SYMBOLIC CALCULUS FOR TOEPLITZ OPERATORS WITH HALF-FORMS

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Abstract. This paper is devoted to the use of half-form bundles in the symbolic calculus of Berezin-Toeplitz operators on Kähler manifolds. We state the Bohr-Sommerfeld conditions and relate them to the functional calculus of Toeplitz operators, a trace formula and the characteristic classes in deformation quantization. We also develop the symbolic calculus of Lagrangian sections, with the crucial estimates of the subprincipal terms.

In semi-classical analysis we usually consider (pseudo) differential operators depending on a small parameter and acting on a L^2 space, the underlying classical limit being a cotangent space with its canonical symplectic structure. In this paper we are interested in a similar theory where the classical phase space is a compact Kähler manifold endowed with a prequantum line bundle L. Here the quantum Hilbert space consists of the holomorphic sections of L^k. The small parameter is the inverse of the power k. The operators of interest are the Berezin-Toeplitz operators. This setting was mainly introduced by Kostant [16], Souriau [19] and Berezin [1] and the suitable microlocal techniques were developed by Boutet de Monvel and Guillemin [5]. Since then many standard results for pseudo-differential operators have been adapted to this context, like for instance the Schnirelman theorem [22], the Gutzwiller trace formula [4] or the Bohr-Sommerfeld conditions [7]. The statements of these results are easily predictable as far as only the symplectic structure of the phase space is concerned, because they are the same for the cotangent and Kähler spaces. But the semi-classical results for the pseudo-differential operators may involve also some invariants, like the subprincipal symbol or the Maslov index, which do not only depend on the symplectic structure and consequently are difficult to identify in the Kähler setting. Furthermore these quantities generally appear as quantum corrections and are difficult to compute. Nevertheless in the papers [6] and [7], we carried out successfully some techniques to handle this. To formulate our result we used the Riemannian metric of the Kähler structure instead of the vertical polarization of the cotangent bundle. Typically we proved some Bohr-Sommerfeld conditions where the Maslov index is replaced with a curvature integral. Actually we missed the right formulation which uses the half-form bundles. The main purpose of this paper and the sequel [8] is to develop this point of view. In this part we focus on the Bohr-Sommerfeld conditions whereas [8] is devoted to the dependence of the quantization on the complex structure.

Concretely we alter the usual setting by defining the quantum space as the space of holomorphic sections of $L^k \otimes L_1 \otimes \delta \rightarrow M$. Here L is the prequantum bundle as previously, L_1 is an auxiliary line bundle and δ is a half-form bundle, *i.e.* a square root of the canonical bundle of M. A priori artificial, this decomposition enlightens the semi-classical results, even in the usual case where $L_1 \otimes \delta$ is the trivial bundle. Roughly speaking the contribution of L_1 in the semi-classical limit is the same as

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the one of L^k (one can view $L^k \otimes L_1$ as a first order deformation of L^k) whereas the half-form bundle contributes in a specific way. This principle will be confirmed in all our results. Another important point is that there is a topological obstruction to the existence of half-form bundles. To avoid this problem we consider globally the bundle $L^k \otimes K$ and write locally $K = L_1 \otimes \delta$. The situation here is analogous to that in Riemannian geometry where we think any Clifford module, at least locally, as the spinor bundle twisted with an auxiliary bundle.

The first section is devoted to basic properties of Toeplitz operators and their symbolic calculus. In particular an important subprincipal symbol is defined. We state the Bohr-Sommerfeld conditions in section 2 and relate them to the symbolic calculus and trace formula by adapting an argument of Colin de Verdière [9]. Here the formulation with half-forms permits to check easily the consistency of the results. The next sections contain the proof of the Bohr-Sommerfeld conditions. In section 4, we introduce the Lagrangian sections, which are similar to the Lagrangian distributions, and develop their symbolic calculus. Bohr-Sommerfeld conditions follows immediately. A comparison with the usual setting is included, where a $\mathbb{Z}_4$-bundle plays a role analogous to the Maslov bundle. In section 5, the technical part of the paper, we provide the proof for the symbolic calculus of the Lagrangian sections. We follow essentially the method of [7] but avoid the complicated computations involving the derivatives of the Kähler metric. These simplifications rely on a version of the stationary phase lemma stated in an appendix of this paper. In view of this proof, we think that our result should generalize mutatis mutandis to the case where the symplectic manifold doesn't admit any integrable complex structure.

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1. The setting

1.1. **Square root of line bundle.** Let M be a manifold and $F \rightarrow M$ be a complex line bundle. A *square root* (δ, φ) of F is a line bundle $\delta \rightarrow M$ together with an isomorphism of line bundle $\varphi : \delta^{\otimes 2} \rightarrow F$. If M is a complex manifold, a square root of its canonical bundle $\Lambda^{n,0} T^* M$ is called a *half-form bundle*. Let us state basic properties of square roots.

If F has a Hermitian structure and (δ, φ) is a square root of F, then δ has a unique Hermitian structure such that φ is a isomorphism of Hermitian line bundle. In the same way, if F is holomorphic or flat, δ inherits the same structure. If D^F is a first order differential operator acting on sections of $F \rightarrow M$, then there exists a unique first order differential operator D^δ acting on section of δ such that

$$D^F \varphi(s \otimes s) = 2\varphi(s \otimes D^\delta s), \quad \forall\, s \in C^\infty(M, \delta).$$

A line bundle admits a square root if and only if its Chern class is divisible by 2 in $H^2(M, \mathbb{Z})$. Two square roots (δ, φ) and (δ', φ') of F are *equivalent* if there exists an isomorphism $\Psi : \delta \rightarrow \delta'$ such that $\varphi' \circ \Psi^2 = \varphi$.

Proposition 1.1. *Assume that F admits a square root. Then the set of equivalence classes of square roots of F is a principal homogeneous space for the first group of cohomology of M with coefficient in $\mathbb{Z}_2$.*

Proof. First, if (δ, φ) is a square root of the trivial line bundle $1_M = M \times \mathbb{C}$, then δ inherits a flat structure from 1_M with structure group $\mathbb{Z}_2$. Furthermore this flat structure determines φ. It is easily proved that this induces an isomorphism between the set of equivalence classes of square roots of 1_M and the set of equivalences of flat line bundles with structure group $\mathbb{Z}_2$. The latter is isomorphic to $H^1(M, \mathbb{Z}_2)$. Now, observe that the tensor product of a square root of L with a square root of 1_M is a square root of L. This defines an action of $H^1(M, \mathbb{Z}_2)$ on $\mathcal{E}_L$, which is easily shown to be free and transitive. $\square$

1.2. **Quantum spaces.** Let M be a connected compact Kähler manifold of complex dimension n. Denote by $\omega \in \Omega^2(M, \mathbb{R})$ the fundamental form of M. Assume M is endowed with a prequantization bundle

$$L \rightarrow M,$$

that is a Hermitian line bundle with a connection ∇^L of curvature $\frac{1}{i}\omega$. Since ω is a $(1,1)$-form, L has a natural holomorphic structure defined in such a way that the (local) holomorphic sections satisfy the Cauchy-Riemann equations: $\nabla_{\bar{Z}} s = 0$ for every holomorphic vector field Z of M.

Let $K \rightarrow M$ be a Hermitian holomorphic line bundle. For every positive integer k define the quantum space $\mathcal{H}_k$:

$$\mathcal{H}_k = \big\{ \text{holomorphic section of } L^k \otimes K \big\}.$$

Assume that M carries a half-form bundle (δ, φ). $\delta \rightarrow M$ inherits a Hermitian scalar product and a holomorphic structure from $\Lambda^{n,0}T^*M$. Introduce the Hermitian holomorphic line bundle L_1 such that

$$K = L_1 \otimes \delta$$

and let $\frac{1}{i}\omega_1$ be the curvature of the Chern connection of L_1.

Since M is compact, $\mathcal{H}_k$ is finite dimensional and it follows from the Riemann-Roch-Hirzebruch theorem and Kodaira vanishing theorem that

$$\dim \mathcal{H}_k = \Big(\frac{k}{2\pi}\Big)^n \int_M (\omega + k^{-1}\omega_1)^{\wedge n}/n! + O(k^{n-2}) \tag{1}$$

To interpret this formula, we consider $L^k \otimes L_1$ and $\omega + \hbar\omega_1$ as deformations of L^k and ω which give the first quantum corrections in the semi-classical limit. Indeed the leading term

$$\big(\tfrac{k}{2\pi}\big)^n \int \omega^{\wedge n}/n!$$

gives the second-order correction when we replace ω with $\omega + k^{-1}\omega_1$. Furthermore in the case M doesn't carry any half-form bundle, equation (1) is still valid if we define ω_1 by

$$\omega_1 := \omega_K - \omega_c/2,$$

where $\frac{1}{i}\omega_K$ and $\frac{1}{i}\omega_c$ are the curvatures of the Chern connections of K and $\Lambda^{n,0}T^*M$.

1.3. **Toeplitz operators.** Denote by Π_k the orthogonal projector of $L^2(M, L^k \otimes K)$ onto $\mathcal{H}_k$, where the scalar product of two sections of $L^k \otimes K$ is defined from the Hermitian structures of L and K and the Liouville form μ_M.

A Toeplitz operator is any sequence $(T_k : \mathcal{H}_k \rightarrow \mathcal{H}_k)$ of operators of the form

$$T_k = \Pi_k f(., k) + R_k,$$

where $f(.,k)$ is a sequence of $C^\infty(M)$ with an asymptotic expansion $f_0 + k^{-1}f_1 + ...$ for the C^∞ topology and the norm of R_k is $O(k^{-\infty})$.

The set $\mathcal{T}$ of Toeplitz operators is a semi-classical algebra associated to (M, ω) in the following sense.

Theorem 1.2. *$\mathcal{T}$ is closed under the formation of product. So it is a star algebra, the identity is (Π_k). The symbol map*

$$\sigma_{\mathrm{cont}} : \mathcal{T} \rightarrow C^\infty(M)[[\hbar]],$$

sending T_k into the formal series $f_0 + \hbar f_1 + ...$ where the functions f_i are the coefficients of the asymptotic expansion of the multiplicator $f(.,k)$, is well defined. It is onto and its kernel is the ideal consisting of $O(k^{-\infty})$ Toeplitz operators. More precisely for any integer ℓ,

$$\|T_k\| = O(k^{-\ell}) \text{ if and only if } \sigma_{\mathrm{cont}}(T_k) = O(\hbar^\ell).$$

Furthermore, the induced product $*_{\mathrm{cont}}$ on $C^\infty(M)[[\hbar]]$ is a star-product.

