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WADGE HIERARCHY OF DIFFERENCES OF CO-ANALYTIC SETS

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Abstract. We begin the fine analysis of non Borel pointclasses. Working in $ZFC+$ DET($\mathbf{\Pi}^1_1$), we describe the Wadge hierarchy of the class of increasing differences of co-analytic subsets of the Baire space by extending results obtained by Louveau ([5]) for the Borel sets.

Introduction. Collections of substets of the Baire space, the "logician's reals", that are closed under continuous preimages have always been ubiquitous in descriptive set theory. It is thus quite remarkable to realize that the concept of *pointclass* has not been singled out and studied for itself before the 1960's and the work of Wadge. In his PhD Thesis ([10]), he was the first to study systematically the concept, via the notion of continuous reducibility. Given two subsets A and B of the Baire space, A is said to be reducible to B, and we write $A \leq_W B$, if and only if A is the preimage of B for some continuous function f from the Baire space to itself. The relation $\leq_W$ is merely by definition a preorder, and its initial segments are exactly the pointclasses of the Baire space. When restricted to a class with suitable closure properties, the preorder induced by $\leq_W$ on its equivalence classes, the Wadge degrees, is in fact a well-quasi-ordering. The study of this well-quasi-ordering, the Wadge hierarchy, and of the Wadge degreees gives thus the finest analysis of the pointclasses of the Baire space.

The Wadge hierarchy of the Borel subsets of the Baire space has been thoroughly studied by Louveau in [5] and Duparc in [3] and [2], in two different manners that were both initiated by Wadge. The former relies on a Theorem proved by Wadge stating that all the non self-dual Borel pointclasses can be obtained by ω-ary Borel boolean operations on open sets - a result later generalized to all non self-dual pointclasses of the Baire space by Van Wesep in [8] under AD, using of course arbitrary ω-ary boolean operations. Louveau's work provides a description of all the Borel pointclasses, and thus of the whole Wadge hierarchy on the Borel sets, by means of boolean operations. The latter approach, followed by Duparc and that we do not pursue here, aims to define and use specific operations on sets in order to give for each Wadge class of Borel subsets a canonical complete set.

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Working in $ZFC + \mathrm{DET}(\underset{\sim}{\mathbf{\Pi}}^1_1)$, we show how to extend the constructions of Louveau on the Borel sets to $\mathrm{Diff}(\underset{\sim}{\mathbf{\Pi}}^1_1)$, the class of increasing differences of co-analytic subsets. This extension provides a complete description of the pointlcasses included in $\mathrm{Diff}(\underset{\sim}{\mathbf{\Pi}}^1_1)$. Surprisingly enough, the set of operations used in the Borel case is sufficient for this task, we so to speak only add the possibility for them to act on $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets. The second part of this article is devoted to the discreapancy between the pointclasses of differences using *increasing* sequences of $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets, and differences using *decreasing* sequences of $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets. We prove, combining our analysis with results from Martin ([6]) and Harrington ([4]), that our determinacy hypothesis is optimal.

As for prerequisites, the reader is expected to be familiar with the basic notions and results of Wadge theory, as exposed for example in [9]. For his convenience, we repeat the relevant material from [5] when we need it, thus keeping our exposition as self-contained as possible, even if we do not recall a big part of the proof of Theorem 1.9 in [5].

§1. The Difference Hierarchy of Co-Analytic Sets.

1.1. Wadge framework and notation.

> "The Wadge Hierarchy is the ultimate analysis of $\mathcal{P}(\omega^\omega)$ in terms of topological complexity [...]"
>
> Alessandro Andretta, Alain Louveau, [1].

The Wadge theory is in essence the theory of *pointclasses*. Let X be a topological space. A pointclass is a collection of subsets of X that is closed under continuous preimages. For Γ a pointclass, we denote by $\check{\Gamma}$ its *dual* class containing all the subsets of X whose complements are in Γ, and by $\Delta(\Gamma)$ the ambiguous class $\Gamma \cap \check{\Gamma}$. If $\Gamma = \check{\Gamma}$, we say that Γ is self-dual.

We only consider the *Baire space* ω^ω in this paper, with the usual topology. The *Wadge preorder* $\leq_W$ on $\mathcal{P}(\omega^\omega)$ is defined as follows: for $A, B \subseteq \omega^\omega$, $A \leq_W B$ if and only if there exists $f : \omega^\omega \longrightarrow \omega^\omega$ continuous such that $f^{-1}(B) = A$. For $A, B \subseteq \omega^\omega$, we write $A <_W B$ if and only if $A \leq_W B$ but $B \nleq_W A$. The Wadge preorder induces an equivalence relation $\equiv_W$ whose equivalence classes are called the *Wadge degrees*, and denoted by $[A]_W$. We say that the set $A \subseteq \omega^\omega$ is *self-dual* if it is Wadge equivalent to its complement, that is if $A \equiv_W A^{\complement}$, and *non-self-dual* if it is not. We use the same terminology for the Wadge degrees.

A useful game characterization is provided by the *Wadge game*, a two players infinite game. Let $A, B \subseteq \omega^\omega$, in the Wadge game $W(A, B)$ player I plays first an integer x_0, II answers with an integer y_0, and so on and so forth. II has the possibility to skip, even ω times, provided she also plays infinitely often. At the end of the game, each player has constructed an infinite sequence, x for I and y for II. II wins the game if and only if $(x \in A \leftrightarrow y \in B)$. Noticing that strategies for II can be viewed as continuous functions, we have:

$$\text{II has a winning strategy in } W(A, B) \quad \longleftrightarrow \quad A \leq_W B.$$

Given a pointclass Γ with suitable closure properties, the assumption of the determinacy of Γ is sufficient to prove that Γ is semi-linearly ordered by $\leq_W$, denoted SLO(Γ), i.e. that for all $A, B \in \Gamma$,

$$A \leq_W B \qquad \text{or} \qquad B \leq_W A^{\complement}.$$

and that $\leq_W$ is well founded when restricted to sets in Γ. Under these conditions, the Wadge degrees of sets in Γ with the induced order is thus a hierarchy called the *Wadge hierarchy.* There is a strong connection between pointclasses included in Γ and Wadge degrees of sets in Γ since all non-self-dual pointclasses are of the form

$$\{B \subseteq \omega^\omega \mid B \leq_W A\}$$

for some non-self-dual set A, while self-dual pointclasses are all of the form

$$\{B \subseteq \omega^\omega \mid B <_W A\}.$$

We have thus a direct correspondance between $(\mathcal{P}(\omega^\omega), \leq_W)$ restricted to Γ and the pointclasses included in Γ with the inclusion: the pointclasses are exactly the initial segments of the Wadge hierarchy. The semi-linear ordering property becomes then: for any pointclasses Γ' and Γ'' included in Γ,

$$\Gamma' \subseteq \Gamma'' \text{ or } \check{\Gamma''} \subseteq \Gamma'.$$

In this paper, following Louveau's framework, we describe the pointclasses. From the works of Martin and Harrington (see [6] and [4]), we know that the class $D^*_{\omega^2}(\boldsymbol{\Pi}^1_1)$ of all decreasing ω^2 differences of co-analytic sets is determined under DET($\boldsymbol{\Pi}^1_1$). Since Diff($\boldsymbol{\Pi}^1_1$) $\subseteq D^*_\omega(\boldsymbol{\Pi}^1_1)$ - see Proposition 3.4, it is sufficient here to assume this determinacy hypothesis.

1.2. General Observations. Notice that every ordinal θ can be written as $\theta = \lambda + n$, where λ is limit and $n < \omega$. We call θ *even* if n is even, and *odd* if n is odd.

