Résolvantes des Laplaciens magnétiques et opérateurs pseudodifférentiels semiclassiques de Heisenberg

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Outline

1. Bochner Laplacien
   a definition and the semiclassical limit.
   b spectral asymptotics: the Weyl law and the clusters at smaller scale.
2. Twisted semiclassical pseudodifferential operators
3. Landau Hamiltonian, spectrum, resolvent and spectral projectors
4. Heisenberg pseudodifferential operators
   a definitions, composition
   b application to the resolvent and spectral projectors of the Bochner Laplacian

Bochner Laplacian

Consider

- (M, g) a compact Riemannian manifold with $\partial M = \emptyset$
- $L \to M$ a Hermitian line bundle with a connection ∇

The Bochner Laplacian, or Schrödinger operator with magnetic field $\omega = i\,\mathrm{courb}(\nabla)$, is

$$\Delta = \tfrac{1}{2}\nabla^*\nabla \qquad \text{acting on } \mathcal{C}^\infty(M, L)$$

In a trivialisation on a coordinate chart U,

$$\Delta = -\tfrac{1}{2\sqrt{g}} \sum_{i,j} \nabla_i (g^{ij}\sqrt{g}\nabla_j) \qquad \text{acting on } \mathcal{C}^\infty(U)$$

with $\nabla_i = \partial_{x_i} + \frac{1}{i}\beta_i$, $\beta_i \in \mathcal{C}^\infty(U, \mathbb{R})$, and $d(\sum_i \beta_i dx_i) = \omega$.

Elliptic positive operator, so discrete spectrum in $[0, \infty[$, eigenvalues with finite multiplicities, smooth eigensections.

Large magnetic field limit

take $k \in \mathbb{N}$ and replace L by $L^k = L^{\otimes k}$, ∇ by ∇^{L^k} and set

$$\Delta_k = \frac{1}{2}(\nabla^{L^k})^* \nabla^{L^k} : \mathcal{C}^\infty(M, L^k) \to \mathcal{C}^\infty(M, L^k).$$

What can we say on eigenvalues and eigensections in the limit $k \to \infty$?
A trivialisation $L|_U \simeq U \times \mathbb{C}$ induces trivialisations $L^k|_U \simeq U \times \mathbb{C}$ so that $\mathcal{C}^\infty(U, L^k) \simeq \mathcal{C}^\infty(U)$ and

$$\Delta_k = -\frac{1}{2\sqrt{g}} \sum_{i,j} \nabla_i^k (g^{ij} \sqrt{g} \nabla_j^k)$$

where $\nabla_i^k = \partial_{x^i} + \frac{k}{i} \beta_i$.

Semiclassical result

Weyl law

One has in the limit $k \to \infty$

$$\sharp\big(\mathrm{sp}(k^{-2}\Delta_k) \cap [E, E']\big) \sim \left(\frac{k}{2\pi}\right)^n (\mathrm{vol}(\{E \leqslant \tfrac{1}{2}|\xi|^2 \leqslant E'\}, \Omega)$$

with $n = \dim M$ and $\Omega \in \Omega^2(T^*M)$ the twisted symplectic form $\Omega = \sum d\xi^i \wedge dx_i + \pi^*\omega \in \Omega^2(T^*M)$.

Smaller scale (Guillemin-Uribe 88, Faure-Tsuji 15)

If ω is non-degenerate and compatible with g, then

$$\mathrm{sp}(k^{-1}\Delta_k) \cap [0, M] \subset (\tfrac{n}{4} + \mathbb{N}) + C_M k^{-\frac{1}{2}}[-1, 1]$$

and for any $m \in \mathbb{N}$,

$$\sharp\big(\mathrm{sp}(k^{-1}\Delta_k) \cap (\tfrac{n}{4} + m + [-\tfrac{1}{6}, \tfrac{1}{6}]\big) \sim \left(\frac{k}{2\pi}\right)^{\frac{n}{2}} \binom{\frac{n}{2}+m-1}{m} \mathrm{Vol}(M, \omega)$$