Following the terminology of Berezin in [1], we call σ_{cont} the contravariant symbol map. This result is essentially a consequence of the works of Boutet de Monvel and Guillemin [5] (cf. also [12], [3] and [6]). Let us recall that equivalence classes of star-products on (M, ω) are parametrized by elements in

$$\frac{1}{i\hbar}[\omega] + H^2(M, \mathbb{C})[[\hbar]]$$

called Fedosov characteristic classes. The following theorem was proved by Karabegov and Schlichenmaier in [14] and [15], in the case K is the trivial line bundle.

Theorem 1.3. *The Fedosov class of the star-product $*_{\mathrm{cont}}$ is $\frac{1}{i\hbar}([\omega] + \hbar[\omega_1])$.*

Again it is interesting to note the appearance of $\omega + \hbar\omega_1$. We do not need this result but some related facts. Let us define the *normalized symbol* of a Toeplitz operator by

$$\sigma_{\mathrm{norm}}(T_k) := (\mathrm{Id} + \tfrac{\hbar}{2}\Delta)\sigma_{\mathrm{cont}}(T_k)$$

where Δ is the holomorphic Laplacian acting on $C^\infty(M)$. Actually we are only interested in the leading and second order terms of $\sigma_{\mathrm{norm}}(T_k)$ and modifying the definition of $\sigma_{\mathrm{norm}}(T_k)$ by a $O(\hbar^2)$ term wouldn't change the statements of our results. To compare with our previous article [7], the Weyl symbol that we introduced when K is the trivial line bundle is equal to the normalized symbol modulo $O(\hbar^2)$.

The map $\sigma_{\mathrm{norm}} : \mathcal{T} \rightarrow C^\infty(M)[[\hbar]]$ satisfies the same properties as σ_{cont} stated in theorem 1.2. Denote by $*_{\mathrm{norm}}$ the associated star product.

Theorem 1.4. *Let f and g belong to $C^\infty(M)[[\hbar]]$, then*

$$f *_{\mathrm{norm}} g = f.g + \tfrac{\hbar}{2i}\langle \pi, df \wedge dg\rangle + O(\hbar^2)$$

and

$$i\hbar^{-1}\big(f *_{\mathrm{norm}} g - g *_{\mathrm{norm}} f\big) = \langle \pi + \hbar\pi_1, df \wedge dg\rangle + O(\hbar^2).$$

where π is the Poisson bivector and π_1 is the bivector such that $\langle \pi_1, df \wedge dg\rangle + \langle X_f \wedge X_g, \omega_1\rangle = 0$ for every $f, g \in C^\infty(M)$.

So $*_{\mathrm{norm}}$ is a normalized star-product, in the sense that the second order term in the first formula is antisymmetric, which explains our terminology. Observe that $\pi + \hbar\pi_1$ is the Poisson bivector associated to $\omega + \hbar\omega_1$ in the sense that

$$\langle \pi + \hbar\pi_1, df \wedge dg\rangle = \langle (X_f + \hbar X_f^1) \wedge (X_g + \hbar X_g^1), \omega + \hbar\omega_1\rangle + O(\hbar^2),$$

where $X_f + \hbar X_f^1 + O(\hbar^2)$ is the Hamiltonian vector field of f with respect to $\omega + \hbar\omega_1$, that is

$$df + \langle \omega + \hbar\omega_1, X_f + \hbar X_f^1\rangle = O(\hbar^2),$$

and the same holds for g and $X_g + \hbar X_g^1$. So it follows from theorem 1.3 that there exists a star product equivalent to $*$ satisfying the formulas of theorem 1.4. This last result is more precise because the equivalence is specified. We can prove it using the methods of [6] or [15]. But this leads to complicated computations. We will present in [8] a more conceptual proof. We stated this result because we can deduce from it a part of the Bohr-Sommerfeld conditions (cf. sections 2.2, 2.3, 2.4).

1.4. **Relation with geometric Quantization.** Our definition of the normalized symbol agrees in some sense with the usual procedure to quantize observables in geometric quantization. Assume that the Hamiltonian flow of $f \in C^\infty(M)$ preserves the complex structure of M. Assume also that $K = L_1 \otimes \delta$, where (δ, φ) is a half-form bundle. Then the following operator is well-defined

$$\mathcal{Q}(f) := f + \frac{1}{ik}\big(\nabla^{L^k \otimes L_1}_{X_f} \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathcal{L}_{X_f}\big) : \mathcal{H}_k \to \mathcal{H}_k. \tag{2}$$

Here $\mathcal{L}_X$ acts on sections of δ by $\varphi((\mathcal{L}_X s) \otimes s) = \frac{1}{2}\mathcal{L}_X \varphi(s^{\otimes 2})$, or equivalently as a Lie derivative where the pull-back of sections of δ by a complex diffeomorphism ζ is defined in such a way that $\varphi((\zeta^* s)^{\otimes 2}) = \zeta^* \varphi(s^{\otimes 2})$.

The definition (2) is natural for the following reason. Denote by Φ_t the Hamiltonian flow of X_f. Let $\tilde{\Phi}_t$ be the lift of Φ_t to $L^k \otimes L_1 \otimes \delta$ defined by the tensor product of the parallel transport along the trajectories of X_f in $L^k \otimes L_1$ and by the pull-back in δ. Then the solution of the Schrödinger equation

$$\frac{1}{ik}\frac{d}{dt}\Psi(.,t) + \mathcal{Q}(f)\Psi(.,t) = 0$$

with initial condition $\Psi \in \mathcal{H}_k$ is given by

$$\Psi(x,t) = e^{\frac{k}{i}\int_0^t f(\Phi_{s-t}(x))ds}\tilde{\Phi}_t\big(\Psi(\Phi_{-t}(x))\big).$$

The important point for us is that $\mathcal{Q}(f)$ is a Toeplitz operators whose normalized symbol is f modulo $O(\hbar^2)$. We will prove a more general result for every smooth function f, which simplifies some further proofs.

If f is an arbitrary smooth function, formula (2) doesn't make sense, because the Lie derivative with respect to X_f doesn't necessarily preserve $\Omega^{n,0}(M)$. So we define for any vector field X the operator D_X,

$$D_X \alpha = p(\mathcal{L}_X \alpha), \qquad \alpha \in \Omega^{n,0}(M)$$

where p is the projection from $\Lambda^n T^*M \otimes \mathbb{C}$ onto $\Lambda^{n,0}T^*M$ with kernel the sum

$$\Lambda^{n-1,1}T^*M \oplus \Lambda^{n-2,2}T^*M \oplus \ldots \oplus \Lambda^{0,n}T^*M.$$

Next we let D_X act on the sections of δ, as the first-order differential operator such that $2\varphi(s \otimes D_X s) = D_X \varphi(s^2)$.

Theorem 1.5. *For any $f \in C^\infty(M)$, the operator*

$$\mathcal{Q}(f) := \Pi_k\Big(f + \frac{1}{ik}\big(\nabla^{L^k \otimes L_1}_{X_f} \otimes \mathrm{Id} + \mathrm{Id} \otimes D_{X_f}\big)\Big) : \mathcal{H}_k \to \mathcal{H}_k$$

is a Toeplitz operator with principal symbol f and vanishing subprincipal symbol.

This theorem is a consequence of the following lemma and an argument of Tuynman [20].

Lemme 1.6. *Let s be a half-form, then*

$$D_{X_f} s = \nabla^{\delta}_{X_f} s + \tfrac{i}{2}(\Delta f)s$$

where ∇^δ is the Chern connection of δ and Δ is the holomorphic Laplacian of M.

It follows that

$$\nabla^{L^k\otimes L_1}_{X_f} \otimes \mathrm{Id} + \mathrm{Id}\otimes D_{X_f} = \nabla^{L^k\otimes L_1\otimes\delta}_{X_f} + \tfrac{i}{2}(\Delta f) = \nabla^{L^k\otimes K}_{X_f} + \tfrac{i}{2}(\Delta f) \tag{3}$$

Now we have for every $\Psi \in \mathcal{H}_k$,

$$\Pi_k(\nabla^{L^k\otimes K}_{X_f}\Psi) = \tfrac{1}{i}\Pi_k(\Delta f.\Psi),$$

cf. [20] or [3] for a proof. Hence

$$\mathcal{Q}(f)\Psi = \Pi_k\big((f - \tfrac{1}{2k}\Delta f)\Psi\big)$$

which proves theorem 1.5.

The last expression in (3) shows that the definition of $\mathcal{Q}(f)$ is independent of the choice of the half-form bundle and generalizes in the cases where no such bundle exists.

Proof of lemma 1.6. It suffices to prove that for every $\alpha \in \Omega^{n,0}(M)$, we have

$$D_{X_f}\alpha = \nabla_{X_f}\alpha + i(\Delta f)\alpha$$

Introduce normal complex coordinates $z^1, ..., z^n$ centered at x_0. So if $\alpha = dz^1\wedge ...\wedge dz^n$, then $\nabla\alpha = 0$ at x_0. Let us write $\omega = iG_{j,k}dz^j\wedge d\bar{z}^k$. Then

$$X_f = -iG^{j,k}(\partial_{z^j} f)\partial_{\bar{z}^k} + iG^{j,k}(\partial_{\bar{z}^k} f)\partial_{z^j}$$

Using that the first derivatives of $G^{j,k}$ vanish at x_0, we obtain easily

$$D_{X_f}\alpha = iG^{j,k}(\partial_{z^j}\partial_{\bar{z}^k} f) = i\Delta f.$$

The result follows. □

2. Bohr-Sommerfeld Conditions

2.1. **The result.** Assume that M is 2-dimensional. Let (δ,φ) be a half-form bundle and let us write $K = L_1\otimes\delta$ as previously. Consider a self-adjoint Toeplitz operator (T_k). Its normalized symbol

$$f_0 + \hbar f_1 + ...$$

is real-valued. Bohr-Sommerfeld conditions give the spectrum of T_k on every open interval I of regular values of f_0 in the semi-classical limit. To simplify the statements, assume that f_1 vanishes.

