SEMI-CLASSICAL PROPERTIES OF GEOMETRIC QUANTIZATION WITH METAPLECTIC CORRECTION

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Abstract. The geometric quantization of a symplectic manifold endowed with a prequantum bundle and a metaplectic structure is defined by means of an integrable complex structure. We prove that its semi-classical limit does not depend on the choice of the complex structure. We show this in two ways. First, by introducing unitary identifications between the quantum spaces associated to the various complex polarizations and second, by defining an asymptotically flat connection in the bundle of quantum spaces over the space of complex structures. Furthermore Berezin-Toeplitz operators are intertwined by these identifications and have principal and subprincipal symbols defined independently of the complex structure. The relation with Schrödinger equation and the group of prequantum bundle automorphisms is considered as well.

1. INTRODUCTION

Geometric Quantization of Kostant [14] and Souriau [17] is a procedure which associates a quantum space to a symplectic manifold endowed with a prequantum bundle and a polarization. Since its introduction, there has been some attempt to find natural *identifications* between quantum spaces associated to different polarizations (cf. [4], [16]). In the case of symplectic compact manifolds with complex polarizations, Ginzburg and Montgomery observed in [10] that a natural identification does not exist for a broad class of manifolds. Recently Foth and Uribe [9] obtained semi-classical results in the same direction.

We prove that there exists a natural **semi-classical** identification when the definition of the quantum spaces is altered with the metaplectic correction. This result is a consequence of our study undertaken in [8] of the symbolic calculus of Toeplitz operators and Lagrangian sections that we extend in this paper to Fourier integral operators. Before we state our results, let us discuss quantization without metaplectic correction.

1.1. Ordinary quantization.

Let (M,ω) be a symplectic compact manifold with a prequantization bundle $L \rightarrow M$, i.e. a Hermitian line bundle with a connection of curvature $\frac{1}{i}\omega$. Denote by $\mathcal{J}_{\text{int}}$ the space of integrable complex structure of M compatible with ω and positive. To any $j \in \mathcal{J}_{\text{int}}$ is associated a sequence of quantum spaces

$$\mathcal{Q}_k(j) := \{j\text{-holomorphic sections of } L^k\}, \qquad k = 1, 2, ...$$

Here the holomorphic structure of L is the one compatible with the connection. The semi-classical limit corresponds to $k \rightarrow \infty$. When k is sufficiently large, the Kodaira vanishing theorem and Riemann-Roch-Hirzebruch theorem imply that the dimension of $\mathcal{Q}_k(j)$ is given by a Riemann-Roch number, which only depends on the symplectic structure of M and k. Assume that we can choose such an integer k

1991 *Mathematics Subject Classification.* 53D22, 53D50, 53D55, 81S30, 47L80.

Key words and phrases. Geometric Quantization, Toeplitz operator, Fourier integral operator, Half-form bundle.

independently of the complex structure j. Then for any $j_a, j_b \in \mathcal{J}_{\rm int}$ we can identify $\mathcal{Q}_k(j_a)$ with $\mathcal{Q}_k(j_b)$ by means of a unitary map

$$U_k(j_a, j_b) : \mathcal{Q}_k(j_a) \rightarrow \mathcal{Q}_k(j_b).$$

These identifications are mutually compatible if they satisfy:

- (functoriality) $U_k(j_b, j_c) \circ U_k(j_a, j_b) = U_k(j_a, j_c)$, for any $j_a, j_b, j_c \in \mathcal{J}_{\rm int}$.

Moreover if these maps are canonical in the sense that they only depend on the complex, symplectic and prequantum structures, they should satisfy:

- (naturality) for any prequantization bundle automorphism Φ of L^k and complex structures $j_a, j_b \in \mathcal{J}_{\rm int}$, the diagram

$$\begin{array}{ccc} \mathcal{Q}_k(j_a) & \xrightarrow{U_k(j_a,j_b)} & \mathcal{Q}_k(j_b) \\ {\scriptstyle \Phi^*}\big\downarrow & & \big\downarrow{\scriptstyle \Phi^*} \\ \mathcal{Q}_k(\Phi^* j_a) & \xrightarrow{U_k(\Phi^* j_a, \Phi^* j_b)} & \mathcal{Q}_k(\Phi^* j_b) \end{array}$$

  commutes.

Here the vertical maps are pull-back by Φ, sending a j-holomorphic section into a section holomorphic with respect to $\Phi^* j := \phi^* j$, where ϕ is the symplectomorphism of M covered by Φ. Sometimes one only requires an identification between the projectivised quantum spaces.

It is important to observe that if there exists such a collection $\{U_k(j_a, j_b), (j_a, j_b) \in \mathcal{J}_{\rm int}^2\}$ which is both functorial and natural, then for any complex structure $j \in \mathcal{J}_{\rm int}$, the quantum space $\mathcal{Q}_k(j)$ becomes a representation of the group $\mathcal{G}$ of prequantization bundle automorphism of L^k. Indeed let us set

$$V_k(\Phi) := U_k(\Phi^* j, j) \circ \Phi^* : \mathcal{Q}_k(j) \rightarrow \mathcal{Q}_k(j), \qquad \Phi \in \mathcal{G}. \tag{1}$$

Then for any prequantization bundle automorphisms Φ_1 and Φ_2, we have

$$\begin{aligned} V_k(\Phi_1) \circ V_k(\Phi_2) =& U_k(\Phi_1^* j, j) \circ \Phi_1^* \circ U_k(\Phi_2^* j, j) \circ \Phi_2^* \\ =& U_k(\Phi_1^* j, j) \circ U_k(\Phi_1^*\Phi_2^* j, \Phi_1^* j) \circ \Phi_1^* \circ \Phi_2^* \quad \text{by naturality,} \\ =& U_k(\Phi_1^*\Phi_2^* j, j) \circ (\Phi_2 \circ \Phi_1)^* \quad \text{by functoriality,} \\ =& V_k(\Phi_2 \circ \Phi_1). \end{aligned}$$

Considering the associated infinitesimal representation, Ginzburg and Montgomery proved in [10] that the existence of such a representation contradicts "no go" theorems in many cases. Indeed one can view the Lie algebra of $\mathcal{G}$ as $C^\infty(M, \mathbb{R})$, the Lie bracket being the Poisson bracket. Then assuming that the maps $U_k(j_a, j_b)$ depend smoothly on j_a and j_b, we obtain a Lie algebra representation

$$C^\infty(M, \mathbb{R}) \rightarrow \operatorname{End}(\mathcal{Q}_k(j)).$$

By [10], since M is compact and $\mathcal{Q}_k(j)$ is finite dimensional, the associated projective representation is trivial. But for a broad class of manifolds M, $\mathcal{G}$ contains a finite dimensional subgroup which preserves a complex structure j and whose induced representation on $\mathcal{Q}_k(j)$ is not projectively trivial. The same arguments contradict also the existence of an identification between the projectivised quantum spaces.

In spite of this result, there is a natural identification of a particular interest which has been introduced for the quantization of the moduli spaces of flat connections (cf. [12] and [1]). To define it consider the quantum spaces $\mathcal{Q}_k(j)$ as the fibers of a bundle $\mathcal{Q}_k \rightarrow \mathcal{J}_{\rm int}$. Then introducing a functorial and natural family $(U_k(j_a, j_b))$

which depends smoothly on j_a and j_b amounts to endowing this bundle with a flat $\mathcal{G}$-invariant connection. Now consider $\mathcal{Q}_k$ as a subbundle of

$$\mathcal{P}_k := C^\infty(M, L^k) \times \mathcal{J}_{\rm int} \to \mathcal{J}_{\rm int}$$

Since $\mathcal{P}_k$ is trivial, it has a natural flat connection and $\mathcal{Q}_k$ is equipped with the projected connection. Because of the previous result, the curvature R_k of $\mathcal{Q}_k$ can not vanish in general. On the other hand by the theory of Boutet de Monvel and Guillemin [5], the Toeplitz operators provide an asymptotic representation of the Poisson algebra $C^\infty(M)$ as operators on $\mathrm{End}(\mathcal{Q}_k(j))$ when $k \to \infty$. So it is possible that the curvature R_k is asymptotically flat (cf. end of section 8.4 for a quantitative argument). Foth and Uribe compute the asymptotics of R_k in [9] and prove the following: for any $j \in \mathcal{J}_{\rm int}$ and tangent vectors $\eta, \mu \in T_j\mathcal{J}_{\rm int}$, there exists a function $f(\eta,\mu) \in C^\infty(M)$ such that

$$R_k(\eta,\mu) = \Pi_k(j) f(\eta,\mu) + O(k^{-1}) : \mathcal{Q}_k(j) \to \mathcal{Q}_k(j)$$

where $\Pi_k(j)$ is the orthogonal projector of $C^\infty(M, L^k)$ onto $\mathcal{Q}^{\rm m}_k(j)$. Furthermore, they give a simple formula for the multiplicator $f(\eta,\mu)$, which shows that it does not vanish for a generic choice of (η,μ). Consequently the curvature is not asymptotically flat. Neither is it asymptotically projectively flat.

1.2. **Main results.** Let us turn to geometric quantization with metaplectic correction. The metaplectic structures were introduced by Kostant in [15] as metaplectic principal bundles lifting the symplectic frame bundle (cf. also [11] and [2]). Here we use the half-form bundle approach (cf. [18]) more convenient for our purpose.

Given a complex structure $j \in \mathcal{J}_{\rm int}$, a half-form bundle (δ, φ) of (M, j) is a line bundle $\delta \to M$ together with an isomorphism of line bundles

$$\varphi : \delta^2 \to \Lambda^{n,0}_j T^*M$$

covering the identity of M. (M, j) admits a half-form bundle if and only if the second Stiefel-Whitney class of M vanishes. From now on, we assume this condition is satisfied and we set

$$\mathcal{Q}^{\rm m}_k(j,\delta,\varphi) := \{j\text{-holomorphic sections of } \delta \otimes L^k\}, \quad k = 1, 2, ...$$

where the superscript "m" stands for metaplectic. Here the holomorphic structure of δ is such that φ is an isomorphism of holomorphic bundles.

There is an obvious notion of isomorphism between two half-form bundles associated to the same complex structure and these isomorphisms give rise to isomorphisms between the associated quantum spaces. Our aim is to extend this to the whole collection $\mathcal{D}$ of triples (j, δ, φ), where j ranges through $\mathcal{J}_{\rm int}$.

In section 5.1, we define a collection $\mathcal{M}$ of morphisms, which makes $(\mathcal{D}, \mathcal{M})$ a category such that every morphism is an isomorphism. Important facts are that the automorphism group of any $a \in \mathcal{D}$ is $\mathbb{Z}_2$ and the isomorphism classes are in one to one correspondence with the elements of $H^1(M, \mathbb{Z}_2)$. Furthermore isomorphism classes correspond to equivalence classes of metaplectic structures of M.

Theorem 1.1. *There exists a family $((U^{\rm m}_k(\Psi))_k;\ \Psi \in \mathcal{M})$ such that for any morphism $\Psi : a \to b$, the sequence $(U^{\rm m}_k(\Psi))_k$ consists of operators*

$$U^{\rm m}_k(\Psi) : \mathcal{Q}^{\rm m}_k(a) \to \mathcal{Q}^{\rm m}_k(b)$$

that are unitary if k is sufficiently large. Furthermore, for any composable morphisms Ψ and $\Psi' \in \mathcal{M}$, we have

$$U^{\rm m}_k(\Psi') \circ U^{\rm m}_k(\Psi) = U^{\rm m}_k(\Psi' \circ \Psi) + O(k^{-1})$$

where the estimate $O(k^{-1})$ is for the uniform norm of operators.

One of the original motivations to introduce the metaplectic correction was to define some natural pairings between the quantum spaces associated to different polarizations, which are called now Blattner-Kostant-Sternberg pairings. Our construction of the operators $U_k(\Psi)$ is rather different. These are Fourier integral operators with a prescribed principal symbol and the functoriality property is a consequence of the symbolic calculus.

We interpret this theorem as a semi-classical functoriality of quantization with half-form bundle. Moreover, the family $((U_k^{\mathrm{m}}(\Psi))_k;\ \Psi \in \mathcal{M})$ is natural with respect to a suitable action of the group $\mathcal{G}$ of prequantization bundle automorphisms of L on $(\mathcal{D}, \mathcal{M})$. We can therefore adapt the previous construction (1) and we obtain for any $a \in \mathcal{D}$ an asymptotic representation on $\mathcal{Q}_k^{\mathrm{m}}(a)$ of a central extension by $\mathbb{Z}_2$ of the identity component of $\mathcal{G}$. This is in some sense a generalization of the standard metaplectic representation.

We will also prove that the operators $U_k^{\mathrm{m}}(\Psi)$ can be defined as parallel transport in an appropriate bundle. Let us consider a smooth family $((\delta_j, \varphi_j),\ j \in \mathcal{J}_{\mathrm{int}})$ of isomorphic half-form bundles. Let $\mathcal{Q}_k^{\mathrm{m}} \to \mathcal{J}_{\mathrm{int}}$ be the quantum space bundle, whose fiber over j is the space of j-holomorphic sections of $L^k \otimes \delta_j$.

Theorem 1.2. *For any positive integer k, the bundle $(\mathcal{Q}_k^{\mathrm{m}} \to \mathcal{J}_{\mathrm{int}})$ has a canonical connection $\nabla^{\mathcal{Q}_k^{\mathrm{m}}}$. The sequence $(\nabla^{\mathcal{Q}_k^{\mathrm{m}}},\ k \in \mathbb{N}^*)$ satisfies*

- *for any $j \in \mathcal{J}_{\mathrm{int}}$ and tangent vectors $\eta, \mu \in T_j\mathcal{J}_{\mathrm{int}}$, the uniform norm of the curvature $R^{\mathcal{Q}_k^{\mathrm{m}}}(\eta, \mu)$ is $O(k^{-1})$,*
- *the parallel transport in $\mathcal{Q}_k^{\mathrm{m}}$ along a curve γ with endpoints j_a and j_b is equal to $U_k^{\mathrm{m}}(\Psi)$ modulo $O(k^{-1})$. Here $U_k^{\mathrm{m}}(\Psi)$ is the sequence of theorem 1.1 and Ψ is the half-form bundle morphism*
$$(j_a, \delta_{j_a}, \varphi_{j_a}) \to (j_b, \delta_{j_b}, \varphi_{j_b})$$
obtained by extending continuously the identity of $(j_a, \delta_{j_a}, \varphi_{j_a})$ in morphisms $(j_a, \delta_{j_a}, \varphi_{j_a}) \to (\gamma(t), \delta_{\gamma(t)}, \varphi_{\gamma(t)})$.

The connection $\nabla^{\mathcal{Q}_k^{\mathrm{m}}}$ is induced by a connection on the prequantum space bundle. However the latter bundle is not trivial contrary to the case without metaplectic correction.

The paper is organized as follows. Section 2 is devoted to preliminary material. Section 3 contains our results about symbolic calculus for Fourier integral operators. These results are reformulated in section 5.3 with the half-form bundle formalism. In section 5, we deduce theorem 1.1 and related facts on the representation of the prequantization bundle automorphisms and the Schrödinger equation. The study of the quantum space bundle and its connection is in sections 7. Section 8 is devoted to the action of the prequantization bundle automorphism group on the quantum space bundle.

2. Preliminaries

Let (E, ω) be a symplectic real vector space of dimension $2n$. Let $\mathcal{J}(E, \omega)$ be the space of complex structures j of E compatible with ω and positive. Given $j \in \mathcal{J}(M, \omega)$, we denote by $\Lambda_j^{n,0}E^*$ the line of complex linear forms of E of type $(n, 0)$ for the complex structure j.

Definition 2.1. *Given j_a and $j_b \in \mathcal{J}(E, \omega)$, let Ψ_{j_a, j_b} be the linear map from $\Lambda_{j_a}^{n,0}E^*$ to $\Lambda_{j_b}^{n,0}E^*$ such that*
$$\Psi_{j_a, j_b}(\alpha) \wedge \bar{\beta} = \alpha \wedge \bar{\beta}, \quad \forall\ \alpha, \beta \in \Lambda_{j_a}^{n,0}E^*.$$

Let us give some elementary properties of these maps. First, Ψ_{j_a,j_b} is well-defined and invertible because the sesquilinear pairing

$$\Lambda^{n,0}_{j_b}E^* \times \Lambda^{n,0}_{j_a}E^* \to \mathbb{C}, \quad (\alpha_b, \alpha_a) \to \alpha_b \wedge \bar{\alpha}_a / \omega^{\wedge n}$$

is non-degenerate, j_a and j_b being positive. Whenever $j_a = j_b$, Ψ_{j_a,j_b} is the identity. With the usual scalar product on $\Lambda^{n,0}_j E^*$ defined by means of ω and j, the adjoint of Ψ_{j_a,j_b} is Ψ_{j_b,j_a}. This is easily checked using that the scalar product of $\alpha, \beta \in \Lambda^{n,0}_j E^*$ is given by

$$i^{n(2-n)} \alpha \wedge \bar{\beta} / \omega^n.$$

Last definition that we need is the following.

Definition 2.2. *Given j_a, j_b, j_c in $\mathcal{J}(E,\omega)$, let $\zeta(j_a,j_b,j_c)$ be the complex number such that*

$$\Psi_{j_a,j_c} = \zeta(j_a,j_b,j_c)\, \Psi_{j_b,j_c} \circ \Psi_{j_a,j_b},$$

As we will see in the next section, the symbols of Fourier integral operators behave in part as square roots of the Ψ_{j_a,j_b}. This will appear first via the continuous square root $\zeta^{\frac{1}{2}}$ of the complex function $\zeta : \mathcal{J}^3(E,\omega) \to \mathbb{C}^*$ determined by $\zeta^{\frac{1}{2}}(j,j,j) = 1$, for any $j \in \mathcal{J}(M,\omega)$. $\zeta^{\frac{1}{2}}$ is well-defined and analytic because $\mathcal{J}^3(E,\omega)$ is contractible and ζ is an analytic function (cf. (5)).

