Chapter 2

Holonomy and Chern classes

2.1 Cheeger-Simons character

Before we define the Cheeger-Simons characters, we recall the definition of singular homology and cohomology. We give more details in degree 0, 1 and 2 since these are the only cases we need in the sequel.

Singular homology

Let M be a manifold. For any non negative integer k, we denote by Δ_k the standard k-dimensional simplex. So

$$\Delta_0 = \{0\}, \quad \Delta_1 = [0,1] \quad \text{and} \quad \Delta_2 = \{(s,t) \in \mathbb{R}^2 / \ s \geqslant 0, \ t \geqslant 0, \ s+t \leqslant 1\}.$$

Let $\Delta_k(M)$ be the space of smooth maps from Δ_k to M. In particular $\Delta_0(M) = M$ and $\Delta_1(M)$ is the space of paths of M. The group of *k-chains* $C_k(M)$ of M is defined as the free $\mathbb{Z}$-module with basis $\Delta_k(M)$. For $k > 0$, the boundary map

$$\partial_k : C_k(M) \to C_{k-1}(M)$$

is the morphism given by $\partial_k c = \sum_{i=0}^{k} (-1)^i c \circ f_{i,k}$ where $f_{i,k} : \Delta_{k-1} \to \Delta_k$ is the i-th face of Δ_k. In particular,

$$\partial_1 \gamma = \gamma(1) - \gamma(0), \qquad \partial_2 S = S \circ e_1 - S \circ e_2 + S \circ e_3$$

for any path $\gamma : [0,1] \to M$ and smooth map $S : \Delta_2 \to M$. Here $e_k : [0,1] \to \Delta_2$, for $k = 1, 2, 3$ is given by

$$e_1(s) = (s, 0), \quad e_2(s) = (s, 1-s) \quad \text{and} \quad e_3(s) = (0, 1-s).$$

For $k = 0$, we define ∂_0 as the morphism $C_0(M) \to \{0\}$. The group of *k-cycles* $Z_k(M)$ is the kernel of ∂_k, the group of *k-boundaries* $B_k(M)$ is the image of ∂_{k+1}. Since $\partial_k \circ \partial_{k+1} = 0$, $B_k(M) \subset Z_k(M)$. The k-th *homology* group of M is $H_k(M) = Z_k(M)/B_k(M)$.

Singular cohomology

For any abelian group G, let $C^k(M,G) = \mathrm{Mor}(C_k(M), G)$ be the group of G-valued cochain. The differential

$$d_k : C^k(M,G) \to C^{k+1}(M,G)$$

is defined by $d_k \alpha = \alpha \circ \partial_k$. The k-th *cohomology* group $H^k(M,G)$ is by definition $\ker d_k / \operatorname{Im} d_{k-1}$. For $G = \mathbb{Z}$, we denote $C^k(M,G)$ and $H^k(M,G)$ by $C^k(M)$ and $H^k(M)$.

De Rham Theorem

When $G = \mathbb{R}$, $C^k(M,G)$ and $H^k(M,G)$ are real vector spaces. We have a one-to-one linear map $\Omega^k(M,\mathbb{R}) \to C^k(M,\mathbb{R})$ sending ω into the cochain $c \to \int_c \omega$. By Stokes Theorem, this morphism commutes with the differential, so that it induces a morphism from $H^k_{dR}(M,\mathbb{R})$ to $H^k(M,\mathbb{R})$.

Theorem 2.1.1. *The morphism $H^k_{dR}(M,\mathbb{R}) \to H^k(M,\mathbb{R})$ is an isomorphism.*

In the sequel we identify these cohomology spaces. We also view $\Omega^k(M,\mathbb{R})$ as a subspace of $C^k(M,\mathbb{R})$.

Cheeger-Simons character

For any $\alpha \in \Omega^k(M,\mathbb{R})$, we denote by $\tilde{\alpha} \in C^k(M,\mathbb{T})$ the cochain given by

$$\tilde{\alpha}(c) = \int_c \alpha \mod \mathbb{Z}, \qquad \forall\, c \in C_k(M).$$

Observe that the map sending α to $\tilde{\alpha}$ is injective. Furthermore $d\tilde{\alpha} = \widetilde{d\alpha}$, by Stokes Theorem.

A *Cheeger-Simons character* of M with degree k is by definition a group morphism $A : Z_k(M) \to \mathbb{T}$ such that there exists $\omega \in \Omega^{k+1}(M)$ satisfying

$$A(\partial c) = \int_c \omega \mod \mathbb{Z}$$

for any smooth map $c : \Delta_{k+1} \to M$. Since $C_{k+1}(M)$ is generated by $\Delta_{k+1}(M)$, this last equation is equivalent to $A \circ \partial = \tilde{\omega}$. So ω is determined by A. Since $\widetilde{d\omega} = d\tilde{\omega} = A \circ \partial^2 = 0$, ω is closed. We call ω the *differential* of A. zz We denote by $\hat{H}^k(M,\mathbb{T})$ the group of degree k Cheeger-Simons characters. For $k = 0$, $\hat{H}^0(M,\mathbb{T})$ identifies with the space of smooth maps from M to $\mathbb{T}$.