Let Γ^1, ..., Γ^m be the components of $f_0^{-1}(I)$. For every $i\in\{1,...,m\}$, the map

$$f_0 : \Gamma^i \to I$$

is a trivial fibration with fiber diffeomorphic to S^1. For every $\lambda\in I$, fix an orientation on the fiber $\Gamma^i_\lambda = f_0^{-1}(\lambda)\cap\Gamma^i$ depending continuously on λ.

Let $a^i\in C^\infty(I)$ be the *principal action*, defined in such a way that the parallel transport in L along Γ^i_λ is the multiplication by $\exp\big(ia^i(\lambda)\big)$. Using L_1 instead of L, define in the same way the *subprincipal action* $a^i_1\in C^\infty(I)$.

Let us define an index ϵ^i from the half-form bundle (δ,φ). Observe that the restriction $\delta_{i,\lambda}$ of δ to Γ^i_λ is a square root of $T^*\Gamma^i_\lambda\otimes\mathbb{C}$. Indeed, let us denote by ι the embedding $\Gamma^i_\lambda\to M$, then the map

$$\varphi_{i,\lambda} : \delta^2_{i,\lambda} \to T^*\Gamma^i_\lambda\otimes\mathbb{C}, \qquad u\to\iota^*\varphi(u)$$

is an isomorphism of line bundle. The set

$$\{u\in\delta_{i,\lambda};\ \varphi_{i,\lambda}(u^{\otimes 2}) > 0\}$$

has one or two connected components. In the first case, we set $\epsilon^i_\lambda = 1$ and in the second case $\epsilon^i_\lambda = 0$. Observe that ϵ^i_λ doesn't depend on λ.

The Bohr-Sommerfeld conditions are

$$a^i(\lambda) + k^{-1}(a_1^i(\lambda) + \epsilon^i \pi) \in \frac{2\pi}{k}\mathbb{Z} \tag{4}$$

Denote by $\Sigma^i(k)$ the set of $\lambda \in I$ satisfying (4). When k is sufficiently large, $\Sigma^i(k)$ is a finite set containing

$$\frac{k}{2\pi}\mathrm{Vol}(\Gamma^i) + O(k^{-1})$$

points. Let $\Sigma(k)$ be the union of the $\Sigma^i(k)$. Define the *multiplicity* of $\lambda \in \Sigma(k)$ as the number of $\Sigma^i(k)$ which contains λ. The points of $\Sigma(k)$ approximate the eigenvalues of T_k in the following sense.

Theorem 2.1. *Let $\lambda_-(k)$ and $\lambda_+(k)$ be two sequences of I such that*

$$d(\lambda_-(k), \Sigma(k)) \geqslant Ck^{-1}, \quad d(\lambda_+(k), \Sigma(k)) \geqslant Ck^{-1} \tag{5}$$

for some positive C. Assume furthermore that there exists $\lambda_-, \lambda_+ \in I$ such that

$$\lambda_- \leqslant \lambda_-(k) \leqslant \lambda_+(k) \leqslant \lambda_+. \tag{6}$$

Denote by $\lambda_1(k) \leqslant \lambda_2(k) \leqslant .... \leqslant \lambda_{N(k)}(k)$ (resp. $\lambda'_1(k) \leqslant \lambda'_2(k) \leqslant .... \leqslant \lambda'_{N'(k)}(k)$) the eigenvalues of (T_k) (resp. points of $\Sigma(k)$) contained in $(\lambda_-(k), \lambda_+(k))$ and counted with multiplicities. Then, when k is sufficiently large, $N(k) = N'(k)$. Furthermore

$$\lambda_j(k) = \lambda'_j(k) + O(k^{-2}) \tag{7}$$

uniformly with respect to j.

The interest of condition (5) is to avoid any ambiguity in the counting of eigenvalues near the endpoints of $(\lambda_-(k), \lambda_+(k))$. It is not restrictive. Indeed if $\lambda_-(k)$, $\lambda_+(k)$ are arbitrary sequences satisfying (6), then by modifying them by suitably chosen $O(k^{-1})$ sequences, we obtain sequences satisfying both estimates (5) and (6).

Since the definition of the Toeplitz operators and of their normalized symbol only depend on K and not on the choice of the half form bundle, it is likely that the same holds for the Bohr-Sommerfeld conditions. This is easily checked using that any other half-form bundle is of the form $\delta' = \delta \otimes F$, where F is a flat Hermitian line bundle with holonomy in $\mathbb{Z}_2$. So $L'_1 = L_1 \otimes F^{-1}$, and straightforward computations show that the functions $a_1^i + \epsilon^i$ do not depend on the choice of δ.

To compare with our previous results in [7], when K is the trivial bundle, we defined the function $a_1^i + \epsilon^i$ as the integral of the geodesic curvature of Γ^i_λ.

We can also approximate the eigenvalues up to a $O(k^{-\infty})$ error. More precisely, there exist sequences $(S^i(.,k))_k$ of $C^\infty(I)$ such that the Bohr-Sommerfeld conditions

$$S^i(.,k) \in \tfrac{2\pi}{k}\mathbb{Z}$$

instead of (4) lead to the same result with a $O(k^{-\infty})$ error in (7). Furthermore, the sequences $S^i(.,k)$ admit asymptotic expansions of the form

$$S_0^i + k^{-1}S_1^i + k^{-2}S_2^i + ...$$

The leading and second order terms are the same like in (4). Applying an argument of Colin de Verdière [9], we can prove that the derivatives of the S_j^i only depend on the star-product $*_{\mathrm{norm}}$ and the normalized symbol of T_k. In particular, assuming that S_0^i is increasing, we will deduce from proposition 1.4 that necessarily

$$S_0^i(\lambda) - S_0^i(\lambda') = \int_{D^i} \omega \quad \text{and} \quad S_1^i(\lambda) - S_1^i(\lambda') = \int_{D^i} \omega_1 \tag{8}$$

if $\lambda, \lambda' \in I$ are such that $\lambda' \leqslant \lambda$ and $D^i = \Gamma^i \cap f_0^{-1}((\lambda, \lambda'))$. This determines the Bohr-Sommerfeld conditions (4) modulo a constant term. Note that only the derivatives of S_0^i and S_1^i are determined by the star-product. Indeed, if we twist L or L_1 by a flat Hermitian line bundle, the actions of the non-contractible loops may change although the star-product remains the same.

As a first step to deduce (8) from proposition 1.4, let us state some results on traces and functional calculus of Toeplitz operators.

2.2. **Traces.** In this section and the next one, we do not necessarily assume that M is 2-dimensional. It is a known result that the trace of a Toeplitz operator T_k with normalized symbol $f_0 + \hbar f_1 + ...$ admits an asymptotic expansion of the following form:

$$\operatorname{Tr}(T_k) = \Big(\frac{k}{2\pi}\Big)^n \int_M (f_0 + k^{-1} f_1 + ...)(1 + k^{-1} d_1 + k^{-2} d_2 + ...)\mu_M + O(k^{-\infty})$$

where d_1, d_2, d_3, ... are functions of $C^\infty(M)$ which do not depend of T_k (cf. for instance [6]).

These functions may be computed in terms of the Kähler metric by using the methods of [6], but it is more convenient to relate them to the star-product $*_{\text{norm}}$. To do this, observe that the $\mathbb{C}[[\hbar]]$-linear map

$$\text{(9)} \qquad \operatorname{tr} : C^\infty(M)[[\hbar]] \to \mathbb{C}[\hbar^{-1}, \hbar]], \quad f(\hbar) \to (2\pi\hbar)^{-n} \int_M f(\hbar) d(\hbar)\; \mu_M$$

where $d(\hbar) = 1 + \hbar d_1 + \hbar^2 d_2 + ...$ is a trace for the star-product $*_{\text{norm}}$, in the sense that it satisfies

$$\operatorname{tr}(f *_{\text{norm}} g) = \operatorname{tr}(g *_{\text{norm}} f).$$

Following Fedosov [11] or Nest-Tsygan [18], such a trace is unique up to multiplication by an element of $\mathbb{C}[\hbar^{-1}, \hbar]]$ and there exists a canonical one determined by the following normalization condition: for every local equivalence Φ between $*_{\text{norm}}$ and the Weyl star-product, we have

$$\text{(10)} \qquad \operatorname{tr}(f) = (2\pi\hbar)^{-n} \int_M \Phi(f)\; \mu_M, \qquad \text{where } \mu_M = \omega^n / n!.$$

We claim that the trace defined in (9) is the canonical trace. This follows from the fact that the quantization by Toeplitz operators is microlocally equivalent to the usual Weyl quantization.

So the functions d_i are determined by $*_{\text{norm}}$. In particular, it follows from proposition 1.4 that

$$\text{(11)} \qquad \operatorname{tr}(f) = (2\pi\hbar)^{-n} \int_M f\; (\omega + \hbar\omega_1)^{\wedge n} / n! + O(\hbar^{-n+2})$$

Proof of formula (11). Consider an equivalence of star-product of the form

$$\Phi = \operatorname{Id} + \hbar X + O(\hbar^2)$$

where X is a vector field. It is easily checked that the star-product

$$f *' g = \Phi\big(\Phi^{-1}(f) *_{\text{norm}} \Phi^{-1}(g)\big)$$

is normalized and satisfies

$$i\hbar^{-1}\big(f *' g - g *' f\big) = \langle \pi + \hbar(\pi_1 + X.\pi), df \wedge dg\rangle + O(\hbar^2).$$

Locally on can choose X in such a way that $\pi_1 + X.\pi = 0$. Indeed this equation is equivalent to $\omega_1 + d\alpha = 0$, where α is the 1-form such that $\omega(X, .) = \alpha$.