Goal of this talk

Explain the smaller scale results and introduce the class $\Psi_{\mathrm{Heis}}(L)$ of semiclassical Heisenberg pseudodifferential operators which contains

- the Laplacian $k^{-1}\Delta_k$
- the resolvent $(z - k^{-1}\Delta_k)^{-1}$ for $z \in \mathbb{C} \setminus (\frac{n}{4} + \mathbb{N})$
- the spectral projector $1_{\Sigma_m}(k^{-1}\Delta_k)$ for $m \in \mathbb{N}$, $\Sigma_m = \frac{n}{4} + m + [-\frac{1}{6}, \frac{1}{6}]$.

based on arxiv:2309.04.

Previous works on the clusters, including their exact dimensions, the Schwartz kernels of the spectral projectors, the associated Toeplitz operators by Boutet-Guillemin, Zelditch, Guillemin-Uribe, Borthwick-Uribe, Ma-Marinescu, Faure-Tsuji, Kordyukov and many other people including myself...

The road from Harmonic oscillator to secmiclassical magnetic Laplacian

1. describe resolvent and spectral projectors of Harmonic oscilator by using Weyl quantization $f^w(\frac{1}{i}\partial_x, x)$
2. pass from Harmonic oscillator to Landau Hamiltonian and from Weyl quantization $f^w(\frac{1}{i}\partial_x, x)$ to $f(\frac{1}{i}\nabla)$
3. rescale to semiclassical Landau Hamiltonian and introduce $f(\frac{k^{-\frac{1}{2}}}{i}\nabla^k)$
4. pass from semiclassical Landau Hamiltonian to semiclassical magnetic Laplacian,
   introduce Heisenberg semiclassical pseudodifferential operator $f(x, \frac{k^{-\frac{1}{2}}}{i}\nabla^k)$.

Landau Hamiltonian vs Harmonic Oscillator

The covariant derivatives $\nabla_1 = \partial_{x_1} - \frac{i}{2}x_2$, $\nabla_2 = \partial_{x_2} + \frac{i}{2}x_1$ satisfy

$$i\left[\tfrac{1}{i}\nabla_1, \tfrac{1}{i}\nabla_2\right] = 1, \qquad i\left[\tfrac{1}{i}\partial_x, x\right] = 1$$

Landau Hamiltonian and Harmonic oscillator are

$$\hat{H} = \tfrac{1}{2}((\tfrac{1}{i}\nabla_1)^2) + (\tfrac{1}{i}\nabla_2)^2) \qquad \text{acting on } \mathcal{S}(\mathbb{R}^2_{x_1,x_2})$$

$$H^w = \tfrac{1}{2}((\tfrac{1}{i}\partial_x)^2 + x^2) \qquad \text{acting on } \mathcal{S}(\mathbb{R}_x)$$

Facts: There exists a unitary operator U of $L^2(\mathbb{R}^2)$ such that

$$U\hat{H}U^* = H^w \boxtimes \mathrm{id}_{L^2(\mathbb{R}_y)}\,.$$

Proof: U is metaplectic representation of the symplectomorphism $\Phi : \mathbb{R}^4_{\xi_1,x_1,\xi_2,x_2} \to \mathbb{R}^4_{\xi,x,\eta,y}$ given by

$$(\xi, x, \eta, y) = (\xi_1 - \tfrac{1}{2}x_2, \xi_2 + \tfrac{1}{2}x_1, \xi_2 - \tfrac{1}{2}x_1, \xi_1 + \tfrac{1}{2}x_2)$$

Consequence: Landau Hamiltonian spectrum

Landau Hamiltonian and Harmonic oscillator are

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$$U\hat{H}U^* = H^w \boxtimes \mathrm{id}_{L^2(\mathbb{R}_y)} \,.$$

Consequently,

- $\mathrm{Spec}(\hat{H}) = \mathrm{Spec}(H^w) = \frac{1}{2} + \mathbb{N}$
- each eigenspaces of $\hat{H}$ is $L^2(\mathbb{R})$.