Definition 1.1. Let $(A_\eta)_{\eta<\theta}$ be an *increasing* sequence of subsets of the Baire space, with $\theta < \omega_1$. Define the set $D_\theta((A_\eta)_{\eta<\theta})$ by:

$$D_\theta((A_\eta)_{\eta<\theta}) = \Big\{ x \in \bigcup_{\eta<\theta} A_\eta : \text{the least } \eta < \theta \text{ with } x \in A_\eta \text{ has parity opposite to that of } \theta \Big\}.$$

For $\theta < \omega_1$, and Γ a pointclass, let

$$D_\theta(\Gamma) = \{D_\theta((A_\eta)_{\eta<\theta}) | A_\eta \in \Gamma, \eta < \theta\}.$$

It is also a pointclass.

We denote the class of all countable differences of co-analytic sets by Diff($\boldsymbol{\Pi}^1_1$) $= \bigcup_{\alpha<\omega_1} D_\alpha(\boldsymbol{\Pi}^1_1)$. Merely by definition, we have $D_\alpha(\boldsymbol{\Pi}^1_1) \subseteq D_\beta(\boldsymbol{\Pi}^1_1)$ and $D_\alpha(\boldsymbol{\Pi}^1_1) \subseteq \check{D}_\beta(\boldsymbol{\Pi}^1_1)$ for all $\alpha < \beta$. Moreover, since there exists a ω^ω-universal set for $\boldsymbol{\Pi}^1_1$, the hierarchy does not collapse, i.e. for all $\alpha < \omega_1$, $D_\alpha(\boldsymbol{\Pi}^1_1) \setminus \check{D}_\alpha(\boldsymbol{\Pi}^1_1) \neq \emptyset$. We have thus the following classical diamond-shape diagram:

$$\begin{array}{ccccccc}
\mathbf{\Pi}^1_1 & & D_2(\mathbf{\Pi}^1_1) & & D_3(\mathbf{\Pi}^1_1) & & \\
 & \Delta(D_2(\mathbf{\Pi}^1_1)) & & \Delta(D_3(\mathbf{\Pi}^1_1)) & & \cdots \\
\mathbf{\Sigma}^1_1 & & \check{D}_2(\mathbf{\Pi}^1_1) & & \check{D}_3(\mathbf{\Pi}^1_1) & &
\end{array}$$

where the pointclasses are strictly included in each other from the left to the right.

1.3. The ambiguous classes. To describe the pointclasses included in $\Delta(D_\alpha(\mathbf{\Pi}^1_1))$, we need a characterization of the $D_\alpha(\mathbf{\Pi}^1_1)$ classes.

1.3.1. *The successor case.*

Proposition 1.2. *For every countable ordinal α, we have:*
(a) $D_{\alpha+1}(\mathbf{\Pi}^1_1) = \check{D}_\alpha(\mathbf{\Pi}^1_1) \cap \mathbf{\Pi}^1_1 = \{D \cap C \mid D \in \check{D}_\alpha(\mathbf{\Pi}^1_1) \text{ and } C \in \mathbf{\Pi}^1_1\}$;
(b) $\check{D}_{\alpha+1}(\mathbf{\Pi}^1_1) = D_\alpha(\mathbf{\Pi}^1_1) \cup \mathbf{\Sigma}^1_1 = \{D \cup C \mid D \in D_\alpha(\mathbf{\Pi}^1_1) \text{ and } C \in \mathbf{\Sigma}^1_1\}$.

Proof. We only prove the first assertion for the finite differences, the other follows by considering the complements, and the generalization to the transfinite is straightforward. Let $n = 2k$ for $k \geq 1$. Observe that for any increasing family $(A_i)_{i<n}$ of co-analytic subsets of the Baire space, we have:

$$D_n((A_i)_{i<n}) = A_{n-1} \setminus D_{n-1}((A_i)_{i<n-1});$$

abd therefore $D_n(\mathbf{\Pi}^1_1) \subseteq \check{D}_{n-1}(\mathbf{\Pi}^1_1) \cap \mathbf{\Pi}^1_1$. For the other inclusion, let $D \in \check{D}_{n-1}(\mathbf{\Pi}^1_1)$ and $B \in \mathbf{\Pi}^1_1$. Then there exists an increasing family of co-analytic sets $(A_i)_{i<n-1}$ such that:

$$D = D_{n-1}((A_i)_{i<n-1})^{\complement}.$$

We obtain:

$$\begin{aligned}
D \cap B &= \bigcap_{i=1}^{k-1} (A_{2i}^{\complement} \cup A_{2i-1}) \cap A_0^{\complement} \cap B \\
&= (B \cap A_{2k-2}^{\complement}) \cup \bigcup_{i=0}^{k-2} (B \cap A_{2i}^{\complement} \cap A_{2i+1}) \\
&= D_n((A_0 \cap B, A_1 \cap B, \ldots, A_{n-2} \cap B, B)),
\end{aligned}$$

where the second equality relies on the fact that the family $(A_i)_{i<n-1}$ is increasing. Thus $D \cap B \in D_n(\mathbf{\Pi}^1_1)$ and $D_n(\mathbf{\Pi}^1_1) = \check{D}_{n-1}(\mathbf{\Pi}^1_1) \cap \mathbf{\Pi}^1_1$. The odd case is similar. ⊣

This result can be illustrated by the following diagram.

$$\begin{array}{ccccccccc}
\mathbf{\Sigma}^1_1 & \subseteq & \check{D}_2(\mathbf{\Pi}^1_1) & \subseteq & \check{D}_3(\mathbf{\Pi}^1_1) & \subseteq & \cdots \\
 & \searrow{\scriptstyle \cap \mathbf{\Pi}^1_1} & & \nearrow{\scriptstyle \cup \mathbf{\Sigma}^1_1} & & & \\
\mathbf{\Pi}^1_1 & \subseteq & D_2(\mathbf{\Pi}^1_1) & \subseteq & D_3(\mathbf{\Pi}^1_1) & \subseteq & \cdots
\end{array}$$

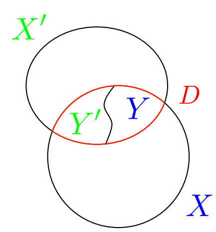
This inductive definition for the successor classes $D_\alpha(\mathbf{\Pi}^1_1)$ and $\check{D}_\alpha(\mathbf{\Pi}^1_1)$ allows us to adapt a result from Louveau, stated in [5].

Proposition 1.3. (Louveau's trick I).
Let $\alpha < \omega_1$, and $D \in \Delta(D_{\alpha+1}(\mathbf{\Pi}^1_1))$. Then there exists $B \in \mathbf{\Delta}^1_1$, $X \in \check{D}_\alpha(\mathbf{\Pi}^1_1)$ and $Y \in D_\alpha(\mathbf{\Pi}^1_1)$ such that

$$D = (X \cap B) \cup (Y \setminus B).$$

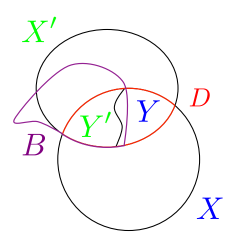
Proof. The set D is both in $D_{\alpha+1}(\mathbf{\Pi}^1_1)$ and in $\check{D}_{\alpha+1}(\mathbf{\Pi}^1_1)$. Proposition 1.2 gives $X' \in \mathbf{\Pi}^1_1$, $X \in \check{D}_\alpha(\mathbf{\Pi}^1_1)$, $Y' \in \mathbf{\Sigma}^1_1$ and $Y \in D_\alpha(\mathbf{\Pi}^1_1)$ such that

$$D = X' \cap X \qquad \text{and} \qquad D = Y' \cup Y.$$



In particular, we have that $Y' \cap X'^{\complement} = \emptyset$. By the separation property for the analytic sets, there exists a Borel subset B such that

$$Y' \subseteq B \qquad \text{and} \qquad B \cap X'^{\complement} = \emptyset.$$



Hence,

$$D = (X \cap B) \cup (Y \setminus B). \qquad \dashv$$

1.3.2. *The limit case.* A similar description of the ambiguous classes can be provided for the limit case, using a countable Borel partition instead of just one Borel set and its complement.

