in \mathbb{N}$  such that, whenever  $W \subset G$  is a symmetric set such that countably many left-translates of  $W$  cover  $G$ , there exists a neighborhood of the identity of  $G$  contained in  $W^n$ .

As shown in [5, Prop. 2], the Steinhaus property implies automatic continuity (via a straightforward Baire category argument). Here we prove the following.

**Theorem 2.** Let  $M$  be a manifold, either compact or homeomorphic to the interior of a compact manifold with boundary. Let  $X \subset M$  be a set consisting of finitely many isolated points, or a Cantor set, or both. Then the group  $\text{Homeo}(M, X)$  of homeomorphisms preserving  $X$  (setwise) is Steinhaus.

In [4], it was shown that if  $M$  is a compact manifold, and  $X$  a submanifold of dimension at least 1, then  $\text{Homeo}(M, X)$  has automatic continuity. Theorem 2 covers the missing cases of  $\dim(X) = 0$ , and deals with noncompact  $M$ .

For any manifold  $M$ , and closed set  $X \subset M$ , the group  $\text{Homeo}(M, X)$  is a closed subgroup of the Polish group  $\text{Homeo}(M)$  (endowed with the compact-open topology), hence is Polish. Since  $\text{Homeo}_0(M, X)$  is a closed, normal subgroup of  $\text{Homeo}(M, X)$ , the quotient  $\text{Homeo}(M, X)/\text{Homeo}_0(M, X)$  is also Polish. (For the case where  $M$  is a surface, this fact was observed in [2] using the property that these groups are the automorphism groups of a countable structure, namely the curve complex of the surface.) An easy argument (see [5, Cor. 3]) shows that, provided that  $G$  is a Polish, Steinhaus group, any Polish quotient group  $H$  of  $G$  is also Steinhaus. This gives the following consequence of Theorem 2, answering Vlamis' question.

**Corollary 3.** Let  $M$  be either compact or homeomorphic to the interior of a compact manifold with boundary, and  $X \subset M$  the union of a Cantor set and finite set. Then the *mapping class group*  $\text{Homeo}(M, X)/\text{Homeo}_0(M, X)$  is Steinhaus, hence has the automatic continuity property.

## 2 General tools

This section contains some standard and some preliminary results to be used in the proof. For a topological manifold  $M$  (we assume that manifolds are metrizable, but otherwise

arbitrary) we take the standard compact-open topology on  $\text{Homeo}(M)$ , which is separable and completely metrizable, i.e. Polish. If  $X \subset M$  is a closed set, then  $\text{Homeo}(M, X)$  is a closed subgroup of  $\text{Homeo}(M)$ , so also Polish. The following lemma is widely used in automatic continuity arguments.

**Lemma 4.** Let  $M$  be a manifold,  $X \subset M$  a closed set, and  $W \subset \text{Homeo}(M, X)$  a symmetric set such that  $\text{Homeo}(M, X)$  is a countable union of left translates of  $W$ . Then there exists a neighborhood  $U$  of the identity in  $\text{Homeo}(M, X)$  such that  $W^2$  is dense in  $U$ .

*Proof.* Since  $\text{Homeo}(M, X)$  is a Baire space, and each left translated of  $W$  is homeomorphic to  $W$ , it follows that the set  $W$  is not meagre, so dense in some open set of  $\text{Homeo}(M, X)$ . Since  $W$  is symmetric, it follows that  $W^2$  is dense in a neighborhood of the identity.  $\square$

The following technical lemma generalizes [4, Lemma 3.8], which is modeled after arguments of Rosendal from [6]. Recall that the *support* of a homeomorphism  $f$  is the closure of the set  $\{x \mid f(x) \neq x\}$ , and a group  $G$  is said to have *commutator length*  $p$  if each element of  $G$  can be written as a product of  $p$  commutators in  $G$ .