Next step is to introduce local Darboux coordinates, which define a Weyl star-product $*_{\text{Weyl}}$. And modifying Φ by a $O(\hbar^2)$ term, one has $*_{\text{Weyl}} = *'$ (cf. [2]). Then it follows from (10) that

$$\operatorname{tr}(f) = (2\pi\hbar)^{-n} \int_M f \left(\mu_M - \hbar(X\mu_M) + O(\hbar^2)\right).$$

By definition of π_1 in theorem 1.4, one has $\langle \pi_1, \omega\rangle + \langle \pi, \omega_1\rangle = 0$. Since $X.\langle \pi, \omega\rangle = 0$, we obtain

$$\langle \pi, X\omega\rangle = -\langle X\pi, \omega\rangle = \langle \pi_1, \omega\rangle = -\langle \omega_1, \pi\rangle.$$

Then a straightforward computation leads to

$$\langle \pi^{\wedge n}, \mu_M - \hbar X\mu_M\rangle = \langle \pi^{\wedge n}, (\omega + \hbar\omega_1)^{\wedge n}/n!\rangle + O(\hbar^2),$$

which proves the result. $\square$

As a consequence of (11), we obtain the estimate (1) of the dimension of $\mathcal{H}_k$, since this dimension is the trace of the projector Π_k, whose normalized symbol is 1. Actually the index theorems of deformation quantization proved in [11] or [18] yield the asymptotic expansion of $\operatorname{Tr}(\Pi_k)$ modulo $O(k^{-\infty})$ in terms of the Fedosov class of $*_{\text{norm}}$. Since this trace is an integer, the $O(k^{-\infty})$ error vanishes when k is sufficiently large. In this way we can deduce Riemann-Roch-Hirzebruch theorem from theorem 1.3 .

2.3. **Functional calculus.** Let T_k be a self-adjoint Toeplitz operator with normalized symbol

$$f = f_0 + \hbar f_1 + \dots.$$

and let g be a function of $C^\infty(\mathbb{R}, \mathbb{C})$. Then it is known that $g(T_k)$ is a Toeplitz operator (cf. for instance [6]). Furthermore, the normalized symbol of $g(T_k)$ is given by the following non-commutative Taylor formula:

$$g^{*_{\text{norm}}}(f)(x) = \sum \frac{1}{\ell!} g^{(\ell)}(f_0(x)) \left(f(y) - f_0(x)\right)^{*_{\text{norm}}\ell}|_{y=x}, \tag{12}$$

where $g^{(\ell)}$ is the ℓth derivative of g and $h^{*_{\text{norm}}\ell} = h *_{\text{norm}} \dots *_{\text{norm}} h$ repeated ℓ times. In particular, an easy computation from proposition 1.4 leads to

$$g^{*_{\text{norm}}}(f) = g(f_0) + \hbar g'(f_0) f_1 + O(\hbar^2). \tag{13}$$

2.4. **On the variation of S_0 and S_1.** Let us deduce formulas (8) on the variations of S_0 and S_1 from the trace formula (11) and the functional symbolic calculus. Assume that $f_0^{-1}(I)$ is connected.

If $g \in C^\infty_o(I, \mathbb{C})$, then we deduce from (13), (11) and the fact that $f_1 = 0$ that

$$\begin{aligned} \operatorname{Tr}(g(T_k)) =& \frac{k}{2\pi} \int_M (g \circ f_0)(\omega + k^{-1}\omega_1) \ + O(k^{-1}) \\ =& \frac{k}{2\pi} \int_I g \left(f_{0*}[\omega + k^{-1}\omega_1]\right) \ + O(k^{-1}). \end{aligned} \tag{14}$$

where f_{0*} is the push-forward $\Omega^2(M) \to \Omega^1(I)$ defined by

$$\int_M (f_0^* h)\ \alpha = \int_I g.f_{0*}\alpha, \qquad \forall h \in C^\infty_o(I).$$

On the other hand, assume the Bohr-Sommerfeld condition is

$$S(\lambda, k) \in 2\pi k^{-1}\mathbb{Z}$$

with $S(\lambda,k) = S_0 + k^{-1}S_1 + O(k^{-2})$. If the derivative of S_0 doesn't vanish, then one can invert the functions $S(.,k)$ when k is sufficiently large and

$$\mathrm{Tr}(g(T_k)) = \sum_{x\in 2\pi k^{-1}\mathbb{Z}} g(S^{-1}(x,k)) + O(k^{-1})$$

Interpreting this as a Riemann sum, it follows that

$$\begin{aligned}\mathrm{Tr}(g(T_k)) =& \frac{k}{2\pi}\int_{\mathbb{R}} g(S^{-1}(x,k))dx + O(k^{-1})\\ =& \frac{k}{2\pi}\int_I g(\lambda)S'(\lambda,k)d\lambda + O(k^{-1})\end{aligned} \tag{15}$$

with the same orientation of I as before if S_0' is positive. Since (14) and (15) hold for any function g of $C^\infty_o(I,\mathbb{C})$, we have

$$dS_0 + \hbar dS_1 = f_{0*}(\omega + \hbar\omega_1)$$

and (8) follows.

3. Lagrangian states

3.1. **Definitions and symbolic calculus.**

First we recall the definition of a local Lagrangian section associated to a closed Lagrangian embedding $\iota : \Gamma \to M$.

Let U be an open set of M such that $U_\Gamma := \iota^{-1}(U)$ is contractible. Since the curvature of ι^*L vanishes, there exists a flat unitary section t_Γ of $\iota^*L \to U_\Gamma$. Introduce a formal series

$$\sum_{\ell=0}^{\infty}\hbar^\ell g_\ell \in C^\infty(U_\Gamma, \iota^*K)[[\hbar]].$$

Let V be an open set of M such that $\overline{V} \subset U$. Then a sequence $\Psi_k \in \mathcal{H}_k$ is a *Lagrangian section* over V associated to (Γ, t_Γ) with symbol $\sum \hbar^\ell g_\ell$ if

$$\Psi_k(x) = \Big(\frac{k}{2\pi}\Big)^{\frac{n}{4}} F^k(x)\tilde{g}(x,k) + O(k^{-\infty}) \text{ over } V,$$

where

- F is a section of $L \to U$ such that
$$\iota^*F = t_\Gamma \quad \text{and} \quad \bar{\partial}F \equiv 0$$
modulo a section which vanishes to every order along $\iota(\Gamma)$. Furthermore $|F(x)| < 1$ if $x \notin \iota(\Gamma)$.
- $\tilde{g}(.,k)$ is a sequence of $C^\infty(U,K)$ with an asymptotic expansion $\sum k^{-\ell}\tilde{g}_\ell$ in the C^∞ topology such that
$$\iota^*\tilde{g}_\ell = g_\ell \text{ and } \quad \bar{\partial}\tilde{g}_\ell \equiv 0$$
modulo a section which vanishes at every order along $\iota(\Gamma)$.

We assume furthermore that Ψ_k is admissible in the sense that $\Psi_k(x)$ is uniformly $O(k^N)$ for some N and the same holds for its successive covariant derivatives.

It is not obvious that such a sequence exists.

Theorem 3.1. *For every series $\sum \hbar^\ell g_\ell$ of $C^\infty(U_\Gamma, \iota^*K)[[\hbar]]$, there exists a Lagrangian section over V associated to (Γ, t_Γ) with symbol $\sum \hbar^\ell g_\ell$. It is unique modulo a section which is $O(k^{-\infty})$ over V.*

In the statement of the following theorems, we consider that $K = L_1 \otimes \delta$ over U, where (δ, φ) is a half-form bundle. Recall that $\iota^* \delta = \delta_\Gamma$ is a square root of $\Lambda^n T^* \Gamma \otimes \mathbb{C}$ through the isomorphism

$$\varphi_\Gamma : \delta_\Gamma^2 \to \Lambda^n T^* \Gamma \otimes \mathbb{C}, \quad u \to \iota^* \varphi(u).$$

Let us associate to the *principal symbol* g_0 of a Lagrangian section a density $m(g_0)$ where m is the map:

$$m : \iota^* L_1 \otimes \delta_\Gamma \to |\Lambda|(\Gamma), \quad u \otimes v \to \|u\|^2_{L_1} \; |\varphi_\Gamma(v^{\otimes 2})|$$

We then have the following estimate of the norm of Ψ_k.

Theorem 3.2. *Let $\xi \in C^\infty_o(V)$, then we have*

$$\int_M \xi \; \|\Psi_k\|^2_{L^k \otimes L_1 \otimes \delta} \; \mu_M = \int_\Gamma (\iota^* \xi) \; m(g_0) + O(k^{-1}).$$

Next results describe how a Toeplitz operator acts on a Lagrangian section.

Theorem 3.3. *Let T_k be a Toeplitz operator with principal symbol f_0. Then $T_k \Psi_k$ is a Lagrangian section over V associated to (Γ, t_Γ) with symbol $(\iota^* f_0) g_0 + O(\hbar)$.*

To prove the Bohr-Sommerfeld conditions, we need to compute the subsequent coefficient of the symbol of $T_k \Psi_k$, in the case where f_0 is constant over Γ.

Theorem 3.4. *Let T_k be a Toeplitz operator with normalized symbol $f_0 + \hbar f_1 + O(\hbar^2)$. Assume that f_0 is constant along Γ. Then the symbol of $T_k \Psi_k$ is*

$$(\iota^*(f_0 + \hbar f_1)).(g_0 + \hbar g_1) + \hbar \tfrac{1}{i} (\nabla^{\iota^* L_1}_X \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathcal{L}^{\delta_\Gamma}_X).g_0 + O(\hbar^2)$$

where

- *X is the Hamiltonian vector field of f_0,*
- *$\nabla^{\iota^* L_1}$ is the pull-back of the Chern connection of L_1,*
- *$\mathcal{L}^{\delta_\Gamma}_X$ is the first order differential operator acting on sections of δ_Γ such that*

  $$\varphi_\Gamma(\mathcal{L}^{\delta_\Gamma}_X g \otimes g) = \tfrac{1}{2} \mathcal{L}_X \varphi_\Gamma(g^{\otimes 2})$$

  for every section g.

It is easily checked that the operator $\nabla^{\iota^* L_1}_X \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathcal{L}^{\delta_\Gamma}_X$ doesn't depend of the choice of the half-form bundle if we consider that it acts on sections of $\iota^* K = \iota^* L_1 \otimes \delta_\Gamma$. The same holds with the map m.