Proposition 1.4. (Louveau's trick II).
Let $D \subseteq \omega^\omega$ be in the $\Delta(D_\delta(\underset{\sim}{\mathbf{\Pi}}^1_1))$ class with $\delta < \omega_1$ limit. Then there exists a countable Borel partition $(C_i)_{i\in\omega}$ of the Baire space such that, for all $j < \omega$,

$$D \cap C_j \in D_{\alpha_j}(\underset{\sim}{\mathbf{\Pi}}^1_1),$$

with $\alpha_j < \delta$.

Proof. We only prove it for $\delta = \omega$. Let $D \subseteq \omega^\omega$ be in the $\Delta(D_\delta(\underset{\sim}{\mathbf{\Pi}}^1_1))$ class. By definition there exists two increasing families $(B_i)_{i\in\omega}$ and $(B'_i)_{i\in\omega}$ of co-analytic subsets of the Baire space such that

$$D = \bigcup_{i\in\omega}(B_{2i+1}\backslash B_{2i}) \qquad \text{and} \qquad D^{\complement} = \bigcup_{i\in\omega}(B'_{2i+1}\backslash B'_{2i}).$$

By the generalized reduction property of the class of co-analytic sets, there exists a disjoint co-analytic family $(C_i)_{i\in\omega}$ such that

- for all $i < \omega$, $C_{2i} \subseteq B_i$ and $C_{2i+1} \subseteq B'_i$, and
- $\bigcup_{i\in\omega} C_i = \bigcup_{i\in\omega} B_i \cup \bigcup_{i\in\omega} B'_i$.

Since $\bigcup_{i\in\omega} B_i \cup \bigcup_{i\in\omega} B'_i = D \cup D^{\complement} = \omega^\omega$, the family $(C_i)_{i\in\omega}$ is in fact an analytic, thus Borel, partition of the Baire space. In addition, the fact that $C_{2i} \subseteq B_i$ and $C_{2i+1} \subseteq B'_i$ hold for all $i \in \omega$ implies that $D \cap C_{2i}$ and $D^{\complement} \cap C_{2i+1}$ are in the class $D_{i+1}(\underset{\sim}{\mathbf{\Pi}}^1_1)$. To prove that our partition is indeed as required, it only remains to show that for all $i \in \omega$, $D \cap C_{2i+1}$ and $D^{\complement} \cap C_{2i}$ are finite differences of co-analytic sets. Fix $i \in \omega$, we have

$$D \cap C_{2i+1} = C_{2i+1} \cap \left(D^{\complement} \cap C_{2i+1}\right)^{\complement}.$$

But $D^{\complement} \cap C_{2i+1}$ is a finite difference of analytic sets, so that $D \cap C_{2i+1}$ is also a finite difference of co-analytic sets. The same argument works for $D^{\complement} \cap C_{2i}$. ⊣

Louveau's tricks I and II provide a bottom up description of the ambiguous classes, and from them we can now derive the complete description *à la* Louveau of the Wadge hierarchy of the class $\mathrm{Diff}(\underset{\sim}{\mathbf{\Pi}}^1_1)$.

§2. The Wadge Hierarchy of the $\mathrm{Diff}(\underset{\sim}{\mathbf{\Pi}}^1_1)$ sets.

2.1. Boolean operations and descriptions. We recall the definitions of the operations used by Louveau in [5]. Besides the differences that we have already introduced, four more operations are needed.

(a) *Separated Unions.* Let Γ and Γ' be two pointclasses. The set A is in $\mathrm{SU}(\Gamma, \Gamma')$ if and only if there exists a disjoint family $(C_n)_{n\in\omega}$ of sets in Γ, and a family $(A_n)_{n\in\omega}$ of sets in Γ' such that

$$A = \mathrm{SU}((C_n)_{n\in\omega}, (A_n)_{n\in\omega}) = \bigcup_{n\in\omega} A_n \cap sC_n.$$

(b) *One-sided Separated Unions.* Let Γ and Γ' be two pointclasses. The set A is in $\mathrm{Sep}(\Gamma, \Gamma')$ if there exists $C \in \Gamma$, $B_1 \in \check{\Gamma}'$ and $B_2 \in \Gamma'$ such that

$$A = \mathrm{Sep}(C, B_1, B_2) = (C \cap B_1) \cup (B_2 \backslash C).$$

(c) *Two-sided Separated Unions.* Let Γ, Γ' and Γ'' be three pointclasses. The set A is in $\mathrm{Bisep}(\Gamma,\Gamma',\Gamma'')$ if there exists C_1, C_2 in Γ disjoint, $A_1 \in \check{\Gamma}'$, $A_2 \in \Gamma'$, and $B \in \Gamma''$ such that

$$A = \mathrm{Bisep}(C_1, C_2, A_1, A_2, B) = (C_1 \cap A_1) \cup (C_2 \cap A_2) \cup (B \backslash (C_1 \cup C_2)).$$

If $\Gamma'' = \{\emptyset\}$, we just write $\mathrm{Bisep}(\Gamma, \Gamma')$.

(d) *Separated Differences.* Let Γ, Γ' and Γ'' be three pointclasses, and $\xi \geq 2$ be countable. The set A is in $\mathrm{SD}_\xi((\Gamma,\Gamma'),\Gamma'')$ if there is an increasing family $(C_\eta)_{\eta<\xi}$ in Γ, an increasing family $(A_\eta)_{\eta<\xi}$ in Γ' and $B \in \Gamma''$ such that, for all $\eta < \xi$, $A_\eta \subseteq C_\eta \subseteq A_{\eta+1}$ and

$$A = \mathrm{SD}_\xi((C_\eta)_{\eta<\xi}, (A_\eta)_{\eta<\xi}, B) = \bigcup_{\eta<\xi} (A_\eta \backslash \bigcup_{\eta'<\eta} C_{\eta'}) \cup (B \backslash \bigcup_{\eta<\xi} C_\eta).$$

These operations, combined and applied in certain ways to certain classes give us all the non self-dual pointclasses included in $\mathrm{Diff}(\underset{\sim}{\mathbf{\Pi}}^1_1)$. But first we need to introduce some notation. Let $u_0, u_1 \in (\omega_1+1)^\omega$, we denote by $\langle u_0, u_1\rangle$ the sequence $u \in (\omega_1+1)^\omega$ such that, for all $n \in \omega$, $u(2n) = u_0(n)$, and $u(2n+1) = u_1(n)$. Similarly, if $(u_i)_{i\in\omega} \subseteq \big((\omega_1+1)^\omega\big)^\omega$, we denote by $\langle (u_i)_{i\in\omega}\rangle$ the sequence $u \in (\omega_1+1)^\omega$ such that for all $n, m \in \omega$, $u(\langle n, m\rangle) = u_n(m)$, where $(n,m) \mapsto \langle n,m\rangle$ is a bijection between $\omega\times\omega$ and ω. We now define inductively the set of *descriptions* $D \subseteq (\omega_1+1)^\omega$, and for each $u \in D$, the class Γ_u it describes.