**Lemma 5.** Let  $M$  be a manifold, and  $W \subset \text{Homeo}(M)$  a symmetric set such that  $\text{Homeo}(M)$  is a countable union of left translates of  $W$ . Let  $\mathcal{A}$  be a family of open subsets of  $M$  satisfying:

- i) There exists an infinite family of pairwise disjoint, closed sets  $U_i \subset M$  such that each set  $U_i$  contains an infinite family of pairwise disjoint sets belonging to  $\mathcal{A}$ .
- ii) There exists  $p \in \mathbb{N}$  such that, for each  $A \in \mathcal{A}$ , the group of homeomorphisms with support on  $A$  has commutator length bounded by  $p$ .

Then there exists  $A \in \mathcal{A}$  such that each homeomorphism supported on  $A$  is contained in  $W^{8p}$ .

*Proof.* Let  $G = \text{Homeo}(M)$ , and for any set  $U \subset M$ , let  $G(U)$  denote the homeomorphisms with support contained in  $U$ . Suppose that  $G = \bigcup_{i=1}^{\infty} g_i W$ , for some symmetric set  $W$ , and let  $U_1, U_2, \dots$  be the closed, disjoint sets from the statement of the lemma.

*Step 1.* We first claim that there is some  $U_i$  such that, for each  $f \in G(U_i)$ , there exists  $w_f \in g_i W$ , with support in the closure of  $\bigcup_i U_i$ , and such that the restriction of  $w_f$  to  $U_i$  agrees with  $f$ . For if this statement does not hold, then there is a sequence of counterexamples  $f_i \in G(U_i)$  such that each  $f_i$  does not agree with the restriction to  $U_i$  of any element of  $g_i W$  supported on  $\bigcup_i U_i$ . Define a homeomorphism  $F(x)$  by

$$F(x) = \begin{cases} f_i(x) & \text{if } x \in U_i \text{ for some } i \\ x & \text{otherwise} \end{cases}$$

By assumption, for some  $i$  we have  $F \in g_i W$ , but  $F$  restricts to  $f_i$  on  $U_i$ . This gives the desired contradiction.

Now, given  $f \in G(U_i)$ , consider the homeomorphisms  $w_{\text{id}}$  and  $w_f$  obtained above. Then the restriction of  $(w_{\text{id}})^{-1}w_f$  to  $U_i$  agrees with  $f$  on  $U_i$ , and  $(w_{\text{id}})^{-1}w_f \in Wg_i^{-1}g_iW = W^2$ . Thus, we conclude that for any  $f \in G(U_i)$ , there exists some element in  $W^2$  agreeing with  $f$  on  $U_i$ .

*Step 2.* Let  $U = U_i$  be the open set obtained from step 1, and let  $A_1, A_2, \dots$  be a countable disjoint family of open sets in  $U$ , with each  $A_i \in \mathcal{A}$ . We may apply the argument from Step 1 above to the family consisting of the closures of the  $A_i$  in place of  $U_1, U_2, \dots$  to conclude that there exists  $i$  such that for every  $f \in G(A_i)$ , there exists an element in  $W^2$ , with support contained in the closure of the union of the  $A_i$ , and agreeing with  $f$  on  $U$ . Forgetting subscripts, let  $A = A_i$  denote this set.

Let  $f \in G(A)$ . Using the bounded commutator length assumption, write  $f = [a_1, b_1] \dots [a_p, b_p]$ , where  $a$  and  $b$  also have support in  $A$ . Since in particular, each  $a_j$  has support in  $U$ , there exists  $w_{a_j} \in W^2$  with  $\text{supp}(w_{a_j}) \subset U$  and such that the restriction of  $w_{a_j}$  to  $U$  agrees with  $a_j$ . There also exists  $w_{b_j} \in W^2$  with  $w_{b_j} \in G(A)$  and such that the restriction of  $w_{b_j}$  to  $A$  agrees with  $b_j$ . Since  $\text{supp}(w_{a_j}) \cap \text{supp}(w_{b_j}) \subset A$ , we have  $[a_j, b_j] = [w_{a_j}, w_{b_j}]$ , hence  $f = [w_{a_1}, w_{b_1}] \dots [w_{a_p}, w_{b_p}] \in W^{8p}$ .  $\square$

The other general tool that we will use in the proof is the following.

