On Unambiguous Regular Tree Languages of Index $(0, 2)$

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Joint work with

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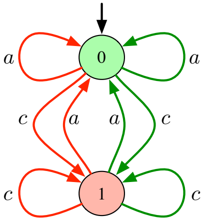
Parity Tree Automata

Parity tree automaton

$$\mathcal{A} = \langle A, Q, I, \delta, r \rangle$$

- a finite input alphabet A,
- a finite set Q of states,
- a set of initial states $I \subseteq Q$,
- a transition relation $\delta \subseteq Q \times A \times Q \times Q$ and
- a priority function $r : Q \to \omega$.



Acceptance condition: the greatest parity seen infinitely often on a branch is even.

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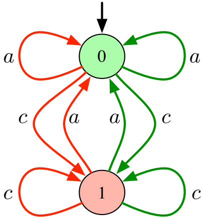

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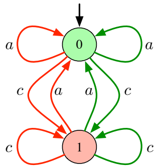
Tree Automata

- Automata that read full infinite binary trees in a top-down manner.
- A *run* of an automaton on an infinite tree is an infinite binary tree.
- A run is accepting if the acceptance condition holds on all its branches.
- An automaton $\mathcal{A}$ accepts a tree t iff there is an accepting run of $\mathcal{A}$ on t.
- The set of trees accepted by an automaton $\mathcal{A}$ is called the *language* recognized by $\mathcal{A}$



Determinism vs. Unambiguity

- An automaton is *deterministic* iff there is a unique run per input.
- An automaton is *unambiguous* iff there is at most one accepting run per input.

Fact

$$\mathit{DET} \subsetneq \mathit{UNAMB} \subsetneq \mathit{REG}$$

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Tree Languages and Topology

- The space T_A (of all full binary trees on the alphabet A) can be endowed with a very natural topology, the *prefix topology*: any basic open set is of the form

$$N_s = \left\{ t \in T_A \mid s \subseteq t \right\}$$

for some finite tree s over A.

- This space is homeomorphic to the Cantor space.

Topological complexity

Definition

A *pointclass* Γ is a collection of subsets closed under continuous preimages: for all continuous function f and all $C \in \Gamma$,

$$f^{-1}(C) \in \Gamma.$$

The *topological complexity* of a subset of a topological space is the smallest pointclass it belongs to.

Definition

Let Γ be a pointclass. The set $C \in \Gamma$ is Γ-*complete* iff for all $B \in \Gamma$ there exists a continuous function f such that

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A set C is of complexity Γ iff C is Γ-complete.

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The Borel Hierarchy

For every countable ordinal $0 < \xi$, we define the pointclasses $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$ and $\mathbf{\Delta}^0_\xi$ of T_A as follows:

- $\mathbf{\Sigma}^0_1$ is the class of all the open subsets of T_A,
- $L \in \mathbf{\Pi}^0_\xi$ iff $L^\complement \in \mathbf{\Sigma}^0_\xi$,
- for $2 \leq \xi$, $L \in \mathbf{\Sigma}^0_\xi$ iff there is a sequence $(L_n)_{n<\omega}$ of elements of $\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta$ such that $L = \bigcup_{n<\omega} L_n$,
- $L \in \mathbf{\Delta}^0_\xi$ iff $L \in \mathbf{\Sigma}^0_\xi$ and $L \in \mathbf{\Pi}^0_\xi$.

Note that, for every $\xi < \omega_1$, $\mathbf{\Sigma}^0_\xi \subseteq \mathbf{\Delta}^0_{\xi+1}$ and $\mathbf{\Pi}^0_\xi \subseteq \mathbf{\Delta}^0_{\xi+1}$.

This hierarchy provides a bottom-up description of the Borel sets since

$$\mathbf{B} = \bigcup_{\xi<\omega_1} \mathbf{\Sigma}^0_\xi = \bigcup_{\xi<\omega_1} \mathbf{\Pi}^0_\xi.$$

This hierarchy is strict, i.e. for every countable ordinal $0 < \xi$, $\mathbf{\Sigma}^0_\xi \setminus \mathbf{\Pi}^0_\xi \neq \emptyset$.

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The Projective Hierarchy

For every positive integer n, we define the pointclasses $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$ and $\mathbf{\Delta}^1_n$ of T_A as follows:

- $C \in \mathbf{\Sigma}^1_1$ iff C is an *analytic* subset of T_A, i.e. the direct image of a Borel set by a continuous function;
- $L \in \mathbf{\Pi}^1_n$ iff $L^\complement \in \mathbf{\Sigma}^1_n$;
- $L \in \mathbf{\Sigma}^1_{n+1}$ iff it is the direct image of some $M \in \mathbf{\Pi}^1_n$ by a continuous function;
- $L \in \mathbf{\Delta}^1_n$ iff $L \in \mathbf{\Sigma}^1_n$ and $L \in \mathbf{\Pi}^1_n$.

Note that, by Suslin's theorem, $\mathbf{B} = \mathbf{\Delta}^1_1$, and that for every positive integer n, the following holds:

$$\mathbf{\Sigma}^1_n \subseteq \mathbf{\Delta}^1_{n+1} \qquad \text{and} \qquad \mathbf{\Pi}^1_n \subseteq \mathbf{\Delta}^1_{n+1}.$$

This hierarchy is called the *projective hierarchy*.

Topological complexity of Regular Tree Languages

Theorem

- *If a language is regular, then it is in the class* $\mathbf{\Delta}^1_2 = \mathbf{\Sigma}^1_2 \cap \mathbf{\Pi}^1_2$.
- *If a language is recognized by a deterministic automaton, then it is in the class* $\mathbf{\Pi}^1_1$.