2.2 Holonomy

Let $\pi : P \to M$ be a $\mathbb{T}$-principal bundle and $\alpha \in \Omega^1(P)$ be a connection. Introduce a smooth path $\gamma : [0,1] \to M$. An *horizontal lift* of γ is a smooth path $\tilde{\gamma} : [0,1] \to P$ such that

$$\pi(\tilde{\gamma}(t)) = \gamma(t) \quad \text{and} \quad \alpha(\tilde{\gamma}'(t)) = 0, \qquad \forall t \in [0,1].$$

Lemme 2.2.1. *For any $v \in P_{\gamma(0)}$, there exists a unique horizontal lift $\tilde{\gamma}$ of γ such that $\tilde{\gamma}(0) = v$. Furthermore, for any $\theta \in \mathbb{T}$, the horizontal lift of γ starting at $\theta.v$ is the path $t \to \theta.\tilde{\gamma}(t)$.*

Proof. We can work in a local trivialisation $U \times \mathbb{T}$ with connection form $\beta + d\theta$. Then a lift $\tilde{\gamma}(t) = (\gamma(t), \tau(t))$ is horizontal if it satisfies $\beta(\gamma'(t)) + \tau'(t) = 0$. Equivalently, we have

$$\tau(t) = \tau(0) - \int_0^t \beta(\gamma'(s))ds \mod \mathbb{Z} \tag{2.1}$$

which proves the result. □

The *parallel transport* along γ is the map $T_\gamma : P_{\gamma(0)} \to P_{\gamma(1)}$ such that for any horizontal lift $\tilde{\gamma}$ of γ, we have $T_\gamma(\tilde{\gamma}(0)) = \tilde{\gamma}(1)$. Observe that T_γ is equivariant and does not depend on the parametrisation of the path. Furthermore, if γ is a constant loop, T_γ is the identity of the corresponding fiber. If $\gamma_1 * \gamma_2$ is the concatenation of γ_1

and γ_2, then $T_{\gamma_1 * \gamma_2} = T_{\gamma_1} \circ T_{\gamma_2}$. If γ^{op} is the opposite of γ, that is $\gamma^{op}(t) = \gamma(1-t)$, then $T_{\gamma^{op}}$ is the inverse of T_γ.

If γ is a loop, that is $\gamma(0) = \gamma(1)$, then the *holonomy* of γ is defined as the angle $\theta \in \mathbb{T}$ such that T_γ is the translation by θ. The holonomy does not depend on the choice of the base point.

Let us define the holonomy of the 1-cycles of M. Choose a map $\varphi : P \to \mathbb{T}$ such that $\varphi(\theta.y) = \theta + \varphi(y)$ for any $\theta \in \mathbb{T}$ and $y \in P$. We do not require that φ is smooth, actually in the case P is not isomorphic to the trivial bundle, such a map can not be continuous. For any 1-chain $c = \sum n_i \gamma_i$, we set

$$\operatorname{hol}(c) = \sum n_i (\varphi(\tilde{\gamma}_i(1)) - \varphi(\tilde{\gamma}_i(0)))$$

where the $\tilde{\gamma}_i$ are horizontal lifts of the γ_i. Then $\operatorname{hol}(c)$ does not depend on the choice of the lifts. If furthermore c is a cycle, $\operatorname{hol}(c)$ does not depend on the choice of φ.

Proposition 2.2.2. *The holonomy map* $\operatorname{hol} : Z_1(M) \to \mathbb{T}$ *is a group morphism. Furthermore, for any smooth map* $S : \Delta_2 \to M$*, we have*

$$\operatorname{hol}(\partial S) = \int_S \omega \mod \mathbb{Z}$$

where ω *is the curvature of* α.

Proof. The holonomy map is clearly a group morphism. If the image of S is contained in the domain of a trivialisation of P, the formula for the holonomy of ∂S follows from Stokes theorem and Equation (2.1). We can deduce the general case by introducing a sufficiently fine subdivision of Δ_2. □

The holonomy of a Hermitian line bundle with connection is defined as the holonomy of the corresponding $\mathbb{T}$-principal bundle. If hol and hol$'$ are the holonomies of $L \to M$ and $L' \to M$, then one easily check that the holonomy of $L \otimes L'$ is the sum $\operatorname{hol} + \operatorname{hol}'$ and that the holonomy of L^{-1} is $-\operatorname{hol}$. If furthermore f is a map from a manifold N to M, the holonomy of f^*L is $\operatorname{hol} \circ f_*$, where f_* is the map $Z_1(N) \to Z_1(M)$ sending c into $c \circ f$.

2.3 A characterization

By proposition 2.2.2, for any $\mathbb{T}$-principal bundle with connection (P, α), the associated holonomy map is a Cheeger-Simons character. We say that two $\mathbb{T}$-principal bundles with connection (P, α) and (P', α') are isomorphic if there exists a $\mathbb{T}$-principal bundle isomorphism $\varphi : P \to P'$ such that $\varphi^* \alpha' = \alpha$.