3.2. **Proof of Bohr-Sommerfeld conditions.** Let us deduce from the previous theorems the Bohr-Sommerfeld conditions for n self-adjoint commuting Toeplitz operators T^1, T^2,..., T^n, which is a slight generalization of (4).

Denote by f^i_0 and f^i_1 the principal and subprincipal symbols of T^i. Let E be a regular value of $f = (f^1_0, ..., f^n_0)$ and $\iota : \Gamma \to M$ be an embedding with image a connected component of $f^{-1}(\{E\})$. Since M is compact, $\iota(\Gamma)$ is a Lagrangian torus. So there exists a half-form bundle (δ, φ) defined over a neighborhood of $\iota(\Gamma)$. It is not unique but as usual the final result doesn't depend on the choice of (δ, φ). Introduce like in the previous section two open sets U, V and a flat section t_Γ. Let us try to solve the eigenvalues equation

$$T^i \Psi = E^i \Psi + O(k^{-\infty}) \text{ over } V \tag{16}$$

where Ψ is a Lagrangian section associated to (Γ, t_Γ). By theorem 3.4, the symbol of $(T^i - E^i)\Psi$ is $O(\hbar)$ because $\iota^* f^i = E^i$. Furthermore it is $O(\hbar^2)$ if and only if it satisfies the following transport equation

$$\big[f^i_1 + (\nabla^{\iota^* L_1}_{X^i} \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathcal{L}^{\delta_\Gamma}_{X^i}) \big] g_0 = 0 \text{ over } V \cap \Gamma \tag{17}$$

where g_0 is the principal symbol of Ψ. This equation can be interpreted as g_0 being flat for a connection on $\iota^* L_1 \otimes \delta_\Gamma$ that we describe now.

First a section g of δ_Γ is flat if $\mathcal{L}^{\delta_\Gamma}_{X^i} g = 0$ for every i. With this definition, $\varphi_\Gamma : \delta_\Gamma^2 \to \Lambda^n T^*\Gamma \otimes \mathbb{C}$ is a morphism of flat bundles, if we endow $\Lambda^n T^*\Gamma \otimes \mathbb{C}$ with the Weinstein connection. Since the form β such that

$$\langle \beta, X^1 \wedge ... \wedge X^n \rangle = 1$$

is a global non vanishing flat section of $\Lambda^n T^*\Gamma \otimes \mathbb{C}$, δ_Γ has holonomy in $\mathbb{Z}_2 \subset U(1)$.

Let $\alpha \in \Omega^1(\Gamma)$ be such that $\frac{1}{i}\langle \alpha, X^j \rangle = f_1^j$. Consider the connection $\nabla^{\iota^* L_1} + \alpha$ on $\iota^* L_1$. Its flat sections satisfy

$$(f_1^j + \tfrac{1}{i} \nabla^{\iota^* L_1}_{X^j}) s = 0.$$

Furthermore its curvature vanishes. Indeed, since $[X^i, X^j] = 0$, we have

$$\begin{aligned} \frac{1}{i}\langle d\alpha, X^i \wedge X^j \rangle =& (X^i.f_1^j - X^j.f_1^i) \\ =& \omega_1(X^i, X^j) \end{aligned}$$

which follows from proposition 1.4 and the fact that $[T^i, T^j] = 0$.

This defines a structure of flat line bundle for $\iota^* L_1 \otimes \delta_\Gamma$, whose flat sections are the solutions of (17). Recall that

$$\Psi(x,k) = t_\Gamma^k(x)(g_0(x) + O(k^{-1})) \text{ over } V \cap \iota(\Gamma),$$

where t_Γ is a flat section of $\iota^* L$. The condition to patch together these sections along Γ is the Bohr-Sommerfeld condition:

$$\iota^*(L^k \otimes L_1) \otimes \delta_\Gamma \to \Gamma \text{ is trivial as a flat bundle.}$$

When M is two-dimensional, this is equivalent to (4). To prove theorem 2.1 or a similar result in the $2n$-dimensional case for the joint spectrum, we should consider Lagrangian sections depending continuously on Γ. Furthermore, we can show by using a local normal form that the solutions of (16) are necessarily Lagrangian sections associated to Γ. A complete proof is in [7]. The only novelty here is the formulation of theorems 3.4 and 3.2, and consequently of the Bohr-Sommerfeld conditions.

3.3. **Comparison with the cotangent case.** To compare theorems 3.2 and 3.4 with the similar statements in the case of pseudo-differential operators, we can introduce some kind of Maslov bundle in the following way. Recall that we denote by φ_Γ the isomorphism $\delta_\Gamma^2 \to \Lambda^n T^*\Gamma \otimes \mathbb{C}$. Introduce

$$P := \big\{ u \in \delta_\Gamma;\ \varphi_\Gamma(u^{\otimes 2}) \in \Lambda^n T^*\Gamma - \{0\} \big\}.$$

Let $\mathbb{Z}_4$ be the subgroup $\{1, -1, i, -i\}$ of $\mathbb{C}^*$. Then P is a principal bundle with structure group $\mathbb{Z}_4 \times \mathbb{R}_+$. Introduce the complex line bundles $|\delta_\Gamma|$ and $\arg(\delta_\Gamma)$ associated to P via the homomorphism $\mathbb{Z}_4 \times \mathbb{R}_+ \to \mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{R}_+ \to \mathbb{R}_+$ respectively. Following Weinstein in [21], we call $\arg(\delta_\Gamma)$ the unitarization of δ_Γ. We have a canonical isomorphism

$$\delta_\Gamma \to |\delta_\Gamma| \otimes \arg(\delta_\Gamma).$$

Furthermore, the map

$$|\delta_\Gamma| = P \times_{\mathbb{R}_+} \mathbb{C} \ni [u,z] \to z.|\varphi_\Gamma(u^{\otimes 2})|^{\frac{1}{2}} \in |\Lambda|^{\frac{1}{2}}(\Gamma)$$

is an isomorphism between $|\delta_\Gamma|$ and the bundle of half-densities of Γ. So we obtain an isomorphism

$$\zeta : \delta_\Gamma \to |\Lambda|^{\frac{1}{2}}(\Gamma) \otimes \arg(\delta_\Gamma).$$

The bundle $\arg(\delta_\Gamma)$ is a line bundle with structure group $\mathbb{Z}_4$ like the Maslov bundle. The isomorphism ζ intertwines the operator $\mathcal{L}_X^{\delta_\Gamma}$ of theorem 3.4 with the Lie derivative of half-densities. So in the case L_1 is trivial, the theorem 3.4 is similar to the formula 1.3.13 in [10] p223, computing the symbol of an oscillatory integral acted on under a differential operator. Furthermore the map $\delta_\Gamma \rightarrow |\Lambda|(\Gamma)$ used in theorem 3.2 is the composition of ζ with the squaring map from half-densities into densities. Again theorem 3.2 is similar to formula 1.3.15 in [10] p224, computing the norm of an oscillatory integral.

To end this comparison, we apply the previous construction to a symplectic vector space E and prove that we obtain the usual Maslov bundle. Consider a one-dimensional vector space δ with an isomorphism $\varphi : \delta^2 \rightarrow \Lambda^{n,0}E^*$. Let $\mathrm{Lag}(E)$ be the Lagrangian Grassmannian and $\eta \rightarrow \mathrm{Lag}(E)$ be the tautological vector bundle, that is the bundle whose fiber over x is x itself. In the same way we defined φ_Γ, one has an isomorphism

$$\varphi_{\mathrm{Lag}} : \mathrm{Lag}(E) \times \delta^2 \rightarrow \Lambda^n \eta \otimes \mathbb{C},$$

sending (x, u) into $\iota_x^* \varphi(u)$ where ι_x is the embedding $\eta_x \rightarrow E$. Introduce the $\mathbb{Z}_4 \times \mathbb{R}_+$ bundle

$$\big\{(x,u) \in \mathrm{Lag}(E) \times \delta;\ \varphi_{\mathrm{Lag}}(x, u^2) \in \Lambda^n_x \eta \text{ and } u \neq 0\big\}$$

Dividing by $\mathbb{R}_+$ we get a $\mathbb{Z}_4$-principal bundle $\mathcal{M}$. We claim that the holonomy in $\mathcal{M}$ of a loop of $\mathrm{Lag}(E)$ is the mod 4 reduction of its Maslov index.

Proof. Introduce linear Darboux coordinates (p^i, q^i) and identify E with $\mathbb{R}^{2n}$. Set $z^j = p^j + iq^j$ and let

$$\delta := (dz^1 \wedge ... \wedge dz^n)^{\frac{1}{2}} \mathbb{C}$$

be the square root of $\Lambda^{n,0}E = (dz^1 \wedge ... \wedge dz^n)\, \mathbb{C}$. Recall that $\mathrm{Lag}(E)$ is isomorphic to $U(n)/O(n)$ (cf. lemma 2.31 of [17]), through the map sending the unitary matrix $U = P + iQ$ into the range of

$$A_U = \begin{pmatrix} P \\ Q \end{pmatrix}$$

Let us denote by α_U^1,...,α_U^n the base of η_x^* dual to the column vectors of A_U. A straightforward computation shows that φ_{Lag} sends the square of

$$([U], (dz^1 \wedge ... \wedge dz^n)^{\frac{1}{2}}) \in \mathrm{Lag}(E) \times \delta$$

into $\det(U)\ \alpha_U^1 \wedge ... \wedge \alpha_U^n$. Consequently,

$$\mathcal{M} \simeq \big\{([U], v) \in \mathrm{Lag}(E) \times \mathbb{C}^*;\ U \in U(n) \text{ and } v^2 \det(U) = \pm 1\big\}$$

Recall now that the Maslov index of a loop $x : \mathbb{R}/\mathbb{Z} \rightarrow \mathrm{Lag}(E)$ is the degree of $\rho \circ x : S^1 \rightarrow S^1$ where ρ is the map

$$\mathrm{Lag}(E) \rightarrow S^1, \quad [U] \rightarrow \det{}^2(U)$$

(cf. [17] page 53). Its mod 4 reduction is the holonomy of x in $\mathcal{M}$. □

This result is related to the paper [21] of Weinstein, where it is observed that the Maslov bundle of $\mathrm{Lag}(E)$ is a unitarization of a square root of $\Lambda^n \eta \otimes \mathbb{C}$.