Landau Hamiltonian vs Harmonic Oscillator

Using that metaplectic representation U of Φ conjugates Weyl quantization of $g(\xi, x, \eta, y)$ and $g \circ \Phi(\xi_1, x_1, \xi_2, x_2)$, we also have

$$U(f^w \boxtimes \mathrm{id})U^* = f(\tfrac{1}{i}\nabla), \qquad \forall f \in \mathcal{S}'(\mathbb{R}^2)$$

where $f^w : \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ is Weyl quantization of $f(\xi, x)$ with Schwartz kernel

$$(2\pi)^{-1} \int e^{i\xi(x-y)} f(\xi, \tfrac{x+y}{2})\, d\xi$$

and $f(\frac{1}{i}\nabla) : \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{R}^2)$ is Weyl quantization of $f(\xi_1 - \frac{1}{2}x_2, \xi_2 + \frac{1}{2}x_1)$ with Schwartz kernel

$$(2\pi)^{-2} \int e^{i\xi_1(x_1-y_1)+\xi_2(x_2-y_2)} f(\xi_1 - \tfrac{1}{2}x_2, \xi_2 + \tfrac{1}{2}x_1)\, d\xi_1 d\xi_2$$

In particular with $H = \frac{1}{2}(\xi^2 + x^2)$, Landau Hamiltonian is $\hat{H} = H(\frac{1}{i}\nabla)$, harmonic oscillator is H^w.

Landau Hamiltonian resolvent and spectral projectors

Since for any $f \in \mathcal{S}'(\mathbb{R}^2)$, one has $U(f^w \boxtimes \mathrm{id})U^* = f(\frac{1}{i}\nabla)$, there exists $R_z, \pi_E \in \mathcal{S}'(\mathbb{R}^2)$ such that

- For any $z \in \mathbb{C} \setminus (\frac{1}{2} + \mathbb{N})$, the resolvent is

$$(\hat{H} - z)^{-1} = R_z(\tfrac{1}{i}\nabla) \qquad \text{where } R_z^w = (H^w - z)^{-1}$$

- the spectral projector of $E \in \frac{1}{2} + \mathbb{N}$ are

$$1_E(\hat{H}) = \pi_E(\tfrac{1}{i}\nabla) \qquad \text{where } \pi_E^w = 1_E(H^w)$$

Explicit formula by Derezinski-Karczmarczyk (2017)

$$R_z = \int_0^1 (1 - \tfrac{1}{2}s)^{\frac{1}{2} - z - 1}(1 + \tfrac{1}{2}s)^{\frac{1}{2} + z - 1} e^{-sH} ds$$

when $\operatorname{Re} z < 1$ and by Unterberger (2016)

$$\pi_E = 2(-1)^m e^{-2H} L_m^{-1}(4H), \qquad \text{where } E = m + \tfrac{1}{2}$$

where L_m is a Laguerre polynomial.

Semiclassical Landau Hamiltonian

With $k > 0$, consider $\nabla_1^k = \partial_{x_1} - \frac{ik}{2} x_2$, $\nabla_2^k = \partial_{x_2} + \frac{ik}{2} x_1$ and

$$\hat{H}_k = \frac{1}{2}((\frac{1}{i}\nabla_1^k)^2 + (\frac{1}{i}\nabla_1^k)^2) \qquad \text{acting on } \mathcal{S}(\mathbb{R}^2)$$

Define the rescaling ρ_k of $L^2(\mathbb{R}^2)$ by $\rho_k u(x) = k^{-1} u(k^{-\frac{1}{2}} x)$, $x \in \mathbb{R}^2$. Then one has

$$\rho_k^* \hat{H} \rho_k = k^{-1} \hat{H}_k, \qquad \text{so } \operatorname{sp}(k^{-1}\hat{H}_k) = \frac{1}{2} + \mathbb{N}$$