It follows from the associativity of the composition that ζ is a cocycle

$$\zeta(j_b,j_c,j_d).\zeta^{-1}(j_a,j_c,j_d).\zeta(j_a,j_b,j_d).\zeta^{-1}(j_a,j_b,j_c) = 1 \tag{2}$$

Furthermore $\Psi_{j,j} = \mathrm{Id}$ implies

$$\zeta(j_a,j_b,j_b) = \zeta(j_a,j_a,j_b) = 1. \tag{3}$$

The function $\zeta^{\frac{1}{2}}$ satisfies the same equations.

To prepare further proofs, we compute the function ζ in the following parametrization of $\mathcal{J}(E,\omega)$. Let us choose a fixed complex structure $j_0 \in \mathcal{J}(E,\omega)$. Then for any $j \in \mathcal{J}(E,\omega)$, the space of linear forms of type $(1,0)$ with respect to j, viewed as a subspace of

$$E^* \otimes \mathbb{C} = \Lambda^{1,0}_{j_0}E^* \oplus \Lambda^{0,1}_{j_0}E^*,$$

is the graph of a complex linear map

$$\mu : \Lambda^{1,0}_{j_0}E^* \to \Lambda^{0,1}_{j_0}E^*. \tag{4}$$

The condition that j is compatible with ω is that

$$\omega(\mu^t X, Y) + \omega(X, \mu^t Y) = 0, \qquad \forall\ X, Y \in E^{0,1}_{j_0}$$

where $\mu^t : E^{0,1}_{j_0} \to E^{1,0}_{j_0}$ is the transposed of μ. And the positivity of j translates into the positivity of the Hermitian map:

$$\mathrm{Id} - \mu^t \bar{\mu}^t : E^{1,0}_{j_0} \to E^{1,0}_{j_0}$$

This defines a one-to-one correspondence between $\mathcal{J}(M,\omega)$ and an open set of a subspace of $\mathrm{Hom}(\Lambda^{1,0}_{j_0}E^*, \Lambda^{0,1}_{j_0}E^*)$.

For any j, let us identify the $(n,0)$-forms with respect to j with the $(n,0)$-forms with respect to j_0 by the map

$$\Lambda^{n,0}_{j_0}E^* \to \Lambda^{n,0}_j E^*, \qquad \alpha^1 \wedge \ldots \wedge \alpha^n \to (\alpha^1 + \mu(\alpha^1)) \wedge \ldots \wedge (\alpha^n + \mu(\alpha^n)).$$

Then straightforward computations prove the following lemma.

Lemme 2.3. *With the previous identifications, Ψ_{j_a,j_b} regarded as a map from $\Lambda^{n,0}_{j_0} E^*$ to itself is the multiplication by*

$$\det \begin{pmatrix} \mathrm{Id} & \bar\mu_a \\ \mu_a & \mathrm{Id} \end{pmatrix} . \det{}^{-1} \begin{pmatrix} \mathrm{Id} & \bar\mu_a \\ \mu_b & \mathrm{Id} \end{pmatrix}$$

where the matrices represent maps from $\Lambda^{1,0}_{j_0} E^* \oplus \Lambda^{0,1}_{j_0} E^*$ to itself. Consequently,

$$\zeta(j_a, j_b, j_c) = \frac{\det \begin{pmatrix} \mathrm{Id} & \bar\mu_a \\ \mu_b & \mathrm{Id} \end{pmatrix} . \det \begin{pmatrix} \mathrm{Id} & \bar\mu_b \\ \mu_c & \mathrm{Id} \end{pmatrix}}{\det \begin{pmatrix} \mathrm{Id} & \bar\mu_a \\ \mu_c & \mathrm{Id} \end{pmatrix} . \det \begin{pmatrix} \mathrm{Id} & \bar\mu_b \\ \mu_b & \mathrm{Id} \end{pmatrix}} . \tag{5}$$

3. Fourier integral operator

Let (M, ω) be a symplectic compact connected manifold with a prequantization bundle (L, ∇), i.e. L is a Hermitian line bundle and ∇ a connection of curvature $\frac{1}{i}\omega$. The quantizations of (M, ω) we will consider depend on two additional datas: a complex structure j of M compatible with ω and positive, and a holomorphic Hermitian line bundle K over the complex manifold (M, j).

Let us denote by $\mathcal{K}$ the collection of such pairs (j, K). To any $a = (j_a, K_a) \in \mathcal{K}$, we associate the sequence of Hilbert spaces

$$\mathcal{H}_k(a) := \{ \text{ holomorphic sections of } L^k \otimes K_a \}, \qquad k = 1, 2, ...$$

where the holomorphic structure of L is the one compatible with the connection ∇ such that $L \to M$ is holomorphic with respect to j_a. The scalar product is defined by means of the Hermitian structure of $L^k \otimes K_a$ and the Liouville measure of M.

For any $a, b \in \mathcal{K}$, let us introduce the space $\mathcal{F}(a, b)$ of Fourier integral operators from $\mathcal{H}(a)$ to $\mathcal{H}(b)$. Their definition is a slight generalization of the one in [7] because of the fiber bundles K_a and K_b. Consider a sequence (S_k) such that for every k, S_k is an operator $\mathcal{H}_k(a) \to \mathcal{H}_k(b)$. The scalar product of $\mathcal{H}_k(a)$ gives us an isomorphism

$$\mathrm{Hom}(\mathcal{H}_k(a), \mathcal{H}_k(b)) \simeq \mathcal{H}_k(b) \otimes \overline{\mathcal{H}}_k(a).$$

The latter space can be regarded as the space of holomorphic sections of

$$(L^k \otimes K_b) \boxtimes (\bar L^k \otimes \overline{K}_a) \to M^2,$$

where M^2 is endowed with the complex structure $(j_b, -j_a)$. The section $S_k(x, y)$ associated in this way to S_k is its Schwartz kernel.

We say that (S_k) is a Fourier integral operator of $\mathcal{F}(a, b)$ if

$$S_k(x, y) = \Big(\frac{k}{2\pi}\Big)^n E^k(x, y) f(x, y, k) + O(k^{-\infty}) \tag{6}$$

where

- E is a section of $L \boxtimes \bar L \to M^2$ such that $\|E(x, y)\| < 1$ if $x \neq y$,
$$E(x, x) = u \otimes \bar u, \quad \forall u \in L_x \text{ such that } \|u\| = 1,$$
and $\bar\partial E \equiv 0$ modulo a section vanishing to any order along the diagonal.
- $f(., k)$ is a sequence of sections of $K_b \boxtimes \bar K_a \to M^2$ which admits an asymptotic expansion in the C^∞ topology of the form
$$f(., k) = f_0 + k^{-1} f_1 + k^{-2} f_2 + ...$$
whose coefficients satisfy $\bar\partial f_i \equiv 0$ modulo a section vanishing to any order along the diagonal.

Let us define the principal symbol of (S_k) to be the map $x \to f_0(x,x)$. Using the Hermitian structure of K_a, we regard it as a section of $\operatorname{Hom}(K_a, K_b) \to M$. The principal symbol map

$$\sigma : \mathcal{F}(a,b) \to C^\infty(M, \operatorname{Hom}(K_a, K_b))$$

satisfies the expected property.

Theorem 3.1. *The following sequence is exact*

$$0 \to \mathcal{F}(a,b) \cap O(k^{-1}) \to \mathcal{F}(a,b) \xrightarrow{\sigma} C^\infty(M, \operatorname{Hom}(K_a, K_b)) \to 0,$$

where the $O(k^{-1})$ is for the uniform norm of operators.

The composition of these operators is also as expected, with some complications regarding the product of the symbols. Given three complex structures j_a, j_b and j_c of M, we denote by $\zeta^{\frac{1}{2}}(j_a, j_b, j_c)$ the function of $C^\infty(M)$ whose values at x is the complex number $\zeta^{\frac{1}{2}}(j_a(x), j_b(x), j_c(x))$ defined in section 2 with $E = T_xM$.

Theorem 3.2. *Let a, b and c belong to $\mathcal{K}$. If $T \in \mathcal{F}(a,b)$ and $S \in \mathcal{F}(b,c)$, then $S \circ T$ is a Fourier integral operator of $\mathcal{F}(a,c)$. Furthermore,*

$$\sigma(S \circ T) = \zeta^{\frac{1}{2}}(j_a, j_b, j_c)\, \sigma(S) \circ \sigma(T).$$

The two previous theorems were essentially proved in the chapter 4.1 of [7] except the formula for the composition of the symbols, which will be proved in chapter 4. Since the composition of operators is associative, the same holds for the symbol. Observe that this can be directly checked with the cocycle relation (2).

$\mathcal{F}(a,a)$ is the space $\mathcal{T}(a)$ of Toeplitz operators of $\mathcal{H}(a)$. Equivalently, a Toeplitz operators is any sequence $(T_k : \mathcal{H}_k(a) \to \mathcal{H}_k(a))$ of operators of the form

$$T_k = \Pi_k g(.,k) + R_k,$$

where Π_k is the orthogonal projector of $L^2(M, L^k \otimes K_a)$ onto $\mathcal{H}_k(a)$, $g(.,k)$ is a sequence of $C^\infty(M)$ with an asymptotic expansion $g_0 + k^{-1}g_1 + \ldots$ in the C^∞ topology and R_k is $O(k^{-\infty})$. As a result the principal symbol $\sigma(T_k)$ is the function g_0. Let us define the *normalized symbol* of (T_k) to be the formal series

$$g(.,\hbar) + \frac{\hbar}{2}\Delta g(.,\hbar),$$

where $g(.,\hbar) = g_0 + \hbar g_1 + \ldots$ and Δ is the holomorphic Laplacian. We are actually only interested in the two first terms of this series, which are the principal symbol and the *subprincipal symbol* $g_1 + \frac{1}{2}\Delta g_0$. As a consequence of the works of Boutet de Monvel and Guillemin, the product of the normalized symbol is a star-product ([5]).

Theorem 3.3. *Let S be a Fourier integral operator of $\mathcal{F}(a,b)$ and $T_a \in \mathcal{T}(a)$, $T_b \in \mathcal{T}(b)$ be two Toeplitz operators with the same principal symbol f. Then*

$$T_b \circ S - S \circ T_a = \tfrac{1}{k} R$$

with $R \in \mathcal{F}(a,b)$. Furthermore the principal symbol of R is

$$\sigma(R) = \big(f_{1,b} - f_{1,a} + \tfrac{1}{2}\langle \alpha_{j_a,j_b}, X_f\rangle\big)\sigma(S) + \tfrac{1}{i}\nabla_{X_f}^{\operatorname{Hom}(K_a,K_b)}\sigma(S)$$

where

- *$f_{1,a}$, $f_{1,b}$ are the subprincipal symbols of T_a and T_b respectively,*
- *X_f is the Hamiltonian vector field of f,*

- α_{j_a,j_b} *is the one-form of* M *such that*
$$\nabla \Psi_{j_a,j_b} = \tfrac{1}{i}\alpha_{j_a,j_b} \otimes \Psi_{j_a,j_b},$$
where Ψ_{j_a,j_b} *is the section of*
$$\operatorname{Hom}(\Lambda^{n,0}_{j_a} T^*M, \Lambda^{n,0}_{j_b} T^*M) \to M$$
whose value at x *is the endomorphism* $\Psi_{j_a(x),j_b(x)}$ *defined in 2.1 with* $E = T_xM$*. The connection* ∇ *is induced by the Chern connections of* $\Lambda^{n,0}_{j_a} T^*M$ *and* $\Lambda^{n,0}_{j_b} T^*M$*.*

By theorem 3.2, $T_bS - ST_a$ is Fourier integral operator of $\mathcal{F}(a,b)$. Since
$$\zeta^{\frac{1}{2}}(j_a, j_a, j_b) = \zeta^{\frac{1}{2}}(j_a, j_b, j_b) = 1,$$
its principal symbol vanishes and consequently $R \in \mathcal{F}(a,b)$. So the proof of the theorem consists in computing the principal symbol of R. This is postponed to chapter 6. Let us deduce some interesting consequences. Applying the theorem with two Toeplitz operators S and T of $\mathcal{T}(a)$, we recover that the principal symbol of $\frac{k}{i}[S,T]$ is the Poisson bracket of the principal symbols of S and T. Actually, we can also compute the subprincipal symbol of $\frac{k}{i}[S,T]$ with the previous theorem.

First, the operators of $\mathcal{F}(a,b)$ may be used to identify $\mathcal{H}(a)$ with $\mathcal{H}(b)$ in a semi-classical sense. More precisely, we consider the space $U\mathcal{F}(a,b)$ consisting of Fourier integral operators of $\mathcal{F}(a,b)$ which are unitary in the sense that
$$S_k S_k^* = \mathrm{Id}_{\mathcal{H}_k(a)} \text{ and } S_k^* S_k = \mathrm{Id}_{\mathcal{H}_k(b)}$$
when k is sufficiently large. By some standard argument that we briefly recall now, $U\mathcal{F}(a,b)$ is not empty if and only if K_a and K_b are isomorphic as line bundles. First it follows directly from the definition of a Fourier integral operator that the adjoint of an operator $S \in \mathcal{F}(a,b)$ belongs to $\mathcal{F}(b,a)$ and its principal symbol is the adjoint of the principal symbol of S. So if $S \in U\mathcal{F}(a,b)$, theorem 3.2 implies that the principal symbol of S is a line bundle isomorphism $K_a \to K_b$. Conversely if K_a and K_b are isomorphic, there exists an elliptic $R \in \mathcal{F}(a,b)$, meaning that its principal symbol doesn't vanish anywhere. Then R^*R is an elliptic Toeplitz operator by theorem 3.2. So $(R^*R)^{-\frac{1}{2}}$ is a Toeplitz operator (cf. as instance [6]). Finally $R(R^*R)^{-\frac{1}{2}}$ belongs to $U\mathcal{F}(a,b)$.

Now if $S \in U\mathcal{F}(a,b)$ and T_a is a Toeplitz operator of $\mathcal{H}(a)$, then by theorem 3.2
$$T_b = ST_aS^*$$
is a Toeplitz operator of $\mathcal{H}(b)$ with the same principal symbol as T_a. Applying theorem 3.3, we compute its subprincipal symbol in terms of the principal and subprincipal symbols of T_a:
$$f_{1,b} = f_{1,a} + \langle \alpha_S - \tfrac{1}{2}\alpha_{j_a,j_b}, X_f \rangle \tag{7}$$
where α_S is such that
$$\nabla^{\operatorname{Hom}(K_a,K_b)} \sigma(S) = \tfrac{1}{i}\alpha_S \otimes \sigma(S).$$
A consequence of this formula is the following result.

Theorem 3.4. *The composition law* $*_a$ *of the normalized symbols of the Toeplitz operators of* $\mathcal{H}(a)$ *satisfies:*
$$f *_a g = fg + \tfrac{\hbar}{2i}\{f,g\} + O(\hbar^2)$$
and
$$\tfrac{i}{\hbar}(f *_a g - g *_a f) = \{f,g\} - \hbar \langle \omega_{j_a} - \tfrac{1}{2}\omega_{K_a}, X_f \wedge X_g \rangle + O(\hbar^2)$$
where $\frac{1}{i}\omega_{j_a}$ *and* $\frac{1}{i}\omega_{K_a}$ *are the Chern curvatures of* $\Lambda^{n,0}_{j_a} T^*M$ *and* K_a *respectively.*

Indeed, if K_a and K_b are isomorphic and $*_a$ satisfies the result, the same holds for $*_b$ because of (7) and the relations

$$\omega_{j_b} = d\alpha_{j_a,j_b} + \omega_{j_a}, \quad \omega_{K_b} = d\alpha_S + \omega_{K_a}.$$

Furthermore we can explicitly compute $*_a$, in the case where M is $\mathbb{C}^n$ with $\mathcal{H}(a)$ the Bargmann space, and the result is satisfied. Of course, this is not sufficient to conclude. But it appears in the proofs of the previous theorems that all the results about the symbolic calculus are completely local and we can really deduce in this way theorem 3.4.

4. Proof of theorem 3.2

The proof relies on the complex stationary phase lemma. We only sketch the first part, because the details appeared in [7], with some typos however. The Schwartz kernel of an operator $S \in \mathcal{F}(a,b)$ is by definition of the form

$$\left(\frac{k}{2\pi}\right)^n E^k_{a,b}(x,y) f(x,y,k) + O(k^{-\infty})$$

Let us write on a neighborhood of the diagonal

$$\nabla^{L\boxtimes\bar{L}} E_{a,b} = \tfrac{1}{i}\alpha_{a,b} \otimes E_{a,b}$$

The following lemma is proved in [7].

Lemme 4.1. *The one-form $\alpha_{a,b}$ vanishes along the diagonal of M^2. Furthermore, for every vector fields X_1, X_2, Y_1, Y_2 of M*

$$\mathcal{L}_{(X_1,X_2)} \langle \alpha_{a,b}, (Y_1,Y_2)\rangle(x,x) = \omega(\bar{q}_{a,b}(X_1 - X_2), Y_1)(x) + \omega(q_{b,a}(X_1 - X_2), Y_2)(x)$$

where $\bar{q}_{a,b}(x)$ and $q_{b,a}(x)$ are respectively the projections onto $(T_xM)^{0,1}_{j_b}$ with kernel $(T_xM)^{1,0}_{j_a}$ and onto $(T_xM)^{1,0}_{j_a}$ with kernel $(T_xM)^{0,1}_{j_b}$.