Definition 2.1. The set of *descriptions* $D \subseteq (\omega_1+1)^\omega$ is the least satisfying the following conditions:

(a) If $u(0) = 0$, then $u \in D$ and $\Gamma_u = \{\emptyset\}$.
(b) If $u(0) = \xi < \omega_1$ with $\xi \neq 0$, $u(1) = 1$ and $u(2) = \eta < \omega_1$, then $u \in D$ and $\Gamma_u = D_\eta(\underset{\sim}{\mathbf{\Sigma}}^0_\xi)$.
(c) If $u(0) = \omega_1$, $u(1) = 1$ and $u(2) = \eta < \omega_1$, then $u \in D$ and $\Gamma_u = D_\eta(\underset{\sim}{\mathbf{\Pi}}^1_1)$.
(d) If $u = \xi\hat{\ }2\hat{\ }\eta\hat{\ }u^*$, where $1 \leq \xi < \omega_1$, $1 \leq \eta \leq \omega_1$, $u^* \in D$ and $u^*(0) > \xi$, then $u \in D$ and $\Gamma_u = \mathrm{Sep}(D_\eta(\underset{\sim}{\mathbf{\Sigma}}^0_\xi), \Gamma_{u^*})$.
(e) If $u = \xi\hat{\ }3\hat{\ }\eta\hat{\ }\langle u_0, u_1\rangle$, where $1 \leq \xi < \omega_1$, $1 \leq \eta \leq \omega_1$, $u_0, u_1 \in D$, $u_0(0) > \xi$, $u_1(0) \geq \xi$ or $u_1(0) = 0$, and $\Gamma_{u_1} \subset \Gamma_{u_0}$, then $u \in D$ and $\Gamma_u = \mathrm{Bisep}(D_\eta(\underset{\sim}{\mathbf{\Sigma}}^0_\xi), \Gamma_{u_0}, \Gamma_{u_1})$.
(f) If $u = \xi\hat{\ }4\hat{\ }\langle (u_n)_{n\in\omega}\rangle$, where $1 \leq \xi < \omega_1$, each $u_n \in D$, and either $u_n(0) = \xi_1 > \xi$ for all $n \in \omega$, and the Γ_{u_n} are strictly increasing, or $u_n(0) = \xi_n$ and the ξ_n are strictly increasing with $\xi < \sup_{n\in\omega} \xi_n$, then $u \in D$ and $\Gamma_u = \mathrm{SU}(\underset{\sim}{\mathbf{\Sigma}}^0_\xi, \bigcup_{n\in\omega} \Gamma_{u_n})$.
(g) If $u = \xi\hat{\ }5\hat{\ }\eta\hat{\ }\langle u_0, u_1\rangle$, where $1 \leq \xi < \omega_1$, $2 \leq \eta \leq \omega_1$, $u_0, u_1 \in D$, $u_0(0) = \xi$, $u_0(1) = 4$, $u_1(0) \geq \xi$ or $u_1(0) = 0$, and $\Gamma_{u_1} \subset \Gamma_{u_0}$, then $u \in D$ and $\Gamma_u = \mathrm{SD}_\eta((\underset{\sim}{\mathbf{\Sigma}}^0_\xi, \Gamma_{u_0}), \Gamma_{u_1})$.

Notice that compared to the Borel case, we only add the classes $D_\eta(\underset{\sim}{\mathbf{\Pi}}^1_1)$, which are given the level ω_1.

Proposition 2.2. *Let $u \in D$ with $u(0) = \xi \neq 0$.*

(a) *If* $\xi < \omega_1$, *then*
   - Γ_u *is closed under union with a* $\underset{\sim}{\mathbf{\Delta}}^0_\xi$ *set.*
   - $\mathrm{SU}(\underset{\sim}{\mathbf{\Sigma}}^0_\xi, \Gamma_u) = \Gamma_u$, *and we say that* Γ_u *is closed under* $\underset{\sim}{\mathbf{\Sigma}}^0_\xi - \mathrm{SU}$.

(b) *If* $\xi = \omega_1$, *then*
   - Γ_u *is closed under union with a* $\underset{\sim}{\mathbf{\Delta}}^1_1$ *set.*
   - $\mathrm{SU}(\underset{\sim}{\mathbf{\Pi}}^1_1, \Gamma_u) = \Gamma_u$, *and we say that* Γ_u *is closed under* $\underset{\sim}{\mathbf{\Pi}}^1_1 - \mathrm{SU}$.

Proof. The only thing left to verify is the case where $\Gamma_u = D_\eta(\underset{\sim}{\mathbf{\Pi}}^1_1)$, the rest is by the same induction as in [5]. Let $0 < \eta < \omega_1$, we have to prove that the class $D_\eta(\underset{\sim}{\mathbf{\Pi}}^1_1)$ is closed under union with a Borel set, and under $\underset{\sim}{\mathbf{\Pi}}^1_1 - \mathrm{SU}$. The first comes from the fact that the class $\underset{\sim}{\mathbf{\Pi}}^1_1$ is closed under union with a Borel set. For the second, let $(C_n)_{n\in\omega}$ be a disjoint family of $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets, and $(A_n)_{n\in\omega}$ a family of $D_\eta(\underset{\sim}{\mathbf{\Pi}}^1_1)$ sets. For all integer n, there exists a family $(A_n^\alpha)_{\alpha<\eta}$ such that $A_n = D_\eta((A_n^\alpha)_{\alpha<\eta})$. Thus:

$$\begin{aligned}
\mathrm{SU}((C_n)_{n\in\omega}, (A_n)_{n\in\omega}) &= \bigcup_{n\in\omega} A_n \cap C_n \\
&= \bigcup_{n\in\omega} D_\eta((A_n^\alpha)_{\alpha<\eta}) \cap C_n \\
&= \bigcup_{n\in\omega} D_\eta((A_n^\alpha \cap C_n)_{\alpha<\eta}) \\
&= D_\eta((\bigcup_{n\in\omega} (A_n^\alpha \cap C_n))_{\alpha<\eta}).
\end{aligned}$$

Notice that the last equality holds because for all $\alpha < \eta$ and all integers n and m, if $n \neq m$ then $(A_n^\alpha \cap C_n) \cap (A_m^\alpha \cap C_m) = \emptyset$. Hence $\mathrm{SU}((C_n)_{n\in\omega}, (A_n)_{n\in\omega}) \in D_\eta(\underset{\sim}{\mathbf{\Pi}}^1_1)$ and the class $D_\eta(\underset{\sim}{\mathbf{\Pi}}^1_1)$ is closed under $\underset{\sim}{\mathbf{\Pi}}^1_1 - \mathrm{SU}$. ⊣

Proposition 2.3. *Let* $u \in D$. *Then* Γ_u *is a non-self-dual pointclass included in* $\mathrm{Diff}(\underset{\sim}{\mathbf{\Pi}}^1_1)$.

Proof. The classes Γ_u are pointclasses merely by definition, as results of Boolean operations on pointclasses. The fact that they are all in $\mathrm{Diff}(\underset{\sim}{\mathbf{\Pi}}^1_1)$ is a consequence of the closure properties proved in Proposition 2.2. The existence of universals for the classes Γ_u provides the non-self-dualness. ⊣

We now give to each description u a type. These types reveal information on the structural properties of the class described. For example the descriptions of type 1 share the property that the classes they describe can be written as $\mathrm{Bisep}(\underset{\sim}{\mathbf{\Sigma}}^0_\xi, \Gamma_{u'})$ or $\mathrm{Bisep}(\underset{\sim}{\mathbf{\Pi}}^1_1, \Gamma_{u'})$ for some ξ and some description u'; the descriptions of type 2 share the property that the classes they describe can be written as $\mathrm{SU}(\underset{\sim}{\mathbf{\Sigma}}^0_\xi, \bigcup_{n\in\omega} \Gamma_{u_n})$ or $\mathrm{SU}(\underset{\sim}{\mathbf{\Pi}}^1_1, \bigcup_{n\in\omega} \Gamma_{u_n})$ for some ξ and some family of descriptions $(u_n)_{n\in\omega}$, etc.

Definition 2.4. Let $u \in D$. The type $t(u)$ of u is 0 if $u(0) = 0$. If $u(0) \geq 1$ then the type $t(u)$ of u is

(a) 1 if:
   - $u(1) = 1$ and $u(2)$ is successor;

  - $u(1) = 3$, $t(u_1) = 0$ and $u(2)$ is successor;
  - $u(1) = 3$, $t(u_1) = 1$ and $u_1(0) = u(0)$;
  - $u(1) = 5$, $t(u_1) = 1$ and $u_1(0) = u(0)$.

