THERE ARE NO EXOTIC ACTIONS OF DIFFEOMORPHISM GROUPS ON 1-MANIFOLDS

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Abstract. Let M be a manifold and N a 1-dimensional manifold. Assuming $r \neq \dim(M)+1$, we show that any nontrivial homomorphism $\rho : \mathsf{Diff}^r_c(M) \to \mathsf{Homeo}(N)$ has a standard form: necessarily M is 1-dimensional, and there are countably many embeddings $\phi_i : M \to N$ with disjoint images such that the action of ρ is conjugate (via the product of the ϕ_i) to the diagonal action of $\mathsf{Diff}^r_c(M)$ on $M \times M \times ...$ on $\bigcup_i \phi_i(M)$, and trivial elsewhere. This solves a conjecture of Matsumoto. We also show that the groups $\mathsf{Diff}^r_c(M)$ have no countable index subgroups.

1. Introduction

Let $\mathsf{Diff}^r_c(M)$ denote the identity component (in the compact-open C^r topology) of the group of compactly supported C^r diffeomorphisms of a manifold M, for $0 \leq r \leq \infty$. These groups are locally contractible, so in fact $\mathsf{Diff}^r_c(M)$ agrees with the group of diffeomorphisms which are isotopic to the identity through a compactly supported isotopy. When we speak of $\mathsf{Diff}^r(M)$, we assume that manifolds admit a C^r structure, and a metric structure in the C^0 case, but are otherwise arbitrary. In this paper, we prove the following statement.

Theorem 1.1. Let M be a connected manifold, and suppose that $\rho : \mathsf{Diff}^r_c(M) \to \mathsf{Homeo}(N)$ is a nontrivial homomorphism, where $N = S^1$ or $N = \mathbb{R}$, $r \neq \dim(M)+1$. Then $\dim(M) = 1$ and there are countably many disjoint embeddings $\phi_i : M \to N$ such that $\rho(g)|_{\phi_i(M)} = \phi_i g \phi_i^{-1}$ and $N - \bigcup_i \phi_i(M)$ is globally fixed by the action.

This proves [12, Conjecture 1.3] and generalizes works of Mann [8], Militon [13] and Matsumoto [12], but with an independent proof. Matsumoto's work [12] proves an analogous result when the target is $\mathsf{Diff}^1(N)$ using rigidity theorems of [3] for solvable affine subgroups of $\mathsf{Diff}^1(\mathbb{R})$. This generalized [8], which proved the result for homomorphisms to $\mathsf{Diff}^2(N)$ using Kopell's lemma. Militon [13] studies homomorphisms where the source is the group of *homeomorphisms* of M. Our proof here is comparatively short, and is self-contained modulo the standard but difficult result that $\mathsf{Diff}^r_c(M)$, for $r \neq \dim(M)+1$ is a simple group, due to Anderson, Mather and Thurston [1, 10, 11, 18]. Whether simplicity holds for $r = \dim(M)+1$ is an open question; this is responsible for our restrictions on dimension in the statement.

Theorem 1.1 is already known in the case where ρ is assumed to be continuous; it is a consequence of the orbit classification theorem of [5], and was likely known to others before. In the case where the target is the group of smooth diffeomorphisms of N, this also follows from work of Hurtado [6] who proves additionally that any such homomorphism is necessarily (weakly) continuous. Here we make no assumptions on continuity, however, our proof suggests that diffeomorphism groups exhibit "automatic continuity"–like properties. Specifically, we show the following *small index property.*

Theorem 1.2 (The small index property of $\mathsf{Diff}^r_c(M)$)**.** If $r \neq \dim(M)+1$, then $\mathsf{Diff}^r_c(M)$ has no proper countable index subgroup. Equivalently, $\mathsf{Diff}^r_c(M)$ has no nontrivial homomorphism to the permutation group S_∞.

This is in stark contrast with the case for finite dimensional Lie groups, where we have the following.

Theorem 1.3 (Thomas [17] and Kallman [7])**.** There is an injective homomorphism $\mathrm{SL}_n(\mathbb{R}) \to S_\infty$.