Theorem 2.3.1. *The application which sends a* $\mathbb{T}$*-principal bundle with connection to its holonomy map induces a bijection from the set of isomorphism classes of* $\mathbb{T}$*-principal bundles with connection, to* $\hat{H}_1(M, \mathbb{T})$.

Recall that the flat connections are characterized by the condition that the holonomies of the boundaries vanish. Since $H_1(M) = Z_1(M)/B_1(M)$, we obtain the

Corollary 2.3.2. *The map which sends a flat* $\mathbb{T}$*-principal bundle to its holonomy map induces a bijection from the set of isomorphism classes of flat* $\mathbb{T}$*-principal bundles to* $\operatorname{Mor}(H_1(M), T)$.

Proof of theorem 2.3.1, injectivity. We may assume that M is connected. Consider two $\mathbb{T}$-principal bundles endowed with a connection (P, α) and (P', α'). Assume that the holonomies of α and α' are the same and let us define an isomorphism $\varphi : P \to P'$ of $\mathbb{T}$-principal bundle. First choose $x_0 \in M$ and define $\varphi : P_{x_0} \to P'_{x_0}$ as any $\mathbb{T}$-equivariant map. Then for any $x_1 \in M$ choose a path γ from x_0 to x_1 and define $\varphi : P_{x_1} \to P'_{x_1}$ so that

$$T'_\gamma \circ \varphi = \varphi \circ T_\gamma : P_{x_0} \to P'_{x_1}$$

where $T_\gamma : P_{x_0} \to P_{x_1}$ and $T'_\gamma : P'_{x_0} \to P'_{x_1}$ denote the parallel transport along γ in P and P' respectively. Because the holonomies of α and α' are the same, this definition does not depend on the choice of γ. Observe also that φ commute with parallel transport along any path of M.

We have to prove that φ is smooth and $\varphi^* \alpha' = \alpha$. One way to do that is to choose local trivialisations of P and P' such that the push-forward of α and α' are the same. This is possible by Lemma 1.1.5 because α and α' have the same curvature. Then using that φ commute with parallel transport, one checks that φ in these trivialisations has the following form $(x, \theta) \to (x, \theta + \tau(x))$, where τ is locally constant. □

We will need the following elementary lemma.

Lemme 2.3.3. *For any abelian group G, any group morphism $\chi : Z_1(M) \to G$ can be extended to a group morphism $C_1(M) \to G$.*

Proof. We may assume that M is connected. Let $x_0 \in M$ and let $(\gamma_x : [0, 1] \to M,\ x \in M)$ be a family of paths with endpoints $\gamma_x(0) = x_0$, $\gamma_x(1) = x$. Let f be the group morphism $C_0(M) \to C_1(M)$ such that $f(x) = \gamma_x$. Let $g : C_1(M) \to C_1(M)$ be given by $g = \mathrm{id} - f \circ \partial$, that is for any path σ

$$g(\sigma) = \sigma - \gamma_{\sigma(1)} + \gamma_{\sigma(0)}.$$

Clearly $\partial g(\sigma) = 0$ so that $\partial \circ g = 0$, that is $\operatorname{Im} g \subset Z_1(M)$. Furthermore, for any 1-cycle c, $g(c) = c$. So we can extend χ to $\chi \circ g$. □

Proof of theorem 2.3.1, surjectivity. For any character χ, we have to construct a $\mathbb{T}$-principal bundle $\pi : P \to M$ with a connection α whose holonomy is χ. We will consider $P = M \times \mathbb{T}$ endowed with a topology and differential structure $\mathcal{D}$ which may be different of the product manifold. These structures will be given by a choice of convenient trivialisations.

Lemme 2.3.4. *Let $(U_i)_{i \in I}$ be an open cover of M. Let $(f_i : U_i \to \mathbb{T})$ be a family of maps (not necessarily continuous). Assume that for any $i, j \in I$, the map $f_i - f_j : U_i \cap U_j \to \mathbb{T}$ is smooth. Then $M \times \mathbb{T}$ has a unique Hausdorff topology and differential structure $\mathcal{D}$ such that for any $i \in I$,*

- *$U_i \times \mathbb{T}$ is an open set of $(M \times \mathbb{T}, \mathcal{D})$,*
- *the map $\varphi_i : (U_i \times \mathbb{T}, \mathcal{D}) \to U_i \times \mathbb{T}$ sending (x, θ) into $(x, \theta + f_i(x))$ is a diffeomorphism. Here the image $U_i \times \mathbb{T}$ of φ_i is the product manifold.*

Proof. We define the open sets of $\mathcal{D}$ as the subsets V of $M \times \mathbb{T}$ such that for any i, $\varphi_i(V \cap (U_i \times \mathbb{T}))$ is an open set of $U_i \times \mathbb{T}$. The proof that these are the open sets of an Hausdorf topology is left to the reader. Then we endow $M \times \mathbb{T}$ with the maximal atlas containing the φ_i's. This is possible because the maps $\varphi_i \circ \varphi_j^{-1}$ are smooth by assumption. □

Observe that a map $f : (M \times \mathbb{T}, \mathcal{D}) \to \mathbb{R}$ is smooth if and only if for any i, $f \circ \varphi_i^{-1}$ is smooth. Furthermore $P = (M \times \mathbb{T}, \mathcal{D})$ is a $\mathbb{T}$-principal bundle where the projection is the projection $M \times \mathbb{T} \to M$ onto the first factor and the action is given by multiplication in the second factor.

