Semiclassical Heisenberg pseudodifferential operators

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Abstract

We prove that the semiclassical Heisenberg pseudodifferential operators form a filtered algebra.

1 Introduction

For any compact manifold M and Hermitian line bundle $L \to M$ with a connection ∇, the space $\Psi^m_{\mathrm{Heis}}(L, \nabla)$ of Heisenberg semiclassical pseudodifferential operator with order m has been defined in [1]. The principal symbol $\sigma(P)$ of such an operator P is a smooth function of T^*M which is a polyhomogeneous symbol.

Write the curvature of ∇ as $\frac{1}{i}\omega$ with $\omega \in \Omega^2(M, \mathbb{R})$. Define the product of symbols of T^*M by

$$(f \sharp_\omega g)(x, \xi) = \Big[e^{-\frac{i}{2}\omega_x(D_\xi, D_\eta)} f(x, \xi) g(x, \eta) \Big]_{\xi = \eta}, \qquad \xi, \eta \in T^*_x M.$$

The goal of this note is to prove the following result.

Theorem 1.1. *For any $P \in \Psi^m_{\mathrm{Heis}}(L, \nabla)$ and $Q \in \Psi^p_{\mathrm{Heis}}(L, \nabla)$, $(P_k Q_k)$ belongs to $\Psi^{m+p}_{\mathrm{Heis}}(L, \nabla)$ and its principal symbol is $\sigma(P) \sharp_\omega \sigma(Q)$.*

An important point is that we do not assume anything on ω, it doesn't have necessarily constant rank. The fact that $\sharp_\omega$ is a well-defined continuous product of symbols, is already a nontrivial result. The proof is given in Section 3, another proof was already given in [2]. In Section 4, we briefly recall the definition of Heisenberg pseudodifferential operators and prove Theorem 1.1.

Applications (not written yet) include:

- Heisenberg Sobolev estimates
- Hermite algebras $\Psi^{-\infty}_{\text{Heis}}(L,\nabla)$ and Toeplitz subalgebras
- functional calculus of Heisenberg elliptic operators

2 Symbols

Same definitions as in [1]: symbols, semiclassical pseudodifferential operators and residual class.

3 Isotropic algebras

Let $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be bilinear antisymmetric and (e_j) be the canonical basis of $\mathbb{R}^n$. Define the covariant derivative

$$\nabla^A = d + \tfrac{1}{2i} \sum_j A(x, e_j) dx_j \tag{1}$$

acting on $\mathbb{R}^n_x$. The curvature of ∇^A is $\frac{1}{i}A$, that is

$$[\nabla^A_j, \nabla^A_k] = \tfrac{1}{i} A(e_j, e_k) \tag{2}$$

where $\nabla^A_j := \nabla^A_{e_j} = \partial_{x_j} + \frac{1}{2i}A(x, e_j)$.

For any tempered distribution $g \in \mathcal{S}'(\mathbb{R}^n)$, we denote its Fourier transform and inverse Fourier transform by $\widehat{g}$ and $g^\vee$,

$$\widehat{g}(t) = \int e^{-it\xi} g(\xi)\, d\xi, \qquad g^\vee(\xi) = (2\pi)^{-n} \widehat{g}(-\xi).$$

We define $g(\frac{1}{i}\nabla^A)$ as the operator $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ with Schwartz kernel

$$\begin{aligned} K_g(x,y) &= (2\pi)^{-n} e^{-\frac{i}{2}A(x,y)} \int e^{i\xi(x-y)} g(\xi)\, d\xi \\ &= e^{-\frac{i}{2}A(x,y)} g^\vee(x-y) \end{aligned} \tag{3}$$

The reason for the notation $g(\frac{1}{i}\nabla^A)$ is that for $g = \xi_j$, $g(\frac{1}{i}\nabla^A) = \frac{1}{i}\nabla^A_j$ and more generally if g is the monomial $\xi_{j_1} \dots \xi_{j_\ell}$ then

$$g(\tfrac{1}{i}\nabla^A) = \frac{(-i)^\ell}{\ell!} \sum_{\sigma \in \Sigma_\ell} \nabla^A_{j_{\sigma(1)}} \dots \nabla^A_{j_{\sigma(\ell)}},$$

cf [1, Proposition 6.2].

We will prove that the space of $f(\frac{1}{i}\nabla_A)$ with f in the Schwartz class, is an algebra and compute the corresponding product of functions. For any g, $h \in \mathcal{S}(\mathbb{R}^n)$, define $g\sharp_A h \in \mathcal{S}(\mathbb{R}^n)$ by

$$(g\sharp_A h)(\xi) = \Big[e^{-\frac{i}{2}A(D_\xi, D_\eta)} g(\xi)h(\eta)\Big]_{\xi=\eta}. \tag{4}$$

Here we use the standard notation: $D = \frac{1}{i}\partial$ and for any real quadratic form B of $\mathbb{R}^m$ and $u \in \mathcal{S}(\mathbb{R}^m)$, $v = e^{iB(D)}u$ is the function of $\mathcal{S}(\mathbb{R}^m)$ such that its Fourier transform is $\widehat{v}(\xi) = e^{iB(\xi)}\widehat{u}(\xi)$. Observe that for $A = 0$, $\sharp_A$ is the pointwise multiplication of functions whereas for n even and A the standard symplectic form, $\sharp_A$ is the Weyl product.

Lemma 3.1. *For every $g, h \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} K_g(x,y)\, K_h(y,z)\, dy = K_f(x,z) \tag{5}$$

where $f = g\sharp_A h$.

Proof. Since g and h are in the Schwartz class, $K_g(x,y)$ and $K_h(x,y)$ are in $\mathcal{O}(\langle x-y\rangle^{-\infty})$. By Peetre inequality,

$$\langle x-y\rangle^{-N}\langle y-z\rangle^{-N-n-1} \leqslant C\langle x-z\rangle^{-N}\langle y-z\rangle^{-n-1}.$$

Since $\int \langle y-z\rangle^{-n-1}\, dy = \int \langle y\rangle^{-n-1} dy$ is finite, we get that

$$\widetilde{K}(x,z) := \int_{\mathbb{R}^n} K_g(x,y)K_h(y,z)\, dy \tag{6}$$

is well-defined, continuous and in $\mathcal{O}(\langle x-z\rangle^{-\infty})$.

We claim that $e^{\frac{i}{2}A(r,z)}\widetilde{K}(z+r,z)$ is independent of z. Indeed, inserting the definitions of K_g and K_h in (6) and using the antisymmetry of A, we have

$$\begin{aligned} e^{\frac{i}{2}A(r,z)}\widetilde{K}(z+r,z) &= \int_{\mathbb{R}^n} e^{-\frac{i}{2}A(r,y-z)} g^\vee(z+r-y)h^\vee(y-z)\, dy \\ &= \int_{\mathbb{R}^n} e^{-\frac{i}{2}A(r,t)} g^\vee(r-t)h^\vee(t)\, dt \end{aligned} \tag{7}$$

by setting $t = y - z$. We recognize here a twisted convolution product.

It remains to prove that the Fourier transform of (7) with respect to t is equal to $g\sharp_A h$, that is

$$\int_{\mathbb{R}^n} e^{i\zeta\cdot r} e^{\frac{i}{2}A(r,z)} \widetilde{K}(z+r,z)\, dr = (g\sharp_A h)(\zeta). \tag{8}$$

The left-hand side of (8) is equal to

$$\begin{aligned}&(2\pi)^{-2n}\int_{\mathbb{R}^{4n}} e^{-i\zeta\cdot r-\frac{i}{2}A(r,t)}e^{i(r-t)\cdot\xi}g(\xi)e^{it\cdot\eta}h(\eta)\, dr\, dt\, d\xi\, d\eta\\ &=(2\pi)^{-2n}\int_{\mathbb{R}^{4n}} e^{-i\zeta\cdot(s+t)-\frac{i}{2}A(s,t)+is\cdot\xi+it\cdot\eta}g(\xi)h(\eta)\, ds\, dt\, d\xi\, d\eta\end{aligned}$$

by setting $s=r-t$. We recognize $g\sharp_A h(\zeta)$. □

Theorem 3.2. *Let $m,p\in\mathbb{R}$. Then the product $\sharp_A$ extends continuously from $S^m(\mathbb{R}^n)\times S^p(\mathbb{R}^n)$ to $S^{m+p}(\mathbb{R}^n)$. More generally, for any $N\in\mathbb{N}$, the map sending g, h to the remainder*

$$r_N(g,h,A)(\xi) = (g\sharp_A h)(\xi) - \sum_{\ell=0}^{N}(\tfrac{-i}{2})^{\ell}(\ell!)^{-1}\Big[A(D_\xi,D_\eta)^{\ell}(g(\xi)h(\eta))\Big]_{\eta=\xi}$$

is continuous from $S^m(\mathbb{R}^n)\times S^p(\mathbb{R}^n)$ to $S^{m+p-2(N+1)}(\mathbb{R}^n)$.