**Proposition 6.** Let  $M$  be a manifold, and suppose that  $\text{Homeo}(M) \subset g_iW$  for some symmetric set  $W$ . Then there exists  $n$ , depending only on the dimension of  $M$ , and a neighborhood  $U$  of the identity in  $\text{Homeo}(M)$  so that the following holds: If  $K \subset M$  is any compact set, then any homeomorphism in  $U$  with support contained in  $K$  is an element of  $W^n$ .

This follows directly from the proof of the main theorem of automatic continuity for compact manifolds in [4].

### 3 First case: $X$ finite or empty.

We begin with the case  $X = \emptyset$ ; given the results of [4], this reduces to the case where  $M$  is homeomorphic to the interior of a compact manifold with boundary. A major technical ingredient for the proof is adapted from Rosendal–Solecki’s proof of automatic continuity for the group of order-preserving automorphisms of  $\mathbb{Q}$  in [5]. The argument here also covers the case where  $X$  is finite, by the following remark.

**Remark 7** ( $X = \emptyset$  case implies  $X$  finite case). Suppose  $M$  is the interior of a compact manifold  $\bar{M}$ , and  $X \subset M$  is a finite set. Since  $\mathbb{R}^n - \{0\}$  is homeomorphic to  $(0, 1) \times S^{n-1}$ , the manifold  $M - X$  is homeomorphic to the interior of a compact manifold  $\bar{N}$  with  $\partial\bar{N}$  the union of  $\partial\bar{M}$  and a disjoint union of spheres, one for each point of  $X$ . Though homeomorphisms of  $N$  do not necessarily extend to homeomorphisms of  $\bar{N}$ , they do extend to the space obtained by one-point-compactifying each spherical end, and this gives a topological isomorphism  $\text{Homeo}(N) \rightarrow \text{Homeo}(M, X)$ .

*Proof of automatic continuity for  $M$  noncompact,  $X = \emptyset$ .* Let  $M$  be homeomorphic to the interior of a compact manifold  $\bar{M}$ . Let  $W \subset \text{Homeo}_0(M)$  be a symmetric set such that  $\text{Homeo}_0(M) = \bigcup_{i \in \mathbb{N}} g_i W$ .

Identify a neighborhood of the ends of  $M$  with  $([0, \infty) \times \partial\bar{M}) \subset M$ . Let  $U$  be a neighborhood of the identity in  $\text{Homeo}_0(M)$  small enough so that, if  $f \in U$ , then  $f$  can be written as  $f = k \circ h$  where  $k$  has support on the compact set  $K = M - ((3, \infty) \times \partial M)$ , and  $h$  has support on  $[2, \infty) \times \partial\bar{M}$ . Proposition 6 shows that, provided  $U$  is chosen small enough, there exists  $n$  depending only on the dimension of  $M$  such that  $k \in W^n$ . We wish to prove the same is true for  $h$ .

Any homeomorphism  $h$  of  $[0, \infty) \times \partial M$  can be factored as  $h_1 h_2$  where each  $h_i$  has support on a set of the form  $X_i \times \partial\bar{M}$ , and  $X_i \subset [0, \infty)$  is an infinite disjoint union of open intervals homeomorphic in  $[0, \infty)$  to  $\bigcup_{n=1}^{\infty} (2n, 2n+1)$ . (See e.g. [7, Prop. 5.1].) Thus, it suffices to find  $n$  such that  $W^n$  contains any homeomorphism with support on a set of the form  $X \times \partial\bar{M} \subset [0, \infty) \times \partial\bar{M}$  where  $X$  is homeomorphic to  $\bigcup_{n \in \mathbb{N}} (2n, 2n+1)$ .