Remark 2.3.5. If the maps $f_i - f_j : U_i \cap U_j \to \mathbb{T}$ are locally constant, then P has a natural flat structure. If more generally, there exist a family of 1-forms $\beta_i \in \Omega^1(U_i)$, $i \in I$ such that $\beta_i - \beta_j + d(f_i - f_j) = 0$ for any i, j, then P inherits a connection α determined by $\varphi_i^*(\beta_i + d\theta) = \alpha$ for any i.

Let χ be a Cheeger-Simons character. By Lemma 2.3.3, we can extend χ to a morphism $C_1(M) \to G$. Assume first that χ vanishes on $B_1(M)$. We will apply Lemma 2.3.4 to the familly (U, f), where U is any open set of M and $f \in \Gamma_U$ defined by

$$\Gamma_U = \{f \in C^0(U, \mathbb{T});\ df + \chi = 0\}.$$

Here $C^0(U, \mathbb{T})$ is not the space of continuous map from U to $\mathbb{T}$, but the space of 0-cochains of U which are $\mathbb{T}$-valued. Since $C_0(M)$ is free with basis $\Delta_0(M) = M$, $C^0(U, \mathbb{T})$ naturally identifies with the space of maps from U to $\mathbb{T}$.

We have to check the hypothesis of Lemma 2.3.4.

Lemme 2.3.6. *If $U \subset V$ and $f \in \Gamma_V$, the restriction of f to U belongs to Γ_U. For any $f, f' \in \Gamma_U$, $f - f'$ is locally constant. Furthermore, if U is contractible, then $\Gamma_U \neq \emptyset$.*

Proof. The first assertion is clear. The second one follows from $d(f - f') = 0$. Let us prove the third one. Let $\varphi : [0,1] \times M \to M$ be a contraction, i.e. $\varphi(0, \cdot) = x_0$ and $\varphi(1, \cdot) = \mathrm{id}$. Let us define $f \in C^0(U, T)$ by $f(x) = \chi(\gamma_x)$ where γ_x is the path

$$\gamma_x(t) = \varphi(t, x).$$

We have to show that $df = \chi$, that is for any path σ, $\chi(\sigma) = f(\sigma(1)) - f(\sigma(0))$. To do that, observe that since χ vanishes on $B_1(M)$, χ vanishes on the constant paths, and consequently $\chi(\gamma^-) = -\chi(\gamma)$ for any path γ. This implies that $\chi(\partial S) = 0$ for any $S : [0,1]^2 \to M$, because we can divide the square into two triangles. Since the boundary of $[0,1]^2 \to M$, $(s,t) = \varphi(s, \sigma(t))$ is $\gamma_{\sigma(0)} + \sigma - \gamma_{\sigma(1)} - x_0$, we have

$$\chi(\sigma) = \chi(\gamma_{\sigma(1)}) - \chi(\gamma_{\sigma(0)}) = f(\sigma(1)) - f(\sigma(0))$$

as was to be proved. □

So we obtain a flat $\mathbb{T}$-principal bundle P. Let us prove that its holonomy map is χ. Let $h : C_1(M) \to \mathbb{T}$ be the group morphism defined on the paths by

$$h(\gamma) = \varphi(\tilde{\gamma}(1)) - \varphi(\tilde{\gamma}(0)) \tag{2.2}$$

where $\tilde{\gamma}$ is any horizontal lift of γ and φ is the projection from $M \times \mathbb{T}$ onto the second factor. The restriction of h to $Z_1(M)$ is the holonomy.

Lemme 2.3.7 (flat case)**.** *We have $h = \chi$.*

Proof. If the image of γ is contained of U and $f \in \Gamma_U$, then $\tilde{\gamma}(t) = (\gamma(t), f(\gamma(t)))$ is an horizontal lift of γ. So

$$h(\gamma) = f(\gamma(1)) - f(\gamma(0)) = \chi(\gamma).$$

If a path γ is the concatenation of two path γ_1 and γ_2, we have $h(\gamma) = h(\gamma_1) + h(\gamma_2)$ and also $\chi(\gamma) = \chi(\gamma_1) + \chi(\gamma_2)$ because χ vanishes on the boundaries. Now, any path can be subdivided into paths, each of them being contained in a contractible open set. □

This concludes the proof for flat bundle. Observe that Γ_U is actually the space of flat sections of P defined on U. Let us consider now the general case, so that χ does not necessarily vanish on $B_1(M)$. For any open set U of M, we set

$$\Gamma_U = \{(f, \beta);\ f \in C^0(U, \mathbb{T}),\ \beta \in \Omega^1(U, \mathbb{R}) \text{ such that } \chi + df + \tilde{\beta} = 0\}$$