(b) 2 if:
  - $u(1) = 1$ and $u(2)$ is limit;
  - $u(1) = 3$, $t(u_1) = 0$ and $u(2)$ is limit;
  - $u(1) = 3$, $t(u_1) = 2$ and $u_1(0) = u(0)$;
  - $u(1) = 4$;
  - $u(1) = 5$ and $t(u_1) = 0$;
  - $u(1) = 5$, $t(u_1) = 2$ and $u_1(0) = u(0)$.

(c) 3 if:
  - $u(1) = 2$;
  - $u(1) = 3$ and $u_1(0) > u(0)$;
  - $u(1) = 3$, $t(u_1) = 3$ and $u_1(0) = u(0)$;
  - $u(1) = 5$ and $u_1(0) > u(0)$;
  - $u(1) = 5$, $t(u_1) = 3$ and $u_1(0) = u(0)$.

Thanks to these types, we can now sort the descriptions in four groups, depending on the position in which their associated class lies in the Wadge hierarchy. $D^0 = \{u \in D : t(u) = 0\}$ is the set of descriptions that code the class $\{\emptyset\}$, which is at the bottom of the hierarchy. $D^+ = \{u \in D : u(0) = 1 \text{ and } t(u) = 1\}$ is the set of descriptions that code classes which are at a successor position in the Wadge hierarchy. $D^\omega = \{u \in D : u(0) = 1 \text{ and } t(u) = 2\}$ is the set of descriptions that code classes which are at a limit of cofinality ω position in the Wadge hierarchy. $D^{\omega_1} = D \setminus (D^0 \cup D^+ \cup D^\omega) = \{u \in D : u(0) = 1 \text{ and } t(u) = 3\} \cup \{u \in D : u(0) > 1\}$ is the set of descriptions that code classes which are at a limit of cofinality ω_1 position in the Wadge hierarchy.

Theorem 2.5. *Let $\mathcal{W} = \{\Gamma_u : u \in D\} \cup \{\check{\Gamma}_u : u \in D\} \cup \{\Delta(\Gamma_u) : u \in D\}$. Then $\mathcal{W}$ is exactly the set of all pointclasses included in* $\mathrm{Diff}(\underset{\sim}{\mathbf{\Pi}}^1_1)$.

The strategy for the proof is the same as in [5], and relies on the determinacy of the class $\mathrm{Diff}(\underset{\sim}{\mathbf{\Pi}}^1_1)$. For each description u that is not in D^0, we find a code that describes the immediate predecessor of Γ_u if it is at a successor position, or a sequence of codes that describe a sequence of classes that is cofinal under Γ_u if it is at a limit position. Formally we have the following.

Lemma 2.6. *Let u be a description.*

(a) *If $u \in D^+$, there exists $\bar{u} \in D$ such that, for any class Γ, $\Gamma_{\bar{u}} \subsetneq \Gamma \subsetneq \Gamma_u$ implies that $\Gamma = \Delta(\Gamma_u)$.*

(b) *If $u \in D^\omega$, there exists a sequence of descriptions $(\bar{u}_n)_{n\in\omega}$ such that, for any class Γ, if for all integer n $\Gamma_{\bar{u}_n} \subset \Gamma \subset \Gamma_u$, then $\Gamma = \Delta(\Gamma_u)$.*

(c) *If $u \in D^{\omega_1}$, there exists a set of descriptions Q_u of cardinality ω_1 such that $\Delta(\Gamma_u) = \bigcup\{\Gamma_{\bar{u}} : \bar{u} \in Q_u\}$.*

The proof of Theorem 2.5 now goes as follows. Suppose, towards a contradiction, that the collection $\tilde{\mathcal{W}}$ of pointclasses included in $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$ that are not in $\mathcal{W}$ is not empty. Using our determinacy hypothesis, the SLO property holds for $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$, and the poinclasses included in $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$ are well-founded for the inclusion. By the definition of $\mathcal{W}$, there is thus either a self-dual class Γ that is the $\subset$-least class in $\tilde{\mathcal{W}}$, or a couple of non-self-dual classes Γ and $\check{\Gamma}$ such that for all $\Gamma' \in \tilde{\mathcal{W}}$, $\Gamma \subseteq \Gamma'$ or $\check{\Gamma} \subseteq \Gamma'$ holds. Both situations lead to the same argument: since the classes $D_\eta(\boldsymbol{\Pi}^1_1)$ are in $\mathcal{W}$ and cofinal in $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$, there exists a description u such that Γ_u is the least class described above Γ, and we have three cases:

- $u \in D^+$, and then $\Gamma = \Delta(\Gamma_u)$ or $\Gamma = \check{\Gamma}_{\bar{u}}$;
- $u \in D^\omega$, and then $\Gamma = \Delta(\Gamma_u)$;
- $u \in D^{\omega_1}$, and then $\Gamma = \Delta(\Gamma_u)$.

Thus $\Gamma \in \mathcal{W}$ in each case, and we reach a contradiction.

So the only thing left to prove here is Lemma 2.6. But most cases are already covered by the proofs in [5], or straightforward extension of those using Proposition 2.2. In this article, we do not go through them again and only take care of the $D_\eta(\boldsymbol{\Pi}^1_1)$ classes.

2.2. The successor case. In this section we look at the classes $D_{\eta+1}(\boldsymbol{\Pi}^1_1)$, with $\eta < \omega_1$. These classes are described by descriptions u such that $u(0) = \omega_1$, $u(1) = 1$ and $u(2) = \eta + 1$, and are of type 1.

Lemma 2.7. *Let $\eta < \omega_1$ and u be a description of the class $D_{\eta+1}(\boldsymbol{\Pi}^1_1)$. Then:*
(a) $\Gamma_u = \mathrm{Bisep}(\boldsymbol{\Pi}^1_1, D_\eta(\boldsymbol{\Pi}^1_1))$;
(b) $\Delta(\Gamma_u) = \mathrm{Bisep}(\boldsymbol{\Delta}^1_1, D_\eta(\boldsymbol{\Pi}^1_1))$.

Proof.

(a) By Proposition 1.2, we know that $D_{\eta+1}(\boldsymbol{\Pi}^1_1) = \check{D}_\eta(\boldsymbol{\Pi}^1_1) \cap \boldsymbol{\Pi}^1_1$ so that $D_{\eta+1}(\boldsymbol{\Pi}^1_1) \subseteq \mathrm{Bisep}(\boldsymbol{\Pi}^1_1, D_\eta(\boldsymbol{\Pi}^1_1))$. For the other inclusion, we use Proposition 2.2.

(b) By Proposition 1.3, we know that if $D \in \Delta(D_{\eta+1}(\boldsymbol{\Pi}^1_1))$, then there exists $B \in \boldsymbol{\Delta}^1_1$, $X \in \check{D}_\eta(\Pi^1_1)$ and $Y \in D_\eta(\Pi^1_1)$ such that

$$D = (X \cap B) \cup (Y \backslash B).$$

Thus $D = \mathrm{Bisep}(B, B^{\complement}, X, Y, \emptyset)$, so that the inclusion from left to right holds. Since $D_{\eta+1}(\boldsymbol{\Pi}^1_1)$ is closed under $\boldsymbol{\Pi}^1_1 - \mathrm{SU}$, $\mathrm{Bisep}(\boldsymbol{\Delta}^1_1, D_\eta(\boldsymbol{\Pi}^1_1)) \subseteq D_{\eta+1}(\boldsymbol{\Pi}^1_1)$. What remains to prove is that the dual class of $\mathrm{Bisep}(\boldsymbol{\Delta}^1_1, D_\eta(\boldsymbol{\Pi}^1_1))$ is also included in $D_{\eta+1}(\boldsymbol{\Pi}^1_1)$. Let A be such that $A^{\complement} \in \mathrm{Bisep}(\boldsymbol{\Delta}^1_1, D_\eta(\boldsymbol{\Pi}^1_1))$. There exists thus $A_1 \in \check{D}_\eta(\boldsymbol{\Pi}^1_1)$, $A_2 \in D_\eta(\boldsymbol{\Pi}^1_1)$, and B_1, B_2 two disjoint Borel sets such that:

$$A^{\complement} = (A_1 \cap B_1) \cup (A_2 \cap B_2).$$

Therefore $A = (B_1 \cap A_1^{\complement}) \cup (B_2 \cap A_2^{\complement}) \cup (B_1 \cup B_2)^{\complement}$ is in $D_{\eta+1}(\boldsymbol{\Pi}^1_1)$ since this class is closed under $\boldsymbol{\Pi}^1_1 - \mathrm{SU}$, and $\mathrm{Bisep}(\boldsymbol{\Delta}^1_1, D_\eta(\boldsymbol{\Pi}^1_1)) \subseteq \Delta(D_{\eta+1}(\boldsymbol{\Pi}^1_1))$.