Consider now $S \in \mathcal{F}(a,b)$ and $S' \in \mathcal{F}(b,c)$. The Schwartz kernel of $S'S$ is

$$\left(\frac{k}{2\pi}\right)^{2n} \int_M E^k_{b,c}(x,y).E^k_{a,b}(y,z) f'(x,y,k).f(y,z,k)\ \mu_M(y) + O(k^{-\infty})$$

with μ_M the Liouville form of M. Since $|E_{b,c}(x,y).E_{a,b}(y,z)| < 1$ outside the diagonal of M^3, this integral is $O(k^{-\infty})$ outside the diagonal of M^2, and to estimate it on a neighborhood of $(x,z) = (x,x)$ it suffices to integrate on a neighborhood of x. We evaluate the result by applying the stationary phase lemma. Let us write

$$E_{b,c}(x,y).E_{a,b}(y,z) = e^{i\Phi(x,y,z)} t(x) \otimes \bar{t}(z) \tag{8}$$

with t a unitary local section of $L \to M$. We deduce from the previous lemma the following facts.

- $d_y\Phi$ vanishes along the diagonal Δ_3 of M^3.
- if Y_1 and Y_2 are two tangent vectors of M at x,

$$d^2_y\Phi(Y_1,Y_2)(x,x,x) = \omega(q_{c,b}Y_1, Y_2)(x) - \omega(\bar{q}_{a,b}Y_1, Y_2)(x), \tag{9}$$

  In particular $d^2_y\Phi$ is non-degenerate along Δ_3.
- the kernel of the tangent map to $d_y\Phi$ at (x,x,x) is

$$\big(T_{(x,x,x)}\Delta_3 \otimes \mathbb{C}\big) \oplus \big((T_xM)^{0,1}_{j_c} \times (0) \times (0)\big) \oplus \big((0) \times (0) \times (T_xM)^{1,0}_{j_a}\big).$$

These ensure that we can apply the stationary phase lemma (cf. [13] or the appendix of [8]). Thus the Schwartz kernel of $S'S$ is of the form

$$\left(\frac{k}{2\pi}\right)^n F^k(x,z) g(x,z,k) + O(k^{-\infty})$$

where $g(.,k)$ is a sequence of sections of $K_c \boxtimes \bar{K}_a$ which admits an asymptotic expansion in negative power of k and

$$F(x,z) = e^{i\Phi^r(x,z)} t(x) \otimes \bar{t}(z)$$

with

$$\Phi^r(x,z) \equiv \Phi(x,y,z) \tag{10}$$

modulo a linear combination with C^∞ coefficient of the functions $\partial_{y^i}\Phi(x,y,z), i = 1,...,2n$.

Let us check that the section F satisfies the assumptions following equation (6). Since $\partial_{y^i}\Phi$ vanishes along the diagonal, it follows from (8) that $\Phi^r(x,x) = 0$. Furthermore, we have

Lemme 4.2. *Consider M^2 as a complex manifold with complex structure $(j_c, -j_a)$. Then $\bar{\partial} F \equiv 0$ modulo a section vanishing to any order along the diagonal.*

Proof. Introduce complex coordinates $x^1,...,x^n$ on M for j_c. Let us write

$$\nabla t = \tfrac{1}{i}\ t \otimes \sum a_j dx^j - \bar{a}_j d\bar{x}^j.$$

Derivating equation (8) and using that $\nabla_{(\partial_{\bar{x}^i},0)} E_{b,c}$ vanishes to any order along the diagonal of M^2, we get

$$\partial_{\bar{x}^i}\Phi(x,y,z) + \bar{a}_i(x) \equiv 0 \tag{11}$$

modulo $\mathcal{I}_{\Delta_3}(\infty)$, i.e. modulo a function vanishing to any order along the diagonal of M^3. Thus

$$\partial_{\bar{x}^i}\partial_{y^j}\Phi(x,y,z) \equiv 0 \mod \mathcal{I}_{\Delta_3}(\infty).$$

In the same way, if $z^1,...,z^n$ are complex coordinates for j_a, we show that

$$\partial_{z^i}\partial_{y^j}\Phi(x,y,z) \equiv 0 \mod \mathcal{I}_{\Delta_3}(\infty).$$

Then we deduce from (10) and (11) that for any multi-index α and β, the function

$$\partial_{\bar{x}^1}^{\alpha(1)}...\partial_{\bar{x}^n}^{\alpha(n)}\partial_{z^1}^{\beta(1)}...\partial_{z^n}^{\beta(n)}\big(\partial_{\bar{x}^i}\Phi^r(x,z) + \bar{a}_i(x)\big)$$

vanishes along the diagonal Δ_2 of M^2. This implies that

$$\nabla^{L\boxtimes\bar{L}}_{(\partial_{\bar{x}^i},0)} F \equiv 0$$

modulo a section vanishing to any order along Δ_2. We treat in the same way the covariant derivatives of F with respect to the vector fields $(0, \partial_{z^i})$. $\square$

Then since the kernel of $S'S$ is a holomorphic section of

$$(L^k \otimes K_b) \boxtimes (\bar{L}^k \boxtimes K_a)$$

the coefficients of the asymptotic expansion of $g(.,k)$ satisfy $\bar{\partial} g_l \equiv 0$ modulo a section vanishing at any order along the diagonal. So we proved that $S'S$ is a Fourier integral operator of $\mathcal{F}(a,c)$.

Final step is to compute its symbol. By the stationary phase lemma, we have

$$g(x,x,k) = f'(x,x,k).f''(x,x,k)\frac{\delta(x)}{\det^{\frac{1}{2}}[-i\partial_{y^j}\partial_{y^k}\Phi(x,x,x)]_{j,k}} + O(k^{-1})$$

where

$$\mu_M(y) = \delta(y).|dy^1...dy^{2n}|$$

We deduce from (9) that

$$\begin{aligned} -id_y^2\Phi(Y_1,Y_2)(x,x,x) &= -\omega(iq_{c,b}Y_1 - i\bar{q}_{a,b}Y_1, Y_2)(x) \\ &= -\omega(j_b q_{c,b}Y_1 + j_b\bar{q}_{a,b}Y_1, Y_2)(x) \\ &= g_b(q_{c,b}Y_1 + \bar{q}_{a,b}Y_1, Y_2)(x) \end{aligned}$$

where g_b is the metric $\omega(X, j_bY)$. Since the Liouville form μ_M is the Riemannian volume for g_b, it comes that

$$\frac{\delta(x)}{\det^{\frac{1}{2}}[-i\partial_{y^j}\partial_{y^k}\Phi(x,x,x)]_{j,k}} = \det{}^{-\frac{1}{2}}[q_{c,b} + \bar{q}_{a,b}](x)$$

Thus to obtain the formula in theorem 3.2, we have to show that

$$\det{}^{-\frac{1}{2}}[q_{c,b} + \bar{q}_{a,b}] = \zeta^{\frac{1}{2}}(j_a, j_b, j_c).$$

To see this, let us choose j_b as the reference complex structure and let us associate to j_a and j_c the bundle maps μ_a and μ_c from $\Lambda^{1,0}_{j_b}T^*M$ to $\Lambda^{0,1}_{j_b}T^*M$ as in (4). On one hand, we have by (5) (since $\mu_b = 0$)

$$\zeta(j_a, j_b, j_c) = \det{}^{-1}\begin{pmatrix} \mathrm{Id} & \bar{\mu}_a \\ \mu_c & \mathrm{Id} \end{pmatrix}$$

On the other hand, $T^{0,1}_{j_a}M$ is the graph of $-\mu_a^t : T^{0,1}_{j_b}M \to T^{1,0}_{j_b}M$. It follows that $\bar{q}_{a,b}$ is the map

$$\begin{pmatrix} 0 & 0 \\ \bar{\mu}_a^t & \mathrm{Id} \end{pmatrix} : T^{1,0}_{j_b}M \oplus T^{0,1}_{j_b}M \to T^{1,0}_{j_b}M \oplus T^{0,1}_{j_b}M$$

Similarly, $q_{c,b}$ is the map

$$\begin{pmatrix} \mathrm{Id} & \mu_c^t \\ 0 & 0 \end{pmatrix} : T^{1,0}_{j_b}M \oplus T^{0,1}_{j_b}M \to T^{1,0}_{j_b}M \oplus T^{0,1}_{j_b}M$$

The result follows.

5. Half-form bundle and quantization

5.1. **Preliminaries on half-form bundle.** Let j be an almost-complex structure of M. Recall that a half-form bundle of (M, j) is a complex line bundle $\delta \to M$ together with a line bundle isomorphism

$$\varphi : \delta^2 \to \Lambda^{n,0}_j T^*M$$

which covers the identity of M. Two half-form bundles (δ_a, φ_a) and (δ_b, φ_b) are isomorphic if there exists a line bundle isomorphism $\Psi : \delta_a \to \delta_b$ covering the identity and such that

$$\varphi_b \circ \Psi^2 = \varphi_a.$$

In the case where there exists a half-form bundle, there are $\#H^1(M, \mathbb{Z}_2)$ isomorphism classes of half-form bundles.

The existence and the choice up to isomorphism of a half-form bundle over a symplectic manifold (M, ω) is in some sense independent of the almost complex structure, providing it is compatible with ω and positive. To see this we extend the previous notion of half-form bundle isomorphisms to the collection $\mathcal{D}$ consisting of the triples (j, δ, φ), where j is an almost-complex structure of M compatible with ω and positive, and (δ, φ) is a half-form bundle for (M, j).

Let us define a half-form bundle morphism $(j_a, \delta_a, \varphi_a) \to (j_b, \delta_b, \varphi_b)$ to be an isomorphism of line bundles $\Psi : \delta_a \to \delta_b$ such that

$$\varphi_b \circ \Psi^2 = \Psi_{j_a, j_b} \circ \varphi_a. \tag{12}$$

Here Ψ_{j_a,j_b} is the morphism $\Lambda^{n,0}_{j_a}T^*M \rightarrow \Lambda^{n,0}_{j_b}T^*M$ defined over $x \in M$ as in definition 2.1 with $E = T_xM$ and the complex structures $j_a(x)$ and $j_b(x)$.

The composition of a morphism $\Psi : (j_a, \delta_a, \varphi_a) \rightarrow (j_b, \delta_b, \varphi_b)$ with a morphism $\Psi' : (j_b, \delta_b, \varphi_b) \rightarrow (j_c, \delta_c, \varphi_c)$ is defined as

$$\Psi' \circ_{\mathrm{m}} \Psi := \zeta^{\frac{1}{2}}(j_a, j_b, j_c)\Psi' \circ \Psi$$

where the product $\circ$ on the right-hand side is the usual composition of maps and the function $\zeta^{\frac{1}{2}}(j_a, j_b, j_c)$ is defined as in section 2. Observe that $\circ_{\mathrm{m}}$ is the product of symbol appearing in theorem 3.2.

It is easily checked that $\mathcal{D}$ with this collection of morphisms is a groupoid. The associativity of $\circ_{\mathrm{m}}$ follows from the cocycle condition (2). Equations (3) imply that the identity 1_a of δ_a is the unit of $(j_a, \delta_a, \varphi_a)$, i.e.

$$1_a \circ_{\mathrm{m}} \Psi = \Psi, \qquad \Psi' \circ_{\mathrm{m}} 1_a = \Psi',$$

if Ψ and Ψ' are any morphisms $(j_b, \delta_b, \varphi_b) \rightarrow (j_a, \delta_a, \varphi_a)$ and $(j_a, \delta_a, \varphi_a) \rightarrow (j_b, \delta_b, \varphi_b)$ respectively. Moreover, for any $(j, \delta, \varphi) \in \mathcal{D}$, define the Hermitian structure of δ in such a way that φ becomes an isomorphism of Hermitian bundles. Then since

$$\Psi^*_{j_a,j_b} = \Psi_{j_b,j_a},$$

the adjoint Ψ^* of any morphism $\Psi : (j_a, \delta_a, \varphi_a) \rightarrow (j_b, \delta_b, \varphi_b)$ is a morphism $(j_b, \delta_b, \varphi_b) \rightarrow (j_a, \delta_a, \varphi_a)$ satisfying

$$\Psi^* \circ_{\mathrm{m}} \Psi = 1_a, \qquad \Psi \circ_{\mathrm{m}} \Psi^* = 1_b. \tag{13}$$

So Ψ is invertible, with inverse Ψ^*.

If a and b in $\mathcal{D}$ are isomorphic, there exists exactly two morphisms $a \rightarrow b$. Observe also that given an almost complex structure j, each isomorphism class of $\mathcal{D}$ has a representative whose almost complex structure is j. So the existence of a half-form bundle doesn't depend on the almost complex structure. And there are $\#H^1(M, \mathbb{Z}_2)$ isomorphism classes in $\mathcal{D}$ if it is not empty.

5.2. **Quantization.** Let us consider now the collection $\mathcal{D}_{\mathrm{int}}$ consisting of triples $(j, \delta, \varphi) \in \mathcal{D}$ with an integrable complex structure j. Given $a \in \mathcal{D}_{\mathrm{int}}$, let us denote by $\mathcal{Q}^{\mathrm{m}}_k(a)$ the Hilbert space of holomorphic sections of $L^k \otimes \delta_a$. With our previous notations

$$\mathcal{Q}^{\mathrm{m}}_k(a) := \mathcal{H}_k(j_a, \delta_a).$$

Here the holomorphic and Hermitian structures of δ_a are such that $\varphi_a : \delta_a^2 \rightarrow \Lambda^{n,0}_{j_a}T^*M$ is an isomorphism of holomorphic Hermitian bundle.

If a and b belongs to $\mathcal{D}_{\mathrm{int}}$, any half-form bundle morphism $\Psi : \delta_a \rightarrow \delta_b$ is the symbol of a unitary Fourier integral operator of $\mathcal{F}((j_a, \delta_a), (j_b, \delta_b))$

$$U^{\mathrm{m}}_k(\Psi) : \mathcal{Q}^{\mathrm{m}}_k(a) \rightarrow \mathcal{Q}^{\mathrm{m}}_k(b), \quad k = 1, 2, ...$$

Indeed if S is a Fourier integral operator with symbol Ψ, it follows from (13) and theorem 3.2 that S^*S is a Toeplitz operator with symbol 1. Hence $S(S^*S)^{-1/2}$ is a unitary Fourier integral operator with symbol Ψ.

Contrary to the notations, $U^{\mathrm{m}}(\Psi) = (U^{\mathrm{m}}_k(\Psi))_k$ is not uniquely determined by Ψ. It is unique modulo multiplication by a unitary Toeplitz operator of symbol 1. So strictly speaking, $U^{\mathrm{m}}(\Psi)$ is an equivalence class of Fourier integral operators. To avoid any confusion we will say that two such operators are equal modulo $O(\hbar)$.

Theorem 5.1. *U^{m} is functorial, that is if Ψ'' is the composition of the morphisms of half-form bundle Ψ' and Ψ, then*

$$U^{\mathrm{m}}(\Psi'') = U^{\mathrm{m}}(\Psi') \circ U^{\mathrm{m}}(\Psi) \text{ modulo } O(\hbar).$$

Furthermore if Ψ is a half-form bundle morphism $a \to b$, the map sending the Toeplitz operator $T : \mathcal{Q}^{\mathrm{m}}(a) \to \mathcal{Q}^{\mathrm{m}}(a)$ into

$$(U^{\mathrm{m}}(\Psi))^* T U^{\mathrm{m}}(\Psi) : \mathcal{Q}^{\mathrm{m}}(b) \to \mathcal{Q}^{\mathrm{m}}(b)$$

preserves the normalized symbols modulo $O(\hbar^2)$.

First part is an immediate consequence of theorem 3.2 because the composition of half-form bundle morphisms is the same as the composition of symbols. Second part follows from theorem 3.3, or more directly from formula (7).

The group $\mathcal{G}$ of connection-preserving Hermitian automorphisms of L acts on the quantum spaces as follows. First an automorphism Φ of $\mathcal{G}$ covers a symplectomorphism ϕ of M. Then Φ acts on $\mathcal{D}_{\mathrm{int}}$ by sending $a = (j, \delta, \varphi)$ into $\Phi^* a := (\phi^* j, \phi^* \delta, \phi^* \varphi)$, where $\phi^* \varphi$ is defined in such a way that the diagram

$$\begin{tikzcd} \Lambda^{n,0}_j T^*_{\phi(x)} M \arrow[r, "\phi^*"] & \Lambda^{n,0}_{\phi^* j} T^*_x M \\ \delta^2_{\phi(x)} \arrow[u, "\varphi"] \arrow[r, "(\phi^*)^2"] & (\phi^*\delta)^2_x \arrow[u, "\phi^*\varphi"'] \end{tikzcd}$$

commutes. Finally the operator

$$C^\infty(M, L^k \otimes \delta_a) \to C^\infty(M, L^k \otimes \phi^* \delta_a), \qquad s \to ((\Phi^k)^{-1} \otimes \phi^*) \circ s \circ \phi$$

restricts to a unitary operator $\mathcal{Q}^{\mathrm{m}}_k(a) \to \mathcal{Q}^{\mathrm{m}}_k(\Phi^* a)$ that we denote by Φ^*.

Let us consider now $a \in \mathcal{D}$, fixed until the end of this section. If Φ belongs to the identity component $\mathcal{G}_o$ of $\mathcal{G}$, then a and $\Phi^* a$ are isomorphic half-form bundles. In this case, we associate to any morphism $\Psi : \Phi^* a \to a$ the sequence of operators

$$V^{\mathrm{m}}_k(\Psi, \Phi) := U^{\mathrm{m}}_k(\Psi) \circ \Phi^* : \mathcal{Q}^{\mathrm{m}}_k(a) \to \mathcal{Q}^{\mathrm{m}}_k(a)$$

As the operators $U^{\mathrm{m}}(\Psi)$, $V^{\mathrm{m}}(\Psi, \Phi)$ is uniquely defined up to multiplication by a unitary Toeplitz operator with symbol 1. Denote by $\mathcal{G}^{\mathrm{m}}_0$ the set of pairs (Ψ, Φ) where $\Phi \in \mathcal{G}_o$ and Ψ is a half-form bundle morphism $\Phi^* a \to a$.