Lemme 2.3.8. *If $U \subset V$ and $(f, \beta) \in \Gamma_V$, then $(f|_U, \beta|_U)$ belongs to Γ_U. If (f, β) and (f', β') belong to Γ_U, then $f - f'$ is smooth with differential $\beta - \beta'$. Furthermore if U is contractible, Γ_U is not empty.*

Proof. Let $\omega \in \Omega^2(M, \mathbb{R})$ be the differential of the character χ, that is $d\chi = \tilde{\omega}$. If $(f, \beta) \in \Gamma_U$, then $d\tilde{\beta} + \tilde{\omega} = 0$ which implies that $d\beta + \omega = 0$. Consequently, if (f', β') also belongs to Γ_U, then $\beta - \beta'$ is closed. So on a neighborhood of each point of U, we can write $\beta - \beta' = dg$ where g is smooth. We obtain $df - df' = dg$, so that $f - f' \equiv \tilde{g}$ up to some locally constant function, which shows that $f - f'$ is smooth.

Assume now that U is contractible. By Poincaré Lemma there exists $\beta \in \Omega^1(M, \mathbb{R})$ such that $d\beta + \omega = 0$. So $d(\chi + \tilde{\beta}) = 0$. By Lemma 2.3.6, there exists $f \in C^0(U, \mathbb{T})$ such that $\chi + \tilde{\beta} + df = 0$. Then $(f, \beta) \in \Gamma_U$. □

By Lemma 2.3.4 and Remark 2.3.5, we obtain a $\mathbb{T}$-principal bundle endowed with a connection α. As in the flat case, consider the morphism $h : C_1(M) \to \mathbb{T}$ defined by (2.2) whose restriction to $Z_1(M)$ is the holonomy map.

Lemme 2.3.9 (general case)**.** *We have $h = \chi$.*

Proof. If the image of γ is contained of U and $(f, \beta) \in \Gamma_U$, then by Equation (2.1) a parallel lift of γ is $(\gamma(t), \theta(t))$ where

$$\theta(t) + f(\gamma(t)) = -\int_0^t \beta(\gamma'(s))ds \mod \mathbb{Z}.$$

So $h(\gamma) + df(\gamma) = -\tilde{\beta}(\gamma)$, hence $h(\gamma) = \chi(\gamma)$. We conclude as in the the flat case, using that if γ is the concatenation of γ_1 and γ_2, then $h(\gamma) = h(\gamma_1) + h(\gamma_2)$ and $\chi(\gamma) = \chi(\gamma_1) + \chi(\gamma_2)$. To prove this last equality, we can not use that χ vanishes on the boundaries. Instead, observe that $d\chi = \tilde{\omega}$ implies that $\chi(\partial S) = 0$ if S is contained in a one dimensional manifold. □

This concludes the proof of Theorem 2.3.1. As in the flat case, observe that Γ_U consists actually in the pairs (f, β) where f is a smooth section $U \to P$ and $\beta = f^*\alpha$. □

2.4 Chern classes

As a corollary of Theorem 2.3.1, we can determine which closed 2-forms of a given manifold are the curvature of the connection of a Hermitian line bundle. We denote by

$$j : H^k(M, \mathbb{Z}) \to H^k(M, \mathbb{R})$$

the map induced by the inclusion $C^k(M, \mathbb{Z}) \subset C^k(M, \mathbb{R})$.

Proposition 2.4.1. *Let ω be closed form in $\Omega^2(M, \mathbb{R})$. Then the following assertions are equivalent*

1. *the cohomology class of ω is in the image of $j : H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$*
2. *there exists a Cheegers-Simons character $\chi \in \hat{H}^1(M, \mathbb{T})$ with differential ω.*

3. *there exists a Hermitian line bundle with a connection of curvature* $\frac{2\pi}{i}\omega$.

Proof. Item 1 is equivalent to the existence of $f \in C^1(M,\mathbb{R})$ such that $\omega - df \in C^2(M,\mathbb{Z})$. If this is satisfied, then $\chi := \tilde{f}|_{Z_1(M)}$ verifies $\chi \circ \partial = \tilde{\omega}$. So 1 implies 2. Conversely, let $\chi \in \hat{H}^1(M,\mathbb{T})$. By Lemma 2.3.3, we can extend χ to $C_1(M)$. Since $C_1(M)$ is free, there exists $h \in C^1(M,\mathbb{R})$ such that $\tilde{h} = \chi$. Then $\omega - dh \in C^2(M,\mathbb{Z})$ which proves that 2 implies 1. By Theorem 2.3.1, 2 and 3 are equivalent. □

In the proof of Proposition 2.4.1, we showed that for any character $\chi \in \hat{H}^1(M,\mathbb{T})$, there exists $f \in C_1(M,\mathbb{R})$ lifting χ and $\mu \in C^2(M,\mathbb{Z})$, such that

$$\mu = \omega - df.$$

Let us denote by $\delta(\chi) \in H^2(M,\mathbb{Z})$ the cohomology class of μ.