Let  $X$  be such a set. Reparametrizing  $\mathbb{R}$ , we may assume that  $X = \bigcup_{n \in \mathbb{N}} (2n, 2n+1) \times \partial\bar{M}$ . We begin by applying Lemma 5 with the class  $\mathcal{A}$  consisting of sets of the form  $A_\Lambda = Y_\Lambda \times \partial\bar{M}$ , where  $Y_\Lambda = \bigcup_{n \in \Lambda} (2n, 2n+1)$  for some infinite set  $\Lambda \subset \mathbb{N}$ . Note that  $\mathcal{A}$  satisfies the hypotheses of the lemma, since

- i) We may write  $\mathbb{N}$  as a countable disjoint union of infinite sets  $\Lambda_i$ , and define  $U_i$  to be the closure of  $Y_{\Lambda_i} \times \partial\bar{M}$ . Each such set contains a countable union of disjoint elements of  $\mathcal{A}$ .
- ii) Any element supported on such a set  $A_\Lambda$  may be written as a single commutator.

The proof of item ii) is a standard argument, we briefly recall this for completeness: given  $f$  supported on such a set  $Y_\Lambda \times \partial\bar{M}$ , let  $I_n \subset (2n, 2n+1)$  be closed intervals such that  $\text{supp}(f) \subset \bigcup_{n \in \Lambda} I_n \times \partial M$ . Let  $T : \mathbb{R} \rightarrow \mathbb{R}$  be a homeomorphism with support in  $Y_\Lambda$  such that  $T(I_n) \cap I_n = \emptyset$ ; abusing notation, identify  $T$  with the homeomorphism  $T \times \text{id}$  of  $[0, \infty) \times \partial\bar{M}$ . Let  $a = \prod_{j \geq 0} T^j f T^{-j}$ , which has support in  $\bigcup_{n \in \Lambda} I_n$ . Then  $a \circ T a^{-1} T^{-1} = f$ .

Thus we conclude (using the notation from Lemma 5) that for some such set  $A_\Lambda \in \mathcal{A}$ , we have  $G(A_\Lambda) \subset W^8$ . Now we apply a trick used in [5]. For  $\alpha \in \mathbb{R}$ , let  $\Lambda_\alpha$  be infinite subsets of  $\Lambda$ , such that  $\Lambda_\alpha \cap \Lambda_\beta$  is finite for all  $\alpha \neq \beta$ . Such a collection may be obtained, for example, by putting  $\Lambda$  in bijective correspondence with  $\mathbb{Q}$ , and choosing  $\Lambda_\alpha$  to be a sequence of distinct rational numbers converging to  $\alpha$ .

For each  $\alpha$ , let  $f_\alpha \in \text{Homeo}[0, \infty)$  satisfy  $f_\alpha(2n) = 2n+1$ , and  $f_\alpha(2n+1) \in 2\Lambda_\alpha$  for all  $n \in \Lambda_\alpha$ , so that  $f_\alpha$  maps each interval of  $Y_{\Lambda_\alpha}$  to the interior of a connected component of  $\mathbb{R} - Y_{\Lambda_\alpha}$ . Again abusing notation, identify these with the homeomorphisms  $f_\alpha \times \text{id}$  of  $[0, \infty) \times \partial\bar{M}$ . Since  $\mathbb{R}$  is uncountable, there is some  $\alpha$  and some  $\beta$  such that  $f_\alpha$  and  $f_\beta$  are in the same left-translate  $g_i W$ . Thus,  $f_\alpha^{-1} f_\beta$  and  $f_\beta^{-1} f_\alpha$  are both in  $W^2$ . We will now use these two elements to conjugate (a suitable decomposition of) homeomorphisms supported on  $X \times \partial\bar{M}$  into  $A_\Lambda$ , this will complete the proof.

