Wadge hierarchy of differences of coanalytic sets

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Pointclasses

Definition

A *pointclass* Γ is a collection of subsets of the Baire space ω^ω closed under continuous preimages: for all continuous function $f : \omega^\omega \to \omega^\omega$ and all $A \in \Gamma$,

$$f^{-1}(A) \in \Gamma.$$

Example

Open, closed, G_δ, Borel, $\mathbf{\Sigma}^0_\alpha$, analytic, etc.

$$\check{\Gamma} = \left\{ A \subseteq \omega^\omega : A^{\complement} \in \Gamma \right\} \qquad \text{and} \qquad \Delta(\Gamma) = \Gamma \cap \check{\Gamma}$$

A pointclass Γ is *self-dual* if $\Gamma = \check{\Gamma}$.

The Wadge Quasi-Order

Definition

Let $A, B \subseteq \omega^\omega$, $A \leq_W B$ iff there exists a continuous function $f : \omega^\omega \to \omega^\omega$ such that

$$f^{-1}(B) = A.$$

The relation $\leq_W$ on $\mathcal{P}(\omega^\omega)$ is a quasi-order (qo) whose equivalence classes are called the *Wadge degrees*.

Theorem

If Γ is a pointclass with appropriate closure and determinacy properties (e.g. the Borel sets, projective sets under PD, etc.), the Wadge qo restricted to Γ is in fact a well-quasi-order:

- $\forall A, B \in \Gamma$, $A \leq_W B$ *or* $B \leq_W A^\complement$. *(Wadge's Lemma)*
- *There is no infinite strictly decreasing chain of elements of* Γ *w.r. to* $\leq_W$. *(Martin-Monk)*

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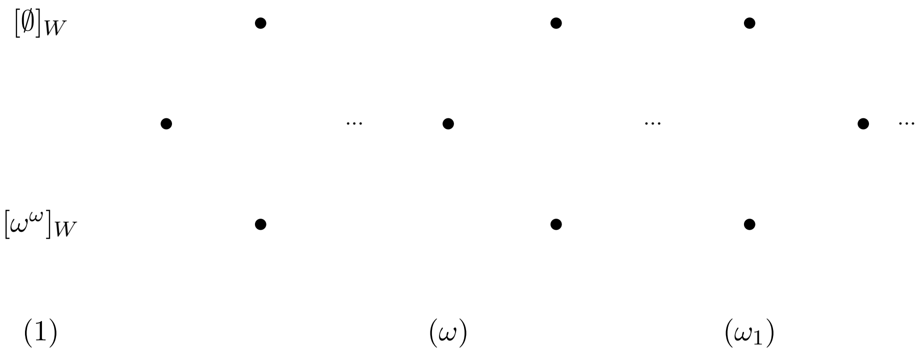
The Wadge Hierarchy



Wadge Hierarchy and Pointclasses

The pointclasses are exactly the *initial segments* of the Wadge hierarchy.

$$(\mathcal{P}(\omega^\omega), \leq_W) \leftrightsquigarrow (\text{Pointclasses}, \subseteq)$$

Two ways of analyzing the Wadge hierarchy:

– from *above* by giving a complete set for each Wadge degree. (Wadge, Duparc)

– from *below* by describing the pointclasses. (Wadge, Louveau)

Pointclasses and Boolean Operations

Definition

An ω-ary operation $\mathcal{O} : \mathcal{P}(\omega^\omega)^\omega \longrightarrow \mathcal{P}(\omega^\omega)$ is a *boolean operation* if there exists a subset $T_\mathcal{O}$ of $\mathcal{P}(\omega)$ such that for any sequence of subsets of the Baire space $(A_n)_{n\in\omega}$, and for all $x \in \omega^\omega$,

$$x \in \mathcal{O}\left((A_n)_{n\in\omega}\right) \leftrightarrow \{n \in \omega : x \in A_n\} \in T_\mathcal{O}.$$

Example

$\bigcup_\omega$, $\bigcap_\omega$, $\boldsymbol{\mathcal{A}}$, etc.