Moreover, $\rho_k^* f(\frac{1}{i}\nabla)\rho_k = f(\frac{k^{-\frac{1}{2}}}{i}\nabla^k)$, the operator with Schwartz kernel

$$(\frac{k}{2\pi})^2 \int e^{ik\xi(x-y)} f(k^{-\frac{1}{2}}(\xi - \beta(\frac{x+y}{2})))d\xi, \quad x, y \in \mathbb{R}^2$$

here $\beta(x_1, x_2) = (\frac{1}{2}x_2, -\frac{1}{2}x_1)$.

Since $\rho_k^* f(\frac{1}{i}\nabla)\rho_k = f(\frac{k^{-\frac{1}{2}}}{i}\nabla^k)$, the resolvent and spectral projectors of $k^{-1}\hat{H}_k$ are

$$(k^{-1}\hat{H}_k - z)^{-1} = R_z(\frac{k^{-\frac{1}{2}}}{i}\nabla^k), \qquad z \in \mathbb{C} \setminus (\frac{1}{2} + \mathbb{N})$$

$$1_{\{E\}}(k^{-1}\hat{H}_k) = \pi_E(\frac{k^{-\frac{1}{2}}}{i}\nabla^k), \qquad E \in \frac{1}{2} + \mathbb{N}$$

From Landau Hamiltonian to Bochner Laplacian

Consider now a metric g on an open set U of $\mathbb{R}^2$ and consider the Bochner Laplacian

$$\Delta_k = -\frac{1}{2\sqrt{g}} \sum_{i,j} \nabla_i^k (g^{ij}\sqrt{g}\nabla_j^k)$$

with ∇_1^k, ∇_2^k as in the definition for $\hat{H}_k$.

To construct resolvent and spectral projectors of $k^{-1}\Delta_k$, we define $f(x, \frac{k^{-\frac{1}{2}}}{i}\nabla^k)$ for symbols $f \in S(U_x, \mathbb{R}^2_\xi)$ as the operator with Schwartz kernel

$$(\tfrac{k}{2\pi})^2 \int e^{ik\xi(x-y)} f(\tfrac{x+y}{2}, k^{-\frac{1}{2}}(\xi - \beta(\tfrac{x+y}{2})))d\xi, \quad x, y \in \mathbb{R}^2$$

Semiclassical Heisenberg pseudodifferential operator

A family $(P_k : \mathcal{C}^\infty(M, L^k) \to \mathcal{C}^\infty(M, L^k),\ k \in \mathbb{N})$ belongs to $\Psi^m_{\mathrm{Heis}}(L, \nabla)$ if locally its Schwartz kernel has the form

$$(2\pi h)^{-n} \int_{\mathbb{R}^n} e^{ih^{-1}\xi(x-y)} f\left(h^{\frac{1}{2}}, \tfrac{x+y}{2}, h^{-\frac{1}{2}}(\xi - \beta(\tfrac{x+y}{2}))\right) d\xi$$

with $h = k^{-1}$, $f(h, x, \xi) = f_0(x, \xi) + hf_1(x, \xi) + \dots$ where the coefficients satisfy $\partial_x^\alpha \partial_\xi^\beta f_\ell = \mathcal{O}(\langle \xi \rangle^{m-\ell-|\beta|})$.

Facts:

- the class $\Psi^m_{\mathrm{Heis}}(L, \nabla)$ is well-defined
- the symbol principal of (P_k) is $f_0 \in \mathcal{C}^\infty(T^*M)$.
- for any vector field X of M, $(\frac{k^{-\frac{1}{2}}}{i} \nabla^k_X) \in \Psi^1_{\mathrm{Heis}}(L, \nabla)$, its principal symbol is $(x, \xi) \to \xi(X(x))$.
- $(k^{-1}\Delta_k) \in \Psi^2_{\mathrm{Heis}}(L, \nabla)$, its principal symbol is $\frac{1}{2}|\xi|^2$.