In the case where A is non-degenerate, so that $\sharp_A$ is the Moyal-Weyl product, this is a well-known result, cf. [4, Section 23] or [3, Section 18.5] for more general symbol classes. For a general A, this has been proved in [2, Chapter 4]. It is possible to deduce the result from the Moyal-Weyl case by decomposing $\mathbb{R}^n$ into $\ker A\oplus S$, but this does not lead to uniform estimates with respect to A, which we need later. Since the result is fundamental in the paper, we provide a proof.

Proof. As in [2, Chapter 4], we will use the following version of stationnary phase lemma. Let q be a non degenerate quadratic form of $\mathbb{R}^d$. Then for any $\ell\in\mathbb{R}$, there exists $C>0$ and $M\in\mathbb{N}$ such that for any $a\in\mathcal{C}_0^\infty(\mathbb{R}^d)$ we have

$$\Big|\int_{\mathbb{R}^d} e^{iq(x)}a(x)\, dx\Big| \leqslant C\max_{|\alpha|\leqslant M}\sup_{x\in\mathbb{R}^d}|\partial^\alpha a(x)|\langle x\rangle^{-\ell}. \tag{9}$$

This is proved by integration by part by using that $e^{iq}=Le^{iq}$ with L the differential operator $L=\langle x\rangle^{-2}(1-i\sum_{j,k}B_{jk}x_j\partial_k)$ where B is the inverse matrix of the matrix of q.

We write $g\sharp_A h$ on the form

$$(g\sharp_A h)(\zeta) = (2\pi)^{-2n} \int_{\mathbb{R}^{4n}} e^{i\xi\cdot s+i\eta\cdot t-\frac{i}{2}A(s,t)} g(\zeta+\xi)h(\zeta+\eta)\, ds\, dt\, d\xi\, d\eta \quad (10)$$

and we will apply (9) with $d = 4n$ and $x = (s, t, \xi, \eta)$, ζ being considered as a parameter. From now on, our proof differs from [2], which discusses whether $|\xi| + |\eta| \leqslant \frac{1}{2}(1 + |\zeta|)$ or not.

Let $f(\zeta, \xi, \eta) := g(\zeta+\xi)h(\zeta+\eta)$ and $\gamma \in \mathbb{N}^n$. Then the derivative $\partial_\zeta^\gamma f$ is a linear combination with constant coefficients of the functions $f_{\alpha,\beta}(\zeta, \xi, \eta) := (\partial^\alpha g)(\zeta + \xi)\, (\partial^\beta h)(\zeta + \eta)$ where $\alpha + \beta = \gamma$. Moreover,

$$\begin{aligned} |\partial_\xi^{\alpha'} \partial_\eta^{\beta'} f_{\alpha,\beta}| &= |(\partial^{\alpha+\alpha'} g)(\zeta + \xi)\, (\partial^{\beta+\beta'} h)(\zeta + \eta)| \\ &\leqslant C\|g\|_{m,M}\|h\|_{p,M}\, \langle\zeta + \xi\rangle^{m-|\alpha|-|\alpha'|}\langle\zeta + \eta\rangle^{p-|\beta|-|\beta'|} \end{aligned}$$

with M sufficiently large. Using $\langle\zeta + \xi\rangle^{-|\alpha'|} \leqslant 1$, Peetre inequality

$$\langle\zeta + \xi\rangle^{m-|\alpha|} \leqslant C\langle\zeta\rangle^{m-|\alpha|}\langle\xi\rangle^{|m|+|\alpha|} \qquad \text{and} \qquad \langle\xi\rangle \leqslant \langle\xi, \eta\rangle$$

we get $\langle\zeta + \xi\rangle^{m-|\alpha|-|\alpha'|} \leqslant C\langle\zeta\rangle^{m-|\alpha|}\langle\xi, \eta\rangle^{|m|+|\alpha|}$. By this and the similar upper bound for $\langle\zeta + \eta\rangle^{p-|\beta|-|\beta'|}$, we get

$$|\partial_\xi^{\alpha'} \partial_\eta^{\beta'} f_{\alpha,\beta}| \leqslant C\|g\|_{m,M}\|h\|_{p,M}\langle\zeta\rangle^{m+p-|\gamma|}\langle\xi, \eta\rangle^{|m|+|p|+|\gamma|} \quad (11)$$

where C and M depend only on γ, α' and β'. Therefore (9) implies that

$$|\partial^\gamma(g\sharp_A h)(\zeta)| \leqslant C\|g\|_{m,M}\|h\|_{p,M}\langle\zeta\rangle^{m+p-|\gamma|},$$

which shows that $\sharp_A$ is continuous $S^m(\mathbb{R}^n) \times S^p(\mathbb{R}^n) \to S^{m+p}(\mathbb{R}^n)$.

To estimate the remainder r_N, we first write it on an integral form similar to (10). Observe that for polynomial functions P, Q on $\mathbb{R}^n$, we have

$$(2\pi)^{-2n} \int_{\mathbb{R}^{4n}} e^{i\xi\cdot s+i\eta\cdot t-\frac{i}{2}A(s,t)} P(\xi)Q(\eta)\, ds\, dt\, d\xi\, d\eta = (P\sharp_A Q)(0)$$

where $P\sharp_A Q$ is given by the (finite) sum

$$(P\sharp_A Q)(\zeta) = \sum_{\ell=0}^{\infty} (\tfrac{-i}{2})^\ell (\ell!)^{-1} \big[A(D_\xi, D_\eta)^\ell (P(\xi)Q(\eta))\big]_{\xi=\eta=\zeta}.$$

So if we replace in (10) the product $g(\zeta+\xi)h(\zeta+\eta)$ by its Taylor expansion in (ξ, η) at the origin at order $2N + 1$, that is

$$\sum_{|\alpha|+|\beta|\leqslant 2N+1} \tfrac{1}{\alpha!}(\partial^\alpha g)(\zeta)\xi^\alpha \tfrac{1}{\beta!}(\partial^\beta h)(\zeta)\eta^\beta, \quad (12)$$

we obtain the sum $\sum_{\ell=0}^{N}(\frac{-i}{2})^{\ell}(\ell!)^{-1}\big[A(D_\xi,D_\eta)^{\ell}(g(\xi)h(\eta))\big]_{\xi=\eta=\zeta}$. Here we have used that for $|\alpha|\neq|\beta|$, $(\zeta^\alpha\sharp_A\zeta^\beta)(0)=0$ so the corresponding terms in (12) do not contribute.