Proposition 2.4.2. *The map* $\delta : \hat{H}^1(M,\mathbb{T}) \to H^2(M,\mathbb{Z})$ *is well-defined. Furthermore, for any Cheeger-Simons character* χ, $j(\delta(\chi)) = [\omega]$ *where* $\omega \in \Omega^2(M,\mathbb{R})$ *is the differential of* χ.

Proof. Let us check that $[\omega - df] \in H^2(M,\mathbb{Z})$ does not depend on the choice of f. Let $f' \in C^1(M,\mathbb{R})$ lifting χ so that $\omega - df' \in C^2(M,\mathbb{Z})$. The restriction of $f - f'$ to $Z_1(M)$ takes its values in $\mathbb{Z}$. By Lemma 2.3.3, there exists $g \in C^1(M,\mathbb{Z})$ such that $f - f' = g$ on $Z_1(M)$. So $d(f - f') = dg$ and $[\omega - df] = [\omega - df']$ in $H^2(M,\mathbb{Z})$. □

Proposition 2.4.3. *We have an exact sequence*

$$0 \to \Omega^1(M,\mathbb{R})/\Omega^1_{\mathbb{Z}}(M) \xrightarrow{i} \hat{H}^1(M,\mathbb{T}) \xrightarrow{\delta} H^2(M,\mathbb{Z}) \to 0$$

where $i(\alpha) = \tilde{\alpha}$ *and* $\Omega^1_{\mathbb{Z}}(M)$ *is the subspace of* $\Omega^1(M,\mathbb{R})$ *consisting of the forms with periods in* $\mathbb{Z}$.

The periods of a form $\alpha \in \Omega^k(M,\mathbb{R})$ are the reals $\int_c \alpha$, where c runs over the cycles of M.

Proof. Let us prove that δ is onto. Let $[c] \in H^2(M,\mathbb{Z})$. By de Rham Theorem, there exists $\omega \in \Omega^2(M,\mathbb{R})$ such that $[\omega] = [c]$ in $H^2(M,\mathbb{R})$. So $\omega - c = df$ with $f \in C^1(M,\mathbb{R})$. Define χ as the restriction of $\tilde{f}$ to $Z_1(M)$. It is a Cheeger-Simons character with differential ω and $\delta(\chi) = [\omega - df] = [c]$.

Assume now that $\delta(\chi) = 0$. So $\omega - df = dg$ with $g \in C^1(M,\mathbb{Z})$. Hence $[\omega] = 0$ in $H^2(M,\mathbb{R})$. By de Rham Theorem, there exists $\alpha \in \Omega^1(M,\mathbb{R})$ such that $\omega = d\alpha$. Then $f + g - \alpha \in C^1(M,\mathbb{R})$ and is closed. Applying again de Rham Theorem, there exists $\beta \in \Omega^1(M,\mathbb{R})$ such that $f + g - \alpha - \beta$ is exact. This implies that $f + g - \alpha - \beta = 0$ on $Z_1(M)$. So $\chi = \tilde{f} = \tilde{\alpha} + \tilde{\beta}$ on $Z_1(M)$. □

Corollary 2.4.4. *Let* (L,∇) *and* (L',∇) *be two Hermitian line bundles with connection over the same base. Denote by* hol *and* hol$'$ *the two holonomy maps. Then* $\delta(\mathrm{hol}) = \delta(\mathrm{hol}')$ *if and only if* L *and* L' *are isomorphic.*

Recall that two line bundles L and L' with the same base M are isomorphic if there exists a diffeomorphism $\varphi : L \to L'$ such that for any $x \in M$, φ_x restricts to an isomorphism from L_x to L'_x.

Proof. Assume that $\mathrm{hol}' = \mathrm{hol} + \tilde{\beta}$ for some $\beta \in \Omega^1(M,\mathbb{R})$. Since the holonomy of $\nabla + \frac{2\pi}{i}\beta$ is $\mathrm{hol} + \tilde{\beta}$, by theorem 2.3.1, (L',∇') is isomorphic to $(L, \nabla + \frac{2\pi}{i}\beta)$. In particular L and L' are isomorphic. Conversely assume that L and L' are isomorphic. Then there exists an isomorphism $\varphi : L \to L'$ such that $\varphi_x : L_x \to L_x$ is unitary for any x. So we can assume that $L = L'$, as Hermitian line bundles. By proposition 1.1.3, $\nabla' = \nabla + \frac{2\pi}{i}\beta$ for some $\beta \in \Omega^1(M,\mathbb{R})$ so that $\mathrm{hol}' = \mathrm{hol} + \tilde{\beta}$ □

This proves that the class $\delta(\text{hol})$ does not depend on the choice of the connection, nor on the choice of the Hermtitian metric. It depends only on the line bundle L.

Definition 2.4.5. *The Chern class $c(L)$ of a line bundle is the cohomology class $\delta(\text{hol}) \in H^2(M, \mathbb{Z})$ defined from any Hermitian metric and compatible connection of L.*

The map sending a Hermitian line bundle to its Chern class induces a bijection from the isomorphism classes of Hermitian line bundles to the elements of $H^2(M, \mathbb{Z})$. Addition in $H^2(M, \mathbb{Z})$ corresponds to tensor product.