If  $n \in \mathbb{N} - \Lambda_\alpha$ , then  $f(2n, 2n + 1) \subset (2m, 2m + 1)$  for some  $m \in \Lambda_\alpha$ . If  $m \notin \Lambda_\beta$ , then  $f_\beta^{-1}f_\alpha(2n, 2n + 1)$  is contained in an interval of the form  $(2k, 2k + 1)$  where  $k \in \Lambda_\beta$ . Since  $\Lambda_\alpha \cap \Lambda_\beta$  is finite, we conclude that, with the exception of finitely many values of  $n$ , the map  $f_\beta^{-1}f_\alpha$  takes intervals of the form  $(2n, 2n + 1)$  where  $n \notin \Lambda_\alpha$  into some interval in  $Y_{\Lambda_\alpha} \subset Y_\Lambda$ . Reversing the role of  $\alpha$  and  $\beta$ , the same argument shows that, with only finitely many exceptions,  $f_\alpha^{-1}f_\beta$  takes every interval of the form  $(2n, 2n + 1)$ ,  $n \notin \Lambda_\beta$ , into some interval of  $Y_\Lambda$ . Let  $F$  denote the union of these two exceptional sets of integers. Thus we can decompose  $X$  as the union of  $X_1 := \bigcup_{n \notin (\Lambda_\alpha \cup F)} (2n, 2n + 1) \times \bar{M}$ , with  $X_2 := \bigcup_{n \notin (\Lambda_\beta \cup F)} (2n, 2n + 1) \times \bar{M}$  and  $X_3 := \bigcup_{n \in F} (2n, 2n + 1) \times \bar{M}$ , hence  $G(X) = G(X_1)G(X_2)G(X_3)$ .

As we have shown above,

$$f_\beta^{-1}f_\alpha G(X_1)(f_\beta^{-1}f_\alpha)^{-1} \subset G(Y_\Lambda) \subset W^8, \text{ and similarly}$$

$$f_\alpha^{-1}f_\beta G(X_2)(f_\alpha^{-1}f_\beta^{-1}) \subset G(Y_\Lambda) \subset W^8.$$

Thus  $G(X_i) \subset W^{12}$  for  $i = 1, 2$ . Since  $X_3$  is compact, we also have that  $G(X_3) \subset W^n$  by Proposition 6. It follows that  $G(X) \subset W^{24+n}$ , which proves the theorem.  $\square$

## 4 General case

Let  $M$  be a topological manifold, either compact or homeomorphic to the interior of a compact manifold with boundary, and let  $X \subset M$  be the union of a Cantor set and a (possibly empty) finite set. Remark 7 reduces the proof to the case where  $X$  is a Cantor set, since if  $X'$  is the set of isolated points of  $X$ , then  $M - X'$  is homeomorphic to interior of a compact manifold with boundary, and  $\text{Homeo}(M, X) \cong \text{Homeo}(M - X', X - X')$ . Thus, we assume going forward that  $X$  is a Cantor set.

*Proof for  $X \subset M$  a Cantor set.* Let  $W \subset \text{Homeo}(M, X)$  be a symmetric set such that  $\text{Homeo}(M, X) = \bigcup_i g_i W$ . For convenience, fix a metric on  $M$ . Then a neighborhood basis of the identity on  $M$  is given by sets of the form  $U_{K, \delta} := \{f \mid d(f(x), x) < \delta \text{ for all } x \in K\}$  as  $K$  ranges over compact sets of  $M$ , and  $\delta > 0$ .