Last remark is that in general the Maslov bundle of a Lagrangian submanifold of a cotangent space can be different of the bundle we construct. Indeed notice that the structure group of $\arg(\delta_\Gamma)$ reduces to $\mathbb{Z}_2$ if and only if Γ is orientable. Consider a non-orientable manifold Q. Then as the null section of T^*Q, Q is a Lagrangian submanifold and its Maslov bundle is the flat trivial bundle. So it can not be a unitarization of a square root of $\Lambda^n T^*Q \otimes \mathbb{C}$.

4. Proof

We assume in the whole section that there exists a globally defined half-form bundle (δ, φ) and $K = \delta$. There is no difficulty to generalize to the case where $K = \delta \otimes L_1$.

4.1. A preliminary result.

Consider a sequence $\Psi_k \in C^\infty(M, L^k \otimes \delta)$ of the form

$$\Psi_k(x) = \Big(\frac{k}{2\pi}\Big)^{\frac{n}{4}} F^k(x)\tilde{g}(x,k) + O(k^{-\infty}) \text{ over } V,$$

where F and $\tilde{g}(.,k)$ satisfies the same assumptions as in section 3.1 except that the coefficients $\tilde{g}_k$ do not necessarily satisfy $\bar{\partial}\tilde{g}_k \equiv 0$. Assume furthermore that Ψ_k is admissible.

Theorem 4.1. *Let T_k be a Toeplitz operator with principal symbol f_0. Then $T_k\Psi_k$ is a Lagrangian section over V with symbol $\iota^*(f_0\tilde{g}_0) + O(\hbar)$.*

Furthermore, if $\tilde{g}_0$ and its first derivatives vanish along $\iota(\Gamma)$, then the symbol of $T_k\Psi_k$ is

$$\hbar(\iota^* f_0)\big(\Box\tilde{g}_0 + \iota^*\tilde{g}_1)\big) + O(\hbar^2)$$

where $\Box\tilde{g}_0 \in C^\infty(\Gamma, \delta_\Gamma)$ and at every $x \in \Gamma$

$$\Box\tilde{g}_0(x) = -\frac{1}{2}\sum \bar{\partial}_i\bar{\partial}_i\tilde{g}_0(\iota(x))$$

if $\partial_1, ..., \partial_n$ is a base of vectors of $T^{1,0}_{\iota(x)}M$ such that $\frac{1}{i}\omega(\partial_i, \bar{\partial}_j) = \delta_{ij}$ and the vectors $\partial_i + \bar{\partial}_i$ are tangent to Γ.

The proof starts from the following representation of the Schwartz kernel of the Toeplitz operator T_k:

$$T_k(x,y) = \Big(\frac{k}{2\pi}\Big)^n E^k(x,y)\tilde{f}(x,y,k) + O(k^{-\infty}) \tag{18}$$

where, if we consider M^2 as a complex manifold with complex structure $(j, -j)$,

- E is a section of $L \boxtimes \bar{L} \to M^2$ satisfying
  $$E(x,x) = u \otimes \bar{u}, \quad \forall u \in L_x \text{ such that } \|u\| = 1,$$
  $\bar{\partial}E \equiv 0$ modulo a section vanishing to every order along the diagonal Δ of M^2 and $\|E(x,y)\| < 1$ if $x \neq y$.
- $\tilde{f}(.,k)$ is a sequence of sections of $\delta \boxtimes \bar{\delta} \to M^2$ with an asymptotic expansion of the form
  $$\tilde{f}(.,k) = \tilde{f}_0 + k^{-1}\tilde{f}_1 + ...$$
  whose coefficient satisfy $\bar{\partial}f_l \equiv 0$ modulo a section vanishing to every order along Δ. Furthermore,
  $$\tilde{f}_0(x,x) = f_0(x),$$
  where f_0 is the principal symbol of T_k.

In other words, $T_k(.,.)$ is a Lagrangian section associated to the diagonal Δ of M^2. This result was proved in [6], without the additional bundle δ. The generalization is straightforward.

Since the norm of E is < 1 outside the diagonal and Ψ_k is admissible, one has for every x in V

$$(T_k\Psi_k)(x) = \Big(\frac{k}{2\pi}\Big)^{n+\frac{n}{4}} \int E^k(x,y).F^k(y)\ \tilde{f}(x,y,k).\tilde{g}(y,k)\ \mu_M(y) + O(k^{-\infty})$$

where we integrate on a neighborhood of x. Introduce a unitary section t of L over U such that $\iota^* t = t_\Gamma$ over U_Γ and let us write

$$E(x,y).F(y) = e^{i\phi(x,y)} t(x). \tag{19}$$

Then the imaginary part of ϕ is non positive and vanishes only if (x,y) belongs to

$$C := \{(x,x) \in M^2;\ x \in \iota(\Gamma)\}.$$

To compute the derivatives of ϕ along C, recall the following lemma proved in [7] (cf. proposition 2.2 p.1535).

Lemme 4.2. *If $\nabla^L F = \frac{1}{i}\alpha_F \otimes F$, then α_F vanishes along $\iota(\Gamma)$ and for every vector field X, Y*

$$\mathcal{L}_X \langle \alpha_F, Y\rangle = \omega(qX, Y)$$

at $x \in \iota(\Gamma)$, where q is the projection onto $T^{0,1}_x M$ with kernel $T_x\iota(\Gamma)\otimes\mathbb{C}$.

As a corollary, we have the following

Lemme 4.3. *If $\nabla^{L\boxtimes\bar{L}} E = \frac{1}{i}\alpha_E \otimes E$, then α_E vanishes along the diagonal, and for every vector fields X_1, Y_1, X_2, Y_2 of M,*

$$\mathcal{L}_{(X_1,Y_1)}.\langle \alpha_E, (X_2,Y_2)\rangle = \omega(X_1^{0,1} - Y_1^{0,1}, X_2) + \omega(X_1^{1,0} - Y_1^{1,0}, Y_2)$$

along the diagonal, where we denoted by $X^{1,0}$ and $X^{0,1}$ the holomorphic and anti-holomorphic parts of X respectively.

We deduce from both lemmas that $d_y\phi$ vanishes along C. Furthermore the kernel of the tangent map to $d_y\phi$ at $(x,x)\in C$ is

$$\big(T^{0,1}_x M \times (0)\big) \oplus \big(T_{(x,x)}C \otimes \mathbb{C}\big)$$

Finally, we have along C,

$$d^2_y\phi(Y_1,Y_2) = \omega(Y_1^{1,0}, Y_2) - \omega(qY_1, Y_2)$$

and $d^2_y\phi$ is non-degenerate. So we can apply the stationary phase lemma (cf. [13] section 7.7 or theorem 5.1). One gets

$$(T_k\Psi_k)(x) = \Big(\frac{k}{2\pi}\Big)^{\frac{n}{4}} e^{ik\phi_r(x)} t^k(x)\ \tilde{h}(x,k) + O(k^{-\infty})$$

where

$$\phi_r(x) \equiv \phi(x,y) \tag{20}$$

modulo a linear combination with C^∞ coefficients of the $\partial_{y^i}\phi(x,y)$.

Lemme 4.4. *$e^{ik\phi_r(x)}t(x)$ satisfies the same assumption as the section F.*

Proof. Since ϕ and $d_y\phi$ vanishes along C, ϕ_r vanishes along $\iota(\Gamma)$ and consequently

$$e^{i\phi_r(x)}t(x) = t_\Gamma(x)$$

for every x in $\iota(\Gamma)$. Introduce complex coordinates $x^1,...,x^n$ and write

$$\nabla t = \tfrac{1}{i} t \otimes \sum a_j(x)dx^j + \bar{a}_j(x)d\bar{x}^j.$$

Derivating (19), it follows from $\bar\partial E \equiv 0$ that

$$\partial_{\bar{x}^i}\phi(x,y) \equiv \bar{a}_j(x) \mod \mathcal{I}_\Delta(\infty),$$

i.e. modulo a function vanishing to infinite order along the diagonal. Derivating again, one has

$$\partial_{\bar{x}^i}\partial_{y^j}\phi(x,y) \equiv 0 \mod \mathcal{I}_\Delta(\infty)$$

Then we deduce from the two previous equations and (20) that for every multi-index α,

$$\partial_{\bar{x}^1}^{\alpha(1)} ... \partial_{\bar{x}^n}^{\alpha(n)} \big(\partial_{\bar{x}^i} \phi_r - \bar{a}_i \big)(x) = 0$$

along $\iota(\Gamma)$. And consequently

$$\partial_{\bar{x}^i} \phi_r \equiv \bar{a}_i \mod \mathcal{I}_{\iota(\Gamma)}(\infty)$$

which proves the result. □

Lemme 4.5. *The sequence $\tilde{h}(.,k)$ admits an asymptotic expansion $\tilde{h}_0 + k^{-1}\tilde{h}_1 + ...$ whose coefficients satisfy $\bar{\partial}\tilde{h}_\ell \equiv 0$ modulo a section vanishing to every order along Γ.*

Proof. First one deduces from lemma 4.2 that the imaginary part of ϕ_r and its first derivatives vanishes along $\iota(\Gamma)$. Furthermore the Hessian of $\Im \phi_r$ along $\iota(\Gamma)$ is non-degenerate in the transverse direction to $\iota(\Gamma)$. As a consequence, if $\tilde{e}(x,k)$ is a sequence with an asymptotic expansion $\tilde{e}_0 + k^{-1}\tilde{e}_1 + ...$ such that

$$e^{ik\phi_r(x)} \tilde{e}(x,k) = O(k^{-\infty})$$

then the coefficients $\tilde{h}_\ell$ vanish to every order along $\iota(\Gamma)$. This was proved in [6] (cf. lemma 1, p.6). We apply this to the sequence $\bar{\partial} T_k \Psi_k$ which vanishes, since $\Pi_k T_k = T_k$ implies that $T_k \Psi_k$ belongs to $\mathcal{H}_k$. □

The two previous lemmas imply that $T_k\Psi_k$ is a Lagrangian section. Then applying theorems 5.1 and 5.2, we obtain the symbol of $T_k\Psi_k$ by computations of linear algebra, which are easily done using the tangent vectors ∂_i introduced in the statement of theorem 4.1.