Exercice 2.4.6. Show that a class of $H^2(M, Z)$ is the Chern class of a flat Hermitian line bundle iff it is in the kernel of $j : H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$.

2.5 Consequences and examples

To understand better what the cohomology groups $H^2(M, \mathbb{R})$ and $H^2(M, \mathbb{Z})$ are, let us recall a particular case of the universal coefficient theorem.

Theorem 2.5.1. *Let G be any abelian group, k be an integer and Φ be the map from $H^k(M, G)$ to $\operatorname{Mor}(H_k(M), G)$ sending $[\omega]$ to the morphism $[c] \to \omega(c)$. Then Φ is onto and its kernel is isomorphic to $\operatorname{Ext}(H_{k-1}(M), G)$. Furthermore, if $H_{k-1}(M)$ is finitely generated, then*

$$\operatorname{Ext}(H_{k-1}(G), G) \simeq \begin{cases} 0 \text{ if } G = \mathbb{R} \text{ or } \mathbb{T} \\ \operatorname{Tor}(H_{k-1}(M)) \text{ if } G = \mathbb{Z}. \end{cases}$$

Observe that the periods of a cohomology class $\Omega \in H^k(M, G)$ are the elements in the image of $\Phi(\Omega) : H_k(M) \to G$.

Assume that $H_1(M)$ is finitely generated. Then by Theorem 2.5.1, we have the following commuting diagram

$$\begin{array}{ccccccccc} & & & & H^2(M,\mathbb{R}) & \xrightarrow{\sim} & \operatorname{Mor}(H_2(M),\mathbb{R}) & & \\ & & & & {\scriptstyle j}\uparrow & & \uparrow & & \\ 0 & \longrightarrow & \operatorname{Tor}(H_1(M)) & \longrightarrow & H^2(M,\mathbb{Z}) & \longrightarrow & \operatorname{Mor}(H_2(M),\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

The top horizontal arrow is an isomorphism, the second line is exact and the second vertical arrow is into. This has the following consequence

Proposition 2.5.2. *Assume that $H_1(M)$ is finitely generated. Then*

- *A class of $H^2(M, \mathbb{R})$ is in the image of j iff its periods are integral.*
- *A class of $H^2(M, \mathbb{Z})$ is in the kernel of j iff its periods vanish iff it is a torsion element of $H^2(M, \mathbb{Z})$.*

The periods of $[\omega] \in H^k(M, \mathbb{R})$ are the reals $\omega(c)$, where $c \in Z_k(M)$. Observe that also under the assumption that $H_2(M)$ is finitely generated so that $H_2(M) \simeq \operatorname{Tor}(H_2(M)) \oplus \mathbb{Z}^r$, the group $\operatorname{Mor}(H_2(M), \mathbb{R})$ is a real n-dimensional vector space and $\operatorname{Mor}(H_2(M), \mathbb{Z})$ is a lattice of it.

In many examples, we know explicitely the group $H_2(M)$ and the previous conditions become very concrete. Recall that for any compact oriented submanifold N of M with dimension k, we define a homology class $[N] \in H_k(M)$ through a triangulation of N. It satisfies $\Phi([\omega])([N]) = \int_N \omega$ for any $\omega \in \Omega(M, \mathbb{R})$.

Example 2.5.3.
- Consider an oriented compact connected surface Σ so that $H_2(\Sigma) = \mathbb{Z}[\Sigma]$. Then a closed 2-form ω of Σ is the curvature of a connection of a Hermitian line bundle iff $\int_\Sigma \omega$ is an integer.
- As another example, let $M = \mathbb{CP}^n$ and let $j : \mathbb{CP}^1 \to \mathbb{CP}^n$ be the embedding sending $[z_0 : z_1]$ to $[z_0 : z_1 : 0 : \ldots : 0]$. Then $H_2(M) = \mathbb{Z}[j(\mathbb{CP}^1)]$. So a 2-form ω of $\mathbb{CP}^n$ is the curvature of a connection of a Hermitian line bundle iff $\int_{\mathbb{CP}^1} j^*\omega$ is an integer. Recall that in section 1.4, we endowed the tautological bundle $T \to \mathbb{CP}^n$ with a connection ∇ and computed that $\int_{\mathbb{CP}^1} j^* \operatorname{curv}(\nabla) = -1$. So the Chern class of T satisfies $c(T)[j(\mathbb{CP}^1)] = -1$. Since there is no torsion, we obtain $H^2(M, \mathbb{Z}) = \mathbb{Z}c(T)$.
- As a more involved example, consider a toric symplectic manifold (M, ω). So the torus $\mathbb{R}^n/\mathbb{Z}^n$ acts on M^{2n} with momentum $\mu : M \to \mathbb{R}^n$ and $\mu(M)$ is a a convex polytope satisfying Delzant's condition. Then the symplectic form ω is the curvature of a $\mathbb{T}$-principal bundle iff there exists $c \in \mathbb{R}^n$ such that $c + \mu(M)$ has integral vertices. One may prove this as follows. $H_2(M)$ is generated by the $[\mu^{-1}(e)]$ where e runs over the edges of of Δ. Furthermore for any edge, $\int_{\mu^{-1}(e)} \omega$ is the real r such that $e = rv$ with v a primitive vector of $\mathbb{Z}^n$. So a necessary and sufficient condition is that the $r(e)$'s are integer. This is easily seen to be equivalent to the above condition.