So the remainder we have to estimate is given by

$$r_N(g,h,A)(\zeta)=(2\pi)^{-2n}\int_{\mathbb{R}^{4n}} e^{i\xi\cdot s+i\eta\cdot t-\frac{i}{2}A(s,t)}\rho_N(\zeta,\xi,\eta)\,ds\,dt\,d\xi\,d\eta \tag{13}$$

where ρ_N is the remainder of the Taylor expansion (12), that is

$$\rho_N(\zeta,\xi,\eta)=2(N+1)\sum_{|\alpha|+|\beta|=2N+2}\frac{\xi^\alpha\eta^\beta}{\alpha!\beta!}\rho_{\alpha,\beta}(\zeta,\xi,\eta)$$

$$\text{with}\quad \rho_{\alpha,\beta}(\zeta,\xi,\eta)=\int_0^1(1-\tau)^{2N+1}(\partial^\alpha g)(\zeta+\tau\xi)(\partial^\beta h)(\zeta+\tau\eta)\,d\tau.$$

We can estimate the integrand by the same method that led to (11), and using that $\langle\tau\xi\rangle\leqslant\langle\xi\rangle$ and $\langle\tau\eta\rangle\leqslant\langle\eta\rangle$ for $\tau\in[0,1]$, we obtain after integrating with respect to τ that

$$\rho_{\alpha,\beta}=\mathcal{O}(\langle\zeta\rangle^{m+p-2(N+1)}\langle\xi,\eta\rangle^{|m|+|p|+2(N+1)}).$$

We can estimate the derivatives similarly:

$$|\partial_\zeta^\gamma\partial_\xi^{\alpha'}\partial_\eta^{\beta'}\rho_{\alpha,\beta}|\leqslant\|g\|_{m,M}\|h\|_{p,M}\langle\zeta\rangle^{m+p-|\gamma|-2(N+1)}\langle\xi,\eta\rangle^{|m|+|p|+|\gamma|+2(N+1)}$$

where C and M depend only of N, γ, α' and β'. By (13) and (9), it follows that

$$|\partial^\gamma r_N(g,h)(\zeta)|\leqslant C\|g\|_{m,M}\|h\|_{p,M}\langle\zeta\rangle^{m+p-(|\gamma|+2(N+1))},$$

which proves the second assertion. ☐

Denote by $\mathcal{A}_n$ the space of antisymmetric real bilinear form of $\mathbb{R}^n$.

Proposition 3.3. *For any $m,p\in\mathbb{R}$, the map from $S^m(\mathbb{R}^n)\times S^p(\mathbb{R}^n)$ to $S^{m+p}(\mathcal{A}_n,\mathbb{R}^n)$ sending (g,h) to $(A,\xi)\to(g\sharp_A h)(\xi)$ is well-defined and continuous.*

We will actually show a better result (15).

Proof. For any f, g in the Schwartz class, it follows directly from the definition (4) that $(A,\zeta)\to f\sharp_A g(\zeta)$ is smooth and we can even compute explicitely the A-derivatives as follows. For $1\leqslant j<k\leqslant n$, the derivative of $e^{-\frac{i}{2}A(\xi,\eta)}$ with respect to $A_{jk}=A(e_j,e_k)$ is $\frac{1}{4i}(\xi_j\eta_k-\xi_k\eta_j)e^{-\frac{i}{2}A(\xi,\eta)}$ so that

$$\frac{\partial(g\sharp_A h)}{\partial A_{jk}}=\tfrac{1}{4i}((D_jg)\sharp_A(D_kh)-(D_kg)\sharp_A(D_jh)). \tag{14}$$

By Theorem 3.2, for any $A \in \mathcal{A}_n$, m, $p \in \mathbb{R}$ and $M \in \mathbb{N}$, there exists $C > 0$ and $M' \in \mathbb{N}$ such that

$$\|g\sharp_A h\|_{m+p,M} \leqslant C \|g\|_{m,M'} \|h\|_{p,M'}.$$

By the proof of Theorem 3.2, C and M' stay bounded when A stays in a compact subset K of $\mathcal{A}_n$. This implies by (14) that for any $A \in K$, $\xi \in \mathbb{R}^n$

$$|\partial_A^\gamma \partial_\zeta^\alpha (g\sharp_A h(\zeta))| \leqslant C \|g\|_{m,M} \|h\|_{p,M} \langle \zeta \rangle^{m+p-2|\gamma|-|\alpha|} \tag{15}$$

with C and M depending only on γ, α and K. This is better than the result we have to prove, because here we gain $\langle \zeta \rangle^{-2}$ for each A-derivative. □

Proposition 3.3 has the following consequence. Consider a real vector bundle $E \to M$ and a section $A \in \mathcal{C}^\infty(M, \wedge^2 E^*)$. Then the pointwise A-product of symbols of E^*

$$(g\sharp_A h)(x, \cdot) = g(x, \cdot)\sharp_{A(x)} h(x, \cdot), \qquad x \in M \tag{16}$$

is a continuous map $S^m(M, E^*) \times S^p(M, E^*) \to S^{m+p}(M, E^*)$, $(g, h) \to g\sharp_A h$.

To end this section, we establish a preparatory lemma for the Heisenberg composition. For any $R > 0$, denote by h_R be the multiplication by R of $\mathbb{R}^n$ and by h_R^* the pull-back operator, so $(h_R^* f)(x) = f(Rx)$. For any operator Q of $\mathcal{S}(\mathbb{R}^n)$, we denote by Q_R the operator $h_R^* \circ Q \circ h_{R^{-1}}^*$ of $\mathcal{S}(\mathbb{R}^n)$. Let ρ, $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp} \rho$ is contained in the interior of $\{\chi = 1\}$. For any functions g, $h \in \mathcal{S}(\mathbb{R}^n)$, define

$$P(R, g, h) := \rho\, g(\tfrac{1}{i} \nabla_A)_R \,(1 - \chi)\, h(\tfrac{1}{i} \nabla_A)_R \,\chi$$

where ρ and χ are identified with the multiplication operators by ρ and χ. Notice that the Schwartz kernel of $P(R, g, h)$ is smooth and supported in $(\operatorname{supp} \rho)^2$.

Lemma 3.4. *For any $m, p \in \mathbb{R}$, $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{2n}$, there exists $C > 0$ and $M \in \mathbb{N}$ such that for any $R > 0$ and $g, h \in \mathcal{S}(\mathbb{R}^n)$,*

$$|\partial_{x,y}^\alpha P(R, g, h)(x, y)| \leqslant C R^{-N} \|g\|_{m,M} \|h\|_{p,M}.$$

This result will be used later in the following form: for any compact subset K of the interior of $\{\chi = 1\}$, we have

$$\big(g(\tfrac{1}{i} \nabla_A)_R \,\chi\, h(\tfrac{1}{i} \nabla_A)_R\big)(x, y) = (g\sharp_A h)(\tfrac{1}{i} \nabla_A)_R (x, y) + \mathcal{O}_\infty(R^{-\infty}) \tag{17}$$

with a $\mathcal{O}$ uniform for $x, y \in K$.

Proof. For any $m \in \mathbb{R}$ and $N \in \mathbb{N}$ such that $m - 2N < -n$, we have

$$|x|^{2N} g^{\vee}(x) = (\Delta^N g)(x) = \mathcal{O}(\|g\|_{m,2N})$$

with a $\mathcal{O}$ uniform in x. So

$$g(\tfrac{1}{i}\nabla_A)_R(x,y) = R^n g(\tfrac{1}{i}\nabla_A)(Rx, Ry) = \mathcal{O}(R^{n-2N}|x-y|^{-2N}\|g\|_{m,2N}).$$

The kernel of $h(\frac{1}{i}\nabla_A)_R$ satisfies the same bound.

On the support of $\rho(x)(1-\chi(y))$, we have $|x| \leqslant C$ and $|x-y| \geqslant C^{-1}$ so that $|x-y|^{-2N} = \mathcal{O}(\langle y \rangle^{-n})$ when $2N \geqslant n$. So for N large enough,

$$\begin{aligned} &\rho(x)\ g(\tfrac{1}{i}\nabla_A)_R(x,y)\ (1-\chi(y))\ h(\tfrac{1}{i}\nabla_A)_R(y,z)\ \rho(z) \\ &\qquad = \mathcal{O}(R^{2n-4N}\langle y \rangle^{-2n}\|g\|_{m,2N}\|h\|_{p,2N}) \end{aligned}$$

which by integrating in y implies

$$P(R,g,h)(x,y) = \mathcal{O}(R^{2n-4N}\|g\|_{m,2N}\|h\|_{p,2N}).$$

The estimates of the derivatives are similar. □

4 Heisenberg pseudodifferential operators

Let $L \to M$ be a Hermitian line bundle with a connection ∇ preserving the metric. Introduce a neighborhood V of the diagonal of M^2 and a section $F \in \mathcal{C}^\infty(V, L \boxtimes \overline{L})$ which is unitary ($|F| = 1$ on V) and satisfies

$$F(x,x) = 1, \quad \nabla F(x,x) = 0, \quad \nabla_Y \nabla_Y F(x,x) = 0, \quad \forall x \in M \tag{18}$$

for any vector field Y of M^2 having the form $Y(x,y) = (X(x), -X(y))$ with $X \in \mathcal{C}^\infty(M, TM)$. In the first equation of (18), we identified $L_x \otimes \overline{L}_x$ with $\mathbb{C}$ through the Hermitian product of L_x. By [1, Lemma 3.1], such a section exists and is unique up to multiplication by a function of the form $\exp(i\varphi)$ where φ is a smooth real valued fonction defined on a neighborhood of the diagonal which vanishes to third order along the diagonal.