Thus, one consequence of Theorem 1.2 and 1.3 is that there is no nontrivial homomorphism from $\mathsf{Diff}^r_c(M)$ into a linear group. Of course, this is nearly immediate if one considers only continuous homomorphisms, since $\mathsf{Diff}^r_c(M)$ is infinite dimensional, and one may simply quote the invariance of domain theorem.

If G is a group with a non-open subgroup H of countable index, then the action of G on the coset space G/H gives a discontinuous homomorphism to S_∞. This is one of very few known general recipes for producing discontinuous group homomorphisms (see [16]), so gives some (weak) evidence that $\mathsf{Diff}^r_c(M)$ might have the automatic continuity property already known to hold for $\mathsf{Homeo}(M)$ by [9]. Automatic continuity also holds for homomorphisms between groups of smooth diffeomorphisms by work of Hurtado [6].

Theorem 1.1 also gives new examples of left orderable groups that do not act on the line. It is a well known fact that any *countable* group with a left-invariant total order admits a faithful homomorphism to $\mathsf{Homeo}_+(\mathbb{R})$. For $r>0$, the groups $\mathsf{Diff}^r_c(\mathbb{R}^n)$ for $r>0$ are known to be left-orderable: the Thurston stability theorem [19] implies that they are locally indicable (any finitely generated subgroup surjects to $\mathbb{Z}$), which implies that they are left-orderable by the Burns-Hale theorem ([4], see also [14, Corollary 2]). Thus, we have the following.

Corollary 1.4. For $r>0$, the group $\mathsf{Diff}^r_c(\mathbb{R}^n)$ is left-orderable but has no faithful action on the line or the circle.

The proof of Theorem 1.2 uses the idea from the first step of the proof of automatic continuity for homeomorphism groups of [9], following Rosendal [15]. This result is then used to prove Theorem 1.1 by constraining the supports and fixed sets of elements for the action on N. We are then able to use this information to build a map from M to N.

2. Proof of the small index property

In this section, we prove Theorem 1.2. The proof follows a strategy used in [9] and [15] used in the proof of automatic continuity of $\mathsf{Homeo}(M)$.

Proof. Let M be a manifold and $r \neq \dim(M)+1$. Let $G = \mathsf{Diff}^r_c(M)$, and for an open subset $U \subset M$, denote by G_U the subgroup of $\mathsf{Diff}^r_c(M)$ consisting of maps with compact support contained in U and isotopic to the identity via an isotopy compactly supported in U. Thus, $G_U \cong \mathsf{Diff}^r_c(U)$. (Note that $\mathsf{Diff}^r_c(U)$ is locally contractible, and in particular path connected, for all $0 \leq r \leq \infty$.)

Suppose for contradiction that $H \subset G$ is a countable index subgroup. We will show in Step 1 that there is some ball U in M such that $G_U \subset H$. After this, we will show (Step 2) that H acts transitively on M, thus every $x \in M$ is contained in some open set U_x such that $G_{U_x} \subset H$. The *fragmentation property* states that $\mathsf{Diff}^r_c(M)$ is generated by the union of such sets G_{U_x} (this is true for any collection of sets U_x which form an open cover of M, see [2, Ch.1]), so this is sufficient to prove $H = G$.

Step 1: There is some open ball U in M such that $G_U \subset H$.

Let $g_1 H, g_2 H, \dots$ denote the left cosets of H. Let $B \subset M$ be an open ball, and take a sequence of disjoint balls $B_i \subset B$ such that $\overline{\cup B_i} \subset B$, with diameter tending to 0 and such that the sequence B_i Hausdorff converges to a point inside B.

We first claim that there exists some $j \in \mathbb{N}$ and a neighborhood U_j of the identity element of G_{B_j} such that the following holds:

($*$) For every $f \in \mathcal{U}_j$, there exists $w_f \in g_j H \cap G_B$ such that the restriction of w_f to $\mathcal{U}_j$ agrees with f.