Recall that $\operatorname{Mor}(H_1(M), \mathbb{T})$ parametrizes the isomorphism classes of flat Hermitian line bundle. If $H_1(M)$ is finitely generated, so that $H_1(M) = \operatorname{Tor}(H_1(M)) \oplus \mathbb{Z}^r$, then we have an isomorphism (not canonical)

$$\operatorname{Mor}(H_1(M), \mathbb{T}) = \mathbb{T}^r \oplus \operatorname{Tor}(H_1(M)).$$

For instance, if Σ is a compact oriented connected surface, $\operatorname{Mor}(H^1(\Sigma), \mathbb{Z}) \simeq \mathbb{T}^{2g}$ where g is the genus. As a consequence, for any closed $\omega \in \Omega^2(\Sigma, \mathbb{R})$, a Hermitian line bundle with connection with curvature ω, if it exists, is unique up to $\mathbb{T}^{2g}$.

Example 2.5.4. Let us consider the case of the symplectic torus. Consider a 2-dimensional symplectic vector space (V, ω). Let β be the primitive of ω given by

$$\beta|_x(y) = \frac{1}{2}\omega(x, y).$$

More concretely, if p, q are linear Darboux coordinates of V so that $\omega = dp \wedge dq$, then $\beta = \frac{1}{2}(pdq - qdp)$. Endow the trivial $\mathbb{T}$-principal bundle $P = V \times \mathbb{T}$ with the connection $\alpha = -\beta + d\theta$. Let G be the group of automorphisms of (P, α) which lift the translations of V. So a diffeomorphism $\varphi : P \to P$ belongs to G if it is $\mathbb{T}$-equivariant, $\varphi^*\alpha = \alpha$ and there exists $u \in V$, such that for any $x \in V$, φ sends P_x into P_{x+u}.

Introduce the reduced Heisenberg group $V \times \mathbb{T}$ with product

$$(u, t).(v, s) = (u + v, t + s + \tfrac{1}{2}\omega(u, v)).$$

Lemme 2.5.5. *G is isomorphic to the reduced Heisenberg group $V \times \mathbb{T}$, through the map sending (u, t) to the automorphism*

$$\varphi_{(u,t)}(x, \theta) = (x + u, \theta + t + \tfrac{1}{2}\omega(u, x))$$

Proof. An automorphism φ lifting the translation by u is necessarily of the form $\varphi(x, \theta) = (x + u, \theta + \tau(x))$. Furthermore $\varphi^*\alpha = \alpha$ if and only if

$$-T_u^*\beta + d\theta + d\tau = -\beta + d\theta,$$

where $T_u(x) = x+u$. This is equivalent to $d_x\tau(y) = \frac{1}{2}\omega(u, y)$, that is $\tau(x) = \frac{1}{2}\omega(u, x)$ up to some constant. ☐

So G is an extension of V by $\mathbb{T}$, but the exact sequence

$$0 \to \mathbb{T} \to G \to V \to 0, \qquad t \to \varphi_{(0,t)}, \quad \varphi_{(u,t)} \to u,$$

does not split.

Consider a lattice Λ of V so that $M = V/\Lambda$ is a torus. The form ω being Λ-invariant it descends to a symplectic form ω_M of M. Observe that two elements (u,t) and (v,s) of the reduced Heisenberg group commute iff $\omega(u,v)$ is integer. Applying this to a basis (u,v) of Λ, we obtain that there exists a morphism $j : \Lambda \to G$ lifting the injection of Λ into V iff the volume $\int_M \omega_M$ is integer. Furthermore, if it is the case, the possible lifts are parametrized by $(t,s) \in \mathbb{T}^2$.

For any lift $j : \Lambda \to G$, we get an action of Λ on P. Then the quotient $P_M = P/\Lambda$ is a $\mathbb{T}$-principal bundle with base M, the projection sending $[x,\theta]$ into $[x]$ and the action of $\mathbb{T}$ being given by $t.[x,\theta] = [x, t.\theta]$. Since the projection p from P to P_M is a local diffeomorphism and α is Λ-invariant, there exists a unique $\alpha_M \in \Omega^1(P)$ such that $p^*\alpha_M = \alpha$. We claim that this form is a connection and its curvature is ω_M. Furthermore, for any $u \in \Lambda$, the holonomy of the cycle $\tau \in [0,1] \to [\tau u] \in M$ is the angle θ such that $j(u) = (u,\theta)$. So the various lifts correspond to the various $\mathbb{T}$-principal bundles with connection over M.