$\dashv$

This allows us to define the set Q_u for a description u of the class $D_{\eta+1}(\boldsymbol{\Pi}^1_1)$:

$$Q_u = \{\xi\hat{}3\hat{}1\hat{}\langle\omega_1\hat{}1\hat{}\eta\hat{}0^\omega, 0^\omega\rangle : \xi < \omega_1\}.$$

We prove now that the family of classes described by Q_u is cofinal below Γ_u.

Proposition 2.8. *Let $\eta < \omega_1$, and u be a description for the class $D_{\eta+1}(\boldsymbol{\Pi}^1_1)$. Then:*

$$\Delta(D_{\eta+1}(\boldsymbol{\Pi}^1_1)) = \bigcup_{u' \in Q_u} \Gamma_{u'}.$$

Proof. Using Lemma 2.7, we have to prove:

$$\mathrm{Bisep}(\boldsymbol{\Delta}^1_1, D_\eta(\boldsymbol{\Pi}^1_1)) = \bigcup_{\xi<\omega_1} \mathrm{Bisep}(\boldsymbol{\Sigma}^0_\xi, D_\eta(\boldsymbol{\Pi}^1_1)).$$

The inclusion from right to left is immediate since each $\mathrm{Bisep}(\boldsymbol{\Sigma}^0_\xi, D_\eta(\boldsymbol{\Pi}^1_1))$ is included in $\mathrm{Bisep}(\boldsymbol{\Delta}^1_1, D_\eta(\boldsymbol{\Pi}^1_1))$. For the other inclusion, we just have to come back to the definition of the operation Bisep. Let $A \in \mathrm{Bisep}(\boldsymbol{\Delta}^1_1, D_\eta(\boldsymbol{\Pi}^1_1))$, then there exists C_1, C_2 two disjoint Borel sets such that:

$$A = (A \cap C_1) \cup (A \cap C_2),$$

with $A \cap C_1 \in \check{D}_\eta(\boldsymbol{\Pi}^1_1)$ and $A \cap C_2 \in D_\eta(\boldsymbol{\Pi}^1_1)$. But if C_1 and C_2 are Borel, there exists $\xi < \omega_1$ such that C_1 and C_2 are in $\boldsymbol{\Sigma}^0_\xi$! Thus $A \in \mathrm{Bisep}(\boldsymbol{\Sigma}^0_\xi, D_\eta(\boldsymbol{\Pi}^1_1))$, and the other inclusion holds. ⊣

This finishes the successor case.

2.3. The limit case. In this section we look at the classes $D_\gamma(\boldsymbol{\Pi}^1_1)$, with $\gamma < \omega_1$ limit. These classes are described by descriptions u such that $u(0) = \omega_1$, $u(1) = 1$ and $u(2) = \gamma$, and are of type 2. First we define a new operation and give a reformulation of Louveau's Trick II.

Definition 2.9. Let Γ and Γ' be two pointclasses. The set A is in $\mathrm{PU}(\Gamma, \Gamma')$ if and only if there exists a partition $(C_n)_{n\in\omega}$ of sets in Γ, and a family $(A_n)_{n\in\omega}$ of sets in Γ' such that

$$A = \mathrm{PU}((C_n)_{n\in\omega}, (A_n)_{n\in\omega}) = \bigcup_{n\in\omega} A_n \cap C_n.$$

This operation is called the *Partitioned Union*. It is of course a special case of SU.

Lemma 2.10. *Let $\gamma < \omega_1$ be a limit ordinal, and u be a description of the class $D_\gamma(\boldsymbol{\Pi}^1_1)$. Then:*

(a) $\Gamma_u = \mathrm{SU}(\boldsymbol{\Pi}^1_1, \Gamma)$*;*

(b) $\Delta(\Gamma_u) = \mathrm{PU}(\boldsymbol{\Pi}^1_1, \Gamma)$.

where $\Gamma = \bigcup_{\eta<\gamma} D_\eta(\boldsymbol{\Pi}^1_1)$.

Proof.

(a) Since $D_\gamma(\boldsymbol{\Pi}^1_1)$ is closed under $\boldsymbol{\Pi}^1_1 - \mathrm{SU}$, the inclusion from right to left is immediate. For the other one, let $(A_\alpha)_{\alpha<\gamma}$ be an increasing family of $\boldsymbol{\Pi}^1_1$ sets, and $D = D_\gamma((A_\alpha)_{\alpha<\gamma})$. By the generalized reduction property of the class of co-analytic sets, there exists a disjoint co-analytic family $(C_\alpha)_{\alpha<\gamma}$ such that

- for all $\alpha < \gamma$, $C_\alpha \subseteq A_\alpha$;
- $\bigcup_{\alpha<\gamma} C_\alpha = \bigcup_{\alpha<\gamma} A_\alpha$.

Now we have $D \cap C_\alpha \subseteq A_\alpha$ for all $\alpha < \gamma$, and thus $D \cap C_\alpha \in \Gamma \cap \mathbf{\Pi}^1_1$. Since

$$D = \bigcup_{\alpha<\gamma} D \cap C_\alpha,$$

$D \in \mathrm{SU}(\mathbf{\Pi}^1_1, \Gamma)$ and the second inclusion is proven.

(b) By Louveau's Trick II, we know that if $D \in \Delta(D_\gamma(\mathbf{\Pi}^1_1))$, there exists a countable Borel partition $(C_i)_{i\in\omega}$ of the Baire space such that, for all $j < \omega$,

$$D \cap C_j \in D_{\eta_j}(\mathbf{\Pi}^1_1),$$

with $\eta_j < \gamma$. Thus $D \in \mathrm{PU}(\mathbf{\Pi}^1_1, \Gamma)$, so that $\Delta(D_\gamma(\mathbf{\Pi}^1_1)) \subseteq \mathrm{PU}(\mathbf{\Pi}^1_1, \Gamma)$. Since $D_\gamma(\mathbf{\Pi}^1_1)$ is closed under $\mathbf{\Pi}^1_1 - \mathrm{SU}$, $\mathrm{PU}(\mathbf{\Pi}^1_1, \Gamma) \subseteq D_\gamma(\mathbf{\Pi}^1_1)$. What remains to prove is that the dual class of $\mathrm{PU}(\mathbf{\Pi}^1_1, \Gamma)$ is also included in $D_\gamma(\mathbf{\Pi}^1_1)$. Let A be such that $A^\complement \in \mathrm{PU}(\mathbf{\Pi}^1_1, \Gamma)$. There exists a partition in co-analytic sets $(C_i)_{i\in\omega}$ such that

$$A^\complement = \bigcup_{i\in\omega} A^\complement \cap C_i,$$

with $A^\complement \cap C_i \in D_{\alpha_i}(\mathbf{\Pi}^1_1)$ and $\alpha_i < \gamma$. Notice that, for all integer i,

$$A \cap C_i = (A^\complement \cap C_i)^\complement \cap C_i.$$

By Proposition 1.2, $(A^\complement \cap C_i)^\complement \cap C_i \in D_{\alpha_i+1}(\mathbf{\Pi}^1_1)$ which is still included in $D_\gamma(\mathbf{\Pi}^1_1)$! The set A is therefore in $D_\gamma(\mathbf{\Pi}^1_1)$, and $\mathrm{PU}(\mathbf{\Pi}^1_1, \Gamma) \subseteq \Delta(D_\gamma(\mathbf{\Pi}^1_1))$. ⊣

This allows us to define the set Q_u for a description u of the class $D_\gamma(\mathbf{\Pi}^1_1)$, it is the set of descriptions

$$\xi \hat{}\, 4 \hat{}\, \langle (u'_n)_{n\in\omega} \rangle$$

for $\xi < \omega_1$, where $u'_n = \omega_1 \hat{}\, 1 \hat{}\, \gamma_n \hat{}\, 0^\omega$, and $(\gamma_n)_{n\in\omega}$ is cofinal in γ. We prove now that the family of classes described by Q_u is cofinal under Γ_u.