To address the question of the position of unambiguous languages between deterministic and nondeterministic ones, we therefore need a finer scale than the Borel or the projective hierarchies. We use the finest topological complexity measure: *the Wadge hierarchy*.

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The Wadge Quasi-Order

Definition

Let $L, M \subseteq T_A$, $L \leq_W M$ iff there exists a continuous function $f : T_A \to T_A$ such that

$$f^{-1}(M) = L.$$

The relation $\leq_W$ on $\mathcal{P}(T_A)$ is a quasi-order (qo) whose equivalence classes are called the *Wadge degrees*.

Theorem

If Γ is a pointclass with appropriate closure and determinacy properties (e.g. the Borel sets, projective sets under PD, etc.), the Wadge qo restricted to Γ is in fact a well-quasi-order:

- $\forall L, M \in \Gamma$, $L \leq_W M$ *or* $M \leq_W L^{\complement}$. *(Wadge's Lemma)*
- *There is no infinite strictly decreasing chain of elements of Γ w.r.t. $\leq_W$. (Martin-Monk)*

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The Wadge Game

- Given $L, M \subseteq T_A$, in the Wadge game $\mathbb{W}(L, M)$:
  - player I builds $t_I \in T_A$ and player $I\!I$ builds $t_{I\!I} \in T_A$;
  - at every round, player I plays first, and both players add a finite number of children to the terminal nodes of their ongoing trees;
  - player $I\!I$ is allowed to skip its turn, but has to produce a tree in T_A throughout a game;
  - Player $I\!I$ wins the game if and only if $t_I \in L \Leftrightarrow t_{I\!I} \in M$.
- $L \leq_W M \Leftrightarrow I\!I$ has a w.s. in $\mathbb{W}(L, M)$.
- $L \equiv_W M \;\Leftrightarrow\; L \leq_W M$ *and* $M \leq_W L$.

The Wadge rank

The wellfoundedness of the Wadge hierarchy ensures that the Wadge rank can be defined by induction as follows:

- $d_w(\emptyset) = d_w(T_A) = 1$
- $d_w(L) = \sup \{d_w(M) + 1 : M$ is non-self-dual, $M <_W L\}$ for $M >_W \emptyset$.

New topological complexity measure! The topological complexity of the set C is the ordinal $d_w(C)$. Our two notions of topological complexity coincide, since pointclasses are exactly the *initial segments* of the Wadge hierarchy.

$$(\mathcal{P}(T_A), \leq_W) \leftrightsquigarrow (\text{Pointclasses}, \subseteq)$$

Hence, to compare the topological complexity of the deterministic and unambiguous languages, we compare the height of their respective Wadge hierarchies.

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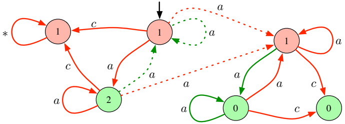
Operations on automata

Three ingredients:

- An unambiguous $\mathbf{\Sigma}_1^1$-complete automaton defined by Hummel.



- Operations on automata defined by Duparc and Murlak:

$$\bullet\,,\ +\,,\ \omega^{\mathcal{A}}.$$

- The adaptation of an operation defined in the DST framework.

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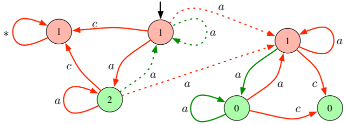

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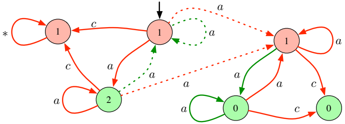

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The Veblen Functions

Let us denote by $\varphi_2(0)$ the first fixed point of the ordinal epsilon function, namely the one that enumerates the fixed points of the exponentiation of base ω:

1. $\varepsilon_0 = \sup_{n<\omega} \underbrace{\omega^{\cdot^{\cdot^{\cdot^{\omega^0}}}}}_{n}$;
2. $\varepsilon_{\alpha+1} = \sup_{n<\omega} \underbrace{\omega^{\cdot^{\cdot^{\cdot^{\omega^{(\varepsilon_\alpha+1)}}}}}}_{n}$;
3. $\varepsilon_\lambda = \sup_{\alpha<\lambda} \varepsilon_\alpha$, for λ some limit ordinal.

$$\varphi_2(0) = \sup_{n<\omega} \underbrace{\varepsilon_{\cdot_{\cdot_{\cdot_{\varepsilon_0}}}}}_{n}$$

A Fragment of the Unambiguous Wadge Hierarchy

Theorem

There exists a family $(\mathcal{A}_\alpha)_{\alpha<\varphi_2(0)}$ *of unambiguous parity tree automata s.t.*

1. *they recognize languages over the alphabet* $\{a, b, c\}$*;*
2. *their priorities are restricted to* $\{0, 1, 2\}$*;*
3. $\alpha < \beta \Leftrightarrow \mathcal{A}_\alpha <_W \mathcal{A}_\beta$*;*
4. *all the underlying winning strategies are* **effective**.

A Fragment of the Unambiguous Wadge Hierarchy

The Wadge hierarchy of **deterministic** tree languages [Filip Murlak] has height precisely

$$(\omega^\omega)^3 + 3$$

The Wadge hierarchy of **unambiguous** tree languages has height at least

$$\varphi_2(0) = \sup_{n<\omega} \underbrace{\varepsilon_{\cdot_{\cdot_{\cdot_{\varepsilon_0}}}}}_{n}$$