Theorem 5.2. *For any half-form bundle $a \in \mathcal{D}_{\mathrm{int}}$, $\mathcal{G}^{\mathrm{m}}_0$ endowed with the product*

$$(\Psi_1, \Phi_1).(\Psi_2, \Phi_2) := (\Psi_2 \circ_{\mathrm{m}} (\Phi_2^* \Psi_1), \Phi_1 \circ \Phi_2)$$

is a central extension of $\mathcal{G}_0$ by $\mathbb{Z}_2$. Furthermore V^{m} is a right-representation of $\mathcal{G}^{\mathrm{m}}_0$ up to $O(\hbar)$ in the sense that

$$V^{\mathrm{m}}(\Psi_2, \Phi_2) \circ V^{\mathrm{m}}(\Psi_1, \Phi_1) \equiv V^{\mathrm{m}}((\Psi_1, \Phi_1).(\Psi_2, \Phi_2)) \mod O(\hbar).$$

In the definition of the product of $\mathcal{G}^{\mathrm{m}}_0$, we used the following action of $\mathcal{G}$ on the half-form bundle morphisms. If Φ is prequantization bundle automorphism of L covering the symplectomorphism ϕ and Ψ is a morphism $a \to b$, then $\Phi^* \Psi$ is the morphism $\Phi^* a \to \Phi^* b$ defined in such a way that the diagram

$$\begin{tikzcd} \delta_a \arrow[r, "\Psi"] \arrow[d, "\phi^*"'] & \delta_b \arrow[d, "\phi^*"] \\ \phi^*\delta_a \arrow[r, "\Phi^*\Psi"] & \phi^*\delta_b \end{tikzcd}$$

commutes. One deduces easily from the relations

$$\Phi_2^*(\Phi_1^* \Psi) = (\Phi_1 \circ \Phi_2)^* \Psi, \qquad \Phi^*(\Psi_1 \circ_{\mathrm{m}} \Psi_2) = (\Phi^* \Psi_1) \circ_{\mathrm{m}} (\Phi^* \Psi_2)$$

that $\mathcal{G}^{\mathrm{m}}_0$ is a group. Furthermore, one has

$$U^{\mathrm{m}}(\Phi^* \Psi) = \Phi^* \circ U^{\mathrm{m}}(\Psi) \circ (\Phi^*)^{-1} \mod O(\hbar)$$

which implies the last part of the theorem:

$$\begin{aligned} V^{\mathrm{m}}(\Psi_2,\Phi_2)\circ V^{\mathrm{m}}(\Psi_1,\Phi_1) =& U^{\mathrm{m}}(\Psi_2)\circ \Phi_2^*\circ U^{\mathrm{m}}(\Psi_1)\circ \Phi_1^* \\ =& U^{\mathrm{m}}(\Psi_2)\circ \Phi_2^*\circ U^{\mathrm{m}}(\Psi_1)\circ (\Phi_2^*)^{-1}\circ (\Phi_1\circ\Phi_2)^* \\ =& U^{\mathrm{m}}(\Psi_2)\circ U^{\mathrm{m}}(\Phi_2^*\Psi_1)\circ (\Phi_1\circ\Phi_2)^* \mod O(\hbar) \\ =& U^{\mathrm{m}}(\Psi_2\circ_{\mathrm{m}}\Phi_2^*\Psi_1)\circ (\Phi_1\circ\Phi_2)^* \mod O(\hbar) \\ =& V^{\mathrm{m}}((\Psi_1,\Phi_1).(\Psi_2,\Phi_2)) \end{aligned}$$

by theorem 5.1.

It is well-known that the Lie algebra of $\mathcal{G}_0$ is $C^\infty(M,\mathbb{R})$, the Lie bracket being the Poisson bracket (cf. (16) for an explicit formula for the exponential map). Let us associate to any $f\in C^\infty(M)$ a Toeplitz operator $\mathcal{Q}^{\mathrm{m}}(f)$ of $\mathcal{Q}^{\mathrm{m}}(a)$ whose normalized symbol is f modulo $O(\hbar^2)$. By theorem 3.4, we obtain a Lie algebra representation up to $O(\hbar)$ in the sense that

$$\tfrac{1}{i}\big[k\mathcal{Q}_k^{\mathrm{m}}(f),k\mathcal{Q}_k^{\mathrm{m}}(g)\big]=k\mathcal{Q}_k^{\mathrm{m}}(\{f,g\})+O(k^{-1})$$

By exponentiating we recover the representation of theorem 5.2.

Theorem 5.3. *For any $f\in C^\infty(M,\mathbb{R})$, we have*

$$\exp\big(it.k\mathcal{Q}_k^{\mathrm{m}}(f)\big)=V_k^{\mathrm{m}}(\Psi_t,\Phi_t)\mod O(\hbar)$$

where $\Phi_t=\exp(tf)$ and (Ψ_t) is the continuous family of half-form bundle morphisms $\Psi_t: a\to\Phi_t^*a$ such that Ψ_0 is the identity of δ_a.

This last result will be proved in section 8 (cf. remark after corollary 8.3).

5.3. **Reformulation of the results of chapter 3.** Assume that (M,ω) admits a unique half-form bundle up to isomorphism. If this is not the case we can still apply what follows by restricting to an open contractible set of M.

Let us return to the quantum spaces $\mathcal{H}(a)$ defined from a complex structure j_a and a Hermitian holomorphic line bundle $K_a\to M$. As in [8], we introduce a half-form bundle (δ_a,φ_a) and a holomorphic Hermitian line bundle $L_{1,a}$ such that

$$K_a=\delta_a\otimes L_{1,a}.$$

For another pair (j_b,K_b), introduce in the same way (δ_b,φ_b) and $L_{1,b}$. Then rewriting the formulas of chapter 3 with these data, we get more transparent results:

- The formula for the commutators in theorem 3.4 becomes
$$\tfrac{i}{\hbar}(f*_a g-g*_a f)=\{f,g\}-\hbar\langle\omega_{1,a},X_f\wedge X_g\rangle+O(\hbar^2),$$
where $\frac{1}{i}\omega_{1,a}$ is the curvature of $L_{1,a}$.
- Denote by $\mathcal{M}_{a,b}$ the set of half-form bundle morphisms $(j_a,\delta_a,\varphi_a)\to(j_b,\delta_b,\varphi_b)$, then
$$\mathrm{Hom}(K_a,K_b)=\mathcal{M}_{a,b}\times_{\mathbb{Z}_2}\mathrm{Hom}(L_{1,a},L_{1,b})$$
where we divided by $\mathbb{Z}_2$ to identify (Ψ,Ψ_1) with $(-\Psi,-\Psi_1)$. The composition of symbols in theorem 3.3 is then the product of the composition of half-form bundle morphisms with the usual composition.
- The symbol of a unitary operator $S\in U\mathcal{F}(a,b)$ is of the form $[\Psi,\Psi_1]$ with $\Psi\in\mathcal{M}_{a,b}$ and Ψ_1 a unitary isomorphism $L_{1,a}\to L_{1,b}$. Furthermore the equivalence of the star-products $*_a$ and $*_b$ induced by S is up to second order
$$f_a\to f_a+\hbar\langle\alpha_1,X_{f_a}\rangle+O(\hbar^2)$$
where α_1 is such that $\nabla^{\mathrm{Hom}(L_{1,a},L_{1,b})}\Psi_1=\frac{1}{i}\alpha_1\otimes\Psi_1$.

This point of view will also be useful to prove theorem 3.3 in the following section.

6. Proof of theorem 3.3

To prove the theorem, we consider the kernels of the Fourier integral operators as Lagrangian sections and interpret $T_b S - S T_a$ as the result of the action of a Toeplitz operator on a Lagrangian section. The computation of the symbol is then a corollary of theorem 3.4 in [8].

Let us regard M^2 as a symplectic manifold with symplectic form

$$\omega_{M^2} = \pi_l^* \omega - \pi_r^* \omega,$$

where π_l and π_r are the projections onto the first and second factor respectively. Then $L \boxtimes \bar{L}$ is a prequantization bundle of M^2 with curvature $\frac{1}{i}\omega_{M^2}$ and the diagonal map $\Delta : M \to M^2$ is a Lagrangian embedding. Furthermore $(j_b, -j_a)$ is a complex structure of M^2 compatible with ω_{M^2} and positive. Denote by $\mathcal{H}(b,-a)$ the associated Hilbert space

$$\mathcal{H}(b,-a) = \left\{ \text{holomorphic sections of } (L^k \boxtimes \bar{L}^k) \otimes (K_b \boxtimes \bar{K}_a) \to M^2 \right\}$$

Then the Fourier integral operators of $\mathcal{F}(a,b)$ are defined in such a way that their kernel is a Lagrangian section of $\mathcal{H}(b,-a)$ associated to the diagonal.

Let $S \in \mathcal{F}(a,b)$ with kernel $S(.)$. Let T_a and T_b be Toeplitz operators of $\mathcal{H}(a)$ and $\mathcal{H}(b)$ with normalized symbols $f_a(.,\hbar)$ and $f_b(.,\hbar)$ respectively. Then it is easily checked that the kernel of $T_b S - S T_a$ is $TS(.)$ where T is a Toeplitz operator of $\mathcal{H}(b,-a)$ with normalized symbol

$$g(x,y,\hbar) = f_b(x,\hbar) - f_a(y,\hbar).$$

Assume now that T_a and T_b have the same principal symbol. Then the principal symbol of T vanishes along the diagonal and consequently the principal symbol of $TS(.)$ vanishes. By applying theorem 3.4 of [8], we obtain the principal symbol of $k^{-1}TS(.)$ which corresponds to the principal symbol of $k^{-1}(T_b S - S T_a)$.

Let us use the half-form bundles as in chapter 5.3. The symbol of S as a Fourier integral operator of $\mathcal{F}(a,b)$ is a class $[\Psi, \Psi_1]$ where Ψ is a half-form bundles morphism $\delta_a \to \delta_b$ and Ψ_1 a bundle morphism $L_{1,a} \to L_{1,b}$. We have to show that the symbol of $k^{-1}(T_b S - S T_a)$ is $[\Psi, \Psi_1']$ with

$$\Psi_1' = (f_{1,b} - f_{1,a})\Psi_1 + \tfrac{1}{i} \nabla_X^{\mathrm{Hom}(L_{1,a}, L_{1,b})} \Psi_1 \tag{14}$$

where X is the Hamiltonian vector field of f and $f_{1,a}$, $f_{1,b}$ are the subprincipal symbols of T_a and T_b respectively.

To the morphisms Ψ and Ψ_1 correspond two sections

$$\tilde{\Psi} \in C^\infty(M, \delta_b \otimes \overline{\delta}_a), \quad \tilde{\Psi}_1 \in C^\infty(M, L_{1,b} \otimes \overline{L}_{1,a}).$$

The principal symbol of the Lagrangian section $S(.)$ is $\tilde{\Psi} \otimes \tilde{\Psi}_1$. The restriction to the diagonal of the Hamiltonian vector field of the principal symbol of T is $\Delta_* X$. Then it follows from theorem 3.4 of [8] that the principal symbol of $k^{-1}T.S(.)$ is

$$(f_{1,b} - f_{1,a})\tilde{\Psi} \otimes \tilde{\Psi}_1 + \tfrac{1}{i}(D_X^\Delta \tilde{\Psi}) \otimes \tilde{\Psi}_1 + \tfrac{1}{i}\tilde{\Psi} \otimes \nabla_X^{L_{1,b} \otimes \overline{L}_{1,a}} \tilde{\Psi}_1$$

It remains to explain how is defined the section $D_X^\Delta \tilde{\Psi}$ of $\delta_b \otimes \overline{\delta}_a$ and to prove that it vanishes. This will imply (14).

Consider the isomorphism

$$\xi : \Lambda_{j_b}^{n,0} T^*M \boxtimes \overline{\Lambda_{j_a}^{n,0} T^*M} \to \Lambda_{j_b,-j_a}^{2n,0} T^*M^2, \quad \beta \boxtimes \overline{\alpha} \to \pi_l^* \beta \wedge \pi_r^* \overline{\alpha}$$

$(\delta_b \boxtimes \bar{\delta}_a, \xi \circ (\varphi_b \boxtimes \bar{\varphi_a}))$ is a half-form bundle of M^2 for the complex structure $(j_b, -j_a)$. Then

$$\delta_b \otimes \bar{\delta}_a = \Delta^*(\delta_b \boxtimes \bar{\delta}_a)$$

is a square root of $\Lambda^{2n} T^* M \otimes \mathbb{C}$ through the map

$$\varphi_\delta : \delta_b^2 \otimes \bar{\delta}_a^2 \to \Lambda^{2n} M \otimes \mathbb{C}, \quad u_b \otimes u_a \to \Delta^*(\xi(\varphi_b(u_b) \boxtimes \bar{\varphi}_a(u_a))).$$

and $D_X^\Delta \tilde{\Psi}$ is defined in such a way that

$$\varphi_\delta(\tilde{\Psi} \otimes D_X^\Delta \tilde{\Psi}) = \tfrac{1}{2} \mathcal{L}_X . \varphi_\delta(\tilde{\Psi}^{\otimes 2})$$

Then $D_X^\Delta \tilde{\Psi} = 0$ follows from the following lemma and Liouville theorem.

Lemme 6.1. $\varphi_\delta(\tilde{\Psi}^{\otimes 2}) = i^{n(n-2)} \omega^n / n!$

Proof. Denote by $\tilde{\Psi}_{j_a, j_b}$ the section of $\Lambda^{n,0}_{j_b} M \otimes \overline{\Lambda^{n,0}_{j_a} M} \to M$ associated to Ψ_{j_a, j_b}. Since Ψ is a half-form bundle morphism, we have

$$(\varphi_a \otimes \bar{\varphi}_b)(\tilde{\Psi}^{\otimes 2}) = \tilde{\Psi}_{j_a, j_b}$$

Introduce a unitary section α of $\Lambda^{n,0}_{j_a} M$. We have $\tilde{\Psi}_{j_a, j_b} = \Psi_{j_a, j_b}(\alpha) \otimes \bar{\alpha}$. Consequently

$$\begin{aligned} \varphi_\delta(\tilde{\Psi}^{\otimes 2}) &= \Delta^*(\pi_l^* \Psi_{j_a, j_b}(\alpha) \wedge \pi_r^* \bar{\alpha})) \\ &= \Psi_{j_a, j_b}(\alpha) \wedge \bar{\alpha} \\ &= \alpha \wedge \bar{\alpha} \end{aligned}$$

by definition of Ψ_{j_a, j_b}

$$= i^{n(n-2)} \omega^n / n!$$

because α is unitary. ☐

7. Geometric interpretation

Consider as previously a symplectic manifold (M, ω) with a prequantization bundle (L, ∇). The space $\mathcal{J}$ of almost complex structures of M compatible with ω and positive may be regarded as the space of sections of a fiber bundle over M, which turns it into an infinite dimensional manifold. Let us fix a isomorphism class D of half-form bundles and choose for any $j \in \mathcal{J}$ a half-form bundle of (M, j) which represents D and depends "smoothly" on j. One way to do that is first to choose $(j_0, \delta_0, \varphi_0)$ representing D and then to set

$$\delta_j := \delta_0, \qquad \varphi_j := \Psi_{j_0, j} \circ \varphi_0, \qquad \forall j \in \mathcal{J}.$$

Let $\mathcal{P}_k^{\mathrm{m}} \to \mathcal{J}$ be the bundle of *prequantum* spaces, whose fiber at j is the space of smooth sections of $L^k \otimes \delta_j$. Let us consider now a submanifold $\mathcal{J}_{\mathrm{int}}$ of $\mathcal{J}$ which contains only integrable complex structures. Assume that the family of Hilbert spaces

$$\mathcal{Q}_{k,j}^{\mathrm{m}} := \{\text{holomorphic sections of } L^k \otimes \delta_j\}, \qquad j \in \mathcal{J}_{\mathrm{int}}$$

defines a smooth subbundle $\mathcal{Q}_k^{\mathrm{m}} \to \mathcal{J}_{\mathrm{int}}$ of $\mathcal{P}_k^{\mathrm{m}} \to \mathcal{J}_{\mathrm{int}}$, when k is sufficiently large. This assumption is satisfied as soon as the dimension of $\mathcal{Q}_{k,j}^{\mathrm{m}}$ is constant when j runs over $\mathcal{J}_{\mathrm{int}}$. This follows from Fredholm theory because $\mathcal{Q}_{k,j}^{\mathrm{m}}$ is the kernel of the holomorphic Laplacian, an elliptic second order differential operator whose coefficient depend smoothly on the complex structure. Furthermore as noticed by Foth and Uribe [9], for any complex structure j_0, there exists an integer N such that the dimension of $\mathcal{Q}_{k,j}^{\mathrm{m}}$ is constant when j describe a C^2 neighborhood of j_0 and k is larger than N. The C^2-topology is involved here to control the curvature term in the Bochner-Kodaira identity and deduce a uniform vanishing theorem. Then the dimension of $\mathcal{Q}^{\mathrm{m}}_{k,j}$ is given by the Riemann-Roch theorem.