Given ($*$), then we have $w_{id}^{-1} w_f \in H g_j^{-1} g_j H = H$, and $w_{id}^{-1} w_f$ restricts to f on B_j. This shows that every element in $\mathcal{U}_j$ agrees with the restriction of an element of H to B_j. Since $\mathcal{U}_j$ is an identity neighborhood of G_{B_j} and G_{B_j} is by definition connected, $\mathcal{U}_j$ generates G_{B_j} and we conclude that every element of G_{B_j} agrees with the restriction of an element of H to B_j.

We prove this claim by contradiction, using a standard diagonal argument. Inductively choose neighborhoods $\mathcal{U}_i$ of the identity in G_{B_i} so that for any sequence of diffeomorphisms $f_i \in \mathcal{U}_i$, the infinite composition $\prod_i f_i$ defines an element of G. is an element of G. Supposing that our claim is not true for any $\mathcal{U}_j$, then for each i we can find $f_i \in \mathcal{U}_i$ such that there does not exist any $w_i \in g_i H$ supported in B satisfying $w_i|_{B_i} = f|_{B_i}$. Let $w = \prod_i f_i$. Then $w \in g_j H$ for some j since $\bigcup_k g_k H = G$. Moreover, the support of w is in B, the restriction of w and f on B_j are the same and we have $w \in g_j H$. This is a contradiction, and proves the claim.

Now we use a commutator trick. Apply the same argument as above using B_j in place of B. We find a smaller ball $B' \subset B_j$ such that every element $f \in G_{B'}$ agrees with the restriction to B' of an element $v_f \in H$, and v_f is supported on B_j. Since $\mathsf{Diff}^r_c(B')$ is perfect [1, 10, 11, 18], any element $f \in \mathsf{Diff}^r_c(B')$ may be written as a product of commutators $f = \prod_{i=1}^k [a_i, b_i]$. The commutator length k of course depends on f, but this is unimportant to us. We have $[a_i, b_i] = [v_{a_i}, w_{b_i}]$ since the supports of v_{a_i} and w_{b_i} intersect only in B', and so $f = \prod [v_{a_i}, w_{b_i}] \in H$. This ends the proof of the first step.

Step 2: transitivity. To prove transitivity, let B' be the ball from step 1, and let $x \in B'$. Suppose $y \in M$ is some point *not* in the orbit of x. Let f_t be a flow such that $f_t(y) \in B'$ for all $t \in (1, 2)$. Such a flow can be defined to have support on a neighborhood of a path from x to y. Since B' lies in the orbit of x under H, we have that $f_t \notin H$ for $t \in (1, 2)$. We know that $H \cap \{f_t : t \in \mathbb{R}\}$ is a countable index subgroup of $\{f_t : t \in \mathbb{R}\} \cong \mathbb{R}$. Thus, it must intersect every open interval of $\mathbb{R}$, this gives the desired contradiction. As explained above, Steps 1 and 2 together with fragmentation complete the proof of the Theorem 1.2. □

As an immediate consequence, we can conclude that any fixed point free action of such a group on the line or circle is minimal.

Corollary 2.1. With the same restrictions on r as above, if $\mathsf{Diff}^r_c(M)$ acts on $\mathbb{R}$ or S^1 without global fixed points, then there are no invariant open sets. In particular, every orbit is dense.

Proof. Suppose the action has an invariant open set. Then $\mathsf{Diff}^r_c(M)$ permutes the (countably many) connected components of U. The stabilizer of an interval is a countable index subgroup, so by Theorem 1.2, the permutation action is trivial. Thus each interval is fixed and their endpoints are global fixed points. □

3. Proof of Theorem 1.1

For the proof of Theorem 1.1, we set the following notation. As in the previous section we fix some $r \neq \dim(M) + 1$ and when $U \subset M$ is an open set we denote by G_U the subgroup of $\mathsf{Diff}^r_c(M)$ consisting of maps with compact support contained in U and isotopic to the identity via an isotopy compactly supported in U. We additionally use the notation $G^U \subset \mathsf{Diff}^r_c(M)$ for the set of elements that pointwise fix U. The *open support* of a homeomorphism g is the set $\mathsf{Osupp}(g) := M - \mathsf{Fix}(g)$; as is standard, the *support* of g is defined to be the closure of $\mathsf{Osupp}(g)$.