For any $m \in \mathbb{R}$, the space $\Psi^m_{\mathrm{Heis}}(L, \nabla)$ of semiclassical pseudodifferential Heisenberg operators of order m consists of the operator families $P = (P_k : \mathcal{C}^\infty(M, L^k) \to \mathcal{C}^\infty(M, L^k),\ k \in \mathbb{N})$ whose Schwartz kernels have the form

$$F^k(x,Y)\phi(x,y)K_{k^{-\frac{1}{2}}}(x,y) + \mathcal{O}_\infty(k^{-\infty}) \tag{19}$$

where $F \in \mathcal{C}^\infty(V, L \boxtimes \overline{L})$ satisfies the previous conditions, $\phi \in \mathcal{C}^\infty_0(V)$ is equal to 1 on a neighborhood of the diagonal and $(K_h, h \in (0,1])$ us the Schwartz kernel family of a semiclassical pseudodifferential operators $(Q_h) \in \Psi^m_{\mathrm{sc}}(M)$. The symbol $\sigma(P)$ of P is by definition the symbol of (Q_h).

The choice of F does not play any role. More precisely, by [1, Lemma 3.3], for any F satisfying (18), for any $P \in \Psi^m_{\mathrm{Heis}}(L,\nabla)$, there exists $Q \in \Psi^m_{\mathrm{sc}}(M)$ such that (19) holds. Moreover, if (19) holds for some F and Q, then $\sigma(P) = \sigma(Q)$.

The curvature of ∇ has the form $\frac{1}{i}\omega$ with ω a real-valued closed 2-form of M. Associated to ω is the product $\sharp_\omega$ of $S^\infty(M, T^*M)$ defined as in (16).

Theorem 4.1. *For any $P \in \Psi^m_{\mathrm{Heis}}(L,\nabla)$ and $Q \in \Psi^p_{\mathrm{Heis}}(L,\nabla)$, $(P_k Q_k)$ belongs to $\Psi^{m+p}_{\mathrm{Heis}}(L,\nabla)$ and its symbol is $\sigma(P_kQ_k) = \sigma(P)\sharp_\omega \sigma(Q)$.*

The remainder of the section is devoted to the proof. First we we write the result as a local statement (25), and then prove the symbol expansion (26) and the remainder estimates (27).

A local statement

Let (U_i) be a finite open cover of M. We claim that it suffices to show the result under the assumption that there exists i_0 such that the Schwartz kernel of P and Q are supported in compact set (independent of k) of $U^2_{i_0}$.

Proof. Introduce a partion of unity (φ_i) subordinated to the cover (U_i). Write PQ as the sum $\sum P\varphi_i Q$. Let $\psi_i \in \mathcal{C}^\infty_0(U_i)$ such that $\operatorname{supp}\varphi_i \subset \{\psi_i = 1\}$. Then

$$P\varphi_i Q = (1-\psi_i)P\varphi_i Q + \psi_i P \varphi_i Q(1-\psi_i) + \psi_i Q\varphi_i Q\psi_i. \tag{20}$$

Since $\operatorname{supp}(1-\psi_i) \times \operatorname{supp}\varphi_i$ does not intersect the diagonal, $(1-\psi_i)P\varphi_i$ and $\varphi_i Q(1-\psi_i)$ are in the residual space $k^{-\infty}\Psi^{-\infty}(L)$. By [1, Lemma 5.2], $k^{-\infty}\Psi^{-\infty}(L)$ is a bilatereal ideal of $\Psi_{\mathrm{Heis}}(L,\nabla)$. So the two first terms in the right-hand side of (20) are in $k^{-\infty}\Psi^{-\infty}(L)$. Now $\psi_i Q \varphi_i P \Psi_i$ is the composition of $\psi_i Q\varphi_i$ with $\psi_i P \psi_i$ whose kernels are both supported in $(\operatorname{supp}\psi_i)^2$. $\square$

Let us introduce a coordinate chart $U, (x_i)$ of M with a unitary section $t \in \mathcal{C}^\infty(U, L)$. By the previous fact, we can assume that the kernels of P and Q are supported in U^2. We have $\nabla t = \frac{1}{i}\beta_i dx_i \otimes t$ for some $\beta_i \in \mathcal{C}^\infty(U,\mathbb{R})$. Then the section

$$F(x,y) = e^{i\beta(\frac{x+y}{2})\cdot(x-y)} t(x) \otimes \bar{t}(y)$$

satisfies (18). So the Schwartz kernel of P has the form $P_k(x,y)t^k(x)\otimes\bar{t}^k(y)$ with

$$P_k(x,y) = e^{ik\beta(\frac{x+y}{2})\cdot(x-y)}\Big(\frac{\sqrt{k}}{2\pi}\Big)^n \int e^{i\sqrt{k}\,\xi\cdot(x-y)} a(k^{-\frac{1}{2}}, x, y, \xi)\, d\xi$$

where $a \in S^m_{\text{sc}}(U^2, \mathbb{R}^n)$ and is supported in $K^2\times\mathbb{R}^n$ with K a compact subset of U. Doing the change of variable $\xi = \sqrt{k}\eta$, we get

$$P_k(x,y) = e^{ik\beta(\frac{x+y}{2})\cdot(x-y)}\Big(\frac{k}{2\pi}\Big)^n \int e^{ik\eta\cdot(x-y)} a(k^{-\frac{1}{2}}, x, y, k^{\frac{1}{2}}\eta))\, d\xi \tag{21}$$

Q_k is given by the same formula with a symbol $b\in S^p_{\text{sc}}(U^2,\mathbb{R}^n)$ such that $\operatorname{supp} b \subset K^2\times\mathbb{R}^{2n}$. The integral (21) being an oscillatory integral, it is convenient to assume first that a and b are compactly supported in ξ as well, and then to deduce the case for general symbols by a density argument. Similarly, we can assume as well that a and b are independent of the small parameter h.

Starting from (21), we obtain the following expression for the Schwartz kernel of P_kQ_k

$$(P_kQ_k)(x,z) = e^{ik\beta(\frac{x+z}{2})\cdot(x-z)}\Big(\frac{k}{2\pi}\Big)^{2n} J_k(f_k)(x,z)$$

where $J_k(f_k)$ is defined as follows. First we have a phase

$$\varphi(x,y,z) = \beta(\tfrac{x+y}{2})\cdot(x-y) + \beta(\tfrac{y+z}{2})\cdot(y-z) - \beta(\tfrac{x+z}{2})\cdot(x-z) \tag{22}$$

from which we define

$$f_k(x,y,z,\xi,\eta) = e^{ik\varphi(x,y,z)} a(x,y,k^{\frac{1}{2}}\xi) b(x,y,k^{\frac{1}{2}}\eta) \tag{23}$$

Finally, for any function $f\in\mathcal{C}^\infty_0(U^3\times\mathbb{R}^{2n})$, we set

$$J_k(f)(x,z) = \int_{\mathbb{R}^n\times U\times\mathbb{R}^n} e^{ik(\xi(x-y)+\eta(y-z))} f(x,y,z,\xi,\eta)\chi(y)\, d\xi d\eta dy \tag{24}$$

Here $\chi\in\mathcal{C}^\infty_0(U)$ is chosen so that the interior of $\{\chi=1\}$ contains K. So for $f=f_k$, we can remove it without modifying the integral but we will need it later.