Proposition 2.11. *Let $\gamma < \omega_1$ be limit, and u be a description for the class $D_\gamma(\mathbf{\Pi}^1_1)$. Then*

$$\Delta(D_\gamma(\mathbf{\Pi}^1_1)) = \bigcup_{u' \in Q_u} \Gamma_{u'}.$$

Proof. Using Lemma 2.10, we have to prove:

$$\mathrm{PU}(\mathbf{\Pi}^1_1, \Gamma) = \bigcup_{\xi<\omega_1} \mathrm{SU}(\mathbf{\Sigma}^0_\xi, \Gamma'),$$

where $\Gamma = \bigcup_{\eta<\gamma} D_\eta(\underset{\sim}{\mathbf{\Pi}}^1_1)$ and $\Gamma' = \bigcup_{n\in\omega} D_{\gamma_n}(\underset{\sim}{\mathbf{\Pi}}^1_1)$. First notice that $\Gamma = \Gamma'$, so that we actually only have to prove:

$$\mathrm{PU}(\underset{\sim}{\mathbf{\Pi}}^1_1, \Gamma) = \bigcup_{\xi<\omega_1} \mathrm{SU}(\underset{\sim}{\mathbf{\Sigma}}^0_\xi, \Gamma).$$

For the first inclusion, from left to right, notice that any co-analytic countable partition is in fact a Borel and hence a $\underset{\sim}{\mathbf{\Sigma}}^0_\xi$ partition for a certain $\xi < \omega_1$. For the other inclusion, let $\xi < \omega_1$ and $D \in \mathrm{SU}(\underset{\sim}{\mathbf{\Sigma}}^0_\xi, \Gamma)$. By definition there exists a disjoint family $(C_n)_{n\in\omega}$ of $\underset{\sim}{\mathbf{\Sigma}}^0_\xi$ sets and a family $(A_n)_{n\in\omega}$ in Γ such that

$$D = \mathrm{SU}((C_n)_{n\in\omega}, (A_n)_{n\in\omega}).$$

But then we have

$$\begin{aligned} D &= \bigcup_{n\in\omega} C_n \cap A_n \\ &= \left((\bigcup_{n\in\omega} C_n)^\complement \cap \emptyset \right) \cup \bigcup_{n\in\omega} C_n \cap A_n \\ &= \mathrm{PU}((C'_n)_{n\in\omega}, (A'_n)_{n\in\omega}); \end{aligned}$$

where $C'_0 = (\bigcup_{n\in\omega} C_n)^\complement$, $C'_{n+1} = C_n$, $A'_0 = \emptyset$ and $A'_{n+1} = A_n$. Since $\emptyset \in \Gamma$ and $C'_0 \in \underset{\sim}{\mathbf{\Pi}}^0_\xi$, $D \in \mathrm{PU}(\underset{\sim}{\mathbf{\Pi}}^1_1, \Gamma)$ and the other inclusion follows. ⊣

This finishes the limit case, and the proof of Lemma 2.6.

§3. Above $\mathrm{Diff}(\underset{\sim}{\mathbf{\Pi}}^1_1)$.

There is another standard way to introduce differences, namely by considering *decreasing* sequence of sets. If $(B_\eta)_{\eta<\theta}$ is a decreasing sequence of subsets of the Baire space, with $1 \leq \theta$, we define the set $D^*_\theta((B_\eta)_{\eta<\theta})$ by

$$D^*_\theta((B_\eta)_{\eta<\theta}) = \bigcup_{\substack{\eta<\theta \\ \eta \text{ even}}} (B_\eta \backslash B_{\eta+1}),$$

where if θ is odd, we let $B_\theta = \emptyset$ by convention.[1] These two definitions coincide up to a certain point.

Facts 3.1. *Let Γ be a pointclass.*

(a) *For every positive integer n, $D_n(\Gamma) = D^*_n(\Gamma)$.*

(b) *For every positive integer n, $D_{2n}(\Gamma) = D_{2n}(\check{\Gamma})$, and $D_{2n+1}(\Gamma) = \check{D}_{2n+1}(\check{\Gamma})$.*

(c) *For every ordinal $0 < \theta$,*

$$D^*_\theta(\check{\Gamma}) = \begin{cases} D_\theta(\Gamma), & \text{if } \theta \text{ is even;} \\ \check{D}_\theta(\Gamma), & \text{if } \theta \text{ is odd.} \end{cases}$$

In this section, we discuss the discrepancy between the pointclasses of differences using *increasing* sequences of co-analytic sets, and differences using

[1] The notation $\theta - \underset{\sim}{\mathbf{\Pi}}^1_1$ can also be found in the litterature for the class of θ decreasing differences of co-analytic sets.

decreasing sequences of co-analytic sets. We prove that the situation is the following:

$$\begin{array}{c} \bigcup_{n\in\omega} D_n(\underset{\sim}{\mathbf{\Pi}}^1_1) \\ \| \\ \bigcup_{n\in\omega} D^*_n(\underset{\sim}{\mathbf{\Pi}}^1_1) \end{array} \subset D_\omega(\underset{\sim}{\mathbf{\Pi}}^1_1) \subset \ldots \subset \mathrm{Diff}(\underset{\sim}{\mathbf{\Pi}}^1_1) \subset \Delta(D^*_\omega(\underset{\sim}{\mathbf{\Pi}}^1_1))$$

This at first sight quite intriguing situation can be explained by a fundamental dissymetry between the two classes of analytics and co-analytic sets. The latter enjoys indeed the generalized reduction property, whereas the former does not.

Lemma 3.2. *Let $(D_i)_{i\in\omega}$ be a family of subsets of the Baire space and $(\alpha_i)_{i\in\omega} \subseteq \omega_1$ such that:*

- *for all $i \in \omega$, $D_i = D^*_{\alpha_i}((A^i_\beta)_{\beta<\alpha_i}) \in D^*_{\alpha_i}(\underset{\sim}{\mathbf{\Pi}}^1_1)$;*
- *if $i \neq j$, then $A^i_0 \cap A^j_0 = \emptyset$.*

Then

$$\bigcup_{i\in\omega} D_i \in D^*_\alpha(\underset{\sim}{\mathbf{\Pi}}^1_1),$$

where $\alpha = \sup_{i\in\omega} \alpha_i$.

Proof. It is sufficient to notice that

$$\bigcup_{i\in\omega} D_i = \bigcup_{\substack{\beta<\alpha \\ \beta \text{ even}}} \left(\left(\bigcup_{i\in\omega} A^i_\beta \right) \setminus \left(\bigcup_{i\in\omega} A^i_{\beta+1} \right) \right).$$

⊣

In fact the classes $D^*_\alpha(\underset{\sim}{\mathbf{\Pi}}^1_1)$ are closed under $\underset{\sim}{\mathbf{\Pi}}^1_1 - \mathrm{SU}$.