Before we continue, let us note that $\mathcal{P}^{\mathrm{m}}_k$ and $\mathcal{Q}^{\mathrm{m}}_k$ depend only on the isomorphism class D, providing we regard them as the orbifold bundles $\mathcal{P}^{\mathrm{m}}_k/\mathbb{Z}_2$ and $\mathcal{Q}^{\mathrm{m}}_k/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts trivially on the base $\mathcal{J}$ and by $\pm\mathrm{Id}$ on the fibers. Indeed, let us consider another smooth family $(\tilde{\delta}_j, \tilde{\varphi}_j)_{j\in\mathcal{J}}$, obtained as above by choosing a half-form bundle $(\tilde{j}_0, \tilde{\delta}_0, \tilde{\varphi}_0)$ representing D and denote by $\tilde{\mathcal{P}}^{\mathrm{m}}_k$ the associated bundle of prequantum spaces. Then there exists exactly two continuous families

$$\big(\Psi_j : (\delta_j, \varphi_j) \rightarrow (\tilde{\delta}_j, \tilde{\varphi}_j);\ j \in \mathcal{J}\big)$$

of half-form bundle morphisms. These families induce isomorphisms $\mathcal{P}^{\mathrm{m}}_k \rightarrow \tilde{\mathcal{P}}^{\mathrm{m}}_k$ and $\mathcal{Q}^{\mathrm{m}}_k \rightarrow \tilde{\mathcal{Q}}^{\mathrm{m}}_k$, which are unique up to the $\mathbb{Z}_2$-action. All the constructions which follow only depend on D in this sense.

First we define a connection on $\mathcal{P}^{\mathrm{m}}_k$. Given a tangent vector μ of $\mathcal{J}$ at j_0, let us introduce a curve j_t of $\mathcal{J}$ tangent to μ at $t = 0$ and consider the continuous family (Ψ_t) of half-form bundle morphism $(\delta_{j_0}, \varphi_{j_0}) \rightarrow (\delta_{j_t}, \varphi_{j_t})$ such that Ψ_0 is the identity of δ_{j_0}. Then we define the covariant derivative of a section Φ of $\mathcal{P}^{\mathrm{m}}_k$ with respect to μ to be

$$\nabla^{\mathcal{P}^{\mathrm{m}}_k}_{\mu} \Phi\ (j_0) := \frac{d}{dt}\Big|_{t=0} \Psi_t^{-1}.\Phi(j_t)$$

where the derivative is in the t-independent space $\mathcal{P}^{\mathrm{m}}_{k,j_0}$. The connection on $\mathcal{Q}^{\mathrm{m}}_k$ is then defined as

$$\nabla^{\mathcal{Q}^{\mathrm{m}}_k} := \Pi_k \circ \nabla^{\mathcal{P}^{\mathrm{m}}_k}$$

where Π_k is the section of $\mathrm{End}(\mathcal{P}^{\mathrm{m}}_k)$ which at j is the orthogonal projector onto $\mathcal{Q}^{\mathrm{m}}_{k,j}$.

Theorem 7.1. *For any k, the connection $\nabla^{\mathcal{Q}^{\mathrm{m}}_k}$ is compatible with the Hermitian structure. Furthermore,*

- *For any $j \in \mathcal{J}_{\mathrm{int}}$ and $\eta, \mu \in T_j\mathcal{J}_{\mathrm{int}}$, the sequence of curvature*
$$R^{\mathcal{Q}^{\mathrm{m}}_k}(\eta, \mu) : \mathcal{Q}^{\mathrm{m}}_{k,j} \rightarrow \mathcal{Q}^{\mathrm{m}}_{k,j}, \quad k = 1, 2, ...$$
is a Toeplitz operator whose principal symbol vanishes.
- *For any curve γ of $\mathcal{J}_{\mathrm{int}}$ with endpoints j_a and j_b, the sequence of parallel transport γ in $\mathcal{Q}^{\mathrm{m}}_k$ is a unitary Fourier integral operator*
$$\mathcal{Q}^{\mathrm{m}}_{k,j_a} \rightarrow \mathcal{Q}^{\mathrm{m}}_{k,j_b}, \quad k = 1, 2, ...$$
of $\mathcal{F}((j_a, \delta_{j_a}), (j_b, \delta_{j_b}))$. Its principal symbol is the half-form bundle morphism $\delta_{j_a} \rightarrow \delta_{j_b}$ obtained by extending continuously the identity of δ_{j_a} in half-form bundle morphisms $\delta_{j_a} \rightarrow \delta_{\gamma(t)}$.

The proof is postponed to section 10. Let us compute the curvature of $\mathcal{P}^{\mathrm{m}}_k$. Given an almost complex-structure $j_0 \in \mathcal{J}$, we can represent any $j \in \mathcal{J}$ as a section

$$\mu \in C^{\infty}(M, \mathrm{Hom}(\Lambda^{1,0}_{j_0}T^*M, \Lambda^{0,1}_{j_0}T^*M))$$

such that the graph of $\mu(x)$ is $\Lambda^{1,0}_j T^*_x M$ for any $x \in M$. In this way, we identify the tangent space to $\mathcal{J}$ at j_0 with

$$T_{j_0}\mathcal{J} \simeq \big\{\mu \in C^{\infty}(M, \mathrm{Hom}(\Lambda^{1,0}_{j_0}T^*M, \Lambda^{0,1}_{j_0}T^*M));\ \omega(\mu^t(.), .) + \omega(.,\ \mu^t(.)) = 0\big\}$$

and $\mathcal{J}$ becomes a neighborhood of the zero section of $T_{j_0}\mathcal{J}$.

Theorem 7.2. *The connection $\nabla^{\mathcal{P}^{\mathrm{m}}_k}$ is compatible with the Hermitian structure. Its curvature at $\eta, \mu \in T_j\mathcal{J}$ is given by*

$$R^{\mathcal{P}^{\mathrm{m}}_k}(\eta, \mu)\ \Phi = \frac{1}{2}\mathrm{tr}(\eta.\bar{\mu} - \mu.\bar{\eta})\ \Phi, \qquad \Phi \in \mathcal{P}^{\mathrm{m}}_{k,j}$$

It is interesting to compare the previous theorems with the results of Foth and Uribe [9]. The curvature of $\mathcal{Q}_k^{\mathrm{m}}$ is the sum of two terms which cancel each other at first order. The first term is the curvature of $\mathcal{P}_k^{\mathrm{m}}$ and the second one is a commutator (cf. lemma 8.6). In the case considered by Foth and Uribe, the prequantum spaces are defined without half-form and consequently don't depend on the complex structure. Then the bundle $\mathcal{J}_{\mathrm{int}} \times C^\infty(M, L^k)$ is endowed with the trivial connection, and composing with the Szegö projector, we obtain a connection on the quantum space bundle. Its curvature equals a commutator (cf. lemma 2.1 of [9]), which is essentially the same as in our situation, and isn't canceled by the curvature of the prequantum bundle, flat in this case.

Proof of theorem 7.2. Let j_0 be a fixed almost-complex structure and let us identify $\mathcal{J}$ with an open convex set O of $T_{j_0}\mathcal{J}$ as previously. Let us compute the connection in the trivialization

$$\mathcal{P}_k^{\mathrm{m}} \simeq O \times \mathcal{P}_{k,j_0}^{\mathrm{m}}$$

induced by the continuous family of half-form bundle isomorphisms $(\delta_{j_0}, \varphi_{j_0}) \to (\delta_j, \varphi_j)$ extending the identity of δ_{j_0}.

Let $\mu(t)$ be a curve of O covered by a section $\Phi(t)$. By lemma 2.3, the continuous curve of half-form bundle morphisms $\Psi_t : \delta_{\mu(0)} \to \delta_{\mu(t)}$ is in the previous trivialization the multiplication by the continuous square root of

$$t \to \det \begin{pmatrix} \mathrm{Id} & \bar{\mu}(0) \\ \mu(0) & \mathrm{Id} \end{pmatrix} . \det{}^{-1} \begin{pmatrix} \mathrm{Id} & \bar{\mu}(0) \\ \mu(t) & \mathrm{Id} \end{pmatrix}$$

equal to 1 at $t = 0$. Then we have

$$\begin{aligned} \nabla^{\mathcal{P}_k^{\mathrm{m}}}_{\dot{\mu}(0)} \Phi(0) &= \frac{d}{dt}\Big|_{t=0} \Psi_t^{-1}.\Phi(t) \\ &= -\frac{1}{2}\mathrm{tr}\Big(\dot{\mu}(0)\bar{\mu}(0) \left(\mathrm{Id} - \mu(0)\bar{\mu}(0)\right)^{-1}\Big)\Phi(0) + \dot{\Phi}(0) \end{aligned}$$

Thus we have $\nabla^{\mathcal{P}_k^{\mathrm{m}}} = d + \alpha$ with

$$\langle \alpha, \dot{\mu}\rangle.\Phi = -\frac{1}{2}\mathrm{tr}\Big(\dot{\mu}\bar{\mu} \left(\mathrm{Id} - \mu\bar{\mu}\right)^{-1}\Big)\Phi, \qquad \forall\ \dot{\mu} \in T_\mu O,\ \Phi \in \mathcal{P}_{k,j_0}^{\mathrm{m}}$$

Finally it is easy to compute the curvature at the origin of O, where α vanishes which leads to the formula of the theorem.

To check the compatibility of the connection with the scalar product, observe that our trivialization doesn't preserve the scalar product. Actually since $\Psi^*_{j_0,j} \circ \Psi_{j_0,j} = \zeta(j_0, j, j_0)$, we have

$$(\Phi, \Phi')_{\mathcal{P}_{k,j}^{\mathrm{m}}} = \int_M \frac{1}{\zeta^{\frac{1}{2}}(j_0, j, j_0)(x)} \big(\Phi(x), \Phi'(x)\big)_{L^k \otimes \delta_{j_0}} \mu_M(x)$$

Then using again that the connection form α vanishes at the origin and that $\zeta(j_0, j, j_0) = 1 + O(|j - j_0|^2)$ we deduce that for every section Φ, Φ' of $O \times \mathcal{P}_{k,j_0}^{\mathrm{m}}$,

$$d(\Phi, \Phi') = (\nabla^{\mathcal{P}_k^{\mathrm{m}}}\Phi, \Phi') + (\Phi, \nabla^{\mathcal{P}_k^{\mathrm{m}}}\Phi')$$

at the origin. □

Remarque 7.3. It is immediate to deduce the first part of theorem 7.1. Since Π is self-adjoint, if Φ and Φ' are section of $\mathcal{Q}_k^{\mathrm{m}} \to \mathcal{J}_{\mathrm{int}}$, then

$$\begin{aligned} (\nabla^{\mathcal{Q}_k^{\mathrm{m}}}\Phi, \Phi') + (\Phi, \nabla^{\mathcal{Q}_k^{\mathrm{m}}}\Phi') &= (\Pi\nabla^{\mathcal{P}_k^{\mathrm{m}}}\Phi, \Phi') + (\Phi, \Pi\nabla^{\mathcal{P}_k^{\mathrm{m}}}\Phi') \\ &= (\nabla^{\mathcal{P}_k^{\mathrm{m}}}\Phi, \Phi') + (\Phi, \nabla^{\mathcal{P}_k^{\mathrm{m}}}\Phi') \\ &= d(\Phi, \Phi') \end{aligned}$$

which proves that $\nabla^{\mathcal{Q}_k^{\mathrm{m}}}$ is Hermitian. □

8. Action of the prequantization bundle automorphisms

Adapting the constructions of section 5, we define an action of the identity component $\mathcal{G}_0$ of the group of prequantization bundle automorphisms of (L, ∇) on $\mathcal{P}^{\mathrm{m}}_k$ and $\mathcal{Q}^{\mathrm{m}}_k$. For any equivariant vector bundle equipped with an invariant connection, one defines a moment (cf. definition 7.5 in [3]). This notion makes sense in our infinite dimensional setting. In the first part of this section, we prove the moment of a function f in the Poisson Lie algebra $C^\infty(M)$ is a Toeplitz operator. From this we compute the solution of Schrödinger equation in terms of parallel transport. This last result was obtained in [9] in the case without metaplectic correction. This enables us to deduce that the quantum propagator is a Fourier integral operator from the fact that parallel transport is such an operator. Next we compute the commutator of Toeplitz operators in terms of the curvature of the quantum space bundle. This prove our estimate of the curvature is sharp. Finally we explain how the same ideas apply in the case without metaplectic correction.

8.1. **The infinitesimal action of $\mathcal{G}_o$ on the bundles $\mathcal{P}^{\mathrm{m}}$ and $\mathcal{Q}^{\mathrm{m}}$.** Let us start with the definition of the action of $\mathcal{G}_0$ on $\mathcal{P}^{\mathrm{m}}_k$. An automorphism $\Phi \in \mathcal{G}_o$ covering the symplectomorphism ϕ acts on the base $\mathcal{J}$ by sending j into $\phi^* j$. Let us lift this action to $\mathcal{P}^{\mathrm{m}}_k$. Given $j \in \mathcal{J}$, there are exactly two bundle maps $\phi^* : \delta_j \to \delta_{\phi^* j}$ covering ϕ such that the diagram

$$\begin{array}{ccc} \Lambda^{n,0}_j T^*M & \xrightarrow{\phi^*} & \Lambda^{n,0}_{\phi^* j} T^*M \\ {\scriptstyle \varphi_j}\uparrow & & \uparrow{\scriptstyle \varphi_{\phi^* j}} \\ \delta_j^2 & \xrightarrow{(\phi^*)^2} & \delta^2_{\phi^* j} \end{array}$$

commutes. Then the pull-back by

$$\Phi^k \otimes (\phi^*)^{-1} : L^k \otimes \delta_j \to L^k \otimes \delta_{\phi^* j}$$

is a linear map $\mathcal{P}^{\mathrm{m}}_{k,j} \to \mathcal{P}^{\mathrm{m}}_{k, \Phi^* j}$. Choosing the bundle map $\delta_j \to \delta_{\Phi^* j}$ in such a way that it depends continuously on j, we obtain the action of Φ on $\mathcal{P}^{\mathrm{m}}_k$. Since this action is only defined up to multiplication by -1, we obtain merely a $\mathcal{G}_o$-action on the orbibundle $(\mathcal{P}^{\mathrm{m}}_k, \mathbb{Z}_2)$.

Given a function $f \in C^\infty(M)$, let us define the operator

$$\mathrm{Op}_k(f) := f + \frac{1}{ik}(\nabla^{L^k}_X \otimes \mathrm{Id} + \mathrm{Id} \otimes D_X) : \mathcal{P}^{\mathrm{m}}_{k,j} \to \mathcal{P}^{\mathrm{m}}_{k,j}, \qquad j \in \mathcal{J}$$

where X is the Hamiltonian vector field of f and D_X is the first order differential operator of $C^\infty(M, \delta_j)$ such that

$$p_j \mathcal{L}_X \varphi_j(\beta^2) = 2\varphi_j\big(\beta \otimes (D_X \beta)\big), \qquad \forall\, \beta \in C^\infty(M, \delta_j) \tag{15}$$

with p_j the projection of $\Lambda^n T^*M \otimes \mathbb{C}$ onto $\Lambda^{n,0}_j T^*M$ with kernel $\Lambda^{n-1,1}_j T^*M \oplus \ldots \oplus \Lambda^{0,n}_j T^*M$.

Recall that the Lie algebra of $\mathcal{G}_o$ may be viewed as $C^\infty(M, \mathbb{R})$. Given $f \in C^\infty(M, \mathbb{R})$, $\Phi_t := \exp(tf)$ is the automorphism of L which covers the Hamiltonian flow ϕ_t of f at time t and is given by

$$\Phi_t(\xi) = e^{itf(x)} \mathcal{T}_t . \xi, \qquad \text{if } \xi \in L_x \tag{16}$$

where $\mathcal{T}_t$ is the parallel transport from L_x to $L_{\phi_t(x)}$ along the Hamiltonian flow.

Theorem 8.1. *Let $f \in C^\infty(M, \mathbb{R})$ and denote by*

$$U_t : \mathcal{P}^{\mathrm{m}}_k \to \mathcal{P}^{\mathrm{m}}_k$$

the action of $\exp(tf)$ *on* $\mathcal{P}^{\mathrm{m}}$. *Let* $j_0 \in \mathcal{J}$ *and* $j : \mathbb{R} \rightarrow \mathcal{J}$ *be the curve* $j_t = \phi_t^* j_0$. *For any* $s_0 \in \mathcal{P}^{\mathrm{m}}_{k,j_0}$ *the section* s *of* $j^*\mathcal{P}^{\mathrm{m}}_k$ *defined by* $s_t = U_t.s_0$ *satisfies*

$$\frac{1}{ik} \nabla^{j^*\mathcal{P}^{\mathrm{m}}_k}_{\partial_t} \; s = \mathrm{Op}_k(f) \; s.$$

The action of the symplectomorphism group on $\mathcal{J}$ preserves the subspace of integrable almost complex structures. Assume that $j_0 \in \mathcal{J}_{\mathrm{int}}$ and that the submanifold $\mathcal{J}_{\mathrm{int}}$ is invariant under the action of $\exp(tf)$. Then the operator U_t restricts to an operator $\mathcal{Q}^{\mathrm{m}}_{k,j_0} \rightarrow \mathcal{Q}^{\mathrm{m}}_{k,j_t}$ and the section s of $j^*\mathcal{Q}^{\mathrm{m}}_k$ defined as above satisfies

$$\frac{1}{ik} \nabla^{j^*\mathcal{Q}^{\mathrm{m}}_k}_{\partial_t} \; s = \mathcal{Q}^{\mathrm{m}}_k(f) \; s$$

where $\mathcal{Q}^{\mathrm{m}}_k(f)$ is defined by

$$\mathcal{Q}^{\mathrm{m}}_k(f) := \Pi_{k,j} \, \mathrm{Op}_k(f) : \mathcal{Q}^{\mathrm{m}}_{k,j} \rightarrow \mathcal{Q}^{\mathrm{m}}_{k,j}.$$

We proved in [8] (cf. theorem 1.5) the the sequence $(\mathcal{Q}^{\mathrm{m}}_k(f))_k$ is a Toeplitz operator whose normalized symbol is $f + O(\hbar^2)$.