Our goal is to prove that for any a, $b\in\mathcal{C}^\infty_0(U^2\times\mathbb{R}^n)$ supported in $K^2\times\mathbb{R}^{2n}$, for any $N\in\mathbb{N}$, we have

$$\begin{aligned}\Big(\frac{k}{2\pi}\Big)^n J_k(f_k)(x,z) = \sum_{\ell=0}^{N} k^{-\frac{\ell}{2}} \int_{\mathbb{R}^n} e^{ik\xi\cdot(x-z)} c_\ell(\tfrac{1}{2}(x+z), k^{\frac{1}{2}}\xi) d\xi \\ + k^{-\frac{1}{2}(N+1)} R_{N,k}(x,z)\end{aligned} \tag{25}$$

where $c_\ell\in S^{-\infty}(U,\mathbb{R}^n)$ and $R_{N,k}\in\mathcal{C}^\infty(U^2)$ satisfy

1. for any $\ell \in \mathbb{N}$,
$$c_\ell(x,\cdot) = \sum_{(\alpha,\beta,\alpha',\beta')\in\Sigma_\ell} \lambda^{\alpha',\beta'}_{\alpha,\beta,\ell}(x)(\partial^\alpha_\xi \partial^{\alpha'}_{x,y} a(x,x,\cdot))\sharp_{\omega(x)}(\partial^\beta_\xi \partial^{\beta'}_{x,y} b(x,x,\cdot)) \tag{26}$$
where Σ_ℓ consists of $(\alpha,\beta,\alpha',\beta') \in \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}^{2n} \times \mathbb{N}^{2n}$ such that $\ell \leqslant |\alpha| + |\beta| \leqslant 3\ell$ and $|\alpha'| + |\beta'| \leqslant \ell$. The coefficients $\lambda^{\alpha',\beta'}_{\alpha,\beta,\ell}$ are independent of a and b and belong to $\mathcal{C}^\infty(U)$.
Moreover, $\lambda^{0,0}_{0,0,0} = 1$ so that $c_0(x,\cdot) = a(x,x,\cdot)\sharp_{\omega(x)} b(x,x,\cdot)$
2. for any $N \in \mathbb{N}$, $m,p \in \mathbb{R}$ and $\alpha \in \mathbb{N}^{2n}$ such that $|\alpha|+m+p < -n+N+1$ there exists $C > 0$ and $M \in \mathbb{N}$ such that
$$k^{-|\alpha|}|\partial^\alpha_{x,z} R_{N,k}(x,z)| \leqslant C\|a\|_{m,M}\|b\|_{p,M} \tag{27}$$

Let us explain the consistency between the formula (26) giving the coefficients c_ℓ and the bounds (27) satisfied by the remainder $R_{N,k}$. For any $a \in \mathcal{C}^\infty_0(U^2 \times \mathbb{R}^n)$, let $I_k(a)(x,z) := \int_{\mathbb{R}^n} e^{ik\xi\cdot(x-z)} a(x,z,k^{\frac{1}{2}}\xi)\, d\xi$.

Lemma 4.2. *For any $\alpha \in \mathbb{N}^{2n}$ and $m \in \mathbb{R}$ such that $|\alpha| + m \leqslant -n$, there exists $C > 0$ and $M \in \mathbb{N}$ such that for all $x,z \in K$, we have*
$$k^{-|\alpha|}|\partial^\alpha_{x,z} I_k(a)(x,z)| \leqslant C\|a\|_{m,M}.$$

Proof. If $m < 0$ and $k \geqslant 1$, then
$$|a(x,z,k^{\frac{1}{2}}\xi)| \leqslant C\|a\|_{m,0}\langle k^{\frac{1}{2}}\xi\rangle^n \leqslant C\|a\|_{m,0}\langle\xi\rangle^n.$$
So $|I_k(a)(x,z)| \leqslant C\|a\|_{m,0}\int_{\mathbb{R}^n}\langle\xi\rangle^n\, d\xi$ and the integral is finite when $m < -n$. The estimates for the derivatives are obtained by derivating under the integral sign and applying the same method. ☐

Now by theorem 3.2 and the fact that $|\alpha| + |\beta| \geqslant \ell$ when $(\alpha,\beta,\alpha',\beta') \in \Sigma_\ell$, the map sending (a,b) into c_ℓ extends continuously from $S^m(U^2,\mathbb{R}^n) \times S^p(U^2,\mathbb{R}^n)$ to $S^{m+p-\ell}(U,\mathbb{R}^n)$. So by Lemma 4.2, $I_k(c_\ell)$ satisfies the same bounds (27) as $R_{N,k}$, so that the expansion (25) is meaningful.

To deduce Theorem 4.1 from (25), introduce $c(h,\cdot) \in S^{m+p}_{\mathrm{sc}}(U \times \mathbb{R}^n)$ having the expansion $\sum_\ell h^\ell c_\ell$, which is possible by Borel Lemma. Then by (26), (27) and Lemma 4.2,
$$\Big(\frac{k}{2\pi}\Big)^n J_k(f_k)(x,z) = \int_{\mathbb{R}^n} e^{ik\xi\cdot(x-z)} c(k^{-\frac{1}{2}}, \tfrac{1}{2}(x+z), k^{\frac{1}{2}}\xi)\, d\xi + \mathcal{O}_\infty(k^{-\infty})$$

In the sequel, we first explain how we obtain the formulas (26) for the coefficients c_ℓ, and then we prove the remainder bounds (27).

Proof of formulas (26)

The first step is to replace φ, a and b by their Taylor expansions along the diagonal $\{x = y = z\}$. To do this, introduce the coordinate system $(\overline{x}_j, u_j, v_j)$ of U^3

$$\overline{x}_j = \tfrac{1}{2}(x_j + z_j), \qquad u_j = x_j - y_j, \qquad v_j = y_j - z_j, \qquad j = 1, \ldots, n$$

The diagonal is $\{u = v = 0\}$. Since the $\overline{x}_j$ are independent of y, they will not be affected during the integration. It could be possible to use x_j or z_j instead of $\frac{1}{2}(x_j + z_j)$ and this would lead to the expansion (25) with coefficients c_ℓ which are functions of x_j (resp. z_j) only. We have

$$x_j = \overline{x}_j + \tfrac{1}{2}(u_j + v_j), \quad y_j = \overline{x}_j + \tfrac{1}{2}(-u_j + v_j), \quad z_j = \overline{x}_j - \tfrac{1}{2}(u_j + v_j)$$

so that the vector field frame associated to $(\overline{x}_j, u_j, v_j)$ is

$$\partial_{u_j} = \tfrac{1}{2}(\partial_{x_j} - \partial_{y_j} - \partial_{z_j}), \quad \partial_{v_j} = \tfrac{1}{2}(\partial_{x_j} + \partial_{y_j} - \partial_{z_j}), \quad \partial_{\overline{x}_j} = \tfrac{1}{2}(\partial_{x_j} + \partial_{y_j} + \partial_{z_j})$$

For any function $f \in \mathcal{C}^\infty(U^3)$, its Taylor expansion with respect to u_j, v_j at $u = v = 0$ with the variables $\overline{x}_j$ fixed is

$$f(x, y, z) = \sum_{\substack{\alpha, \beta \in \mathbb{N}^n \\ |\alpha| + |\beta| \leqslant N}} (\alpha! \beta!)^{-1} (\partial_u^\alpha \partial_v^\beta f)(\overline{x}, \overline{x}, \overline{x}) u^\alpha v^\beta + r_{N+1}(x, y, z) \tag{28}$$

with r_{N+1} in $\mathcal{O}(|u, v|^{N+1})$. Later, in the proof of the remainder bounds, we will need the integral expression of the remainder

$$\begin{aligned} r_{N+1} &= (N+1) \sum_{|\alpha| + |\beta| = N+1} \frac{u^\alpha v^\beta}{\alpha! \beta!} \int_0^1 (1-t)^N c_{\alpha,\beta}(t)\, dt, \\ c_{\alpha,\beta}(t) &= (\partial_u^\alpha \partial_v^\beta f)(\overline{x} + \tfrac{t}{2}(u+v), \overline{x} + \tfrac{t}{2}(-u+v), \overline{x} - \tfrac{t}{2}(u+v)). \end{aligned} \tag{29}$$