4.2. **Proofs of the theorems of part 3.1.** A first corollary of theorem 4.1 is the existence of a Lagrangian section with an arbitrary symbol: applying theorem 4.1 with the Toeplitz operator Π_k, we construct a Lagrangian section with a prescribed principal symbol, then theorem 3.1 follows from Borel resummation. Theorem 3.3 is a particular case of theorem 4.1.

To prove theorem 3.4, we can assume that f_0 vanishes along Γ. Since we compute the symbol of $T_k\Psi_k$ modulo $O(\hbar^2)$, we can replace T_k with every Toeplitz operators of symbol $f_0 + \hbar f_1 + O(\hbar^2)$. So by theorem 1.5, we can choose

$$T_k = \Pi_k \Big(f_0 + k^{-1} f_1 + \frac{1}{ik} (\nabla_X^{L^k} \otimes \mathrm{Id} + \mathrm{Id} \otimes D_X) \Big)$$

where X is the Hamiltonian vector field of f_0. So $T_k\Psi_k$ is equal to

$$\Pi_k \Big[\big(\tfrac{k}{2\pi}\big)^{\frac{n}{4}} \big([f_0 + \tfrac{1}{ik} \nabla_X^{L^k}] F^k \big) \tilde{g}(.,k) \Big] + k^{-1} \Pi_k \Big[\big(\tfrac{k}{2\pi}\big)^{\frac{n}{4}} F^k \big([f_1 + \tfrac{1}{i} D_X] \tilde{g}(.,k) \big) \Big] \tag{21}$$

By theorem 4.1, each term of the sum is a Lagrangian section. Furthermore by the first part of this theorem, the symbol of the second one is

$$\hbar \, \big((\iota^* f_1).g_0 + \iota^* (\tfrac{1}{i} D_X \tilde{g}_0) \big) + O(\hbar^2).$$

Since X is tangent to $\iota(\Gamma)$, $\iota^*(D_X \tilde{g}_0)$ only depends on the restriction of $\tilde{g}_0$ to $\iota(\Gamma)$. So we can define the operator $\iota^* D_X$ acting on $C^\infty(\Gamma, \delta_\Gamma)$ which sends g_0 to $\iota^*(D_X \tilde{g}_0)$. And the previous symbol is

$$\hbar \, \big((\iota^* f_1) + \tfrac{1}{i} (\iota^* D_X) \big).g_0 + O(\hbar^2). \tag{22}$$

To compute the symbol of the first term of (21), let us write

$$[f_0 + \tfrac{1}{ik} \nabla_X^{L^k}] F^k = F^k a$$

with a defined on a neighborhood of $\iota(\Gamma)$.

Lemme 4.6. *The function a and its first derivatives vanish along $\iota(\Gamma)$. If Z and W are holomorphic vector fields of M, then*
$$\mathcal{L}_{\bar{W}}\mathcal{L}_{\bar{Z}} a = \omega(\bar{W}, \mathcal{L}_X \bar{Z})$$
on $\iota(\Gamma)$.

Proof. Denote by α_F be the one-form such that $\nabla^L F = \frac{1}{i}\alpha_F \otimes F$. Then
$$a = f_0 - \langle \alpha_F, X\rangle.$$
This vanishes along $\iota(\Gamma)$ because α_F vanishes along $\iota(\Gamma)$ (cf. lemma 4.2).

Since X is the Hamiltonian vector field of f_0, one has $\mathcal{L}_Y f_0 + \omega(X,Y) = 0$. Since the curvature of L is $\frac{1}{i}\omega$, one has $d\alpha_F = \omega$ and consequently
$$\mathcal{L}_Y \langle \alpha_F, X\rangle = \mathcal{L}_X \langle \alpha_F, Y\rangle + \omega(Y,X) + \langle \alpha_F, [Y,X]\rangle.$$
It follows that
$$\mathcal{L}_Y a = -\mathcal{L}_X \langle \alpha_F, Y\rangle - \langle \alpha_F, [Y,X]\rangle.$$
This vanishes along $\iota(\Gamma)$, because α_F vanishes along $\iota(\Gamma)$ and X is tangent to $\iota(\Gamma)$.

Since $\bar{\partial} F \equiv 0$ modulo a flat section and Z is holomorphic, $\langle \alpha_F, \bar{Z}\rangle$ vanishes to every order along $\iota(\Gamma)$. So choosing $Y = \bar{Z}$ in the previous equation, we obtain
$$\mathcal{L}_{\bar{W}}.\mathcal{L}_{\bar{Z}} a = -\mathcal{L}_{\bar{W}} \langle \alpha_F, [\bar{Z}, X]\rangle \qquad \text{along } \iota(\Gamma).$$
Using again that $\omega = d\alpha_F$ and α_F vanishes along $\iota(\Gamma)$, it follows that
$$\mathcal{L}_{\bar{W}}.\mathcal{L}_{\bar{Z}} a = -\omega(\bar{W}, [\bar{Z}, X]) - \mathcal{L}_{[\bar{Z},X]} \langle \alpha_F, \bar{W}\rangle$$
along $\iota(\Gamma)$. The second term of the right side vanishes along $\iota(\Gamma)$ because $\bar{W}$ is an anti-holomorphic vector field. This gives the result. ☐

Since $\iota^* D_X$ and $\mathcal{L}_X^{\delta_\Gamma}$ are first order differential operators which have the same symbol,
$$\iota^* D_X - \mathcal{L}_X^{\delta_\Gamma} = b$$
where b is a function of Γ.

Lemme 4.7. *The symbol of $\Pi_k \big[F^k a\tilde{g}(.,k)\big]$ is $i\hbar b.g_0 + O(\hbar^2)$*

So the symbol of $T_k \Psi_k$ is the sum of (22) and $i\hbar b g_0 + O(\hbar^2)$, which is equal to
$$\hbar(\iota^* f_1 + \tfrac{1}{i}\mathcal{L}_X^{\delta_\Gamma}).g_0 + O(\hbar^2).$$
Theorem 3.4 follows.

Proof. Let us start with a local computation of the function b. Let u be a non-vanishing section of $\delta_\Gamma \to \Gamma$. Then one has
$$b = \frac{\big(\iota^* D_X - \mathcal{L}_X^{\delta_\Gamma}\big).u}{u}$$
Let β be a non-vanishing $(n,0)$-form of M such that $\varphi(u^{\otimes 2}(x)) = \beta(\iota(x))$ if x belongs to Γ. Then $\iota^* D_X$ is defined in such a way that
$$\frac{(\iota^* D_X).u}{u} = \frac{1}{2}\iota^*\Big(\frac{p\mathcal{L}_X \beta}{\beta}\Big) = \frac{1}{2}\frac{\iota^*(p\mathcal{L}_X\beta)}{\iota^*\beta}.$$
where p is the projection from $\Lambda^n M \otimes \mathbb{C}$ onto $\Lambda^{n,0} M$ with kernel $\Lambda^{0,n} M \oplus ... \oplus \Lambda^{n-1,1} M$. On the other hand, since $\varphi_\Gamma(u^{\otimes 2}(x)) = \iota^*\varphi(u^{\otimes 2}(x)) = \iota^*\beta(x)$, one has
$$\frac{\mathcal{L}_X^{\delta_\Gamma}.u}{u} = \frac{1}{2}\frac{\mathcal{L}_X \iota^*\beta}{\iota^*\beta} = \frac{1}{2}\frac{\iota^*\mathcal{L}_X\beta}{\iota^*\beta}.$$

Consequently

$$b = \frac{1}{2} \frac{\iota^*\big(p\mathcal{L}_X\beta - \mathcal{L}_X\beta\big)}{\iota^*\beta}.$$

Now let us choose a frame $(\partial_1, ..., \partial_n)$ of holomorphic vector fields of M such that the vectors $\partial_i + \bar{\partial}_i$ are tangent to $\iota(\Gamma)$. Denote by $(\theta^1, ..., \theta^n)$ the dual frame and set

$$\beta = \theta^1 \wedge ... \wedge \theta^n.$$

Then using that $\mathcal{L}_X\theta^i \equiv -\sum \langle \theta^i, \mathcal{L}_X\bar{\partial}_j\rangle \bar{\theta}^j$ modulo a linear combination of the θ^i, we obtain that $\iota^*(p\mathcal{L}_X\beta - \mathcal{L}_X\beta)$ is equal to

$$\sum \langle \theta^1, \mathcal{L}_X\bar{\partial}_j\rangle \bar{\theta}^j \wedge \theta^2 \wedge ... \wedge \theta^n + \langle \theta^2, \mathcal{L}_X\bar{\partial}_j\rangle \theta^1 \wedge \bar{\theta}^j \wedge \theta^3 \wedge ... \wedge \theta^n + ... \\ + \langle \theta^n, \mathcal{L}_X\bar{\partial}_j\rangle \theta^1 \wedge ... \wedge \theta^{n-1} \wedge \bar{\theta}^j$$

It follows then from $\iota^*\theta^i = \iota^*\bar{\theta}^i$ that

$$b = \frac{1}{2} \sum \langle \theta^i, \mathcal{L}_X\bar{\partial}_i\rangle.$$

To end the proof, assume furthermore that $\frac{1}{i}\omega(\partial_i, \bar{\partial}_j) = \delta_{ij}$. Then it follows from the second part of theorem 4.1 that the symbol of $\Pi_k\big[F^k a\tilde{g}(., k)\big]$ is

$$-\hbar g_0 \frac{1}{2}\ \iota^* \sum \mathcal{L}_{\bar{\partial}_i}\mathcal{L}_{\bar{\partial}_i} a$$

which by lemma 4.6 is equal to

$$-\hbar g_0 \frac{1}{2}\ \iota^* \sum \omega(\bar{\partial}_i, \mathcal{L}_X\bar{\partial}_i)$$

Using again that $\frac{1}{i}\omega(\partial_i, \bar{\partial}_j) = \delta_{ij}$, we obtain

$$\hbar g_0 \frac{i}{2}\ \iota^* \sum \langle \theta^i, \mathcal{L}_X\bar{\partial}_i\rangle.$$

The final result follows. □

Finally, let us prove theorem 3.2. One has

$$\int_M \xi\ \big\|\Psi_k\big\|^2_{L^k\otimes\delta}\ \mu_M = \Big(\frac{k}{2\pi}\Big)^{\frac{n}{2}} \int_M e^{-kc}\ \xi\ \big\|\tilde{g}(., k)\big\|^2_\delta\ \mu_M + O(k^{-\infty})$$

where $c(x) = -2\ln\|F(x)\|_L$. The following is a consequence of lemma 4.2.