Call a closed ball in  $M$  a *separating ball* if its boundary is disjoint from  $X$  and separates  $X$  into two components. By Lemma 4,  $W^2$  is dense in a neighborhood of the identity of  $\text{Homeo}(M, X)$ . Thus, we may choose  $\epsilon > 0$  such that the following holds:

**Lemma 8.** Let  $x_1, x_2, \dots, x_n$  be any collection of points in  $X$ , with  $d(x_i, x_j) > 2\epsilon$  for all  $i \neq j$ . If  $D_i$  and  $E_i$  are both separating balls contained in the  $\epsilon$ -ball about  $x_i$ , then for any  $\delta > 0$ , there exists  $f \in W^2$  such that, for all  $i = 1, 2, \dots, n$  the ball  $f(D_i)$  is hausdorff distance at most  $\delta$  from  $E_i$ .

The proof is immediate, and the statement also holds if we replace the condition that  $D_i$  and  $E_i$  are separating balls with the condition that  $\overline{D_i} \cap X = \emptyset$  and  $\overline{E_i} \cap X = \emptyset$  holds for all  $i$ .

Now take two  $2\epsilon$ -separated sets, say  $\{x_1, x_2, \dots, x_m\}$  and  $\{x'_1, x'_2, \dots, x'_{m'}\}$ , as in Lemma 8, and let  $D_i \subset B_\epsilon(x_i)$  and  $D'_i \subset B_\epsilon(x'_i)$  be separating balls such that

- i) The union of the  $D_i$  and the  $D'_i$  cover  $X$ , and
- ii) Each point of  $X$  is contained in at most one element of  $\{D_1, \dots, D_n, D'_1, \dots, D'_{m'}\}$ .

We now apply Lemma 5 to sets consisting of separating balls. Let  $\mathcal{A}$  be the collection of sets satisfying the following condition:  $A \in \mathcal{A}$  iff  $A$  consists of a union of disjoint separating balls, with exactly one ball in each metric ball  $B_\epsilon(x_i)$ . Let  $\mathcal{A}'$  be the analogous set for the metric balls  $B_\epsilon(x'_i)$ . It is easily checked that  $\mathcal{A}$  and  $\mathcal{A}'$  satisfy the conditions of Lemma 5, so we conclude that there exists  $A \in \mathcal{A}$  and  $A' \in \mathcal{A}'$  so that  $G(A) \subset W^8$  and  $G(A') \subset W^8$ .

Let  $Z$  be a connected, simply connected neighborhood of  $X$  such that the interior of  $M - Z$  together with the union of the discs  $D_i$  and  $D'_i$  covers  $M$ . By the fragmentation lemma (following Edwards–Kirby [1]), there exists a neighborhood  $U$  of the identity in  $M$  such that any  $f \in U$  can be written as  $g_0 g_1 g_2$  with  $g_0$  supported on  $Z$ ,  $g_1$  supported on  $\bigcup_i D_i$  and  $g_2$  supported on  $\bigcup_i D'_i$ .

Since each point of  $X$  is contained in at most one such supporting set, we additionally have that if  $f(X) = X$ , then each  $g_i$  must preserve  $X$  as well. Repeating the proof (verbatim) from the first case where  $X = \emptyset$  shows that  $g_0 \in W^n$ , where  $n$  is a constant that depends only on the dimension of  $M$ . By Lemma 8, we can find  $h_1$  and  $h_2$  in  $W^2$  such that  $D \subset h_1(A)$  and  $D' \subset h_2(A')$ . Thus, for  $i = 1, 2$  we have  $f_i^{-1} g_i h_i \in W^8$ , hence  $g_i \in W^{10}$ . This shows that  $g \in W^{20+n}$ , which is what we needed to show.  $\square$

The argument we have given here does not seem to readily generalize to other noncompact manifolds. We ask the following:

**Question 9.** Let  $M$  be a surface of infinite genus. Is it true that  $\text{Homeo}(M)$  (and therefore necessarily the mapping class group of  $M$ ) is Steinhaus?

**Question 10.** Let  $M$  be a manifold, and  $X \subset M$  an arbitrary closed, totally disconnected set. Is  $\text{Homeo}(M, X)$  always Steinhaus?

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