Lemma 4.3. *We have $\varphi = \varphi_2 + r$ with $r(x, y, z) = \mathcal{O}(|u, v|^3)$ and*

$$\varphi_2(x, y, z) = \sum_{i,j=1}^n A_{ij}(\overline{x}) u_i v_j, \qquad A_{ij}(x) = \tfrac{1}{2}(\partial_{x_i} \beta_j(x) - \partial_{x_j} \beta_i(x)).$$

Proof. Writing $\varphi(x, y, z)$ in terms of $\overline{x}$, u and v, we get

$$\begin{aligned} \varphi(x, y, z) &= \beta(\overline{x} + \tfrac{v}{2}) \cdot u + \beta(\overline{x} - \tfrac{u}{2}) \cdot v - \beta(\overline{x}) \cdot (u + v) \\ &= \tfrac{1}{2}(\beta'(\overline{x})(v) \cdot u - \beta'(\overline{x})(u) \cdot v) + \mathcal{O}(|u, v|^3) \end{aligned}$$

after the cancellation of the linear terms in u, v. □

For any function $d \in \mathcal{C}^\infty(U^3 \times \mathbb{R}^{2n})$, define

$$\widetilde{d}_k(x,y,z,\xi,\eta) = d(x,y,z,k^{\frac{1}{2}}\xi,k^{\frac{1}{2}}\eta). \tag{30}$$

The isotropic algebra product appears through the following Lemma.

Lemma 4.4. *For any a_0, $b_0 \in \mathcal{C}_0^\infty(U \times \mathbb{R}^n)$, we have with $d(x,y,z,\xi,\eta) = a_0(\overline{x},\xi)b_0(\overline{x},\eta)$ that*

$$\Big(\frac{k}{2\pi}\Big)^n J_k(e^{ik\varphi_2}\widetilde{d}_k)(x,z) = \int_{\mathbb{R}^n} e^{ik\xi\cdot(x-z)} c_0(\overline{x},k^{\frac{1}{2}}\xi)\, d\xi + r_k(a_0,b_0)(x,z)$$

where $c_0(\overline{x},\cdot) = a_0(\overline{x},\cdot)\sharp_{A(\overline{x})}b_0(\overline{x},\cdot)$ and $r_k(a_0,b_0) = \mathcal{O}_\infty(k^{-\infty})$ on K^2.

Proof. For $R = \sqrt{k}$ and $g \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, we have with the notations used for Lemma 3.4 that

$$g(\tfrac{1}{i}\nabla_A)_R(x,y) = e^{-\frac{i}{2}kA(x,y)}\Big(\frac{k}{2\pi}\Big)^n \int_{\mathbb{R}^n} e^{ik\xi\cdot(x-y)} g(k^{\frac{1}{2}}\xi)\, d\xi \tag{31}$$

So identifying U with an open set of $\mathbb{R}^n$ through the coordinates (x_j) and introducing $h \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, we have for $x,z \in U$ that

$$\big(g(\tfrac{1}{i}\nabla_A)_R\, \chi\, h(\tfrac{1}{i}\nabla_A)_R\big)(x,z) = e^{-\frac{i}{2}kA(x,z)}\Big(\frac{k}{2\pi}\Big)^{2n} J_k(e^{ik\varphi_A}\widetilde{e}_k)(x,z) \tag{32}$$

where $e(x,y,z,\xi,\eta) = g(\xi)h(\eta)$ and

$$\varphi_A(x,y,z) = -\tfrac{1}{2}(A(x,y) + A(y,z) - A(x,z)) = -\tfrac{1}{2}A(x-y,y-z).$$

By (17), the left-hand side of (32) is equal to $(g\sharp_A h)(\frac{1}{i}\nabla_A)_R(x,z) + \mathcal{O}_\infty(k^{-\infty})$ when $x,z \in K$. Applying (31) to $g\sharp_A h$ instead of g, we get

$$\Big(\frac{k}{2\pi}\Big)^n J_k(e^{ik\varphi_A}\widetilde{e}_k)(x,z) = \int_{\mathbb{R}^n} e^{ik\xi\cdot(x-z)}(g\sharp_A h)(k^{\frac{1}{2}}\xi)d\xi + \mathcal{O}_\infty(k^{-\infty}). \tag{33}$$

Observe now that $\varphi_2(x,y,z) = \varphi_{A(\overline{x})}(x,y,z)$. So to conclude it suffices to apply (33) to $A = A(s)$, $g = a_0(s,\cdot)$, $h = b_0(s,\cdot)$ for $s \in U$ and to evaluate the result at $s = \overline{x}$. □

Remark 4.5. With the same proof, we can estimate the remainder $r_k(a_0,b_0)$ in terms of the semi-norms of a_0, b_0 by using Lemma 3.4 instead of (17). We obtain that for any m, $p \in \mathbb{R}$, $\alpha \in \mathbb{N}^{2n}$ and $N \in \mathbb{N}$, there exists $C > 0$ and $M \in \mathbb{N}$ such that for any a_0, $b_0 \in \mathcal{C}_0^\infty(U \times \mathbb{R}^n)$,

$$|\partial^\alpha r_k(a_0,b_0)(x,z)| \leqslant Ck^{-N}\|a_0\|_{m,M}\|b_0\|_{p,M}$$

for any $x,z \in K$. This will be used later to prove (27). □

To apply Lemma 4.4, we first write the Taylor expansions of a and b on the form

$$a(x,y,\xi) = \sum a_{\alpha_1,\beta_1}(\overline{x},\xi) u^{\alpha_1} v^{\beta_1}, \qquad b(y,z,\eta) = \sum b_{\alpha_2,\beta_2}(\overline{x},\eta) u^{\alpha_2} v^{\beta_2}.$$

Then take their product and remove the monomials $u^\alpha v^\beta$ by using the following identities

$$J_k(du_j) = ik^{-1} J_k(\partial_{\xi_j} d), \qquad J_k(dv_j) = ik^{-1} J_k(\partial_{\eta_j} d).$$

which hold for any $d \in \mathcal{C}_0^\infty(U^3 \times \mathbb{R}^{2n})$ and are simple integrations by part. For $\widetilde{d}_k$ given by (30), these identities write

$$J_k(\widetilde{d}_k u_j) = ik^{-\frac{1}{2}} J_k(\widetilde{\partial_{\xi_j} d}_k), \qquad J_k(\widetilde{d}_k v_j) = ik^{-\frac{1}{2}} J_k(\widetilde{\partial_{\eta_j} d}_k). \tag{34}$$

So for $d(x,y,z,\xi,\eta) = a(x,y,\xi)\, b(y,z,\eta)$

$$J_k(e^{ik\varphi_2}\widetilde{d}_k) = \sum_{\ell=0}^{\infty} i^\ell k^{-\frac{\ell}{2}} \sum_{|\alpha_1|+|\beta_1|+|\alpha_2|+|\beta_2|=\ell} J_k(e^{ik\varphi_2}(\widetilde{d_{\alpha_1,\beta_1,\alpha_2,\beta_2}})_k) \tag{35}$$

with $d_{\alpha_1,\beta_1,\alpha_2,\beta_2}(x,y,z,\xi,\eta) = (\partial_\xi^{\alpha_1+\alpha_2} a_{\alpha_1,\beta_1})(\overline{x},\xi)(\partial_\eta^{\beta_1+\beta_2} a_{\alpha_2,\beta_2})(\overline{x},\eta)$. Now we can apply Lemma 4.4 to each term in the right-hand side of (35). The estimates of the remainders when we truncate the infinite sum (35), will be given later.