Proof. We will assume the action on N has no global fixed points, since if the action does have fixed points, then $N - \mathsf{Fix}(\rho)$ is a union of open intervals, each with a fixed-point free action of $\mathsf{Diff}^r_c(M)$, so it suffices to understand such actions. In this case, we will show that there is a single homeomorphism $\phi : M \to N$ such that the action on N is induced by conjugation by ϕ.

Lemma 3.1. For any action, if $U \cap V = \emptyset$, then $\mathsf{Osupp}(\rho(G_U)) \cap \mathsf{Osupp}(\rho(G_V)) = \emptyset$.

Proof. Since G_U and G_V commute, $\rho(G_V)$ preserves $\mathsf{Osupp}(\rho(G_U))$, permuting its connected components. By Theorem 1.2, this action is trivial. Let I be a connected component of $\mathsf{Osupp}(\rho(G_U))$. Suppose $\rho(G_V)$ acts nontrivially on I. Since G_V is simple group, its action on I is faithful. Since G_V is *not* abelian, Hölder's theorem implies that some nontrivial $\rho(g) \in \rho(G_V)$ acts with a fixed point. But then $\rho(G_U)$ permutes the connected components of $I - \mathsf{Osupp}(\rho(g))$, and this permutation action is trivial. Thus, $\rho(G_U)$ has a fixed point in I, contradicting that $I \subset \mathsf{Osupp}(\rho(G_U))$. □

We observe the following consequence of the fragmentation property:

Observation 3.2. If $\bar{U} \cap \bar{V} = \emptyset$ then G^U and G^V generate $\mathsf{Diff}^r_c(M)$ because $G^U \supset G_{M-\bar{U}}$ and $G^V \supset G_{M-\bar{V}}$, and $M - \bar{V}$ and $M - \bar{U}$ cover M. Consequently, our assumption that there are no global fixed points for the action implies that $\mathsf{Fix}(\rho(G^U)) \cap \mathsf{Fix}(\rho(G^V)) = \emptyset$.

Our next goal is to define a map from M to N. For each $x \in M$ pick a neighborhood basis $U_n(x)$ of x so $\bigcap_n U_n(x) = \{x\}$. Let $S_x = \bigcap_n \mathsf{Osupp}(\rho(G_{U_n(x)}))$ and let $T_x = \bigcap_n \mathsf{Fix}(\rho(G^{U_n(x)}))$. Note that the sets S_x and T_x are independent of the choice of neighborhood basis.

Lemma 3.3. If $x \neq y$, then $S_x \cap S_y = \emptyset$ and $T_x \cap T_y = \emptyset$. Also, S_x and T_x have empty interior.

Proof. The first assertion follows immediately from Lemma 3.1 and the second because $T_x \cap T_y$ would be globally fixed by ρ by our observation above. Furthermore, if $g(x) = y$, then $\rho(g)(U_n(x))$ is a neighborhood basis of y, so we have

$$\rho(g)S_x = \bigcap_n \rho(g)\,\mathsf{Osupp}(\rho(G_{U_n(x)})) = \bigcap_n \mathsf{Osupp}(\rho(gG_{U_n(x)}g^{-1})) = \bigcap_n \mathsf{Osupp}(\rho(G_{g(U_n(x))})) = S_y.$$

Similarly we have $T_y = \rho(g)T_x$. Thus, if some S_x has nonempty interior, disjointness of S_x and S_y would give an uncountable family of disjoint open sets in N, a contradiction. The same applies to the sets T_x. □

We next prove these sets, though defined differently, are in fact the same.