Lemme 4.8. *The function c and its first derivatives vanish along $\iota(\Gamma)$. Furthermore, its Hessian at $x \in \iota(\Gamma)$ is definite positive on $JT_x\iota(\Gamma)$ and is given by*

$$X.Y.c = 2\omega(X, JY), \quad X, Y \in JT_x\iota(\Gamma).$$

So integrating along transversal directions to Γ, it follows from the stationary phase lemma that

$$\int_M \xi\ \big\|\Psi_k\big\|^2_{L^k\otimes\delta}\ \mu_M = \int_\Gamma (\iota^*\xi)\ d + O(k^{-1})$$

where d is the density of Γ such that

$$d|_x(X) = \big\|g_0(x)\big\|^2_\delta\ \mu_M|_x(X \wedge Y)\ 2^{-\frac{n}{2}}\Big(\det\big[\omega(Y_i, JY_j)\big]\Big)^{-\frac{1}{2}}.$$

Here (X_i) and (Y_i) are bases of $T_x\Gamma$ and $JT_x\Gamma$ respectively, and $X = X_1 \wedge ... \wedge X_n$, $Y = Y_1 \wedge ... \wedge Y_n$. To deduce theorem 3.2, we have to check that

$$d = m(g_0)$$

This is easily done, by introducing the same vector fields ∂_i and forms θ^i like in the proof of lemma 4.7, setting $X_i = \frac{1}{\sqrt{2}}(\partial_i + \bar{\partial}_i)$, $Y_j = JX_j$ and choosing g_0 such that $\varphi(g_0^{\otimes 2}) = \theta_1 \wedge ... \wedge \theta^n$.

5. Appendix

Let W be an open set of $\mathbb{R}^n \times \mathbb{R}^k \ni (x, y)$. Denote by p the projection from W onto $\mathbb{R}^n$. Let $\varphi(x, y)$ be a C^∞ function on W whose imaginary part is $\geqslant 0$. Let $a(x, y)$ be a C^∞ function with compact support in W. Stationary phase lemma gives the asymptotic expansion of

$$I(a, \varphi)(x, \tau) = \int_{\mathbb{R}^k} e^{i\tau\varphi(x,y)} a(x, y) |dy|$$

when $\tau \to \infty$. First, if the support of a doesn't meet the critical locus

$$C := \{(x, y) \in W;\ d_y\varphi(x, y) = 0 \text{ and } \Im\varphi(x, y) = 0\},$$

then $I(a, \varphi)$ is $O(\tau^{-\infty})$. Introduce the functions

$$\varphi_{i,j}(x, y) = \partial_{y^i}\partial_{y^j}\varphi(x, y), \quad i, j = 1, ..., k.$$

The following theorem is proved in [13] section 7.7.

Theorem 5.1. *Assume that at $(x_0, y_0) \in C$, the matrix $\big(\varphi_{i,j}(x_0, y_0)\big)$ is invertible. Then there exists a neighborhood U of (x_0, y_0) such that if the support of a is a subset of U, one has*

$$I(\varphi, a)(x, \tau) = \left(\tfrac{2\pi}{\tau}\right)^{\frac{k}{2}}\ d(x)\ e^{i\tau\varphi_r(x)}\ b(x, \tau) + O(\tau^{-\infty}) \qquad \text{over } p(U)$$

where d, φ_r et $b(., \tau)$ are C^∞ functions such that

- *d only depends on φ. In particular,*
$$d(x) = \det{}^{-\frac{1}{2}}[\tfrac{1}{i}\varphi_{j,k}(x, y)]_{j,k}, \quad \text{if } (x, y) \in C \cap U.$$
- *φ_r is such that $\varphi(x, y) \equiv \varphi_r(x)$ on U modulo a linear combination with C^∞ coefficient of the functions $\partial_{y^j}\varphi$.*
- *$b(., \tau)$ has an asymptotic expansion for the C^∞ topology of the form*
$$b_0(x) + \tau b_1(x) + \tau^2 b_2(x) + ...$$
Furthermore, $b_0(x) = a(x, y)$ if $(x, y) \in C \cap U$.

In [13] the various terms of the asymptotic expansion are completely determined and not only their restriction at $p(C)$. But in the applications this leads to complicated computations that we prefer to avoid. Let us introduce an additional assumption.

Denote by $E_{(x,y)}$ the complexification of the tangent space to the fiber of p at (x, y). At $(x, y) \in C$ the tangent map to the section $d_y\varphi$ of E^*

$$T_{(x,y)}d_y\varphi : T_{(x,y)}W \otimes \mathbb{C} \to E^*_{(x,y)}$$

is well-defined. Assume that $(\varphi_{i,j})$ is invertible along C, that is the kernel $F_{(x,y)}$ of $T_{(x,y)}d_y\varphi$ satisfies

$$\forall\ (x, y) \in C, \quad F_{(x,y)} \oplus E_{(x,y)} = T_{(x,y)}W \otimes \mathbb{C}. \tag{23}$$

Assume furthermore that

$$C \text{ is a submanifold of } W \text{ and } TC \otimes \mathbb{C} = F \cap \bar{F}. \tag{24}$$

Finally, these two assumptions imply that the restriction $p : C \to \mathbb{R}^n$ is an immersion. We assume it is an embedding.

Observe that when the phase takes real values, the assumption (24) is a consequence of (23). We are interested in the opposite case, typically when the Hessian of the imaginary part of the phase is non-degenerate in the transverse directions to C, for instance with

$$\varphi(x,y) = xy + \tfrac{i}{2}(x^2+y^2).$$

We can also consider intermediary cases, for example $\varphi(x,y) = xy + \frac{i}{2}y^2$.

Under the previous assumptions, when the amplitude a vanishes to order m along C, i.e. when the partial derivatives of a of order $\leqslant m-1$ vanish along C, it follows from the result of [13] that the functions b_i vanish to order $m-2i$ along C. Furthermore one can easily compute b_i modulo a function vanishing to order $m-2i+1$ along C.

To state the result, consider a free family $\partial_1, ..., \partial_l$ of complex tangent vectors to W at $(x,y) \in C$ such that

$$\mathrm{Vect}_{\mathbb{C}}(\partial_1, ..., \partial_l) \oplus (TC \otimes \mathbb{C}) = F_{(x,y)}.$$

If a vanishes to order m along C, we define the polynomial

$$[a](Z,Y) = \sum_{|\alpha|+|\beta|=m} \frac{1}{\alpha!\beta!}\Big(\partial_1^{\alpha(1)}...\partial_l^{\alpha(l)}\partial_{y^1}^{\beta(1)}...\partial_{y^k}^{\beta(l)}a(x,y)\Big)\ Z^\alpha Y^\beta$$

at $(x,y) \in C$. Similarly, if $b(x)$ vanishes to order l along $p(C)$, we set

$$[b](Z) = \sum_{|\alpha|=l} \frac{1}{\alpha!\beta!}\Big((p_*\partial_1)^{\alpha(1)}...(p_*\partial_l)^{\alpha(l)}b(x)\Big)\ Z^\alpha.$$

at $x \in p(C)$.

Theorem 5.2. *Under the assumptions* (23) *and* (24)*, if a vanishes to order m along C, then for every $i \leqslant \frac{m}{2}$ the function b_i vanishes to order $m-2i$ along $p(C)$. Furthermore*

$$[b_i](Z) = \frac{1}{i!}\Delta^i A_{2i}(Z,Y)$$

at $(x,y) \in C$, where

- *$[a](Z,Y) = \sum_{l=0}^m A_l(Z,Y)$ and A_l is homogeneous of degree l in Y and of degree $m-l$ in Z.*
- *$\Delta = \frac{i}{2}\sum_{j,k}\varphi^{j,k}(x,y)\partial_{Y^j}\partial_{Y^k}$, with $(\varphi^{j,k}(x,y))$ the inverse of $(\varphi_{j,k}(x,y))$.*

More intrinsically, denote by $\mathcal{I}^m(C) \subset C^\infty(W)$ the ideal of functions vanishing to order l along C. Then $\mathcal{I}^m(C)/\mathcal{I}^{m-1}(C)$ is isomorphic to the space of sections of the m-th symmetric power of the complex conormal bundle $\mathcal{N}^*(C)$. By (23), we have an isomorphism of vector bundle over C,

$$\mathcal{N}^*(C) \simeq \mathcal{N}^*(p(C)) \oplus E^*,$$

which associates γ and $\alpha \oplus \beta$ if

$$\langle \gamma, U+V\rangle = \langle \alpha, p_*U\rangle + \langle \beta, V\rangle, \quad U \in F \text{ and } V \in E$$

Consequently,

$$\mathrm{Sym}_m(\mathcal{N}^*(C)) = \bigoplus_{l=0}^m \mathrm{Sym}_{m-l}\big(\mathcal{N}^*(p(C))\big) \otimes \mathrm{Sym}_l(E^*)$$

$\Delta = \frac{i}{2}\sum_{j,k}\varphi^{j,k}\partial_{y^j}\partial_{y^k}$ defines a section of $\mathrm{Sym}_2(E)$, so Δ^i acts as an operator

$$\mathrm{Sym}_{2i}(E^*) \to \mathbb{C}.$$

In theorem 5.2, we consider $[a]$ as a section of $\mathrm{Sym}_m(\mathcal{N}^*(C))$ and $[b_i]$ as a section of $\mathrm{Sym}_{m-2i}(\mathcal{N}^*(p(C)))$. Then we have

$$[b_i] = \frac{1}{i!}(\mathrm{Id} \otimes \Delta^i)A_{2i}$$

where

$$[a] = \sum_{l=0}^{m} A_l, \quad A_l \in C^\infty(C, \mathrm{Sym}_{m-l}\big(\mathcal{N}^*(p(C))\big) \otimes \mathrm{Sym}_l(E^*)).$$

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