Proof. First we may assume that $s_0 = \alpha \otimes \beta$ with $\alpha \in C^\infty(M, L^k)$ and $\beta \in C^\infty(M, \delta_{j_0})$. Furthermore since $U_t \circ U_s = U_{t+s}$, it is sufficient to prove the result at $t = 0$. Let us write

$$\alpha_t = \Phi_t^* \alpha, \qquad \beta_t = \phi_t^* \beta \in C^\infty(M, \delta_{j_t}).$$

Then $U_t.(\alpha \otimes \beta) = \alpha_t \otimes \beta_t$ and consequently

$$\nabla^{j^*\mathcal{P}^{\mathrm{m}}_k}_{\partial_t} \; s = \dot{\alpha}_t \otimes \beta_t + \alpha_t \otimes \dot{\beta}_t$$

where the derivative $\dot{\alpha}_t$ is in the t-independent vector space $C^\infty(M, L^k)$ and

$$\dot{\beta}_0 := \frac{d}{dt}\Big|_{t=0} \Psi_t^{-1} \Phi_t^* \beta \tag{17}$$

with $\Psi_t : \delta_{j_0} \rightarrow \delta_{j_t}$ the continuous family of half-form bundle morphisms such that Ψ_0 is the identity of δ_{j_0}. It is a classical result that

$$\dot{\alpha}_t = ik \Big(f + \frac{1}{ik} \nabla^{L^k}_{X_f} \Big) \alpha_t$$

So it remains to prove that

$$\dot{\beta}_0 = D_X \beta$$

Denote by $\tilde{D}$ the map sending $\beta \in \delta_{j_0}$ into $\dot{\beta}_0$ defined in (17). $\tilde{D}$ is a first order differential operator. We have to prove that

$$p_{j_0} \mathcal{L}_X \varphi_{j_0}(\beta^2) = 2 \varphi_{j_0}(\beta \otimes \tilde{D}\beta)$$

We have

$$\begin{aligned} p_{j_0} \mathcal{L}_X \varphi_{j_0}(\beta^2) &= p_{j_0} \frac{d}{dt}\Big|_{t=0} \phi_t^* \varphi_{j_0}(\beta^2) \\ &= p_{j_0} \frac{d}{dt}\Big|_{t=0} \varphi_{j_t}((\phi_t^*\beta)^2) \\ &= \frac{d}{dt}\Big|_{t=0} p_{j_0} \varphi_{j_t}((\phi_t^*\beta)^2) \end{aligned}$$

Now it follows from the definition of Ψ_{j_0,j_t} that

$$p_{j_0} \gamma = \Psi^{-1}_{j_0,j_t} \gamma, \qquad \forall \; \gamma \in \Lambda^{n,0}_{j_t} T^*M$$

Since Ψ_t is a half-form bundle morphism, we obtain

$$\begin{aligned} p_{j_0}\mathcal{L}_X\varphi_{j_0}(\beta^2) =& \frac{d}{dt}\Big|_{t=0}\Psi^{-1}_{j_0,j_t}\varphi_{j_t}((\phi_t^*\beta)^2) \\ =& \frac{d}{dt}\Big|_{t=0}\varphi_{j_0}((\Psi_t^{-1}\phi_t^*\beta)^2) \\ =& \varphi_{j_0}\frac{d}{dt}\Big|_{t=0}((\Psi_t^{-1}\phi_t^*\beta)^2) \\ =& 2\varphi_{j_0}(\beta\otimes\tilde{D}\beta) \end{aligned}$$

which was to be proved. □

8.2. **Schrödinger equation.** As a corollary of theorem 8.1, we obtain the relation between the parallel transport in $\mathcal{P}_k^{\mathrm{m}}$, the action of $\mathcal{G}_0$ and the Schrödinger equation with Hamiltonian $\mathrm{Op}_k(f)$.

Corollary 8.2. *Let $\mathcal{T}_t^{\mathcal{P}_k^{\mathrm{m}}} : \mathcal{P}^{\mathrm{m}}_{k,j_0} \rightarrow \mathcal{P}^{\mathrm{m}}_{k,j_t}$ be the parallel transport along the curve $t \rightarrow j_t$. Then the family of operators*

$$P_t := (\mathcal{T}_t^{\mathcal{P}_k^{\mathrm{m}}})^{-1}\circ U_t : \mathcal{P}^{\mathrm{m}}_{k,j_0}\rightarrow \mathcal{P}^{\mathrm{m}}_{k,j_0}$$

satisfies

$$\frac{1}{ik}\frac{d}{dt}P_ts_0 = \mathrm{Op}_k(f)\ P_ts_0$$

for any $s_0\in\mathcal{P}^{\mathrm{m}}_{k,j_0}$.

Proof. Since $\mathcal{T}_t^{\mathcal{P}_k^{\mathrm{m}}}$ is parallel transport,

$$\begin{aligned} \frac{1}{ik}\frac{d}{dt}P_ts_0 =& \frac{1}{ik}(\mathcal{T}_t^{\mathcal{P}_k^{\mathrm{m}}})^{-1}\nabla^{j^*\mathcal{P}_k^{\mathrm{m}}}_{\partial_t}U_ts_0 \\ =& (\mathcal{T}_t^{\mathcal{P}_k^{\mathrm{m}}})^{-1}\,\mathrm{Op}_k(f)U_ts_0 \end{aligned}$$

by theorem 8.1. Furthermore for any function $g\in C^\infty(M)$, we have

$$U_t\,\mathrm{Op}_k(g) = \mathrm{Op}_k(\Phi_t^*g)U_t.$$

So U_t and $\mathrm{Op}_k(f)$ commutes, because f is preserved by its Hamiltonian flow. Consequently

$$\frac{1}{ik}\frac{d}{dt}P_ts_0 = P_t\,\mathrm{Op}_k(f)s_0. \tag{18}$$

To conclude, we prove that P_t and $\mathrm{Op}_k(f)$ commute. We have

$$\begin{aligned} \frac{d}{dt}\big(P_t\,\mathrm{Op}_k(f)(P_t)^{-1}\big) =& \dot{P}_t\,\mathrm{Op}_k(f)(P_t)^{-1} - P_t\,\mathrm{Op}_k(f)(P_t)^{-1}\dot{P}_t(P_t)^{-1} \\ =& ik\big(P_t\,\mathrm{Op}_k^2(f)(P_t)^{-1} - P_t\,\mathrm{Op}_k(f)(P_t)^{-1}P_t\,\mathrm{Op}_k(f)(P_t)^{-1}\big) \\ =& 0 \end{aligned}$$

because of (18). □

Let us assume again that $j_0\in\mathcal{J}_{\mathrm{int}}$ and that $\mathcal{J}_{\mathrm{int}}$ is preserved by the action of $\exp(tf)$. Then arguing as in the previous proof, we can deduce the similar result for the bundle $\mathcal{Q}_k^{\mathrm{m}}$.

Corollary 8.3. *Let $\mathcal{T}_t^{\mathcal{Q}_k^{\mathrm{m}}} : \mathcal{Q}^{\mathrm{m}}_{k,j_0}\rightarrow\mathcal{Q}^{\mathrm{m}}_{k,j_t}$ be the parallel transport in $\mathcal{Q}_k^{\mathrm{m}}$ along the curve $t\rightarrow j_t$. Then we have*

$$\frac{1}{ik}\frac{d}{dt}P_t^{\mathcal{Q}_k^{\mathrm{m}}}s_0 = \mathcal{Q}_k^{\mathrm{m}}(f)P_t^{\mathcal{Q}_k^{\mathrm{m}}}s_0, \qquad \forall s_0\in\mathcal{Q}^{\mathrm{m}}_{k,j_0}$$

where $P_t^{\mathcal{Q}_k^{\mathrm{m}}}$ is the operator $(\mathcal{T}_t^{\mathcal{Q}_k^{\mathrm{m}}})^{-1}\circ U_t : \mathcal{Q}^{\mathrm{m}}_{k,j_0}\rightarrow\mathcal{Q}^{\mathrm{m}}_{k,j_0}$.

Recall that $\mathcal{Q}^{\mathrm{m}}(f)$ is a Toeplitz operator whose normalized symbol is $f + O(\hbar^2)$. Then theorem 5.3 follows from the fact that the parallel transport in $\mathcal{Q}^{\mathrm{m}}$ is a unitary Fourier integral operator (cf. theorem 7.1).

8.3. **Commutators and curvature.** In the next theorem, we compute the commutator of $\mathrm{Op}_k(f)$ and $\mathrm{Op}_k(g)$ (resp. $\mathcal{Q}^{\mathrm{m}}_k(f)$ and $\mathcal{Q}^{\mathrm{m}}_k(g)$) in terms of the curvature of $\mathcal{P}^{\mathrm{m}}_k \to \mathcal{J}$ (resp. $\mathcal{Q}^{\mathrm{m}}_k \to \mathcal{J}$).

Theorem 8.4. *Let f and g be two functions of $C^\infty(M,\mathbb{R})$. Then*

$$ik[\mathrm{Op}_k(f), \mathrm{Op}_k(g)] = \mathrm{Op}_k(\{f,g\}) + (ik)^{-1} R^{\mathcal{P}^{\mathrm{m}}_k}(\eta,\mu)$$

where η and μ are the vector fields of $\mathcal{J}$ corresponding to the infinitesimal action of f and g on $\mathcal{J}$. Furthermore,

$$ik[\mathcal{Q}^{\mathrm{m}}_k(f), \mathcal{Q}^{\mathrm{m}}_k(g)] = \mathcal{Q}^{\mathrm{m}}_k(\{f,g\}) + (ik)^{-1} R^{\mathcal{Q}^{\mathrm{m}}_k}(\eta,\mu)$$

when η and μ are tangent to $\mathcal{J}_{\mathrm{int}}$.

Since $\mathcal{Q}^{\mathrm{m}}(f)$ is a Toeplitz operator with normalized symbol $f + O(\hbar^2)$, it follows from theorem 3.4 that

$$ik[\mathcal{Q}^{\mathrm{m}}_k(f), \mathcal{Q}^{\mathrm{m}}_k(g)] = \mathcal{Q}^{\mathrm{m}}_k(\{f,g\}) + O(k^{-2}) \tag{19}$$

This is consistent with the second equation of the previous theorem and the fact that $R^{\mathcal{Q}^{\mathrm{m}}_k}(\eta,\mu)$ is $O(k^{-1})$. Moreover, on can prove in this way that $R^{\mathcal{Q}^{\mathrm{m}}_k}$ can't be $O(k^{-2})$ except for particular sub manifolds $\mathcal{J}_{\mathrm{int}}$. Indeed given a complex structure j, there is a star-product $*_j$ such that for any functions f and g,

$$\mathcal{Q}^{\mathrm{m}}_k(f)_j \circ \mathcal{Q}^{\mathrm{m}}_k(g)_j \equiv \mathcal{Q}^{\mathrm{m}}_k(h(.,k))_j + O(k^{-\infty})$$

where $h(.,k)$ has an asymptotic expansion $h_0 + k^{-1}h_1 + ..$ whose coefficients satisfy $f *_j g = \sum \hbar^l h_l$. One can prove that $*_j$ is a Vey star-product, i.e. the bidifferential operators defining $*_j$ have the same principal symbol than the bidifferential operators defining the Moyal-Weyl star-product. Hence

$$i\hbar^{-1}\big(f *_j g - g *_j f\big) = \{f,g\} + \hbar^2 A(f,g) + O(\hbar^3)$$

where A is a non-vanishing bidifferential operator. So if η and μ are the infinitesimal actions of f and g,

$$R^{\mathcal{Q}^{\mathrm{m}}_k}(\eta,\mu)_j = ik^{-1}\mathcal{Q}^{\mathrm{m}}_k(A(f,g))_j + O(k^{-2}).$$

and $\mathcal{Q}^{\mathrm{m}}_k(A(f,g))_j$ is not $O(k^{-1})$ as soon as $A(f,g)$ doesn't vanish.

The first equation of the theorem can be deduced from the expression of the curvature in theorem 7.2 as follows. First recall that

$$ik\big[f + \tfrac{1}{ik}\nabla^{L^k}_X, g + \tfrac{1}{ik}\nabla^{L^k}_Y\big] = \{f,g\} + \tfrac{1}{ik}\nabla^{L^k}_{[X,Y]},$$

where X and Y are the Hamiltonian vector fields of f and g. Then we compute the bracket of the operators D_X, D_Y entering in the definition of $\mathrm{Op}_k(f)$ and $\mathrm{Op}_k(g)$ (cf. (15)) in terms of the infinitesimal actions η and μ on $\mathcal{J}$ of X and Y respectively.

Lemme 8.5. *We have $\big[D_X, D_Y\big] = D_{[X,Y]} + \frac{1}{2}\mathrm{tr}(\bar\mu\eta - \bar\eta\mu)$*

Proof. Given any one-form β and complex structure j, denote by $p^{1,0}_j\beta$ and $p^{0,1}_j\beta$ the component of β of type $(1,0)$ and $(0,1)$ for j. The vector field η is given at j by

$$\eta_j \in C^\infty(M, \mathrm{Hom}(\Lambda^{1,0}_j T^*M, \Lambda^{0,1}_j T^*M)), \qquad \eta_j(\alpha) = p^{0,1}_j \mathcal{L}_X\alpha.$$

Consequently, if $\alpha \in \Omega^{1,0}_j M$

$$\begin{aligned}[][p_j^{1,0}\mathcal{L}_X, p_j^{1,0}\mathcal{L}_Y]\alpha &= p_j^{1,0}\mathcal{L}_X(\mathcal{L}_Y - p_j^{0,1}\mathcal{L}_Y)\alpha - p_j^{1,0}\mathcal{L}_Y(\mathcal{L}_X - p_j^{0,1}\mathcal{L}_X)\alpha \\ &= (p_j^{1,0}\mathcal{L}_{[X,Y]} - \bar{\eta}_j\mu_j + \bar{\mu}_j\eta_j)\alpha\end{aligned}$$

So if p_j is the projection from $\Lambda^n T^*M \otimes \mathbb{C}$ onto $\Lambda_j^{n,0}T^*M$ with kernel $\Lambda_j^{n-1,1}T^*M \oplus ... \oplus \Lambda_j^{0,n}T^*M$, we have for any $(n,0)$-form α

$$[p_j\mathcal{L}_X, p_j\mathcal{L}_Y]\alpha = (p_j\mathcal{L}_{[X,Y]} + \operatorname{tr}(\bar{\mu}_j\eta_j - \bar{\eta}_j\mu_j))\alpha$$

which implies the result. $\square$

Consequently,

$$ik[\mathrm{Op}_k(f), \mathrm{Op}_k(g)] = \mathrm{Op}_k(\{f,g\}) + \tfrac{1}{2ik}\operatorname{tr}(\bar{\mu}\eta - \bar{\eta}\mu)$$

and we deduce the first equation of theorem 8.4 from theorem 7.2.

To prove the second equation, we start with the following relation between the curvatures of $\mathcal{P}_k^{\mathrm{m}}$ and $\mathcal{Q}_k^{\mathrm{m}}$.

Lemme 8.6. *For every vector fields η and μ of $\mathcal{J}_{\mathrm{int}}$, we have*

$$R^{\mathcal{Q}_k^{\mathrm{m}}}(\eta,\mu) = \Pi_k\big[\nabla_\eta^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k, \nabla_\mu^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k\big] + \Pi_k R^{\mathcal{P}_k^{\mathrm{m}}}(\eta,\mu)$$

where $\nabla^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k$ is the commutator $[\nabla^{\mathcal{P}_k^{\mathrm{m}}}, \Pi_k]$.

Proof. We have

$$\begin{aligned}\nabla_\eta^{\mathcal{Q}_k^{\mathrm{m}}}\nabla_\mu^{\mathcal{Q}_k^{\mathrm{m}}} =&\Pi_k\nabla_\eta^{\mathcal{P}_k^{\mathrm{m}}}\Pi_k\nabla_\mu^{\mathcal{P}_k^{\mathrm{m}}}\Pi_k \\ =&\Pi_k\big(\nabla_\eta^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k\big)\nabla_\mu^{\mathcal{P}_k^{\mathrm{m}}}\Pi_k + \Pi_k\nabla_\eta^{\mathcal{P}_k^{\mathrm{m}}}\nabla_\mu^{\mathcal{P}_k^{\mathrm{m}}}\Pi_k \\ =&\Pi_k\big(\nabla_\eta^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k\big)\big(\nabla_\mu^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k\big) + \Pi_k\big(\nabla_\eta^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k\big)\Pi_k\nabla_\mu^{\mathcal{P}_k^{\mathrm{m}}} \\ &+ \Pi_k\nabla_\eta^{\mathcal{P}_k^{\mathrm{m}}}\nabla_\mu^{\mathcal{P}_k^{\mathrm{m}}}\Pi_k\end{aligned}$$

Since $\Pi_k^2 = \Pi_k$, we have $\Pi_k\big(\nabla^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k\big)\Pi_k = 0$. So the second term of the sum vanishes. Using this it is easy to compute the curvature of $\mathcal{Q}_k^{\mathrm{m}}$

$$\begin{aligned}R^{\mathcal{Q}_k^{\mathrm{m}}}(\eta,\mu) &= [\nabla_\eta^{\mathcal{Q}_k^{\mathrm{m}}}, \nabla_\mu^{\mathcal{Q}_k^{\mathrm{m}}}] - \nabla_{[\eta,\mu]}^{\mathcal{Q}_k^{\mathrm{m}}} \\ &= \Pi_k\big[\nabla_\eta^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k, \nabla_\mu^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k\big] + \Pi_k[\nabla_\eta^{\mathcal{P}_k^{\mathrm{m}}}, \nabla_\mu^{\mathcal{P}_k^{\mathrm{m}}}] - \Pi_k\nabla_{[\eta,\mu]}^{\mathcal{P}_k^{\mathrm{m}}} \\ &= \Pi_k\big[\nabla_\eta^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k, \nabla_\mu^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k\big] + \Pi_k R^{\mathcal{P}_k^{\mathrm{m}}}(\eta,\mu)\end{aligned}$$

which proves the result. $\square$

On the other hand we can compute the commutator of Π_k with $\mathrm{Op}_k(f)$ in terms of the covariant derivative of Π_k.