Heisenberg composition

Theorem

If $(P_k) \in \Psi^m_{\mathrm{Heis}}(L, \nabla)$ *and* $(Q_k) \in \Psi^\ell_{\mathrm{Heis}}(L, \nabla)$*, then* $(P_k \circ Q_k) \in \Psi^{m+\ell}_{\mathrm{Heis}}(L, \nabla)$*.*

The corresponding product $\sharp$ *of principal symbols is*

$$(f \sharp g)(x, \zeta) = e^{\frac{i}{2}\omega_x(\partial_\xi, \partial_\eta)} f(x, \xi) g(x, \eta)|_{\xi=\eta=\zeta}$$

The product $\sharp$ is fibered. Its restriction to $T^*_x M$ is the usual multiplication when $\omega_x = 0$, the Weyl product when ω_x is non-degenerate. So it is non-commutative in general.

Resolvent and spectral projector of the Bochner Laplacian

Assume that ω is non-degenerate and compatible with g.
Let $d = \frac{n}{2} \in \mathbb{N}$ with $n = \dim M$.

Theorem

For any $z \in \mathbb{C} \setminus (\frac{d}{2} + \mathbb{N})$, there exists $(Q_k(z)) \in \Psi^{-2}_{\mathrm{Heis}}(L, \nabla)$ such that

- $(k^{-1}\Delta_k - z)Q_k(z) \equiv \mathrm{id}$, $Q_k(z)(k^{-1}\Delta_k - z) \equiv \mathrm{id}$ *modulo smoothing operators,*
- $Q_k(z) = (k^{-1}\Delta_k - z)^{-1}$ *when k is large,*
- *the principal symbol of $(Q_k(z))$ is $R_{d,z}(\xi)$.*

For any $E \in \frac{d}{2} + \mathbb{N}$, $(1_{[E-\frac{1}{6}, E+\frac{1}{6}]}(k^{-1}\Delta_k)) \in \Psi^{-\infty}_{\mathrm{Heis}}(L, \nabla)$ and its symbol is $\pi_{d,E}(\xi)$.

Resolvent and Spectral projectors of the Harmonic oscillator in dimension d

For any $f \in \mathcal{S}'(\mathbb{R}^{2d})$, let f^w be its Weyl quantization, so the operator with Schwartz kernel $(2\pi)^{-d} \int e^{i\xi(x-y)} f(\frac{x+y}{2}, \xi) d\xi$.

The d-dimensional quantum harmonic oscillator is H^w with $H(\xi) = \frac{1}{2}|\xi|^2$, $\xi \in \mathbb{R}^{2d}$. Its spectrum is $\frac{d}{2} + \mathbb{N}$.
Define $R_{d,z}$ and $\pi_{d,E}$ in $\mathcal{S}'(\mathbb{R}^{2d})$ by

- $R^w_{d,z} = (H^w - z)^{-1}$ for any $z \in \mathbb{C} \setminus (\frac{d}{2} + \mathbb{N})$
- $\pi^w_{d,E} = 1_{\{E\}}(H^w)$ for any $E \in \frac{d}{2} + \mathbb{N}$.

By the Weyl calculus, $R_{d,z} \in S^{-2}(\mathbb{R}^{2d})$ and $\pi_{d,E} \in \mathcal{S}(\mathbb{R}^{2d})$.

Actually, by Derezinski-Karczmarczyk (2017)

$$R_{d,z} = \int_0^1 (1 - \tfrac{1}{2}s)^{\frac{d}{2}-z-1} (1 + \tfrac{1}{2}s)^{\frac{d}{2}+z-1} e^{-sH} ds$$

when $\operatorname{Re} z < d$ and by Unterberger (2016)

$$\pi_{d,E}(\xi) = 2^d (-1)^m e^{-|\xi|^2} L^{d-1}_m (2|\xi|^2), \qquad \text{where } m = E - \tfrac{d}{2}$$

where L_m is a Laguerre polynomial.