To handle the remainder in the phase $r = \varphi - \varphi_2$, we expand $\exp(ikr)$ and write

$$J_k(e^{ik\varphi}\widetilde{d}_k) = \sum_{\ell'=0}^{\infty} \frac{(ik)^{\ell'}}{\ell'!} J_k(e^{ik\varphi_2} r^{\ell'} \widetilde{d}_k). \tag{36}$$

Here again, the control of the remainders are postponed to the next section. Then we Taylor expand $\frac{i^{\ell'}}{\ell'!} r^{\ell'} = \sum r_{\ell',\alpha_3,\beta_3}(\overline{x}) u^{\alpha_3} v^{\beta_3}$, we multiply by the Taylor expansion of a and b and follow the same method as before. We claim that this lead to the formula (26) for the coefficients c_ℓ.

Indeed, observe that $a_{\alpha_1,\beta_1}(\overline{x},\xi)$ (resp. $b_{\alpha_2,\beta_2}(\overline{x},\eta)$) is a linear combination with constant coefficients of the $\partial_{x,y}^{\alpha'} a(\overline{x},\overline{x},\xi)$ with $|\alpha'| = |\alpha_1 + \beta_1|$ (resp. $\partial_{y,z}^{\beta'} b(\overline{x},\overline{x},\eta)$ with $|\beta'| = |\alpha_2 + \beta_2|$). Then we consider the sum of the products

$$a_{\alpha_1,\beta_1} u^{\alpha_1} v^{\beta_1} b_{\alpha_2,\beta_2} u^{\alpha_2} v^{\beta_2} k^{\ell'} r_{\ell',\alpha_3,\beta_3} u^{\alpha_3} v^{\beta_3}$$

which after the integrations by parts gives

$$k^{-\frac{\ell}{2}} i^{|\alpha+\beta|} J_k(e^{ik\varphi_2}(\widetilde{\partial_\xi^\alpha a_{\alpha_1,\beta_1}})_k(\widetilde{\partial_\eta^\beta b_{\alpha_2,\beta_2}})_k r_{\ell',\alpha_3,\beta_3}(\overline{x}))$$

with $\ell = -2\ell' + |\alpha| + |\beta|$, $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ and $\beta = \beta_1 + \beta_2 + \beta_3$. Then on one hand, $0 \leqslant \ell'$ implies that $\ell \leqslant |\alpha| + |\beta|$. On the other hand, r vanishes to third order along the diagonal, so $3\ell' \leqslant |\alpha_3 + \beta_3|$ which is equivalent to $|\alpha + \beta| + 2|\alpha_1 + \beta_1 + \alpha_2 + \beta_2| \leqslant 3\ell$. So we have that

$$\ell \leqslant |\alpha + \beta| \leqslant 2\ell \qquad \text{and} \qquad |\alpha' + \beta'| = |\alpha_1 + \beta_1 + \alpha_2 + \beta_2| \leqslant \ell$$

the inequalities entering in the definition of Σ_ℓ.

Proof of remainder bounds (27)

In the previous argument, we used two infinite expansions (35) and (36). We need to truncate them and estimate the corresponding remainders. For any $d \in \mathcal{C}_0^\infty(U^3 \times \mathbb{R}^{2n})$, define the remainders $R'_{N+1,k}(d)$ and $R''_{N+1,k}(d)$ in $\mathcal{C}^\infty(U_x \times U_z)$ such that

$$J_k(e^{ik\varphi}\widetilde{d}_k) = J_k\big(e^{ik\varphi_2} \textstyle\sum_{\ell=0}^N \frac{(ikr)^\ell}{\ell!} \widetilde{d}_k\ \big) + k^{-\frac{1}{2}(N+1)} R'_{N+1,k}(d) \tag{37}$$

$$\begin{aligned} J_k(e^{ik\varphi_2}\widetilde{d}_k) = &\sum_{|\gamma|+|\delta|\leqslant N} \frac{(i)^{|\gamma|+|\delta|}}{\gamma!\delta!} k^{-\frac{1}{2}(|\gamma|+|\delta|)} J_k\big(e^{ik\varphi_2}(\widetilde{\partial_\xi^\gamma \partial_\eta^\delta d_{\gamma,\delta}})_k\big) \\ &+ k^{-\frac{1}{2}(N+1)} R''_{N+1,k}(d) \end{aligned} \tag{38}$$

where in the second equation $d_{\gamma,\delta}(x,y,z,\xi,\eta) := (\partial_u^\gamma \partial_v^\delta d)(\overline{x},\overline{x},\overline{x},\xi,\eta)$ for $\gamma, \delta \in \mathbb{N}^n$. Introduce for any m, $p \in \mathbb{R}$, the norm

$$\|d\|_{m,p} = \sup_{\substack{(x,y,z)\in U^3 \\ (\xi,\eta)\in\mathbb{R}^{2n}}} |d(x,y,z,\xi,\eta)| \langle\xi\rangle^{-m} \langle\eta\rangle^{-p} \tag{39}$$

and for any $M \in \mathbb{N}$

$$\|d\|_{m,p,M} = \max_{\substack{\beta\in\mathbb{N}^{3n},\gamma,\delta\in\mathbb{N}^n \\ |\alpha|+|\beta|+|\gamma|\leqslant M}} \|\partial_{x,y,z}^\beta \partial_\xi^\gamma \partial_\eta^\delta d\ \|_{m-|\gamma|,p-|\delta|} \tag{40}$$

Lemma 4.6. *$\forall m$, $p \in \mathbb{R}$, $\alpha \in \mathbb{N}^{2n}$ and $N \in \mathbb{N}$, there exists $C > 0$ and $M \in \mathbb{N}$ such that for every $d \in \mathcal{C}_0^\infty(U^3 \times \mathbb{R}^{2n})$, we have*

1. *if $m + p + |\alpha| \leqslant 3(N+1) - 2n - 1$, then*

$$k^{-|\alpha|} |\partial_{x,z}^\alpha R'_{N+1}(d)(x,z)| \leqslant C k^{\frac{1}{2}(m_+ + p_+)} \|d\|_{m,p,M}$$

2. *if* $m+p+|\alpha| \leqslant N-2n-1$, *then*

$$k^{-|\alpha|}|\partial^{\alpha}_{x,z} R''_{N+1}(d)(x,z)| \leqslant C k^{\frac{1}{2}(m_+ + p_+)} \|d\|_{m,p,M}$$

For the proof we need the following lemma.

Lemma 4.7. *For any* $m, p \in \mathbb{R}$ *and* $\alpha \in \mathbb{N}^{2n}$ *such that* $m+p+|\alpha| < -2n-1$, *there exists* $C>0$ *and* $M \in \mathbb{N}$ *such that for any* $c \in \mathcal{C}^{\infty}_0(U^3 \times \mathbb{R}^{2n})$, *we have*

$$k^{-|\alpha|}|\partial^{\alpha}_{x,z} J_k(c)(x,z)| \leqslant C \max_{\beta \in \mathbb{N}^{3n},\ |\beta| \leqslant M} k^{-|\beta|} \|\partial^{\beta}_{x,y,z} c\|_{m,p}$$

for any $(x,z) \in K$.

Proof. 1. Assume that $m < -n$ and $p < -n$. Then

$$|J_k(c)(x,z)| \leqslant \|c\|_{m,p} \Big(\int \langle \xi \rangle^m \ d\xi \Big) \Big(\int \langle \eta \rangle^p \ d\eta \Big) \Big(\int \rho(y) \ dy \Big)$$

and the three integrals are finite.