Lemma 3.3. *For all $\alpha < \omega_1$, the class $D^*_\alpha(\underset{\sim}{\mathbf{\Pi}}^1_1)$ is closed under $\underset{\sim}{\mathbf{\Pi}}^1_1 - \mathrm{SU}$.*

Proof. Let $(D_i)_{i\in\omega}$ be a family of $D^*_\alpha(\underset{\sim}{\mathbf{\Pi}}^1_1)$ sets. By definition, there exists for each integer i a decreasing family of $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets $(A^i_\xi)_{\xi<\alpha}$ such that $D_i = D^*_\alpha((A^i_\xi)_{\xi<\alpha})$. Let now $(C_i)_{i\in\omega}$ be a disjoint family of $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets.

$$\begin{aligned} \mathrm{SU}((C_i)_{i\in\omega}, (D_i)_{i\in\omega}) &= \bigcup_{i\in\omega} (C_i \cap D_i) \\ &= \bigcup_{i\in\omega} (D^*_\alpha((C_i \cap A^i_\xi)_{\xi<\alpha}), \end{aligned}$$

And we conclude by Lemma 3.2. ⊣

We now give the proof of the inclusion of the classes $D_\alpha(\underset{\sim}{\mathbf{\Pi}}^1_1)$ in the class $D^*_\omega(\underset{\sim}{\mathbf{\Pi}}^1_1)$.

Proposition 3.4. *For every $\alpha < \omega_1$, $D_\alpha(\underset{\sim}{\mathbf{\Pi}}^1_1) \subseteq D^*_\omega(\underset{\sim}{\mathbf{\Pi}}^1_1)$.*

Proof. We proceed by induction on $\alpha < \omega_1$. If α is finite, we conclude by Facts 3.1.

For ω, let $(A_i)_{i\in\omega}$ be an increasing sequence of co-analytic sets, and consider $D_\omega((A_i)_{i\in\omega})$. Using the generalized reduction property on the family $(A_{2i+1})_{i\in\omega}$, we get a new sequence of disjoint co-analytic sets $(B_i)_{i\in\omega}$ such that

– for all $i \in \omega$, $B_i \subseteq A_{2i+1}$;
– $\bigcup_{i\in\omega} B_i = \bigcup_{i\in\omega} A_{2i+1}$.

Thus

$$\begin{aligned} D_\omega((A_i)_{i\in\omega}) &= \bigcup_{i\in\omega}\left(B_i \cap \left(\bigcup_{j\in\omega} A_{2j+1}\setminus A_{2j}\right)\right) \\ &= \bigcup_{i\in\omega}\left(B_i \cap \underbrace{\Big(\bigcup_{j\leq i} A_{2j+1}\setminus A_{2j}\Big)}_{\in D_{2i+2}\subseteq D^*_{2i+3}}\right). \end{aligned}$$

Since the family $(B_i)_{i\in\omega}$ is disjoint, we conclude by Lemma 3.3. The general proof for $\gamma < \omega_1$ limit is mutatis mutandis the same.

Suppose now that there exists $\beta < \omega_1$ such that $D_\alpha(\mathbf{\Pi}^1_1) \subseteq D^*_\omega(\mathbf{\Pi}^1_1)$ for all $\alpha < \beta+1$. Since the odd case is similar, we assume that $\beta+1$ is even. Let $(A_\alpha)_{\alpha\in\beta+1}$ be an increasing sequence of co-analytic sets, and consider $D_{\beta+1}((A_\alpha)_{\alpha<\beta+1})$. By our induction hypothesis, there exists a decreasing sequence of co-analytic sets $(B_i)_{i\in\omega}$ such that:

$$D_{\beta-1}((A_\alpha)_{\alpha<\beta-1}) = D^*_\omega((B_i)_{i\in\omega}).$$

In particular since the family $(A_\alpha)_{\alpha\in\beta+1}$ is increasing, $D_{\beta-1}((A_\alpha)_{\alpha<\beta-1})$, and thus $D^*_\omega((B_i)_{i\in\omega})$ are included in $A_{\beta-1}$. Hence:

$$\begin{aligned} D_{\beta+1}((A_\alpha)_{\alpha<\beta+1}) &= D^*_\omega((B_i)_{i\in\omega}) \cup A_\beta\setminus A_{\beta-1} \\ &= D^*_\omega((B_i\cap A_{\beta-1})_{i\in\omega}) \cup A_\beta\setminus A_{\beta-1} \\ &= D^*_\omega((A_\beta, A_{\beta-1}, B_0\cap A_{\beta-1}, B_1\cap A_{\beta-1},\dots)), \end{aligned}$$

which completes the proof. ⊣

Our determinacy hypothesis is therefore sufficient. Moreover, it follows from the compositions of work by Harrington and Martin that the determinacy of the Wadge games of co-analytic sets is equivalent to $\mathrm{DET}(\mathbf{\Pi}^1_1)$, so that $\mathrm{DET}(\mathbf{\Pi}^1_1)$ is in fact optimal for our work.

The gap between $\mathrm{Diff}(\mathbf{\Pi}^1_1)$ and $\Delta(D^*_\omega(\mathbf{\Pi}^1_1))$ has, to our knowledge, not been investigated yet. The only piece of information on that matter is given by a result from Kechris and Martin mentionned by Steel in [7].

Theorem 3.5. (Kechris - Martin).
Under (AD), the order type of the Wadge hierarchy on $\Delta(D^*_\omega(\mathbf{\Pi}^1_1))$ subsets of the Baire space is ω_2.

Combined with our results, it appears thus that under (AD) the inclusion between $\mathrm{Diff}(\mathbf{\Pi}^1_1)$ and $\Delta(D^*_\omega(\mathbf{\Pi}^1_1))$ is *strict*.

Question. *Is the equality $\mathrm{Diff}(\mathbf{\Pi}^1_1) = \Delta(D^*_\omega(\mathbf{\Pi}^1_1))$ consistent under weaker determinacy hypothesis?*

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References

[1] Alessandro Andretta and Alain Louveau, *Wadge degrees and pointclasses*, ***Wadge degrees and projective ordinals: The cabal seminar, volume ii*** (Alexander S. Kechris, Benedikt Löwe, and John R. Steel, editors), Cambridge University Press, 2012.

[2] Jacques Duparc, *Wadge hierarchy and Veblen hierarchy, Part II : Borel sets of infinite rank*, Unpublished.

[3] ———, *Wadge hierarchy and Veblen hierarchy, Part I : Borel sets of finite rank*, this Journal, vol. 66 (2001), no. 1, pp. 56 – 86.

[4] Leo A. Harrington, *Analytic determinacy and* $0^{\#}$, this Journal, vol. 43 (1978), no. 4, pp. 685 – 693.

[5] Alain Louveau, *Some Results in the Wadge hierarchy of Borel sets*, ***Wadge degrees and projective ordinals: The cabal seminar, volume ii*** (Alexander S. Kechris, Benedikt Löwe, and John R. Steel, editors), Cambridge University Press, 2012.

[6] Donald A. Martin, *Measurable cardinals and analytic games*, ***Fundamenta Mathematicae***, vol. 66 (1970), pp. 287–291.

[7] John R. Steel, *Closure properties of pointclasses*, ***Wadge degrees and projective ordinals: The cabal seminar, volume ii*** (Alexander S. Kechris, Benedikt Löwe, and John R. Steel, editors), Cambridge University Press, 2012.

[8] Robert Van Wesep, *Subsystems of second-order arithmetic, and descriptive set theory under the axiom of determinateness*, ***Ph.D. thesis***, University of California, Berkeley, 1977.

[9] ———, *Wadge degrees and descriptive set theory*, ***Wadge degrees and projective ordinals: The cabal seminar, volume ii*** (Alexander S. Kechris, Benedikt Löwe, and John R. Steel, editors), Cambridge University Press, 2012.

[10] William W. Wadge, *Reducibility and determinateness on the Baire space*, ***Ph.D. thesis***, University of California, Berkeley, 1984.

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