Lemma 3.4. For all x, we have $S_x = T_x$

Proof. Fix x and let $U_n = U_n(x)$ be a neighborhood basis of x with the property that $U_n \supset \overline{U_{n+1}}$ for all n. Thus, by Lemma 3.1, $\rho(G_{U_{n+1}})$ and $\rho(G_{M-\overline{U_n}})$ have disjoint open supports. Since $G_{M-\overline{U_n}} \supset \rho(G^{U_n})$ we conclude

$$\mathsf{Osupp}(\rho(G_{U_{n+1}})) = N - \mathsf{Fix}(\rho(G_{U_{n+1}})) \subset \mathsf{Fix}(\rho(G_{M-\overline{U_n}})) \subset \mathsf{Fix}(\rho(G^{U_n}))$$

Also, since U_n and $M - Un - 1$ have disjoint closures, Observation 3.2 implies that $\mathsf{Fix}(\rho(G^{U_n})) \cap \mathsf{Fix}(\rho(G^{M-Un-1})) = \emptyset$, so

$$\mathsf{Fix}(\rho(G^{U_n})) \subset \mathsf{Osupp}(\rho(G^{M-Un-1})) \subset \mathsf{Osupp}(\rho(G_{U_{n-2}}))$$

Combining the two equations above and taking a limit as $n \to \infty$ shows that $S_x \subset T_x \subset S_x$, as desired.

Thus $S_x \subset T_x$. For the reverse inclusion, suppose $z \in T_x - S_x$. Then $z \notin \mathsf{Osupp}(\rho(G_{U_n}))$ for some n; i.e., $z \in \mathsf{Fix}(\rho(G_{U_n}))$. Also $z \in \mathsf{Fix}(\rho(G^{U_{n+1}}))$ by the definition of T_x. But G_{U_n} and $G^{U_{n+1}}$ together generate $\mathsf{Diff}^r_c(M)$ (this again is the *fragmentation property*), so this implies that z is a global fixed point. □

Lemma 3.5. S_x is nonempty.

Proof. If $N = S^1$, this follows immediately since $S_x = T_x$ is the intersection of nested, nonempty closed sets. If $N = \mathbb{R}$, the same is true provided that $\mathsf{Fix}(\rho(G^{U_n(x)}))$, (or equivalently $\mathsf{Osupp}(\rho(G_{U_n(x)}))$, does not leave every compact set as $n \to \infty$. Note that this holds for some x if and only if it holds for all x because $\rho(g)S_x = S_y$ when $g(x) = y$.

Suppose for contradiction that, for each $x \in M$,as $n \to \infty$ we have that $\mathsf{Osupp}(\rho(G_{U_n(x)})$ does leave every compact set. Fixing some compact $K \subset \mathbb{R}$, this means that for each $x \in M$ there is a neighborhood $U(x)$ of x such that $\mathsf{Osupp}(\rho(G_{U(x)})) \cap K = \emptyset$. Let $\mathcal{O}$ denote the open cover formed by such sets $U(x)$. By fragmentation, $\mathsf{Diff}^r_c(M)$ is generated by the subgroups $G(U(x))$. Thus, $\mathsf{Osupp}(\rho(\mathsf{Diff}^r_c(M)) \cap K = \emptyset$ contradicting the fact that ρ has no global fixed points.

□

Construction of ϕ. To finish the proof, we wish to show that S_x is a singleton, and the assignment $\phi : x \mapsto S_x$ is a homeomorphism conjugating ρ with the standard action of $\mathsf{Diff}^r_c(M)$ on M. We will actually show first that $x \mapsto S_x$ is a local homeomorphism, use this to *conclude* that S_x is discrete, and proceed from there.

Step 1: definition of ϕ locally Let $I = (a, b)$ be a connected component of $N - S_x$, chosen so that $a \neq -\infty$ if $N = \mathbb{R}$. If $N = S^1$ and S_x is a singleton, it is possible that both "endpoints" of this interval agree. For simplicity, we treat the case where $a \neq b$, the case $a = b$ on the circle can be handled with exactly the same strategy, and in fact the argument simplifies quite a bit since S_x is already a singleton.

Fix a neighborhood basis $U_n \supset U_{n+1} \supset ...$ of x. For $n \in \mathbb{N}$, denote by O_n the connected component of $\mathsf{Osupp}(\rho(G_{U_n}))$ that contains a. Since $\cap_k O_k \subset S_x$ and contains a, and since

$(a,b) \subset N - S_x$, we can conclude that for all k sufficiently large a is the rightmost point of $S_x \cap O_k$.