Theorem (Wadge, Van Wesep)

Let Γ be a pointclass with appropriate closure and determinacy properties, and let $\Gamma' \subseteq \Gamma$ be a non-self-dual pointclass. Then there exists a boolean operation $\mathcal{O}_{\Gamma'}$ such that

$$\Gamma' = \mathcal{O}_{\Gamma'}(\boldsymbol{\Sigma}^0_1).$$

Differences of coanalytic sets

There are two different types of difference: increasing (D_α) and decreasing (D^*_α). They coincide in the finite case, but not in general.

$$\begin{array}{l} \bigcup_{n\in\omega} D_n(\mathbf{\Pi}^1_1) \\ \qquad \| \qquad\qquad \subset \ D_\omega(\mathbf{\Pi}^1_1) \ \subset \ \dots \ \subset \ \bigcup_{\alpha<\omega_1} D_\alpha(\mathbf{\Pi}^1_1) \ \subseteq \ \Delta(D^*_\omega(\mathbf{\Pi}^1_1)) \\ \bigcup_{n\in\omega} D^*_n(\mathbf{\Pi}^1_1) \end{array}$$

Theorem (Martin, Harrington)

$\mathrm{DET}(\mathbf{\Pi}^1_1) \Rightarrow \mathrm{DET}(D^*_{\omega^2}(\mathbf{\Pi}^1_1))$

Hence $\mathrm{DET}(\mathbf{\Pi}^1_1)$ is sufficient for the study of the Wadge Hierarchy of the class $\bigcup_{\alpha<\omega_1} D_\alpha(\mathbf{\Pi}^1_1)$.

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The Veblen hierarchy of basis ω_1

The Veblen hierarchy of base ω_1 consists of functions $(V^\xi)_{\xi<\omega_2}$ from $\omega_2 \smallsetminus \{0\}$ to ω_2:

(i) V^0 is almost the exponentiation of base ω_1:
- $V^0(1) = 1$;
- $V^0(\alpha+1) = V^0(\alpha) \cdot \omega_1$ for all $0 < \alpha < \omega_2$;
- $V^0(\alpha) = \omega_1^\alpha$ for all $\alpha < \omega_2$ limit of cofinality ω_1;
- $V^0(\alpha) = \omega_1^{\alpha+1}$ for all $\alpha < \omega_2$ limit of cofinality ω.

(ii) For $\lambda > 0$, V^λ is the function that enumerates the fixpoints of cofinality ω_1 of the Veblen functions of lesser degrees:
- $V^\lambda(1) = 1$;
- $V^\lambda(1+\alpha)$ is the α^{th} fixpoint of cofinality ω_1 of all V^ξ with $\xi < \lambda$.

The Wadge Hierarchy of $\boldsymbol{\Delta}^1_2$ sets

$\boldsymbol{\Delta}^1_2$ — $\aleph_{\omega+1}$ (AD)

$\cup$

$\vdots$

$\cup$

$\bigcup_{\alpha<\omega_1} D^*_\alpha(\boldsymbol{\Pi}^1_1)$

$\cup$

$\vdots$

$\cup$

$D^*_\omega(\boldsymbol{\Pi}^1_1)$ — $\aleph_2$ (AD)

$\cup$

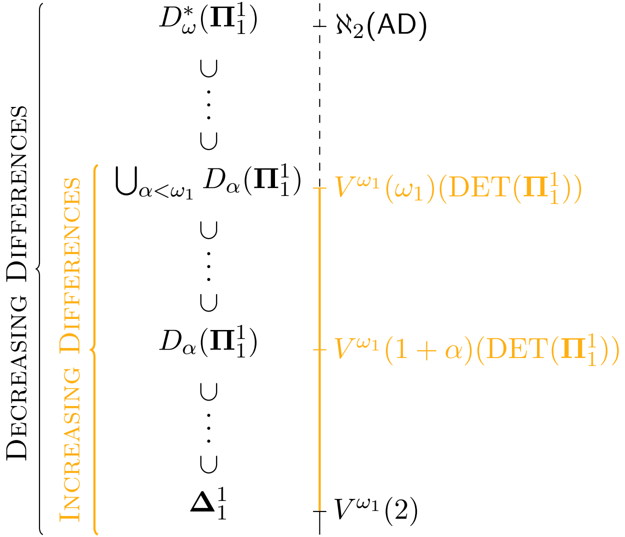
$\vdots$

$\cup$

$\boldsymbol{\Delta}^1_1$ — $V^{\omega_1}(2)$

(Difference Hierarchy: from $\boldsymbol{\Delta}^1_1$ up to $\bigcup_{\alpha<\omega_1} D^*_\alpha(\boldsymbol{\Pi}^1_1)$)

Increasing differences of coanalytic sets



Open Questions

- Is there still a gap between $\bigcup_{\alpha<\omega_1} D_\alpha(\mathbf{\Pi}^1_1)$ and $D^*_\omega(\mathbf{\Pi}^1_1)$ under weaker determinacy hypothesis? What's in the gap under (AD)? How can we reach $\aleph_2$?
- There is some kind of tension between what would be expected given the determinacy hypothesis used and what we get.

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