Lemme 8.7. *Let $f \in C^\infty(M,\mathbb{R})$ and η be the infinitesimal action of f on $\mathcal{J}$, then*

$$\tfrac{1}{ik}\nabla_\eta^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k = \big[\mathrm{Op}_k(f), \Pi_k\big].$$

Proof. This follows from theorem 8.1 by derivating the relation $\Pi_{k,j_t}U_t = U_t\Pi_{k,j_0}$. $\square$

Applying twice this last lemma, we obtain

$$\begin{aligned}\Pi_k\,\mathrm{Op}_k(f)\Pi_k\,\mathrm{Op}_k(g)\Pi_k =&\Pi_k\,\mathrm{Op}_k(f)\Pi_k(\mathrm{Op}_k(g) + \tfrac{1}{ik}\nabla_\mu^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k) \\ =&\Pi_k(\mathrm{Op}_k(f) + \tfrac{1}{ik}\nabla_\eta^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k)(\mathrm{Op}_k(g) + \tfrac{1}{ik}\nabla_\mu^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k)\end{aligned}$$

Hence

$$\big[\Pi_k \operatorname{Op}_k(f)\Pi_k, \Pi_k \operatorname{Op}_k(g)\Pi_k\big] = \Pi_k\big[\operatorname{Op}_k(f) + \tfrac{1}{ik}\nabla_\eta^{\operatorname{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k,\ \operatorname{Op}_k(g) + \tfrac{1}{ik}\nabla_\mu^{\operatorname{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k\big] \tag{20}$$

Similarly, we have

$$\begin{aligned}\Pi_k \operatorname{Op}_k(f)\Pi_k \operatorname{Op}_k(g)\Pi_k =&(\operatorname{Op}_k(f) - \tfrac{1}{ik}\nabla_\eta^{\operatorname{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k)\Pi_k \operatorname{Op}_k(g)\Pi_k \\ =&(\operatorname{Op}_k(f) - \tfrac{1}{ik}\nabla_\eta^{\operatorname{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k)(\operatorname{Op}_k(g) - \tfrac{1}{ik}\nabla_\mu^{\operatorname{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k)\Pi_k\end{aligned}$$

So

$$\big[\Pi_k \operatorname{Op}_k(f)\Pi_k, \Pi_k \operatorname{Op}_k(g)\Pi_k\big] = \big[\operatorname{Op}_k(f) - \tfrac{1}{ik}\nabla_\eta^{\operatorname{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k,\ \operatorname{Op}_k(g) - \tfrac{1}{ik}\nabla_\mu^{\operatorname{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k\big]\Pi_k \tag{21}$$

Now equations (20) and (21) imply

$$\big[\Pi_k \operatorname{Op}_k(f)\Pi_k, \Pi_k \operatorname{Op}_k(g)\Pi_k\big] = \Pi_k\big[\operatorname{Op}_k(f), \operatorname{Op}_k(g)\big]\Pi_k + \Pi_k\big[\tfrac{1}{ik}\nabla_\eta^{\operatorname{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k,\ \tfrac{1}{ik}\nabla_\mu^{\operatorname{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k\big]\Pi_k$$

And we deduce the second equation of theorem 8.4 from the first one and lemma 8.6.

8.4. **An analog result in finite dimension.** It is interesting to note that the expression for the curvature in theorem 7.2 is a direct consequence at least formally of theorem 8.1 on the infinitesimal action. Consider a finite dimensional vector bundle $E \to X$ endowed with a connection ∇. Assume a Lie group G acts on E preserving the connection. Given η in the Lie algebra $\mathfrak{g}$ of G, denote by η_X the vector field corresponding to the infinitesimal action on the base X and by $\mathcal{L}_\eta$ the infinitesimal action on $C^\infty(X, E)$. Then $\mathcal{L}_\eta - \nabla_{\eta_X}$ acts by exterior multiplication by a section

$$M(\eta) \in C^\infty(M, \operatorname{End}(E))$$

called the moment of η.

Proposition 8.8. *For any vectors $\eta, \mu \in \mathfrak{g}$, we have*

$$[M(\eta), M(\mu)] = M([\eta, \mu]) + R^E(\eta_X, \mu_X),$$

where R^E is curvature of ∇.

Proof. Since the connection in invariant, we have

$$[\mathcal{L}_\eta, \nabla] = 0$$

Replacing $\mathcal{L}_\eta$ with $\nabla_{\eta_X} + M(\eta)$, we obtain that

$$\nabla^{\operatorname{End}(E)} M(\eta) = R^E(\eta_X, .). \tag{22}$$

Since $\eta \to \nabla_{\eta_X} + M(\eta)$ is a Lie algebra representation, we have for any $\eta, \mu \in \mathfrak{g}$

$$\begin{aligned}[\nabla_{\eta_X} + M(\eta), \nabla_{\mu_X} + M(\mu)] =&\nabla_{[\eta,\mu]_X} + M([\eta, \mu]) \\ =&\nabla_{[\eta_X, \mu_X]} + M([\eta, \mu])\end{aligned}$$

Assuming that ∇ is G-invariant, we can compute by the lemma the commutators

$$[\nabla_{\eta_X}, M(\mu)] = R^E(\mu_X, \eta_X), \qquad [M(\eta), \nabla_{\mu_X}] = -R^E(\eta_X, \mu_X) = R^E(\mu_X, \eta_X).$$

Then using that

$$[\nabla_{\eta_X}, \nabla_{\mu_X}] = \nabla_{[\eta_X, \mu_X]} + R^E(\eta_X, \mu_X)$$

we obtain the proposition. ∎

If we apply this equation in our infinite dimensional setting with E the bundle of quantum spaces or prequantum spaces and G the group of prequantization bundle automorphism, we obtain theorem 8.4.

It is also interesting to consider the situation the introduction without half-form bundle (cf. section 1.1). Let

$$\mathcal{P}_k := C^\infty(M, L^k) \times \mathcal{J}_{\rm int}$$

be the prequantum space bundle and $\mathcal{Q}_k$ be the subbundle of quantum spaces. As explained in the introduction the Group $\mathcal{G}$ of prequantization bundle automorphisms acts on these bundle. Moreover these bundles are endowed with invariant connection. Then one proves that the moment of $f \in C^\infty(M)$ on $\mathcal{Q}_k$ is the operator $(ik)\mathcal{Q}_k(f)$, where $\mathcal{Q}_k(f)$ is the Toeplitz operator

$$\mathcal{Q}_k(f) := \Pi_k\big(f + \frac{1}{ik}\nabla^{L^k}_X\big)$$

with X the Hamiltonian vector field of f. Consequently, one has

$$ik[\mathcal{Q}_k(f), \mathcal{Q}_k(g)] = \mathcal{Q}_k(\{f,g\}) + (ik)^{-1} R^{\mathcal{Q}_k}(\eta, \mu)$$

where η and μ are the infinitesimal actions of f and g respectively on $\mathcal{J}_{\rm int}$. Then we recover the main point of the argument of Ginzburg and Montgomery: if the curvature vanishes, the map

$$f \to (ik)\mathcal{Q}_k(f)$$

is a Lie algebra representation. Furthermore, the result of Foth and Uribe gives the first correction terms in the computation of the commutator of two Toeplitz operators.

9. Preliminaries for the proof of theorem 7.1

Given two complex structures j_a, j_b, we introduce a class of operators from $\mathcal{P}^{\rm m}_{k,j_a}$ to $\mathcal{P}^{\rm m}_{k,j_b}$ extending the class of Fourier integral operator we considered previously. First by using the scalar product of $\mathcal{P}^{\rm m}_{k,j_a}$, the Schwartz kernels of these operators can be regarded as C^∞ sections of the bundle

$$\big(L^k \otimes \delta_{j_b}\big) \boxtimes \big(\bar{L}^k \otimes \bar{\delta}_{j_a}\big) \to M^2.$$

Let N be a non-negative integer. We say that $(T_k)_{k\in\mathbb{N}^*}$ is an operator of $\mathcal{A}_N(j_a, j_b)$ if its Schwartz kernel is of the form

$$T_k(x,y) = \Big(\frac{k}{2\pi}\Big)^n E^k(x,y) f(x,y,k) + O(k^{-\infty})$$

where

- E is a section of $L \boxtimes \bar{L} \to M^2$ such that $\|E(x,y)\| < 1$ if $x \neq y$,
$$E(x,x) = u \otimes \bar{u}, \quad \forall u \in L_x \text{ such that } \|u\| = 1,$$
and $\bar{\partial} E \equiv 0$ modulo a section vanishing to any order along Δ.
- $f(.,k)$ is a sequence of sections of $\delta_{j_b} \boxtimes \bar{\delta}_{j_a} \to V$ which has an asymptotic expansion in the C^∞ topology
$$f(.,k) = k^N f_{-N} + k^{N-1} f_{-N+1} + ...$$
where for all $0 \leqslant l \leqslant N$, f_{-l} vanishes to order $2l$ along the diagonal of M^2.

As a result, the Schwartz kernel of T_k is uniformly $O(k^{n+N})$. It is $O(k^{n+N-\frac{1}{2}})$ if and only if f_l vanishes at order $2l+1$ along the diagonal whenever $2l+1 \geqslant 0$. This follows from the fact that $\ln\|E\| < 0$ outside the diagonal and its Hessian along the diagonal is non-degenerate in the transverse directions (cf. lemma 1 in [6]). We define the symbol of (T_k) as

$$\hbar^{-N}[f_{-N}]_{2N} + \hbar^{-N+1}[f_{-N+1}]_{2(N-1)} + ... + \hbar^{-1}[f_{-1}]_2 + [f_0]$$

where $[f_{-N+l}]_{2(N-l)}$ is the equivalence class of f_{N-l} modulo the functions vanishing at order $2(N-l)+1$ along the diagonal. So with the usual identification, the space of symbol is the space of sections of

$$S_N^{j_a,j_b} := \delta_{j_b} \otimes \bar{\delta}_{j_a} \otimes \Big(\hbar^{-N} \operatorname{Sym}_{2N} \mathcal{C} \oplus \hbar^{-N+1} \operatorname{Sym}_{2(N-1)} \mathcal{C} \oplus ... \oplus \operatorname{Sym}_0 \mathcal{C} \Big)$$

where $\mathcal{C}$ is the conormal bundle of the diagonal of M^2.

Theorem 9.1. *The composition of $S \in \mathcal{A}_N(j_b, j_c)$ with $S' \in \mathcal{A}_{N'}(j_a, j_b)$ is an operator of $\mathcal{A}_{N+N'}(j_a, j_c)$. Furthermore there exists a bilinear bundle map*

$$L_{N,N'}^{j_a,j_b,j_c} : S_N^{j_b,j_c} \times S_{N'}^{j_a,j_b} \to S_{N+N'}^{j_a,j_c}$$

such that the principal symbol of SS' is $L_{N,N'}^{j_a,j_b,j_c}(\sigma, \sigma')$ if σ and σ' are the symbols of S and S' respectively.

Proof. The proof is essentially the same as the one of theorem 3.2 about the composition of Fourier integral operators. The computation of the symbol follows from the version of the stationary phase lemma stated in the appendix of [8]. □

By extending the Fourier integral operators of $\mathcal{F}((j_a, \delta_{j_a}), (j_b, \delta_{j_b}))$ to operators $\mathcal{P}_{k,j_a}^{\mathrm{m}} \to \mathcal{P}_{k,j_b}^{\mathrm{m}}$ in such a way that they satisfy

$$\Pi_{k,j_a} T_k \Pi_{k,j_b} = T_k, \quad k = 1, 2, ...$$

$\mathcal{F}((j_a, \delta_{j_a}), (j_b, \delta_{j_b}))$ becomes a subspace of $\mathcal{A}_0(j_a, j_b)$. Both definitions of principal symbols are the same if we identify the sections of $S_0^{j_a,j_b} = \delta_{j_b} \otimes \bar{\delta}_{j_a}$ with the fiber bundle morphisms $\delta_{j_a} \to \delta_{j_b}$ by using the scalar product of δ_{j_a}.

Theorem 9.2. *If T is an operator of $\mathcal{A}_N(j_a, j_b)$ with symbol σ, then $\Pi_{k,j_a} T \Pi_{k,j_b}$ is a Fourier integral operator of $\mathcal{F}((j_a, \delta_{j_a}), (j_b, \delta_{j_b}))$. Furthermore the symbol of $\Pi_{k,j_a} T \Pi_{k,j_b}$ is $L_N^{j_a,j_b}(\sigma)$, where*

$$L_N^{j_a,j_b} : S_N^{j_a,j_b} \to S_0^{j_a,j_b}$$

is a fiber-bundle morphism.

Proof. Again the proof relies on the methods of section 4. To show that $\Pi_{k,j_a} S \Pi_{k,j_b}$ is a Fourier integral operator, we argue as in the following of lemma 4.2. The other part is an application of the stationary phase lemma in the appendix of [8]. □

Finally let us describe explicitly the symbol product for the composition of operators of $\mathcal{A}_N(j_a, j_a)$. To do this it is convenient to introduce complex coordinates (U, z^i) for j_a and write the symbol in the following way

$$\sigma(\hbar, \bar{Z}, Z, x) = \sum_{l=0}^{N} \hbar^{-l} \sigma_l(\bar{Z}, Z)(x), \qquad x \in U$$

with

$$\sigma_l(\bar{Z}, Z)(x) = \sum_{|\alpha|+|\beta|=2l} \frac{1}{\alpha!\beta!} \Big(\nabla^{\alpha(1)}_{(\partial_{\bar{z}^1},0)} ... \nabla^{\alpha(n)}_{(\partial_{\bar{z}^n},0)} \ \nabla^{\beta(1)}_{(0,\partial_{z^1})} ... \nabla^{\beta(n)}_{(0,\partial_{z^n})} f_l \Big)(x, x) \ \bar{Z}^\alpha Z^\beta$$

and ∇ a covariant derivation of $\delta_{j_a} \boxtimes \bar{\delta}_{j_a}$.

Theorem 9.3. *With the previous notations, the map of theorem 9.1 is*

$$L^{a,a,a}_{N,N'}(\sigma,\sigma')(\hbar,\bar{Z},Z,x) = \sum_{l=0}^{N+N'} \frac{\hbar^l}{l!}\Big[\Delta^l\big(\sigma(\hbar,\bar{Z}-\bar{Y},Y,x).\sigma'(\hbar,\bar{Y},Z-Y,x)\big)\Big]_{\bar{Y}=Y=0}$$

where Δ *is the operator*

$$\Delta := \sum_{i,j} G^{i,j}(x)\partial_{Y^i}\partial_{\bar{Y}^j}$$

with $(G^{i,j})$ *the inverse matrix of* $(G_{j,i})$ *whose coefficients are such that* $\omega = i\sum G_{i,j}dz^i\wedge d\bar{z}^j$.

There isn't any difficulty to extend these results to the case where the complex structures depends smoothly on a parameter. We end these preliminaries with the variations of the section E as a function of the complex structure. Let $x\in M$ and Γ be a germ at x of a Lagrangian submanifold of M. Let us fix a unitary section s of $L\to\Gamma$. Let j_t be a curve in $\mathcal{J}_{\mathrm{int}}$. Then consider a smooth family E_t of sections of $L\to M$ such that $E_t = s$ along Γ and

$$\bar{\partial}_{j_t}E_t \equiv 0$$

modulo a section vanishing to any order along Γ. Let us write

$$\frac{d}{dt}E_t = f_tE_t.$$

on a neighborhood of x.

Proposition 9.4. *The function* f_0 *and its first derivatives vanish over* Γ. *Furthermore, if* Z *and* Z' *are holomorphic vector fields for the complex structure* j_0, *then*

$$\bar{Z}.\bar{Z}'.f_0 = \frac{1}{i}\omega(\bar{Z},\mu(\bar{Z}'))$$

along Γ, *with* $\mu\in C^\infty(M,\mathrm{Hom}(\Lambda^{1,0}_{j_0}T^*M,\Lambda^{0,1}_{j_0}T^*M))$ *the tangent vector to* j_t *at* j_0.

Since Γ is Lagrangian, $T^{0,1}_{j_t}M$ and $T\Gamma\otimes\mathbb{C}$ are transverse. So the result gives the Hessian of f along Γ.

Proof. Since $E_t(y)$ is constant for all $y\in\Gamma$, f_t vanishes along Γ. Let us write

$$\nabla E_t = \tfrac{1}{i}E_t\otimes\alpha_t.$$

We can prove that α_t vanishes along Γ and

$$\bar{Z}.\langle\alpha_0,W\rangle = \omega(\bar{Z},W) \tag{23}$$

if Z is a holomorphic vector fields for j_0 (cf. Lemma 4.2 in [8]). Since $\frac{d}{dt}$ et ∇ commute, we have $df_t = \frac{1}{i}\dot{\alpha}_t$. So df_t vanishes along Γ because the same holds for α_t.