Fix such a k. We will show that, for $y \in U_k$, the set $S_y \cap O_k$ also has a rightmost point. This allows us to define a map from U_k to O_k, sending y to this rightmost point, which we will then show is the desired local homeomorphism. First, to see that $S_y \cap O_k$ has a rightmost point, take some $g \in G_{U_k}$ with $g(x) = y$. Thus $\rho(g)(S_x) = S_y$. Since $\rho(g)$ fixes endpoints of O_k by definition, we know that $\rho(g)(a) \in S_y$ and is the rightmost point of $S_y \cap O_k$. This proves our claim.

Define $\phi : U_k \to O_k$ by setting $\phi(y)$ to be the rightmost point of $S_y \cap O_k$. An equivalent definition of ϕ is that $\phi(y) := \rho(g)(a)$, where g is any diffeomorphism in G_{U_k} such that $g(x) = y$. Our argument above shows this is independent of choice of g. Furthermore, if we repeat the definition using U_{k+1} instead of U_k, the map we will obtain is simply the restriction of ϕ to U_{k+1}.

Step 2: local continuity of ϕ on U_k

We first show that ϕ is continuous at x. Suppose $x_n \to x$ is a convergent sequence. Passing to a subsequence and reindexing if needed, we may assume that $x_n \in U_n$ and that our index set starts at k. Then we may take $g \in G_{U_n}$ so that $g(x) = x_n$, so $\phi(x_n) = \rho(g)(a)$. Since the sequence of connected components of $\mathsf{Osupp}(\rho(G_{U_n}))$ containing x converges to x, we get that $\phi(x_n) \to a$.

To show that ϕ is continuous on U_k, let $x' \in U_k$, and take a sequence $x'_n \to x'$ in U_k. There exists $g \in G_{U_k}$ such that $g(x) = x'$ and $g^{-1}(x'_n)$ is a sequence converging to x. It follows from continuity at x that $\phi(g^{-1}(x'_n))$ converges to $\phi(x)$. By definition, $\rho(g)\phi(g^{-1}(x'_n)) = \phi(x'_n)$, so we conclude that $\phi(x'_n)$ converges to $\phi(x')$.

Note also that ϕ is injective by Lemma 3.3. Thus, by invariance of domain, we conclude that M is one-dimensional so equal to $\mathbb{R}$ or S^1, and ϕ gives a homeomorphism from U_k onto an open interval A containing a in N. In particular, this shows that a is an isolated point of S_x.

Step 3: extension of ϕ globally

The last step is to show that ϕ extends to a globally defined homeomorphism $M \to N$; to do this we actually work with the inverse of ϕ. First, note that the orbit of A under $\rho(G)$ is an open, $\rho(G)$-invariant set, so by Corollary 2.1, $\rho(G)(A) = N$.

This topological transitivity implies that, for all x, every point of S_x is an isolated point, i.e. S_x is discrete. Extend ϕ^{-1} to a map ψ defined on N by setting $\psi(S_x) = x$. The work in step 2 and the fact that $\rho(g)(S_x) = S_{g(x)}$ implies that ψ is a local homeomorphism, hence a covering map, and is equivariant with respect to the actions of $\mathsf{Diff}^r_c(M)$ by its standard action on M and by ρ on N. If $M = \mathbb{R}$, we immediately conclude that $N = \mathbb{R}$, and ψ conjugates ρ to the standard action of $\mathsf{Diff}^r_c(\mathbb{R})$.

If $M = S^1$, we can also conclude that $N = S^1$ because $\mathsf{Diff}^r_c(S^1)$ contains torsion, so cannot faithfully act on $\mathbb{R}$. Thus, $\psi : S_x \mapsto x$ is a finite cover, and ρ is a lift of the standard action of $\mathsf{Diff}^r_c(S^1)$ on S^1. Identifying the rotation subgroup $\mathrm{SO}(2)$ with S^1, and considering $\rho(\mathrm{SO}(2))$ which is a continuous lift, covering space theory tells us the degree of the cover must be 1. Alternatively, one can derive a contradiction by looking at the action of finite order elements: an order two rotation lifted to a degree d cover will have order $2d$.

□

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