2. Assume now that $m+p < -2n-1$. So there exists $q \in \mathbb{Z}$ such that $m-q < -n$ and $p+q < -n$. We assume $q \geqslant 0$, otherwise the argument is similar by exchanging ξ and η. We have by integration by part that $J_k(Lc) = J_k(c)$ where $L^t = \langle \xi - \eta \rangle^{-2}(1 + (ik)^{-1} \sum_j (\eta_j - \xi_j) \partial_{y_j})$. Furthermore

$$\begin{aligned} |L^q(c)(x,y,z,\xi,\eta)| &\leqslant \langle \xi - \eta \rangle^{-q} \langle \xi \rangle^m \langle \eta \rangle^p \max_{|\gamma| \leqslant q} k^{-|\gamma|} \|\partial^{\gamma}_y c\|_{m,p} \\ &\leqslant C \langle \xi \rangle^{m-q} \langle \eta \rangle^{p+q} \max_{|\gamma| \leqslant q} k^{-|\gamma|} \|\partial^{\gamma}_y c\|_{m,p} \end{aligned}$$

and we conclude as in Part 1 that

$$|J_k(c)(x,z)| \leqslant C \max_{|\gamma| \leqslant q} k^{-|\gamma|} \|\partial^{\gamma}_y c\|_{m,p}. \tag{41}$$

3. It remains to estimates the derivatives. Let $\alpha, \beta \in \mathbb{N}^n$ be such that $m+p+|\alpha|+|\beta| < -2n-1$. Since

$$\begin{aligned} (ik)^{-1} \partial_{x_j} J_k(c) &= J_k((\xi_j + (ik)^{-1} \partial_{x_j})c), \\ (ik)^{-1} \partial_{z_j} J_k(c) &= J_k((-\eta_j + (ik)^{-1} \partial_{z_j})c). \end{aligned}$$

we have $k^{-|\alpha|-|\beta|} \partial^{\alpha}_x \partial^{\beta}_z J_k(c) = J_k(d)$ where d is a linear combination with constant coefficients (depending only of α, β) of the functions

$$c^{\alpha'',\beta''}_{\alpha',\beta'} = \xi^{\alpha''} \eta^{\beta''} k^{-|\alpha'|-|\beta'|} \partial^{\alpha'}_x \partial^{\beta'}_z c, \qquad \alpha', \alpha'', \beta', \beta'' \in \mathbb{N}^n$$

where $\alpha' + \alpha'' = \alpha$ and $\beta' + \beta'' = \beta$. Then we apply (41) to the $c^{\alpha'',\beta''}_{\alpha',\beta'}$. Using that $m + |\alpha''| + p + |\beta''| \leqslant m + p + |\alpha| + |\beta| < -2n - 1$ and

$$\|\partial_y^\gamma c^{\alpha'',\beta''}_{\alpha',\beta'}\|_{m+|\alpha''|,p+|\beta''|} \leqslant Ck^{-|\alpha'|-|\beta'|}\|\partial_y^\gamma \partial_x^{\alpha'}\partial_z^{\beta'} c\|_{m,p},$$

we obtain

$$k^{-|\alpha|-|\beta|}|\partial_x^\alpha \partial_z^\beta J_k(c)(x,z)| \leqslant C \max_{\substack{\alpha',\,\beta',\,\gamma\in\mathbb{N}^n \text{ s.t.}\\ \alpha'\leqslant\alpha,\,\beta'\leqslant\beta,\,|\gamma|\leqslant M}} k^{-|\alpha'|-|\beta'|-|\gamma|}\|\partial_x^{\alpha'}\partial_y^\gamma\partial_z^{\beta'} c\|_{m,p}$$

for some M, which implies the Lemma. □

Proof of Lemma 4.6. Let $(f_k, k \in \mathbb{N})$ be a family of $\mathcal{C}^\infty(U^3)$ such that for any $\alpha \in \mathbb{N}^{3n}$, $k^{-|\alpha|}\partial^\alpha f_k = \mathcal{O}(1)$ on any compact subset of U^3. For instance we can choose $f_k = e^{ik\varphi}$. Using Leibniz rule and $\langle k^{\frac{1}{2}}\xi\rangle^m \leqslant k^{\frac{1}{2}m_+}\langle\xi\rangle^m$ for any $k \geqslant 1$ and $m \in \mathbb{R}$, we deduce from Lemma 4.7 that for any m, $p \in \mathbb{R}$ and $\alpha \in \mathbb{N}^{2n}$ such that $m+p+|\alpha| < -2n-1$, there exists $C > 0$ and $M \in \mathbb{N}$ such that for any $d \in \mathcal{C}_0^\infty(U^3 \times \mathbb{R}^{2n})$, we have

$$k^{-|\alpha|}|\partial^\alpha_{x,z} J_k(f_k\widetilde{d}_k)(x,z)| \leqslant Ck^{\frac{1}{2}(m_+ + p_+)} \max_{\substack{\beta\in\mathbb{N}^{3n},\\ |\beta|\leqslant M}} k^{-|\beta|}\|\partial^\beta_{x,y,z} d\|_{m,p} \qquad (42)$$

for any $(x,z) \in K$.

Let us prove the estimate for $R'_{N+1,k}$. Applying Taylor formula to $t \in [0,1] \to \exp(ik(\varphi_2 + tr))$, we get

$$e^{ik\varphi} = e^{ik\varphi_2}\sum_{\ell=0}^{N}\frac{(ikr)^\ell}{\ell!} + (kr)^{N+1} f_{N,k}, \quad f_{N,k} := \tfrac{i^{N+1}}{N!}\int_0^1 (1-t)^N e^{ik\varphi_2}\, dt$$

Derivating under the integral sign, we get $k^{-|\alpha|}\partial^\alpha_{x,y,z} f_{N,k} = \mathcal{O}(1)$ for any $\alpha \in \mathbb{N}^{3n}$. Since r^{N+1} vanishes to order $3N+3$ along $\{u = v = 0\}$, we have

$$r^{N+1} = \sum_{|\gamma|+|\delta|=3N+3} u^\gamma v^\delta \mu_{\gamma,\delta}, \qquad \text{with } \mu_{\gamma,\delta} \in \mathcal{C}^\infty(U^3).$$

Collecting together the previous formula, we obtain

$$\begin{aligned} R'_{N+1,k}(d) &= k^{\frac{3}{2}(N+1)}\sum J_k(u^\gamma v^\delta \mu_{\gamma,\delta} f_{N,k}\widetilde{d}_k) \\ &= i^{3(N+1)}\sum J_k(\mu_{\gamma,\delta} f_{N,k}\widetilde{\partial_\xi^\gamma\partial_\eta^\delta d}_k) \end{aligned}$$

where we have used (34) and the fact that $|\gamma| + |\delta| = 3N + 3$, so that the powers of k cancel. We conclude with (42) and $\|\partial^{\beta}_{x,y,z}\partial^{\gamma}_{\xi}\partial^{\delta}_{\eta} d\|_{m-|\gamma|,p-|\delta|} \leqslant \|d\|_{m,p,|\beta|+3N+3}$.

To prove the estimate for $R''_{N+1,k}$, we write the Taylor expansion of d as in (28)

$$d = \sum_{|\gamma|+|\delta|\leqslant N} u^{\gamma} v^{\delta} \frac{d_{\gamma,\delta}}{\gamma!\delta!} + \sum_{|\gamma|+|\delta|=N+1} u^{\gamma} v^{\delta} d_{N+1,\gamma,\delta}$$

The coefficients $d_{N+1,\gamma,\delta}$ of the remainder are functions of (x, y, z, ξ, η) and using the formula (29), we get

$$\|d_{N+1,\gamma,\delta}\|_{m,p,M} \leqslant C\|d\|_{m,p,M+N+1} \tag{43}$$

By using (34), it comes that

$$J_k(e^{ik\varphi_2} u^{\gamma} v^{\delta} \widetilde{d_{\gamma,\delta}}_k) = i^{|\gamma|+|\delta|} k^{-\frac{1}{2}(|\gamma|+|\delta|)} J_k(e^{ik\varphi_2} (\widetilde{\partial^{\gamma}_{\xi}\partial^{\delta}_{\eta} d_{\gamma,\delta}})_k)$$

and similarly

$$J_k(e^{ik\varphi_2} u^{\gamma} v^{\delta} \widetilde{d_{N+1,\gamma,\delta}}_k) = i^{N+1} k^{-\frac{1}{2}(N+1)} J_k(e^{ik\varphi_2} (\widetilde{\partial^{\gamma}_{\xi}\partial^{\delta}_{\eta} d_{N+1,\gamma,\delta}})_k)$$

Comparing with (38), we get

$$R''_{N+1,d}(d) = i^{N+1} \sum_{|\gamma|+|\delta|=N+1} J_k(e^{ik\varphi_2} (\widetilde{\partial^{\gamma}_{\xi}\partial^{\delta}_{\eta} d_{N+1,\gamma,\delta}})_k)$$

and we conclude with (42) and (43). □

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