Now let us associate to $j_t\in\mathcal{J}$ the section μ_t of $\mathrm{Hom}(\Lambda^{1,0}_{j_0}T^*M,\Lambda^{0,1}_{j_0}T^*M)$ as we did in (4). Then $T^{0,1}_{j_t}M$ is the graph of

$$-\mu^t_t : T^{0,1}_{j_0}M\to T^{1,0}_{j_0}M.$$

So if Z' is a section of $T^{1,0}_{j_0}M$, $\bar{Z}'-\mu^t_t(\bar{Z}')$ is a section of $T^{0,1}_{j_t}M$ and

$$\langle\alpha_t,\bar{Z}'-\mu^t_t(\bar{Z}')\rangle$$

vanishes to any order along Γ. Thus the same holds for the derivative

$$\langle\dot{\alpha}_t,\bar{Z}'-\mu^t_t(\bar{Z}')\rangle - \langle\alpha_t,\dot{\mu}^t_t(\bar{Z}')\rangle.$$

In particular, we have along Γ,

$$\bar{Z}.\langle \dot{\alpha}_0, \bar{Z}' \rangle = \bar{Z}.\langle \alpha_0, \dot{\mu}_0^t(\bar{Z}') \rangle$$

Here we used that $\mu_0 = 0$. Finally the results follows from $df_t = \frac{1}{i}\dot{\alpha}_t$ and (23). $\square$

10. Proof of theorem 7.1

Let us start with the computation of the curvature. By lemma 8.6,

$$R^{\mathcal{Q}_k^{\mathrm{m}}}(\eta,\mu) = \Pi_k \big[\nabla_\eta^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})} \Pi_k, \nabla_\mu^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})} \Pi_k \big] + \Pi_k R^{\mathcal{P}_k^{\mathrm{m}}}(\eta,\mu).$$

By theorem 7.2, $\Pi_k R^{\mathcal{P}_k^{\mathrm{m}}}(\eta,\mu)$ is at j a Toeplitz operator of $\mathcal{Q}_j^{\mathrm{m}}$ with principal symbol $\frac{1}{2}\mathrm{tr}(\eta.\bar{\mu} - \mu.\bar{\eta})(j)$. Then it follows from the following proposition that $R^{\mathcal{Q}_k^{\mathrm{m}}}(\eta,\mu)(j)$ is a Toeplitz operator with vanishing principal symbol.

Proposition 10.1. *For any tangent vector $\eta, \mu \in T_j \mathcal{J}_{\mathrm{int}}$, the operator*

$$\Pi_k \big[\nabla_\eta^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})} \Pi_k, \nabla_\mu^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})} \Pi_k \big]$$

is a Toeplitz operator of $\mathcal{Q}_j^{\mathrm{m}}$ with principal symbol $-\frac{1}{2}\mathrm{tr}(\eta.\bar{\mu} - \mu.\bar{\eta})(j)$.

Proof. First we prove that $\nabla_\eta^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})} \Pi_k$ is an operator of $\mathcal{A}_2(j,j)$ and compute its symbol. Let j_t be a curve of $\mathcal{J}_{\mathrm{int}}$ whose tangent vector at 0 is η. Let $\Psi_t : \delta_{j_0} \to \delta_{j_t}$ be the continuous family of half-form bundle morphisms such that Ψ_0 is the identity of δ_{j_0}. We have at j_0

$$\nabla_\eta^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})} \Pi_k.\Phi = \frac{d}{dt}\Big|_{t=0} \Psi_t^{-1} \Pi_{k,j_t} \Psi_t \; \Phi, \qquad \Phi \in \mathcal{P}_{k,j_0}^{\mathrm{m}}$$

Recall that Π_{k,j_t} is an operator of $\mathcal{A}_0(j_t,j_t)$ with symbol 1. Thus its kernel is of the form

$$\Big(\frac{k}{2\pi}\Big)^n E_t^k(x,y) f_t(x,y,k) + O(k^{-\infty})$$

where $f_t(.,k)$ is a sequence of sections of $\delta_{j_t} \boxtimes \bar{\delta}_{j_t}$ equal to $1 + O(k^{-1})$ on the diagonal. We obtain the kernel of $\Psi_t^{-1}\Pi_{k,j_t}\Psi_t$ by replacing f_t with

$$\Psi_t^{-1}(x) f_t(x,y,k) \Psi_t(y)$$

which again is equal to $1 + O(k^{-1})$ on the diagonal. Derivating with respect to t, we deduce that $\nabla_\eta^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})} \Pi_k$ is an operator of $\mathcal{A}_2(j,j)$ with symbol

$$\hbar^{-1}[g]$$

where g is the function of M^2 such that $\frac{d}{dt}\Big|_{t=0} E_t = g E_0$.

Let us compute $[g]$. Let (z^i) be a complex coordinates system for j_0 such that $\omega = i \sum dz^j \wedge d\bar{z}^j$ at x. Denote by $U(t)$ the symmetric matrix such that the family

$$dz^i + \textstyle\sum_j U_{ij}(t) d\bar{z}^j, \qquad i = 1, ..., n$$

is a base of $\Lambda^{1,0}_{j_t} T^*_x M$. So the derivative $\dot{U}$ of $U(t)$ at $t=0$ is the matrix of $\eta(j_0)$. By proposition 9.4, the symbol of $\nabla_\eta^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})} \Pi_k$ at x is

$$\hbar^{-1}[g](\bar{Z}, Z, x) = -\frac{1}{2\hbar} \sum (\dot{U}_{ij} \bar{Z}^i \bar{Z}^j + \dot{\bar{U}}_{ij} Z^i Z^j)$$

where we used the notations of theorem 9.3.

Then it follows from theorems 9.1 and 9.2 that

$$\Pi_k \big[\nabla_\eta^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})} \Pi_k, \nabla_\mu^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})} \Pi_k \big]$$

is a Toeplitz operator and we compute its symbol by applying theorem 9.3. First the symbol of $\big[\nabla_\eta^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k, \nabla_\mu^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k\big]$ is at x

$$\hbar^{-1}\sum\big(\dot{U}_{ik}\dot{\bar{V}}_{jk} - \dot{V}_{ik}\dot{\bar{U}}_{jk}\big)\bar{Z}^i Z^j - \frac{\hbar^{-2}}{4}\sum\big(\dot{U}_{ij}\dot{\bar{V}}_{kl} - \dot{V}_{ij}\dot{\bar{U}}_{kl}\big)\bar{Z}^i\bar{Z}^j Z^k Z^l$$

where $\dot{V}$ is associated to μ as $\dot{U}$ to η. Then the symbol of $\Pi_k\big[\nabla_\eta^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k, \nabla_\mu^{\mathrm{End}(\mathcal{P}_k^{\mathrm{m}})}\Pi_k\big]$ is at x

$$-\frac{1}{2}\mathrm{tr}\ (\dot{U}\dot{\bar{V}} - \dot{V}\dot{\bar{U}})$$

Since $\dot{U}$ and $\dot{V}$ are the matrices of $\eta(j_0)$ and $\mu(j_0)$, we obtain the result. □

Let us prove now the last part of theorem 7.1. Consider a curve $j : [0,1] \to \mathcal{J}_{\mathrm{int}}$. Denote by $\Psi_t : \delta_{j_0} \to \delta_{j_t}$ the continuous family of half-form bundle morphisms such that Ψ_0 is the identity.

Proposition 10.2. *Consider a smooth family of Fourier integral operator*

$$\Big(P_t \in \mathcal{F}((j_0,\delta_{j_0}),(j_t,\delta_{j_t}));\ t\in[0,1]\Big)$$

with symbol $(\sigma_t\Psi_t)_t$. Then the operator

$$\big(\nabla_{\partial_t}^{j^*\mathcal{Q}_k^{\mathrm{m}}}\circ P\big)_t : \mathcal{Q}_{k,j_0}^{\mathrm{m}} \to \mathcal{Q}_{k,j_t}^{\mathrm{m}}, \quad \Phi_0 \to (\nabla_{\partial_t}^{j^*\mathcal{Q}_k^{\mathrm{m}}}\Phi)(t) \ \textit{with}\ \Phi(t) = P_t\Phi_0$$

is a Fourier integral operator of $\mathcal{F}((j_0,\delta_{j_0}),(j_t,\delta_{j_t}))$ and its symbol is $\dot{\sigma}_t\Psi_t$

Proof. The Schwartz kernel of P_t is of the form

$$\Big(\frac{k}{2\pi}\Big)^n E_t^k(x,y)f_t(x,y,k)$$

with $f_t(x,x,k) = \sigma_t(x)\Psi_t(x) + O(k^{-1})$. So the kernel of $(\nabla_{\partial_t}^{j^*\mathcal{P}_k^{\mathrm{m}}}\circ P)_t$ is

$$\Big(\frac{k}{2\pi}\Big)^n\Big[\frac{d}{dt}E_t^k(x,y)\Big]f_t(x,y,k) + \Big(\frac{k}{2\pi}\Big)^n E_t^k(x,y)\Big[\frac{d}{ds}(\Psi_{t,s}^{-1}(x).f_s(x,y,k))\Big]_{s=t}$$

where $(\Psi_{t,s} : \delta_{j_t} \to \delta_{j_s})_s$ is the continuous family of half-form bundle morphisms such that $\Psi_{t,t}$ is the identity of δ_{j_t}. By proposition 9.4,

$$\frac{d}{dt}E_t = g_tE_t,$$

where g_t and its first derivatives vanish along the diagonal. So $(\nabla_{\partial_t}^{j^*\mathcal{P}_k^{\mathrm{m}}}\circ P\big)_t$ belongs to $\mathcal{A}_2(j_0,j_t)$. Since

$$\Psi_{j_t,j_s}\circ\Psi_{j_0,j_t} = \zeta(j_0,j_t,j_s)\Psi_{j_0,j_s}$$

we have

$$\Psi_{t,s}^{-1}\circ\Psi_{0,s} = \frac{1}{\zeta^{\frac{1}{2}}(j_0,j_t,j_s)}\Psi_{0,t}$$

Thus the symbol of $(\nabla_{\partial_t}^{j^*\mathcal{P}_k^{\mathrm{m}}}\circ P\big)_t$ is

$$\Big[\Big(\hbar^{-1}[g_t] + \frac{d}{ds}\Big(\frac{1}{\zeta^{\frac{1}{2}}(j_0,j_t,j_s)}\Big)\Big|_{s=t}\Big)\sigma_t + \dot{\sigma}_t\Big]\Psi_{0,t}. \tag{24}$$

Then it follows from theorem 9.2 that

$$(\nabla_{\partial_t}^{j^*\mathcal{Q}_k^{\mathrm{m}}}\circ P\big)_t = \Pi_{k,j_t}\circ(\nabla_{\partial_t}^{j^*\mathcal{Q}_k^{\mathrm{m}}}\circ P\big)_t$$

belongs to $\mathcal{F}((j_0,\delta_{j_0}),(j_t,\delta_{j_t}))$. Furthermore its symbol is of the form $(a_t\sigma_t + \dot{\sigma}_t)\Psi_t$ with a_t a C^∞ function. To end the proof, it suffices to show that if $\sigma_t = 1$ for every t, then the symbol of $(\nabla_{\partial_t}^{j^*\mathcal{Q}_k^{\mathrm{m}}}\circ P\big)_t$ vanishes.

Let us check it at $t=0$. Since $\zeta^{\frac{1}{2}}(j_0, j_0, j_s) = 1$ for every s and $\Psi_{0,0}$ is the identity, formula (24) simplifies into

$$\hbar^{-1}[g_0]\sigma_0 + \dot{\sigma}_0$$

which is equal to $\hbar^{-1}[g_0]$ because $\sigma_t = 1$. Introduce complex coordinates (z^i) for the complex structure j_0 such that $\omega = i dz^i \wedge d\bar{z}^i$ at x. Let μ be the tangent vector of j_t at $t=0$. Let U be the matrix at x of μ in the bases dz^i, $d\bar{z}^i$. Then by proposition 9.4

$$[g_0] = -\frac{1}{2}\sum U_{ij} \bar{Z}^i \bar{Z}^j$$

Finally an application of theorem 9.3 proves that the symbol of $\nabla^{\mathcal{Q}_k^{\mathrm{m}}}_{\mu_t} \circ P_t$ vanishes at $t=0$.

Let us compute now the symbol at any t. Since $\sigma_t = 1$, the operator P_t is invertible with an inverse in $\mathcal{F}((j_t, \delta_{j_t}), (j_0, \delta_{j_0}))$. So the operator

$$P_{t,s} = P_s \circ P_t^{-1} : \mathcal{Q}^{\mathrm{m}}_{k,j_t} \rightarrow \mathcal{Q}^{\mathrm{m}}_{k,j_s}$$

belongs to $\mathcal{F}((j_t, \delta_{j_t}), (j_s, \delta_{j_s}))$ and by theorem 5.2, its symbol is $\Psi_{t,s}$. It follows from the previous computation that

$$\big(\nabla^{j^*\mathcal{Q}_k^{\mathrm{m}}}_{\partial_s} \circ P_{t,.}\big)_s$$

belongs to $\mathcal{F}((j_t, \delta_{j_t}), (j_s, \delta_{j_s}))$ and its symbol vanishes at $t=s$. Consequently, the symbol of

$$\big(\nabla^{j^*\mathcal{Q}_k^{\mathrm{m}}}_{\partial_t} \circ P\big)_t = \big(\nabla^{j^*\mathcal{Q}_k^{\mathrm{m}}}_{\partial_s} \circ P_{t,.}\big)_{s=t} \circ P_t$$

vanishes. □

Then it is easy to construct by successive approximations a smooth family of operators

$$P_t : \mathcal{Q}^{\mathrm{m}}(j_0) \rightarrow \mathcal{Q}^{\mathrm{m}}(j_t)$$

in $\mathcal{F}((j_0, \delta_{j_0}), (j_t, \delta_{j_t}))$ such that P_0 is the identity of $\mathcal{Q}^{\mathrm{m}}_{k,j_0}$ and that the total symbol of $(\nabla^{j^*\mathcal{Q}_k^{\mathrm{m}}}_{\partial_t} \circ P)_t$ vanishes. Consequently,

$$\big(\nabla^{j^*\mathcal{Q}_k^{\mathrm{m}}}_{\partial_t} \circ P\big)_t = O(k^{-\infty})$$

where the big O is for the uniform norm of operators and is uniform with respect to t. If $T_t : \mathcal{Q}^{\mathrm{m}}_{k,j_0} \rightarrow \mathcal{Q}^{\mathrm{m}}_{k,j_t}$ is the parallel transport along j_t, then

$$T_t = P_t - T_t \int_0^t T_{-s} \big(\nabla^{j^*\mathcal{Q}_k^{\mathrm{m}}}_{\partial_s} \circ P\big)_s \, ds.$$

By the first part of theorem 7.1, T_t is unitary. Consequently

$$T_t = P_t + O(k^{-\infty}).$$

Then using that

$$\Pi_{k,j_t}(T_t - P_t)\Pi_{k,j_0} = T_t - P_t$$

and $\Pi_{k,j} \in \mathcal{F}((j, \delta_j), (j, \delta_j))$, we show that the Schwartz kernel of $T_t - P_t$ is uniformly $O(k^{-\infty})$ with its successive covariant derivatives. This proves theorem 7.1.

References

[1] Scott Axelrod, Steve Della Pietra, and Edward Witten. Geometric quantization of Chern-Simons gauge theory. *J. Differential Geom.*, 33(3):787–902, 1991.

[2] Sean Bates and Alan Weinstein. *Lectures on the geometry of quantization*, volume 8 of *Berkeley Mathematics Lecture Notes*. American Mathematical Society, Providence, RI, 1997.

[3] Nicole Berline, Ezra Getzler, and Michèle Vergne. *Heat kernels and Dirac operators*, volume 298 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[4] Robert J. Blattner. Quantization and representation theory. In *Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972)*, pages 147–165. Amer. Math. Soc., Providence, R. I., 1973.

[5] L. Boutet de Monvel and V. Guillemin. *The spectral theory of Toeplitz operators*, volume 99 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1981.

[6] L. Charles. Berezin-Toeplitz operators, a semi-classical approach. *Comm. Math. Phys.*, 239(1-2):1–28, 2003.

[7] L. Charles. Quasimodes and Bohr-Sommerfeld conditions for the Toeplitz operators. *Comm. Partial Differential Equations*, 28(9-10):1527–1566, 2003.

[8] L. Charles. Symbolic calculus for Toeplitz operators with half-forms. http://www.institut.math.jussieu.fr/~charles/Articles/Half1.pdf, 2006.

[9] Tatiana Foth and Alejandro Uribe. Remarks on the naturality of quantization. http://arxiv.org/abs/math/0410016, 2004.

[10] Viktor L. Ginzburg and Richard Montgomery. Geometric quantization and no-go theorems. In *Poisson geometry (Warsaw, 1998)*, volume 51 of *Banach Center Publ.*, pages 69–77. Polish Acad. Sci., Warsaw, 2000.

[11] Victor Guillemin and Shlomo Sternberg. *Geometric asymptotics*. American Mathematical Society, Providence, R.I., 1977. Mathematical Surveys, No. 14.

[12] N. J. Hitchin. Flat connections and geometric quantization. *Comm. Math. Phys.*, 131(2):347–380, 1990.

[13] Lars Hörmander. *The analysis of linear partial differential operators. I*, volume 256 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1990. Distribution theory and Fourier analysis.

[14] Bertram Kostant. Quantization and unitary representations. I. Prequantization. In *Lectures in modern analysis and applications, III*, pages 87–208. Lecture Notes in Math., Vol. 170. Springer, Berlin, 1970.

[15] Bertram Kostant. Symplectic spinors. In *Symposia Mathematica, Vol. XIV (Convegno di Geometria Simplettica e Fisica Matematica, INDAM, Rome, 1973)*, pages 139–152. Academic Press, London, 1974.

[16] J. H. Rawnsley. On the pairing of polarizations. *Comm. Math. Phys.*, 58(1):1–8, 1978.

[17] J.-M. Souriau. *Structure des systèmes dynamiques*. Maîtrises de mathématiques. Dunod, Paris, 1970.

[18] N. M. J. Woodhouse. *Geometric quantization*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1992. Oxford Science Publications.

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